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Quaternionic Geometry and 3-Sasakian Manifolds

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Abstract. We give a survey of some recent results concerning geometric and topological properties of 3-Sasakian manifolds and the rôle played by such manifolds in the context of quaternionic Kähler and hypercomplex geometry.

Introduction

In 1960 Sasaki [Sas1] introduced a geometric structure related to an almost contact structure. This geometry became known as Sasakian geometry and has been studied extensively ever since. Today the importance of Sasakian manifolds is recognized in many different areas of Riemannian geometry. In the late sixties Kuo [Kuo] and Udrişte [Ud] refined this notion and introduced manifolds with Sasakian 3-structures. A 3-Sasakian manifold $(\mathcal{S},g)$ is a $(4n+3)$-dimensional Riemannian manifold with three orthonormal Killing vector fields $\{\xi^i\}_{i=1,2,3}$ satisfying the Lie algebra $\mathfrak{sp}(1)$ along with certain curvature conditions. This provides $\mathcal{S}$ with a 3-dimensional foliation $\mathcal{F}_3$. If the 3-Sasakian vector fields $\{\xi^i\}_{i=1,2,3}$ are complete, there is a locally free action of $Sp(1)$ on $\mathcal{S}$, and we say that $\mathcal{S}$ is *regular* if the foliation $\mathcal{F}_3$ is regular. From 1970-1975 this new kind of geometry was investigated almost exclusively by Japanese school. There are two main properties of 3-Sasakian manifolds that make them a particularly interesting object of study. First, they are Einstein spaces of positive scalar curvature [Ka]. Secondly, Konishi and Ishihara [IKon] noticed that if a 3-dimensional foliation $\mathcal{F}_3$ is regular then the space of leaves has the structure of a $4n$-dimensional quaternionic Kähler manifold M with positive scalar curvature. In particular, the restricted holonomy group of such M reduces to a subgroup of $Sp(n)\cdot Sp(1)$. Conversely, Konishi [Kon] proved the existence of a Sasakian 3-structure on a natural principal $SO(3)$-bundle over any quaternionic Kähler manifold of positive scalar curvature.

Early results of Ishihara and Konishi clearly demonstrate that there is at least one 3-Sasakian manifold associated to every compact quaternionic Kähler space of positive scalar curvature. Yet until recently, for unclear reasons, manifolds with 3-Sasakian structure were relegated to a relative obscurity. In the last few years they resurfaced independently in two different areas. In 1990 Friedrich and Kath studied compact Riemannian 7-manifolds admitting three Killing spinors showing that this condition is equivalent to the existence of a 3-Sasakian structure [FrKat]. Assuming regularity they were able to combine the result of Hitchin [Hi] and Friedrich and Kurke [FrKur] and obtain a classification of all regular complete 7-manifolds with 3-Sasakian structure.

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More recently, in [BGM1] we found that 3-Sasakian manifolds provided a natural piece of a puzzle that links together four different geometric structures. In particular, for any compact quaternionic Kähler manifold M of positive scalar curvature there is a commutative diagram

0.1
$$\begin{tikzcd}
 & \mathcal{U}\simeq \mathbb{R}_+\times\mathcal{S} \arrow[dl, "\mathbb{C}^*/\mathbb{Z}_2"'] \arrow[dd, "\mathbb{H}^*/\mathbb{Z}_2"] \arrow[dr, "\mathbb{R}^*/\mathbb{Z}_2"] & \\
\mathcal{Z} \arrow[dr, "\mathbb{C}P^1"'] & & \mathcal{S}, \arrow[dl, "\mathbb{R}P^3"] \\
 & M &
\end{tikzcd}$$

where $\mathcal{U}$ is the cone on $\mathcal{S}$ with a canonical hyperkähler structure (the Swann bundle [Sw]), $\mathcal{Z}$ is Kähler-Einstein (the twistor space [Sal1-2]), and $\mathcal{S}$ is 3-Sasakian (the Konishi bundle [Kon]). In addition, both $\mathcal{Z}$ and $\mathcal{S}$ are compact, of positive scalar curvature, and $\mathcal{S}$ is a principal circle bundle over $\mathcal{Z}$. It is important to observe that all four geometries in diagram 0.1 are Einstein.

In 1982 generalizing Penrose's twistor construction in 4 dimensions, Salamon showed that every quaternionic Kähler manifold of positive scalar curvature has an auxiliary complex manifold known as the "twistor space". In a sense, diagram 0.1 shows that every positive quaternionic Kähler manifold has not one auxiliary space but three different yet related auxiliary spaces. This is what originally motivated our interest in the geometry and topology of 3-Sasakian manifolds. This article surveys several new results presented earlier in [BGM1-7, GS] as well as many older results in the subject. After setting our notation and introducing necessary definition we first show how Ishihara and Konishi's results generalize to the case of orbifold fibration [BGM1,2]. In particular, we summarize many known facts about the geometry and topology of 3-Sasakian manifolds in section 1. In section 2 we discuss all known examples of compact regular 3-Sasakian manifolds and present the classification of compact 3-Sasakian homogeneous spaces. We briefly describe some relations between Betti numbers of spaces in diagram 0.1 [GS]. Section 3 introduces the 3-Sasakian quotient construction [BGM1,2] which can be applied to obtain a family of inhomogeneous, irregular 3-Sasakian manifolds. In particular, these are the only explicit examples of manifolds with Einstein metrics of positive scalar curvature and of arbitrary cohomogeneity. Section 4 describes a new construction of hypercomplex structures on certain circle bundles over 3-Sasakian manifolds. The method yields a very interesting family of inhomogeneous hypercomplex structures on complex Stiefel manifolds of 2-frames in $\mathbb{C}^n$. Finally, the last section describes how 3-Sasakian manifolds enter naturally into the study of locally conformally hyperkähler spaces.

Update: For further developments in the area, see [BG,BGM9-10,BGMR1-2].

1. Sasakian 3-Structure and Quaternionic Kähler Orbifolds

We begin this section with the definition of a Sasakian structure and some of it basic properties.

Definition 1.1: *Let $(\mathcal{S}, g)$ be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. Then $(\mathcal{S}, g)$ has a Sasakian structure if there exists a Killing vector field ξ of unit length on $\mathcal{S}$ so that the tensor field Φ of type $(1,1)$, defined by*

$$(i) \qquad \Phi(X) = \nabla_X \xi$$

satisfies the condition

$$(ii) \qquad (\nabla_X \Phi)(Y) = \eta(Y)X - g(X,Y)\xi$$

for any pair of vector fields X and Y on $\mathcal{S}$. Here η denotes the 1-form dual to ξ with respect to g; i.e., $g(Y,\xi) = \eta(Y)$ for any vector field Y, and satisfies the dual equation to (i); namely,

$$(iii) \qquad (\nabla_X \eta)(Y) = g(Y, \Phi X).$$

We write (Φ, ξ, η) to denote the specific Sasakian structure on $(\mathcal{S}, g)$ and will refer to $\mathcal{S}$ with such a structure as a Sasakian manifold.

Proposition 1.2: *Let $(\mathcal{S}, g, \xi)$ be a Sasakian manifold and X and Y any pair of vector fields on $\mathcal{S}$. Then*

$$(i) \qquad \Phi \circ \Phi(Y) = -Y + \eta(Y)\xi,$$

$$(ii) \qquad \Phi\xi = 0,$$

$$(iii) \qquad \eta(\Phi Y) = 0,$$

$$(iv) \qquad g(X, \Phi Y) + g(\Phi X, Y) = 0,$$

$$(v) \qquad g(\Phi Y, \Phi Z) = g(Y,Z) - \eta(Y)\eta(Z),$$

$$(vi) \qquad d\eta(Y,Z) = 2g(\Phi Y, Z).$$

Furthermore, the Nijenhuis torsion tensor

$$N_\Phi(Y,Z) = [\Phi Y, \Phi Z] + \Phi^2[Y,Z] - \Phi[Y, \Phi Z] - \Phi[\Phi Y, Z],$$

of Φ satisfies

$$(vii) \qquad N_\Phi(Y,Z) = -d\eta(Y,Z) \otimes \xi.$$

DEFINITION 1.3: *Let $(\mathcal{S}, g)$ be a Riemannian manifold that admits three distinct Sasakian structures $\{\Phi^a, \xi^a, \eta^a\}_{a=1,2,3}$ such that*

$$g(\xi^a, \xi^b) \;=\; \delta^{ab} \qquad \text{and} \qquad [\xi^a, \xi^b] \;=\; 2\epsilon^{abc}\xi^c$$

for $a, b, c = 1, 2, 3$. Then $(\mathcal{S}, g)$ is a 3-Sasakian manifold with Sasakian 3-structure $(\mathcal{S}, g, \xi^a)$.

It follows directly from the definition that every 3-Sasakian manifold admits a local action of either $Sp(1)$ or $SO(3)$ as local isometries and, if the vector fields ξ^a are complete, then these are global isometries. We refer to this action as the *standard* $Sp(1)$ action on $\mathcal{S}$. It is well-known that every 3-Sasakian manifold $(\mathcal{S}, g, \xi^a)$ has dimension $4n+3$ and is foliated by Riemannian foliations

$$\mathcal{F}_1^\tau \subset \mathcal{F}_3, \qquad \tau \in \mathfrak{sp}(1), \tag{1.4}$$

of codimensions $4n+2$ and $4n$ respectively. The foliation $\mathcal{F}_1^\tau$ is defined by a choice of the circle subgroup $U(1)_\tau \subset Sp(1)$ generated by the Killing vector $\tau \in \mathfrak{sp}(1)$, where $\mathfrak{sp}(1)$ the Lie algebra of Definition 1.2 with generators $\{\xi^1, \xi^2, \xi^3\}$. Notice that $\mathcal{F}_1^\tau$ depends only on the direction of τ and not on its norm, and gives $\mathcal{S}$ the structure of a Seifert fibration over the twistor space $\mathcal{Z}$.

DEFINITION 1.5: *A 3-Sasakian manifold is said to be regular if the foliation $\mathcal{F}_3$ is regular.*

Note that if the foliation $\mathcal{F}_3$ is regular then $\mathcal{F}_1^\tau$ is regular for all τ. But the converse is also true, more precisely if the vector fields ξ^i are complete and any of the foliations $\mathcal{F}_1^\tau$ is regular than so is $\mathcal{F}_3$ [Tan2, TachYu]. In the regular case we obtain a diagram of locally trivial fibrations introduced in (0.1). More generally the arrows in (0.1) should be interpreted in the sense of V-bundles (or orbifold bundles) [Sat, Bai1-2].

Some of the basic properties of 3-Sasakian manifolds together with there foliations $\mathcal{F}_1^\tau \subset \mathcal{F}_3$ are summarized in the theorem below.

THEOREM 1.6: *Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian manifold of dimension $4n+3$ such that the Killing vector fields ξ^a are complete for $a = 1, 2, 3$. Then*

(i) $(\mathcal{S}, g, \xi^a)$ is an Einstein manifold of positive scalar curvature equal to $2(2n+1)(4n+3)$. Hence, every complete 3-Sasakian manifold is compact and has finite fundamental group.

(ii) $\mathcal{S}$ admits a second Einstein metric g' of positive scalar curvature which is not homothetic to g.

(iii) The metric g is bundle-like with respect to all of the foliations $\mathcal{F}_1^\tau$ and $\mathcal{F}_3$.

(iv) Each leaf $\mathcal{L}$ of the foliation $\mathcal{F}_3$ is a 3-Sasakian submanifold that is totally geodesic and of constant curvature 1. Furthermore, $\mathcal{L}$ is diffeomorphic to a 3-dimensional homogeneous spherical space form.

(v) The space of leaves $\mathcal{O} = \mathcal{S}/\mathcal{F}_3$ is a quaternionic Kähler orbifold of dimension $4n$ with positive scalar curvature equal to $16n(n+2)$.

(vi) The space of leaves $\mathcal{Z} = \mathcal{S}/\mathcal{F}_1^\tau$ is a complex orbifold of complex dimension $2n+1$ that is independent of τ and has a Kähler-Einstein metric of positive scalar curvature $8(2n+1)(n+1)$. Furthermore, $\mathcal{Z}$ is the twistor space of $\mathcal{O}$, and is a projective algebraic variety.

(vii) The product manifold $M=\mathcal{S}\times\mathrm{R}^+$ *with the cone metric* $g_M=dr^2+r^2g$ *has holonomy* $Sp(n+1)$, *that is, the metric* g_M *is hyperkähler.*

The fact that every 3-Sasakian manifold $\mathcal{S}$ is Einstein and has positive scalar curvature $2(2n+1)(4n+3)$ is due to Kashiwada [Ka]. That $\mathcal{S}$ admits a second Einstein metric is a consequence of the canonical variation [Bes] and was given in [BGM2]. The notion of a bundle-like metric is due to Reinhart [Rei]. That each leaf of $\mathcal{F}_3$ is a totally geodesic 3-Sasakian manifold of constant curvature 1 is due to Kuo and Tachibana [KuTach], and that all 3-Sasakian 3-manifolds are precisely the homogeneous spherical space forms is due to Sasaki [Sas2]. The regular case of (v) is due to Ishihara and Konishi [IKon,I2], whereas the general case appears in [BGM2]. (vi) is given in [BGM1,BGM2] and the proof that $\mathcal{Z}$ is a projective algebraic variety is given in [BG]. (vii) is given independently in [Bär] and [BGM2].

We now recall some old results about harmonic forms on compact Sasakian manifolds due to Tachibana [Tach]. Consider $\{\mathcal{S},g,\xi\}$ such that $\dim\mathcal{S}=2m+1$, let $\Lambda^p(\mathcal{S})$ be the space of p-form on $\mathcal{S}$, and let $\mathcal{H}^p(\mathcal{S})$the vector space of harmonic p-forms. We have

THEOREM 1.7 [TACH]: *If* $p\leq m$ *then* $u\in\mathcal{H}^p(\mathcal{S})$ *is horizontal with respect to the distribution* $\mathcal{F}_1$ *defined by the Killing vector* ξ, *i.e.,* $\xi\lfloor u\equiv 0$.

Now, using Φ we can define an operator $\Phi:\Lambda^p(\mathcal{S})\to\Lambda^p(\mathcal{S})$ by

$$(\Phi u)(X_1,X_2,\ldots,X_p)=\sum_{i=1}^{p}u\Big(X_1,\ldots,\Phi(X_i),\ldots,X_p\Big). \tag{1.8}$$

Then we have

THEOREM 1.9 [TACH]: *If* $p\leq m$ *and* $u\in\mathcal{H}^p(\mathcal{S})$ *then* $\Phi u\in\mathcal{H}^p(\mathcal{S})$.

Applying these two results to the 3-Sasakian case in combination with the fact that harmonic forms must be invariant under the $Sp(1)$ action [YB] one obtains the following vanishing theorem:

THEOREM 1.10 [GS]: *Let* $(\mathcal{S},g,\xi^a)$ *be a compact 3-Sasakian manifold of dimension* $4n+3$. *Then the odd Betti numbers* $b_{2k+1}(\mathcal{S})=0$ *if* $k\leq n$.

2. Regular Case

As every compact regular 3-Sasakian manifold fibers over a compact positive quaternionic Kähler manifold the two geometries are intimately related to one another. In particular, we always have the diagram of fibration

2.1
$$\begin{tikzcd} \mathcal{S} \arrow[rr, "S^1"] \arrow[d, "\mathrm{RP}^3"] & & \mathcal{Z}. \arrow[dll, "\mathrm{CP}^1"] \\ M & & \end{tikzcd}$$

The RP^3-bundle over M can be lifted to an S^3-bundle only in one case, when $M=\mathrm{HP}^n$, which follows from the result of Salamon [Sal1]. In particular we have

Theorem 2.2 [BGM1]: *Let $(\mathcal{S}, g, \xi^a)$ be a regular compact 3-Sasakian manifold of dimension $4n+3$. Then $\pi_1(\mathcal{S})$ is of order at most 2 and $\pi_1(\mathcal{S}) = \mathrm{Z}_2$ if and only if $\mathcal{S} \simeq \mathrm{RP}^{4n+3}$.*

Applying standard Gysin sequence arguments to the fibration of diagram 2.1 we find that the Betti numbers of $\mathcal{S}$ can be regarded as "primitive Betti numbers" of both M and $\mathcal{Z}$. In particular

Theorem 2.3 [GS]: *Let $(\mathcal{S}, g, \xi^a)$ be a compact regular 3-Sasakian manifold of dimension $4n+3$ with quotients $M, \mathcal{Z}$ as in diagram 2.1. Then*

(i) $b_{2k}(\mathcal{S}) = b_{2k}(\mathcal{Z}) - b_{2k-2}(\mathcal{Z}), \qquad k \leq n.$

(ii) $b_{2k}(\mathcal{S}) = b_{2k}(M) - b_{2k-4}(M), \qquad k \leq n.$

We now consider homogeneous 3-Sasakian spaces. To begin we recall some old results of Tanno [Tan1]. First, notice that the Killing vector fields ξ^1, ξ^2, and ξ^3 which give a Riemannian manifold $(\mathcal{S}, g)$ a 3-Sasakian structure generate non-trivial isometries. Thus, every 3-Sasakian manifold $(\mathcal{S}, g, \xi^a)$ has a nontrivial isometry group $I(\mathcal{S}, g)$. Let $I_0(\mathcal{S}, g)$ denote the subgroup of $I(\mathcal{S}, g)$ consisting of those isometries that leave the tensor fields Φ^a invariant for all $a = 1, 2, 3$. We refer to elements of $I_0(\mathcal{S}, g)$ as 3-Sasakian isometries. The following theorem was proven by Tanno.

Theorem 2.4 [Tan1] *Let $(\mathcal{S}, g, \xi^a)$ be a complete 3-Sasakian manifold which is not of constant curvature. Then*

$$\mathit{dim}\, I(\mathcal{S}, g) \;=\; \mathit{dim}\, I_0(\mathcal{S}, g) + 3.$$

Furthermore, the Killing vector fields ξ^a generate the three dimensional subspace of isometries that are not 3-Sasakian isometries. Let $\mathfrak{i}$ and $\mathfrak{i}_0$ denote the Lie algebras of $I(\mathcal{S}, g)$ and $I_0(\mathcal{S}, g)$, respectively. Hence, Tanno's theorem says that if $(\mathcal{S}, g)$ is not of constant curvature, then

2.5
$$\mathfrak{i} = \mathfrak{i}_0 + \mathfrak{sp}(1),$$

where $+$ indicates vector space direct sum. However, more is true, namely

Lemma 2.6: *The direct sum in equation 2.5 is a direct sum of Lie algebras, i.e.,*

$$\mathfrak{i} = \mathfrak{i}_0 \oplus \mathfrak{sp}(1).$$

Using the well-know theorem of Alekseevski [A] which asserts that every compact positive homogeneous quaternionic Kähler manifold has to be symmetric, together with the classification of the quaternionic Kähler symmetric spaces obtained by Wolf [Wo], one can prove the following result [BGM2]:

Theorem 2.7: *Let $(\mathcal{S}, g, \xi^a)$ be a compact 3-Sasakian homogeneous space. Then $\mathcal{S}$ is precisely one of the following:*

$$\frac{Sp(n)}{Sp(n-1)} \simeq S^{4n-1}, \qquad \frac{Sp(n)}{Sp(n-1)\times \mathrm{Z}_2} \simeq \mathrm{RP}^{4n-1},$$

$$\frac{SU(m)}{S\big(U(m-2)\times U(1)\big)}, \qquad \frac{SO(k)}{SO(k-4)\times Sp(1)},$$

$$\frac{G_2}{Sp(1)}, \qquad \frac{F_4}{Sp(3)}, \qquad \frac{E_6}{SU(6)}, \qquad \frac{E_7}{\mathrm{Spin}(12)}, \qquad \frac{E_8}{E_7}.$$

Here $n \geq 1$, $Sp(0)$ denotes the trivial group, $m \geq 3$, and $k \geq 7$. Furthermore, the fiber F over the Wolf space is $Sp(1)$ in only one case which occurs precisely when $(\mathcal{S}, g, \xi^a)$ is simply connected with constant curvature; that is, when $\mathcal{S} = S^{4n-1}$. In all other cases $F = SO(3)$.

The proof is based on an observation that every 3-Sasakian homogeneous space must be regular. Now, Theorem 2.7 implies that any homogeneous 3-Sasakian manifold has to be either 3-Sasakian homogeneous or it is covered by a sphere which reduces the classification of all homogeneous spaces to considering all discrete quotients of S^{4n+3} which preserve the standard 3-Sasakian structure.

In dimension 4 there are only two compact quaternionic Kähler manifolds of positive scalar curvature: the round sphere S^4 and the complex projective plane CP^2 with the $SU(3)$-invariant Fubini-Study metric. This result is due to Hitchin [Hi] and Friedrich and Kurke [FrKur]. (In dimension 4 *quaternionic Kähler* is equivalent to *self-dual and Einstein*). In particular both manifolds are symmetric. The latter is also true in dimension 8, where there are only three models of compact positive quaternionic Kähler manifolds: HP^2, $\mathrm{Gr}_{2,4}(\mathrm{C})$, and $G_2/SO(4)$ [PoSal]. Using these two classification results we can classify all compact regular 3-Sasakian manifolds in dimension 7 and 11. We have

THEOREM 2.8 [BGM1]: *Let $\mathcal{S}$ be a compact regular 3-Sasakian manifold. Then,*

1. *if S has dimension 7, then S is either S^7, RP^7, or $SU(3)/U(1)$.*
2. *if S has dimension 11, then S is either S^{11}, RP^{11}, $SU(4)/S(U(2)\times U(1))$, or $G_2/SU(2)$.*

Part (i) of Theorem 2.8 was obtained earlier in [FrKat, BFGK]. It would be very interesting to extend this classification to the case when the foliation $\mathcal{F}_3$ is not regular. However, we shall see in the next section that such a classification is much more difficult. Both the topology and the geometry are less restricted in such cases and infinitely many homotopy types occur in any allowable dimension. Even though 3-Sasakian and quaternionic Kähler geometries are very much related the latter seems to be much more rigid. Many of the results of LeBrun and Salamon that hold for positive quaternionic Kähler manifolds [Le1, Le2, LeSal, Sal3] do not hold in the case of compact 3-Sasakian spaces [BGMR1]. In general, the strong rigidity results of [LeSal] can be translated into the 3-Sasakian language only in the regular case [GS].

3. 3-Sasakian Reduction and Inhomogeneous 3-Sasakian Manifolds

In [BGM2] we described a 3-Sasakian version of the reduction procedure analogous to the symplectic reduction of Marsden and Weinstein [MW], hyperkähler reduction of Hitchin et al. [HKLR], and quaternionic reduction of Galicki and Lawson [GL]. Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries; that is, isometries that commute with the vector fields ξ^a for $a = 1, 2, 3$. One can define a unique 3-Sasakian momentum mapping

$$\mu : \mathcal{S} \longrightarrow \mathfrak{g}^* \otimes \mathrm{R}^3,$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra of G, by setting

$$< \mu^a, \tau > = \frac{1}{2}\eta^a(X^\tau),$$

where $\tau \in \mathfrak{g}$, X^τ is the corresponding infinitesimal 3-Sasakian isometry, and $< \cdot, \cdot >$ denotes the natural pairing on $\mathfrak{g}\times\mathfrak{g}^*$. Using standard techniques for Riemannian submersions one has

THEOREM 3.1 [BGM2]: *If $\hat{\mathcal{S}} = \mu^{-1}(0)/G$ is a smooth manifold then it has a natural 3-Sasakian structure induced by the inclusion map $\iota : \mu^{-1}(0) \hookrightarrow \mathcal{S}$ and the projection map $\pi : \mu^{-1}(0) \rightarrow \hat{\mathcal{S}}$.*

REMARK: In discussions with D.V. Alekseevski we noticed that it follows from a result of Ishihara and Konishi [IKon] that, if G is one-dimensional then 0 is always a regular value of the moment map μ. Thus, if G is a circle subgroup of the group of 3-Sasakian isometries which acts freely on $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/G$ will be a manifold.

One can use Theorem 3.1 to construct examples of compact simply connected irregular 3-Sasakian manifolds. To begin take $\mathcal{S} = S^{4n-1}(1)$ to be the round unit sphere of dimension $4n-1$. This is well-known to carry a Sasakian 3-structure coming from the canonical embedding $S^{4n-1} \subset \mathrm{H}^n$. The group of 3-Sasakian isometries is $Sp(n) \subset SO(4n)$ and we take $G = U(1)_{\mathbf{p}}$ to be in the maximal torus T^n in $Sp(n)$ contained in $U(n) \subset Sp(n)$, acting with weights $\mathbf{p}$. If $\mathbf{u} = (u_1, \ldots, u_n) \in \mathrm{H}^n$ is the quaternionic coordinate system on the sphere then T^n can be represented as diagonal matrices and the momentum map can be written as

3.2
$$\mu(\mathbf{u}) = \sum_{\alpha=1}^{n} p_\alpha \overline{u}_\alpha i u_\alpha,$$

where $\overline{w}$ denotes the quaternionic conjugate of $w \in \mathrm{H}$ and "i" is the quaternionic unit defining the inclusion $U(n) \subset Sp(n)$. When the weights are chosen to be pairwise coprime then the quotient $\hat{\mathcal{S}} = \mu^{-1}(0)/S^1 \equiv \mathcal{S}(\mathbf{p})$ is a smooth manifold with a natural Sasakian 3-structure given by Theorem 3.1. For any such $\mathbf{p}$ we identify the level set of the momentum map

3.3
$$\mathcal{N}(\mathbf{p}) \ = \ \mu^{-1}(0) \in S^{4n-1}$$

with the complex Stiefel manifold $V^{\mathrm{C}}_{n,2} = \frac{U(n)}{U(n-2)}$, via a smooth map F, and the quotient manifold $\hat{\mathcal{S}} = \mathcal{S}(\mathbf{p})$ with the bi-quotient of $U(n)$ by $U(1) \times U(n-2) \subset U(n)^2 = U(n)_L \times U(n)_R$, where the action is given by the formula

3.4
$$\mathrm{W} \xrightarrow{(\tau,\mathrm{B})} \begin{pmatrix} \tau^{p_1} & & \\ & \ddots & \\ & & \tau^{p_n} \end{pmatrix} \mathrm{W} \begin{pmatrix} \mathrm{I}_2 & \mathrm{O} \\ \mathrm{O} & \mathrm{B} \end{pmatrix}.$$

Here $\mathrm{W} \in U(n)$ and $(\tau, \mathrm{B}) \in U(1) \times U(n-2)$.

In particular, the metric $\hat{g}(\mathbf{p})$ on $\mathcal{S}(\mathbf{p})$ is obtained explicitly and we have the following diagram

$$
\begin{tikzcd}
U(1)_{\mathbf{p}} \arrow[r, hook] & (V^{\mathbb{C}}_{n,2}, F^*g(\mathbf{p})) \arrow[d, "\pi'_{\mathbf{p}}"] \arrow[r, "F", "\simeq"'] & (\mathcal{N}(\mathbf{p}), g(\mathbf{p})) \arrow[dl, "\pi_{\mathbf{p}}"] \arrow[r, hook, "\iota_{\mathbf{p}}"] & (S^{4n-1}, g_{can}) \\
& (\mathcal{S}(\mathbf{p}), \hat{g}(\mathbf{p})) \arrow[d, "\pi_0"] & & \\
& \mathcal{O}(\mathbf{p}) \arrow[r, phantom, "="] & \mathcal{S}(\mathbf{p})/\mathcal{F}_3. &
\end{tikzcd}
\tag{3.5}
$$

Equivalently, $\mathcal{S}(\mathbf{p})$ is the quotient of the complex Stiefel manifold $V^{\mathbb{C}}_{n,2}$ of 2-frames in $\mathbb{C}^n$ by the specific free left circle action which depends on $\mathbf{p}$. Here are some of the results which describe the geometry and topology of these examples.

Theorem 3.6 [BGM2]: *Let $n \geq 3$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n_+$ be an n-tuple of non-decreasing, pairwise relatively prime, positive integers. Then $\mathcal{S}(\mathbf{p})$ is a compact, simply connected, $(4n-5)$-dimensional smooth manifold which admits an Einstein metric $\hat{g}(\mathbf{p})$ of positive scalar curvature and a compatible Sasakian 3-structure. In addition, $\mathcal{S}(\mathbf{p})$ admits a second Einstein metric of positive scalar curvature, $g'(\mathbf{p})$, which is non-homothetic to $\hat{g}(\mathbf{p})$. Furthermore, both $\big(\mathcal{S}(\mathbf{p}), \hat{g}(\mathbf{p})\big)$ and $\big(\mathcal{S}(\mathbf{p}), g'(\mathbf{p})\big)$ are inhomogeneous Einstein manifolds as long as $\mathbf{p} \neq (1, \ldots, 1)$.*

In fact, the computation of the connected component of the isometry group of the metric $\hat{g}(\mathbf{p})$ shows that, depending on $\mathbf{p}$, one can construct metrics of arbitrary cohomogeneity. More precisely we have

Theorem 3.7 [BGM3]: *Let I_0 be the group of 3-Sasakian isometries of $\big(\mathcal{S}(\mathbf{p}), \hat{g}(\mathbf{p})\big)$ and let k be the number of 1's in $\mathbf{p}$. Then the connected component of I_0 is $S(U(k) \times U(1)^{n-k})$, where we define $U(0) = \{e\}$. Thus, the connected component of the isometry group is the product $S(U(k) \times U(1)^{n-k}) \times SO(3)$ if the sums $p_i + p_j$ are even for all $1 \leq i, j \leq n$, and $S(U(k) \times U(1)^{n-k}) \times Sp(1)$ otherwise.*

In the case that $\mathbf{p}$ has no repeated 1's, the cohomogeneity can easily be determined, viz.

Corollary 3.8 [BGM3]: *If the number of 1's in $\mathbf{p}$ is 0 or 1 then the dimension of the principal orbit in $\mathcal{S}(\mathbf{p})$ equals $n+2$ and the cohomogeneity of $\hat{g}(\mathbf{p})$ is $3n-7$. In particular, the 7-dimensional $\mathcal{S}(\mathbf{p})$ family contains metrics of cohomogeneity 0,1, and 2.*

We are not aware of any other **explicit** examples of positive scalar curvature Einstein metric with cohomogeneity greater than 1.

Using techniques developed by Eschenburg [E] in the study of certain 7-dimensional bi-quotients of $SU(3)$ one can compute the integral cohomology ring of $\mathcal{S}(\mathbf{p})$.

Theorem 3.9 [BGM2]: *Let $n \geq 3$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n_+$ be an n-tuple of non-*

decreasing, pairwise relatively prime, positive integers. Then, as rings,

$$H^*(\mathcal{S}(\mathbf{p}), \mathrm{Z}) \;\cong\; \left(\frac{\mathrm{Z}[b_2]}{[b_2^n = 0]} \otimes E[f_{2n-1}]\right)/\mathcal{R}(\mathbf{p}).$$

Here the subscripts on b_2 and f_{2n-1} denote the cohomological dimension of each generator. Furthermore,

1. *The relations $\mathcal{R}(\mathbf{p})$ are generated by $\sigma_{n-1}(\mathbf{p})b_2^{n-1} = 0$ and $f_{2n-1}b_2^{n-1} = 0$.*
2. *$\sigma_{n-1}(\mathbf{p}) = \sum_{j=1}^{n} p_1 \cdots \hat{p_j} \cdots p_n$ is the $(n-1)^{st}$ elementary symmetric polynomial in the entries of $\mathbf{p}$.*

Notice that Theorem 3.9 shows that $H^{2n-2}(\mathcal{S}(\mathbf{p}); \mathrm{Z}) = \mathrm{Z}_{\sigma_{n-1}(\mathbf{p})}$ and hence has the following corollary.

COROLLARY 3.10 [BGM2]: *Both $(\mathcal{S}(\mathbf{p}), \hat{g}(\mathbf{p}))$ and $(\mathcal{S}(\mathbf{p}), g'(\mathbf{p}))$ give infinitely many non-homotopy equivalent simply-connected compact inhomogeneous Einstein manifolds of positive scalar curvature in dimension $4n-5$ for every $n \geq 3$.*

To illustrate just how far some of these examples are from homogeneous Einstein geometry we consider the 7-dimensional case $\mathcal{S}(p_1, p_2, p_3)$. The following theorem is a generalization of a result of Eschenburg [E].

THEOREM 3.11 [BGM2]: *Let $\mathcal{S}(\mathbf{p})$ be any 3-Sasakian space defined above, with $n = 3$. Then either $\mathcal{S}(\mathbf{p})$ is homotopy equivalent to a homogeneous Aloff-Walach space $M_{kl} = SU(3)/T_{kl}$, where $\gcd(k,l) = 1$ and T_{kl} acts as $\mathrm{diag}(\tau^k, \tau^l, \overline{\tau}^{k+l}) \subset SU(3)$, or $\mathcal{S}(\mathbf{p})$ is strongly inhomogeneous, i.e. it is not homotopy equivalent to any compact Riemannian homogeneous space.*

Since $H^4(M_{kl}, \mathrm{Z})$ is a cyclic group of order $k^2 + l^2 + kl$ and for k, l relatively prime is never equal to 2 mod(3) we have, among $\mathcal{S}(\mathbf{p})$, many families of strongly inhomogeneous examples. In particular

COROLLARY 3.12 [BGM2]: *$\mathcal{S}(c, c+1, c+2)$ is strongly inhomogeneous for every positive odd integer c. Similarly, $\mathcal{S}(d, d+3, d+5)$ is strongly inhomogeneous for every positive integer $d \equiv 2 \pmod 6$ such that d is not divisible by 5.*

Theorem 3.11 has an analogue for any $n > 3$ with M_{kl} being replaced by another family of $SU(n)$-homogeneous spaces [BGM3].

4. Hypercomplex Geometry of a Circle Bundle over 3-Sasakian Manifold

A hypercomplex structure on a smooth manifold M is a G-structure where $G = GL(n, \mathrm{H})$ that admits a necessarily unique torsion free connection, the Obata connection [Bon,Ob]. In particular, every such M has three complex structures I, J, and K which satisfy the relations of the algebra of imaginary quaternions and thus generate an entire two-sphere's worth of complex structures on M. Until recently, there were few known examples of compact, irreducible, hypercomplex manifolds in dimension 8 and higher. The first class of such examples are the hyperkähler twisted products of $K3$ surfaces constructed by Beauville [Bea]. Examples of hypercomplex manifolds that are not hyperkähler were very scarce, the simplest ones being the Hopf manifolds $S^{4n+3} \times S^1$ which are locally conformally hyperkähler. Recently the authors [BGM2] gave a class of new compact locally conformally hyperkähler manifolds by replacing S^{4n+3} with any 3-Sasakian manifold.

Similar examples involving the quaternionic Heisenberg group were found by Hernandez [Her]. None of these examples, however, are simply connected.

In contrast to the 4-dimensional case, where all compact hypercomplex manifolds are locally conformally hyperkähler [Boy], in higher dimensions this is no longer true. A class of hypercomplex manifolds that are not locally conformally hyperkähler was studied by physicists interested in supersymmetric σ-models. In this regard, Spindel et. al. [SSTP] classified compact Lie groups which admit hypercomplex structures. This is the generalization to the hypercomplex category of the classic work of Samelson [Sam] and Wang [W] on the classification of compact Lie groups [Sam] and homogeneous spaces [W] admitting complex structures, respectively. Using different methods, Joyce [Joy1] later recovered this [SSTP] classification and developed a theory of homogeneous hypercomplex manifolds which generalizes Wang's [W] result.

In this section we will describe a new construction of compact hypercomplex manifolds using 3-Sasakian geometry. We then use these techniques to obtain uncountably many distinct hypercomplex structures on certain Stiefel manifolds. However, the construction can be used to obtain many other new examples [BGM8]. Let $\pi : P \to \mathcal{S}$ be a circle bundle over $\mathcal{S}$ and let $\hat{g}$ be a Riemannian metric on P such that $\pi : (P, \hat{g}) \to (\mathcal{S}, g)$ is a Riemannian submersion. Let $\mathcal{V}_1$ denote the vertical subbundle of the tangent bundle TP to P. Let Ξ be a nowhere vanishing smooth section of $\mathcal{V}_1$ that generates the S^1 action on P. The almost contact 3-structure on $\mathcal{S}$ allows us to define an almost hypercomplex structure on P as follows. The metric $\hat{g}$ on P splits the tangent bundle TP as $TP \simeq \hat{\mathcal{H}} \oplus \mathcal{V}_1$ and π_* induces an isometry between the horizontal vector space $\hat{\mathcal{H}}_p$ at a point $p \in P$ and the tangent space $T_{\pi(p)}\mathcal{S}$. For any vector field X on $\mathcal{S}$, we denote by $\hat{X}$ its horizontal lift to P, that is, $\hat{X}$ is the unique *basic* vector field that is π-related to X. In particular, the three vector fields $\hat{\xi}^a$ generate a subbundle $\hat{\mathcal{V}}_3$ of $\hat{\mathcal{H}}$ that is isometric at every point to the bundle $\mathcal{V}_3$ on $\mathcal{S}$. Let $\tilde{\mathcal{H}}$ denote the orthogonal complement to $\hat{\mathcal{V}}_3$ in $\hat{\mathcal{H}}$, so that we have the further splitting $TP \simeq \hat{\mathcal{H}} \oplus \mathcal{V}_1 \simeq \tilde{\mathcal{H}} \oplus \hat{\mathcal{V}}_3 \oplus \mathcal{V}_1$. Since the Φ^a 's are sections of End $\mathcal{H} \oplus$ End $\mathcal{V}_3$ on $\mathcal{S}$ they lift to sections $\hat{\Phi}^a$ of End $\tilde{\mathcal{H}} \oplus$ End $\hat{\mathcal{V}}_3$ on P defined on basic vector fields by $\hat{\Phi}^a\hat{X} = \widehat{\Phi^a X}$ and extended to arbitrary sections of End $\tilde{\mathcal{H}} \oplus$ End $\hat{\mathcal{V}}_3$ by linearity. Hence, we can define endomorphisms $\mathcal{I}^a$ on TP by

4.1 $$\mathcal{I}^a X = -\hat{\Phi}^a X + \pi^*\eta^a(X)\Xi \quad \text{and} \quad \mathcal{I}^a\Xi = -\hat{\xi}^a,$$

where X is any horizontal vector field on P. One easily sees that this defines an almost hypercomplex structure on P which is, in general, not integrable. We shall denote by $H(\mathcal{S})$ this circle bundle with its almost hypercomplex structure.

Theorem 4.2 [BGM4]: *Let $H(\mathcal{S})$ be a circle bundle over a 3-Sasakian manifold $\mathcal{S}$, and let Ξ be a nowhere vanishing vertical vector field on $H(\mathcal{S})$ which generates the circle action. We call the pair $(H(\mathcal{S}), \Xi)$ a framed circle bundle on $\mathcal{S}$. The almost hypercomplex structure given above is integrable if and only if the horizontal subbundle $\hat{\mathcal{H}}$ defines a $\mathfrak{u}(1)$ connection with curvature 2-form ω and the following three conditions hold:*

(i) $\omega(\hat{\xi}^a, \hat{\xi}^b) = 0$ *for any* $a, b = 1, 2, 3$.

(ii) $\omega(\hat{\xi}^a, \tilde{X}) = 0$ *for all* $a = 1, 2, 3$ *and any section* $\tilde{X}$ *of* $\tilde{\mathcal{H}}$.

(iii) $\omega(\hat{\Phi}^a\tilde{X}, \hat{\Phi}^a\tilde{Y}) = \omega(\tilde{X}, \tilde{Y})$ *for all* $a = 1, 2, 3$ *and for any sections* $\tilde{X}, \tilde{Y}$ *of* $\tilde{\mathcal{H}}$.

We first apply this theorem to the homogeneous case. Recall that a complex manifold M is called a *homogeneous complex manifold* if there is a transitive action of a real Lie group on M that acts by biholomorphic diffeomorphisms. We now define a *homogeneous hypercomplex manifold* to be a manifold that admits a transitive action of a real Lie group that preserves the hypercomplex structure.

THEOREM 4.3: *Let $H(\mathcal{S})$ be a circle bundle over a 3-Sasakian homogeneous space with almost complex structure given by 4.1 and satisfying the conditions of Theorem 4.2. Then $H(\mathcal{S})$ is a compact homogeneous hypercomplex manifold. Moreover, $H(\mathcal{S})$ is precisely one of the following manifolds: $\mathcal{S}\times S^1$, $H_2(\mathrm{RP}^{4n-1})$, $\mathrm{V}^{\mathrm{C}}_{n,2}$ or $\mathrm{V}^{\mathrm{C}}_{n,2}/\mathrm{Z}_k$. Here $\mathcal{S}$ is one of the 3-Sasakian homogeneous spaces listed in 2.7, $H_2(\mathrm{RP}^{4n-1})$ is the non-trivial S^1 bundle over RP^{4n-1}, $\mathrm{V}^{\mathrm{C}}_{n,2}$ is the complex Stiefel manifold of complex 2 planes in n-space, and k is a positive integer. Moreover, each $H(\mathcal{S})$ admits a one-parameter family of inequivalent homogeneous hypercomplex structures.*

Now results of Borel and Remmert [BR], and Tits [T] say that any compact homogeneous complex manifold X is the total space of a bundle (a Tits bundle) with parallelizable fibers over a generalized flag manifold G/P, where G is a complex Lie group acting transitively and holomorphically on X, and P is a parabolic subgroup. Since all of the hypercomplex manifolds $H(\mathcal{S})$ described in Theorem 4.3 determine a unique homogeneous complex structure, we see that all such homogeneous complex manifolds $H(\mathcal{S})$ are Tits bundles over twistor spaces $\mathcal{Z}=G/P$ that are generalized flag manifolds, and with fibers that are elliptic curves. Moreover, it can be shown that $H(\mathcal{S})$ is a rational non-algebraic manifold of algebraic dimension $2n-1$.

We can now apply Theorem 4.2 to the framed circle bundles $\mathcal{N}(\mathbf{p})\rightarrow\mathcal{S}(\mathbf{p})$ considered in section 3. Note that $\mathcal{N}(\mathbf{p})$ is naturally defined for all $(\mathrm{R}^*)^n$ and the analysis in Theorem 4.2 directly generalizes. We get

THEOREM 4.4 [BGM4]: *Let $n>2$ and $\mathbf{p}=(p_1,\ldots,p_n)\in(\mathrm{R}^*)^n$ be an $n-$tuple of non-zero real numbers. For each such $\mathbf{p}$ there is a compact hypercomplex manifold $(\mathcal{N}(\mathbf{p}),\mathcal{I}^a(\mathbf{p}))$, where $\mathcal{N}(\mathbf{p})$ is diffeomorphic to $\mathrm{V}^{\mathrm{C}}_{n,2}$ the Stiefel manifold of 2-frames in C^n. Moreover, $\mathcal{N}(\mathbf{p})$ is not locally conformally hyperkähler.*

When the compatibility conditions of Theorem 4.2 are satisfied, there is a commutative diagram of foliations:

4.5
$$\begin{tikzcd} & H(\mathcal{S}) \arrow[dl, "S^1"'] \arrow[dr, "E"] \arrow[dd] & \\ \mathcal{S} \arrow[dr, "S^3/\Gamma"'] & & \mathcal{Z}, \arrow[dl, "\mathbb{CP}^1"] \\ & \mathcal{O} & \end{tikzcd}$$

where $\mathcal{O}$ is a quaternionic Kähler orbifold and $\mathcal{Z}$ is its twistor space. Now $\mathcal{Z}$ is actually a projective algebraic variety with a Kähler-Einstein metric of positive scalar curvature, and E is an elliptic curve. Furthermore, the leaves of the vertical foliation in diagram 4.5 are Hopf surfaces $S^1\times S^3/\Gamma$.

Permuting the coordinates of $\mathbf{p}$ or changing their signs in Theorem 4.4 yields the same hypercomplex structure. Thus, we assume that $\mathbf{p}$ is an element in the positive cone $C_n = \{\mathbf{p} \in \mathrm{R}^n \mid 0 < p_1 \leq p_2 \leq \cdots \leq p_n\}$. However, we prove

THEOREM 4.6 [BGM4]: *If $\mathbf{p}$ and $\mathbf{q}$ are both commensurable sequences in the positive cone C_n then the hypercomplex manifolds $\mathcal{N}(\mathbf{p})$ and $\mathcal{N}(\mathbf{q})$ are hypercomplex equivalent if and only if $\mathbf{p} = \mathbf{q}$. Here $\mathbf{p}$ is said to be commensurable if each of the ratios $\frac{p_i}{p_j}$ is a rational number. Furthermore, the manifold $\mathcal{N}(\mathbf{p})$ is hypercomplex homogeneous if and only if $\mathbf{p} = \lambda(1, \ldots, 1)$ for some $\lambda \in \mathrm{R}^*$.*

While the one parameter family of distinct $U(n)$-homogeneous hypercomplex structures on $\mathrm{V}^{\mathrm{C}}_{n,2}$ given in Theorem 4.6 was known [Joy1,Bat], the remaining inhomogeneous hypercomplex structures on $\mathrm{V}^{\mathrm{C}}_{n,2}$ are new. They are analogous to the inhomogeneous complex structures found by Griffiths [Gr] in the versal deformation space of homogeneous complex structures.

Theorem 4.6 is proved by a detailed investigation of the hypercomplex geometry of the $\mathcal{N}(\mathbf{p})$ manifolds. Assuming that $\mathbf{p}$ is commensurable one first shows that any hypercomplex equivalence $F : \mathcal{N}(\mathbf{p}) \to \mathcal{N}(\mathbf{q})$ would have to induce orbifold diffeomorphisms F_1, F_2, F_4 such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{N}(\mathbf{p}) & \xrightarrow{F} & \mathcal{N}(\mathbf{q}) \\
\downarrow & & \downarrow \\
\mathcal{S}(\mathbf{p}) & \xrightarrow{F_1} & \mathcal{S}(\mathbf{q}) \\
\downarrow & & \downarrow \\
\mathcal{Z}(\mathbf{p}) & \xrightarrow{F_2} & \mathcal{Z}(\mathbf{q}) \\
\downarrow & & \downarrow \\
\mathcal{O}(\mathbf{p}) & \xrightarrow{F_4} & \mathcal{O}(\mathbf{q}).
\end{array}$$

Careful analysis of the above orbifold foliations shows that this is possible if and only if $\mathbf{p} = \mathbf{q}$. One can also prove that the connected component of the group of hypercomplex automorphisms of $\mathcal{N}(\mathbf{p})$ depends only on the number of equalities among the components of $\mathbf{p}$. More precisely, we rewrite $\mathbf{p} = (p_1^{m_1}, \cdots, p_k^{m_k})$, where the component p_i occurs m_i times in $\mathbf{p}$. m_i is called the *multiplicity* of p_i and the positive integer $k = k(\mathbf{p})$ counts the number of distinct values taken by the p_i's. We have

THEOREM 4.7 [BGM4]: *For all $\mathbf{p} \in (\mathrm{R}^*)^n$ the Lie algebra $\mathfrak{h}(\mathbf{p})$ of infinitesimal hypercomplex automorphisms of $\mathcal{N}(\mathbf{p})$ is isomorphic to $\bigoplus_{i=1}^{k} \mathfrak{u}(m_i)$.*

Notice that Theorem 4.7 implies that the set of multiplicities in $\mathbf{p}$ is an invariant of the hypercomplex structure even when $\mathbf{p}$ is not a commensurable sequence and that if $\mathbf{p} \neq$

$\lambda(1,\ldots,1)$ then $\mathfrak{h}(\mathbf{p})$ is strictly smaller than the Lie algebra of infinitesimal hypercomplex automorphisms of the classical homogeneous structure on $\mathrm{V}^{\mathrm{C}}_{n,2}$. However, the equivalence problem in the general case is more subtle and we do not know if C_n is indeed a subspace of the moduli space of hypercomplex structures on $\mathrm{V}^{\mathrm{C}}_{n,2}$.

The proof of Theorem 4.7 entails the interplay between our hypercomplex Stiefel manifolds $\mathcal{N}(\mathbf{p})$ and equivalent models using Joyce's [Joy2] hypercomplex quotient procedure. This enables one to exhibit an invariance under scaling in $\mathbf{p}$ of the Lie algebra of the infinitesimal hypercomplex automorphisms $\mathfrak{h}(\mathbf{p})$ of $\mathcal{N}(\mathbf{p})$. This scale invariance in turn allows one to prove that any infinitesimal hypercomplex automorphism must also be an infinitesimal isometry with respect to a suitable metric. Then $\mathfrak{h}(\mathbf{p})$ can be calculated explicitly using standard results about isometric embeddings in R^l.

We now explain how these hypercomplex structures on $\mathrm{V}^{\mathrm{C}}_{n,2}$ are related to some non-simply connected examples. To do so we first need to establish the following notation:

Definition 4.8 [BGM4]: *A commensurable sequence $\mathbf{p}\in C_n$ is called basic if all the coordinates are integers and the greatest common divisor of all the coordinates is one. A basic sequence is said to be coprime if the coordinates are pairwise relatively prime. If $\mathbf{p}$ is an integer multiple of a basic sequence and if the triples (p_i,p_j,k) have no common factor for all $1\le i<j\le n$ then $\mathbf{p}$ is called k-coprime.*

In [BGM2] we showed that for all coprime sequences $\mathbf{p}$ there is a 3-Sasakian manifold $\mathcal{S}(\mathbf{p})$ such that the product $\mathcal{S}(\mathbf{p})\times S^1$ is a hypercomplex manifold. As mentioned above, these examples should be thought of as generalizations of Hopf manifolds as a 3-Sasakian manifold should be thought of as a generalization of the $(4n+3)$-sphere. Moreover, there is a principal fibration $S^1\rightarrow\mathcal{N}(\mathbf{p})\rightarrow\mathcal{S}(\mathbf{p})$. Thus, both $\mathcal{N}(\mathbf{p})$ and $\mathcal{S}(\mathbf{p})\times S^1$ are hypercomplex manifolds which fiber over $\mathcal{S}(\mathbf{p})$ with circle fibers. The following theorem shows that these two spaces are the two extremes of new families of examples as their fundamental groups are 0 and Z, respectively.

Theorem 4.9 [BGM4]: *Let $\mathbf{p}$ be k-coprime. There there is a compact hypercomplex manifold $\mathcal{H}(\mathbf{p},k)$ with universal cover $\rho_k:\mathcal{N}(\mathbf{p})\rightarrow\mathcal{H}(\mathbf{p},k)$ such that $\pi_1(\mathcal{H}(\mathbf{p},k))\cong\mathrm{Z}_k$ and ρ_k is a hypercomplex map. Moreover, $\mathcal{H}(\mathbf{p},k)$ is never locally conformally hyperkähler and is hypercomplex homogeneous if and only if $\mathbf{p}=(p,p,\ldots,p)$.*

Remark: If $\mathbf{p}$ is commensurable but not k-coprime it is still possible to construct $\mathcal{H}(\mathbf{p},k)$ and obtain a hypercomplex *orbifold*. However, if $\mathbf{p}$ is not commensurable then $\mathcal{H}(\mathbf{p},k)$ is non-Hausdorff. While the topology of $\mathcal{N}(\mathbf{p})$ is well-known and independent of $\mathbf{p}$ we have

Theorem 4.10 [BGM4]: *Let $\mathbf{p}$ be coprime and k a positive integer. Then, as graded rings,*

$$H^*(\mathcal{H}(\mathbf{p},k),\mathrm{Z})\;\cong\;\left(\frac{\mathrm{Z}_k[x_2]}{[x_2^n=0]}\otimes E[y_{2n-3},z_{2n-1}]\right)/\mathcal{R}(\mathcal{H}(\mathbf{p},k)),$$

where the subscripts on x_2, y_{2n-3} and z_{2n-1} denote the cohomological dimension of each generator. The relations $\mathcal{R}(\mathcal{H}(\mathbf{p},k))$ are given by

$$d(\mathbf{p},k)x_2^{n-1}=d(\mathbf{p},k)x_2y_{2n-3}=x_2^2y_{2n-3}=x_2^{n-1}z_{2n-1}=x_2y_{2n-3}z_{2n-1}=0.$$

Here $\sigma_{n-1}(\mathbf{p})$ is the $(n-1)^{st}$ elementary symmetric polynomial in the coordinates of $\mathbf{p}$ and $d(\mathbf{p},k)=\gcd(\sigma_{n-1}(\mathbf{p}),k)$. The convention here is that $d(\mathbf{p},0)=\sigma_{n-1}(\mathbf{p})$.

Note that the topology of $\mathcal{H}(\mathbf{p},k)$ depends on both k and $\mathbf{p}$. For example,

COROLLARY 4.11 [BGM4]: *For all coprime* $\mathbf{p}$, $n\geq 3$, *and* $k>1$ *there is one cohomological invariant of* $\mathcal{H}(\mathbf{p},k)$ *that depends on* $\mathbf{p}$; *namely, the integer* $d(\mathbf{p},k)$ *which is the order of the torsion subgroups of the* $2n-2$ *and* $2n-1$ *integral cohomology groups of* $\mathcal{H}(\mathbf{p},k)$.

5. Applications to Other Geometries

As mentioned in the previous section, it was first noticed in [BGM2] that for any 3-Sasakian manifold $\mathcal{S}$ the trivial circle bundle $\mathcal{H}=S^1\times\mathcal{S}$ is an example of a compact hypercomplex manifold which is locally conformally hyperkähler. In this case one has the following diagram of (orbifold) fibrations

5.1
$$\begin{tikzcd} & \mathcal{H} \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathcal{Z} \arrow[dr] & & \mathcal{S}. \arrow[dl] \\ & \mathcal{O} & \end{tikzcd}$$

Very recently it has been observed that the diagram of orbifold fibrations in 5.1 holds true for an arbitrary compact manifold $(\mathcal{H},h)$ which is locally conformally hyperkähler but has no hyperkähler metric in the conformal class $[h]$ of h [PePoSw, OrPi]. In general, however, $\mathcal{S}$ in 5.1 can also be an orbifold. The result follows from the recent work of Gauduchon [Gau1, Gau2] and an earlier work of Vaisman on generalized Hopf surfaces [Vai]. We briefly describe some of Vaisman's results here. First recall

DEFINITION 5.2: *Let* $(\mathcal{H},J,h)$ *be a hermitian manifold with a complex structure* J, *hermitian metric* h, *and the* 2*-form* $\omega(X,Y)=g(JX,Y)$. *Then* h *is locally conformally Kähler if* $d\omega=\omega\wedge\alpha$ *for some close* 1*-form* α. *The* 1*-form* α *is called the Lee form. If* α *is not exact and* $\nabla\alpha=0$ *then* $\mathcal{H}$ *is called a generalized Hopf manifold.*

Note that if $\mathcal{H}$ is a generalized Hopf manifold then there is no Kähler metric in the conformal class of h, as this would imply exactness of the Lie form α. There are two natural foliations of any such $\mathcal{H}$: (i) the foliation $\mathcal{F}_\alpha$ defined by the vector field $\alpha^\sharp$ dual to α, with one-dimensional compact leaves and (ii) the foliation $\mathcal{E}$ defined by $\{\alpha^\sharp,J\alpha^\sharp\}$, with two-dimensional compact leaves. Assuming the leaf compactness, $\mathcal{H}$ is the total space of an analytic V-submersion onto a Kähler V-manifold in the sense of Satake [Sat]. Furthermore, all the fibers of the submersion are complex 1-dimensional tori $T^1_{\mathbb{C}}$. Similarly, assuming that all leaves of the foliation $\mathcal{F}_\alpha$ are compact $\mathcal{H}$ is a Seifert fibered space of an analytic V-submersion onto a Sasakian V-manifold. In particular, if the foliation $\mathcal{F}_\alpha$ is regular than $\mathcal{H}$ is a flat circle bundle over a Sasakian space. If the foliation $\mathcal{E}$ is regular (Vaisman refers to this case as *strongly regular*), then we get the diagram of fibrations

5.3
$$\begin{tikzcd} \mathcal{H} \arrow[r, "S^1"] \arrow[dd, "T^1_{\mathbb{C}}"] & \mathcal{S}. \arrow[ddl, "S^1"] \\ & \\ \mathcal{Z} & \end{tikzcd}$$

Theorem 5.4 [Vai]: *Let $(\mathcal{H}, J, g)$ be a compact strongly regular generalized Hopf surface of dimension $2m$. Then $\mathcal{H}$ is a smooth $T^1_{\mathbb{C}}$-principal fiber bundle over a compact Hodge manifold $\mathcal{Z}$. Furthermore, $\mathcal{H}$ is a flat principal circle bundle over compact Sasakian space $\mathcal{S}$ and $\mathcal{S}$ is a circle bundle over $\mathcal{Z}$ whose Chern class differs only by a torsion element form the Chern class induced by the Hopf fibration.*

This theorem has an orbifold generalization in the case the foliation $\mathcal{E}$ is not regular [Vai]. Next recall

Definition 5.5: *Let $(\mathcal{H}, J_1, J_2, J_3, h)$ be a hyperhermitian manifold with a hypercomplex structure $\{J_1, J_2, J_3\}$, hyperhermitian metric h, and the 2-forms $\omega_i(X, Y) = g(J_iX, Y)$. Let $\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i$. Then h is locally conformally hyperkähler if $d\Omega = \Omega \wedge \alpha$ for some closed 1-form α.*

The result of Gauduchon [Gau1-2] implies that any locally conformally hyperkähler manifold which is not hyperkähler is automatically a generalized Hopf manifold, *i.e.* there exists a metric h_0 in the conformal class of h whose Lee form α_0 is ∇^0-parallel (see also [OrPi]). This is certainly not true in the locally conformally Kähler case. Now the generalization of diagram 5.3 to the locally conformally hyperkähler case is automatic. The hypercomplex structure allows us to define not two but three different foliations of $\mathcal{H}$: (i) the one-dimensional foliation $\mathcal{F}_\alpha$ defined by α, (ii) two-dimensional foliations $\mathcal{F}^i_\alpha$ defined by $\{\alpha^\sharp, J_i\alpha^\sharp\}$ for each $i = 1, 2, 3$, (all three are equivalent), and (iii) a 4-dimensional foliation $\mathcal{F}_4$ defined by $\{\alpha^\sharp, J_1\alpha^\sharp, J_2\alpha^\sharp, J_3\alpha^\sharp\}$. Altogether they yield diagram 5.1. The leaf space $\mathcal{S} = \mathcal{H}/\mathcal{F}_\alpha$ is easily seen to have a natural Sasakian 3-structure.

This observation can be combined with the results on 3-Sasakian geometry described in the previous sections to obtain many new theorems about locally conformally hyperkähler spaces. Some of them, like the classification of strongly regular homogeneous spaces, were recently presented in [OrPi]. For further extensions of these results see [GS].

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