SASAKIAN GEOMETRY, HYPERSURFACE SINGULARITIES, AND EINSTEIN METRICS

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1. Introduction

This review article has grown out of notes for the three lectures the second author presented during the XXIV-th Winter School of *Geometry and Physics* in Srni, Czech Republic, in January of 2004. Our purpose is twofold. We want give a brief introduction to some of the techniques we have developed over the last 5 years while, at the same time, we summarize all the known results. We do not give any technical details other than what is necessary for the clarity of the exposition. In conclusion we would like to argue that Sasakian geometry has emerged as one of the most powerful tools of constructing and proving existence of special Riemannian metrics, such as Einstein metrics or metrics of positive Ricci curvature, on a wide range of odd-dimensional manifolds. The key geometric object in the theory is that of a contact structure (hence, only odd dimensions) together with a Riemannian metric naturally adapted to the contact form. Sasakian metrics in contact geometry are analogous to the Kähler metrics in the symplectic case.

We begin with basic facts about contact and Sasakian manifolds after which we focus on exploring the fundamental relation between the Sasakian and the transverse Kähler geometry. In this context positive Sasakian, Sasakian-Einstein, and 3-Sasakian manifolds are introduced. In Section 3 we present all known constructions of 3-Sasakian manifolds. These come as V-bundles over compact quaternion Kähler orbifolds and large families can be explicitly obtained using symmetry reduction. We also discuss Sasakian-Einstein manifolds which are not 3-Sasakian. Here there has been only one effective method of producing examples, namely by representing a Sasakian-Einstein manifold as the total space of an S^1 Seifert bundle over a Kähler-Einstein orbifold. In the smooth case with a trivial orbifold structure, this construction goes back to Kobayashi [Kob56]. Any smooth Fano variety $\mathcal{Z}$ which admits a Kähler-Einstein metric can be used for the base of a unique simply connected circle bundle P which is Sasakian-Einstein. Any Sasakian-Einstein manifold obtained this way is automatically regular. It is clear that, in order to get non-regular examples of Sasakian-Einstein structures, one should replace the smooth Fano structure with a Fano orbifold. This was done in [BG00] where we generalized the Kobayashi construction to V-bundles over Fano orbifolds. However, at that time, with the exception of twistor spaces of known 3-Sasakian metrics, compact Fano orbifolds known to admit orbifold Kähler-Einstein metrics were rare. The first examples of non-regular Sasakian-Einstein manifolds which are not 3-Sasakian were obtained in [BG00]. There we observed that Sasakian-Einstein manifolds have the structure of a monoid under a certain "join" operation. A join $M_1 \star M_2$ of non-regular 3-Sasakian manifold and a regular Sasakian-Einstein manifold (say an odd-dimensional sphere) is automatically a non-regular Sasakian-Einstein space. The problem is that our join construction produced new examples starting in dimension 9 and higher. It gave nothing new in dimensions 5 and 7. The construction of 5-dimensional examples followed, however, a year later, and with that, new non-regular Sasakian-Einstein examples in every dimension could be obtained.

In [BG01] the authors constructed three 5-dimensional examples using the results of Demailly and Kollár [DK01]. At the same time it became clear that the so-called continuity method as being applied by Demailly and Kollár to Fano orbifolds gave a cornucopia of new examples of Kähler-Einstein orbifolds [Ara02, BG03b, BG03a, BGN02b, BGN02a, BGN03b, BGK03, BGKT03, JK01b, JK01a, Kol04]. A particular illustration of how the method works comes from one example in classical differential topology. It is well-known [Tak78, BG01] that any link of isolated hypersurface singularities has a natural Sasakian structure. The transverse Kähler geometry in such a situation is induced by a Kählerian embedding of a complex hypersurface in an appropriate weighted projective space. Section 6 describes Sasakian geometry of links while reviewing basic facts about their differential geometry and topology. At the end of this section we are left with a powerful method of producing positive Sasakian structure on links.

In Section 7 we begin to discuss the famous Calabi Conjecture proved in 1978 by Yau [Yau78]. Yau's proof uses the continuity method and works equally well for compact orbifolds. This fact has important consequences for Sasakian manifolds: every positive Sasakian manifold admits a metric of positive Ricci curvature. This observation offers a very effective tool of proving the existence of such metrics on many odd-dimensional manifolds. One interesting example is a theorem of Wraith [Wra97], proved originally by surgery methods, which asserts that all homotopy spheres which bound parallelizable manifolds admit metrics of positive Ricci curvature. The authors together with M. Nakamaye [BGN03c] recently gave an independent proof of this result using the methods described here.

In the next section we turn our attention to positive Kähler-Einstein metrics. Even in the smooth Fano category tractable necessary and sufficient conditions for such a metric to exist are not known. After disproving one of the Calabi conjectures asserting that in the absence of holomorphic vector fields Kähler-Einstein metrics should exist, Tian proposed his own conjecture [Tia97] proving it in one direction. Even assuming the conjecture to be true, in general it is not easy to check if a particular Fano manifold (orbifold) satisfies the required stability conditions. On the other hand, in some cases, the continuity method has been used effectively to check sufficient conditions. In this respect the method of Demailly and Kollár mentioned before draws on earlier results of Nadel [Nad90], [Siu88], Tian [Tia87a, Tia87b, Tia90], and Tian and Yau [TY87].

The last two sections give a summary of what has been accomplished to date by applying the continuity method to Fano orbifolds. We begin with a brief discussion of the method itself. We follow with two important examples of how the method applies to orbifolds constructed as hypersurfaces in weighted projective spaces. We review our recent work [BGK03, BGKT03] in collaboration with J. Kollár which shows that standard odd-dimensional spheres have Einstein metrics with one-dimensional isometry group and very large moduli spaces. There we also proved that all homotopy spheres in dimensions $4n+1, 7, 11$, and 15 that bound parallelizable manifolds admit Sasakian-Einstein metrics. We discuss a conjecture that the last statement is true in any odd dimension. Furthermore, it is shown that in each odd dimension starting with $n = 5$ there are infinitely many rational homology spheres which admit Einstein metrics [BG03a]. We close with the discussion of Sasakian-Einstein geometry of Barden manifolds.

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2. Contact Structures and Sasakian Metrics

Contact transformations arose in the theory of Analytical Mechanics developed in the 19th century by Hamilton, Jacobi, Lagrange, and Legendre. But its first systematic treatment was given by Sophus Lie. Consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x^1, \cdots, x^n; y^1, \cdots, y^n; z)$, and a 1-form η given by

$$\eta = dz - \sum_i y^i dx^i \tag{1}$$

It is easy to see that η satisfies $\eta \wedge (d\eta)^n \neq 0$. A 1-form on $\mathbb{R}^{2n+1}$ that satisfies this equation is called a *contact form.* Locally we have the following

Theorem 1. *Let η be a 1-form on $\mathbb{R}^{2n+1}$ that satisfies $\eta \wedge (d\eta)^n \neq 0$. Then there is an open set $U \subset \mathbb{R}^{2n+1}$ and local coordinates $(x^1, \cdots, x^n; y^1, \cdots, y^n; z)$ such that η has the form (1) in U.*

Definition 2. A $(2n+1)$-dimensional manifold M is a *contact manifold* if there exists a 1-form η, called a *contact 1-form*, on M such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A *contact structure* on M is an equivalence class of such 1-forms, where $\eta' \sim \eta$ if there is a nowhere vanishing function f on M such that $\eta' = f\eta$.

Lemma 3. *On a contact manifold (M, η) there is a unique vector field ξ, called the Reeb vector field, satisfying the two conditions*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

Definition 4. An *almost contact structure* on a differentiable manifolds M is a triple (ξ, η, Φ), where Φ is a tensor field of type $(1,1)$ (i.e. an endomorphism of TM), ξ is a vector field, and η is a 1-form which satisfy

$$\eta(\xi) = 1 \text{ and } \Phi \circ \Phi = -\mathbb{I} + \xi \otimes \eta,$$

where $\mathbb{I}$ is the identity endomorphism on TM. A smooth manifold with such a structure is called an *almost contact manifold.*

Let (M,η) be a contact manifold with a contact 1-form η and consider $\mathcal{D} = \ker\,\eta \subset TM$. The subbundle $\mathcal{D}$ is maximally non-integrable and it is called the *contact distribution.* The pair $(\mathcal{D},\omega)$, where ω is the restriction of $d\eta$ to $\mathcal{D}$ gives $\mathcal{D}$ the structure of a symplectic vector bundle. We denote by $\mathcal{J}(\mathcal{D})$ the space of all almost complex structures J on $\mathcal{D}$ that are compatible with ω, that is the subspace of smooth sections J of the endomorphism bundle $\mathrm{End}(\mathcal{D})$ that satisfy

$$J^2 = -\mathbb{I}, \quad d\eta(JX,JY) = d\eta(X,Y), \quad d\eta(X,JX) > 0 \tag{2}$$

for any smooth sections X,Y of $\mathcal{D}$. Notice that each $J \in \mathcal{J}(\mathcal{D})$ defines a Riemannian metric $g_{\mathcal{D}}$ on $\mathcal{D}$ by setting

$$g_{\mathcal{D}}(X,Y) = d\eta(X,JY). \tag{3}$$

One easily checks that $g_{\mathcal{D}}$ satisfies the compatibility condition $g_{\mathcal{D}}(JX,JY) = g_{\mathcal{D}}(X,Y)$. Furthermore, the map $J \mapsto g_{\mathcal{D}}$ is one-to-one, and the space $\mathcal{J}(\mathcal{D})$ is contractible. A choice of J gives M an almost CR structure.

Moreover, by extending J to all of TM one obtains an almost contact structure. There are some choices of conventions to make here. We define the section Φ of $\mathrm{End}(TM)$ by $\Phi = J$ on $\mathcal{D}$ and $\Phi\xi = 0$, where ξ is the Reeb vector field associated to η. We can also extend the transverse metric $g_{\mathcal{D}}$ to a metric g on all of M by

$$g(X,Y) = g_{\mathcal{D}} + \eta(X)\eta(Y) = d\eta(X,\Phi Y) + \eta(X)\eta(Y) \tag{4}$$

for all vector fields X,Y on M. One easily sees that g satisfies the compatibility condition $g(\Phi X,\Phi Y) = g(X,Y) - \eta(X)\eta(Y)$.

Definition 5. A contact manifold M with a contact form η, a vector field ξ, a section Φ of $\mathrm{End}(TM)$, and a Riemannian metric g which satisfy the conditions

$$\eta(\xi) = 1, \qquad \Phi^2 = -\mathbb{I} + \xi\otimes\eta,$$
$$g(\Phi X,\Phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

is known as a *metric contact structure* on M.

Definition–Theorem 6. *A Riemannian manifold (M,g) is called a Sasakian manifold if any one, hence all, of the following equivalent conditions hold:*

(1) *There exists a Killing vector field ξ of unit length on M so that the tensor field Φ of type $(1,1)$, defined by $\Phi(X) = -\nabla_X\xi$, satisfies the condition*
$$(\nabla_X\Phi)(Y) \;=\; g(X,Y)\xi - g(\xi,Y)X$$
for any pair of vector fields X and Y on M.

(2) *There exists a Killing vector field ξ of unit length on M so that the Riemann curvature satisfies the condition*
$$R(X,\xi)Y \;=\; g(\xi,Y)X - g(X,Y)\xi,$$
for any pair of vector fields X and Y on M.

(3) *The metric cone $(\mathcal{C}(M),\bar{g}) \;=\; (\mathbb{R}_+\times M,\; dr^2 + r^2g)$ is Kähler.*

We refer to the quadruple $\mathcal{S} = (\xi,\eta,\Phi,g)$ as a *Sasakian structure* on M, where η is the 1-form dual vector field ξ. It is easy to see that η is a contact form whose Reeb vector field is ξ. In particular $\mathcal{S} = (\xi,\eta,\Phi,g)$ is a special type of *metric contact structure.*

The vector field ξ is nowhere vanishing, so there is a 1-dimensional foliation $\mathcal{F}_\xi$ associated with every Sasakian structure, called the *characteristic foliation.* We will denote the space of leaves of this foliation by $\mathcal{Z}$. Each leaf of $\mathcal{F}_\xi$ has a holonomy group associated to it. The dimension of the closure of the leaves is called the *rank* of $\mathcal{S}$. We shall be interested in the case $\mathrm{rk}(\mathcal{S}) = 1$. We have

Definition 7. The characteristic foliation $\mathcal{F}_\xi$ is said to be *quasi-regular* if there is a positive integer k such that each point has a foliated coordinate chart (U, x) such that each leaf of $\mathcal{F}_\xi$ passes through U at most k times. Otherwise $\mathcal{F}_\xi$ is called irregular. If $k = 1$ then the foliation is called *regular*, and we use the terminology *non-regular* to mean quasi-regular, but not regular.

3. Transverse Kähler Geometry

Let (M, ξ, η, Φ, g) be a Sasakian manifold, and consider the contact subbundle $\mathcal{D} = \ker\,\eta$. There is an orthogonal splitting of the tangent bundle as

$$TM = \mathcal{D} \oplus L_\xi, \tag{5}$$

where L_ξ is the trivial line bundle generated by the Reeb vector field ξ. The contact subbundle $\mathcal{D}$ is just the normal bundle to the characteristic foliation $\mathcal{F}_\xi$ generated by ξ. It is naturally endowed with both a complex structure $J = \Phi|\mathcal{D}$ and a symplectic structure $d\eta$. Hence, $(\mathcal{D}, J, d\eta)$ gives M a *transverse Kähler* structure with Kähler form $d\eta$ and metric $g_\mathcal{D}$ defined as in (3) which is related to the Sasakian metric g by $g = g_\mathcal{D} \oplus \eta \otimes \eta$ as in (4). We have the following fundamental structure theorems [BG00]:

Theorem 8. *Let (M, ξ, η, Φ, g) be a compact quasi-regular Sasakian manifold of dimension $2n+1$, and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation. Then the leaf space $\mathcal{Z}$ is a Hodge orbifold with Kähler metric h and Kähler form ω which defines an integral class $[\omega]$ in $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$ so that $\pi : (M, g) \longrightarrow (\mathcal{Z}, h)$ is an orbifold Riemannian submersion. The fibers of π are totally geodesic submanifolds of M diffeomorphic to S^1.*

Theorem 9. *Let $(\mathcal{Z}, h)$ be a Hodge orbifold. Let $\pi : M \longrightarrow \mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $[\omega]$, and let η be a connection 1-form in M whose curvature is $2\pi^*\omega$, then M with the metric $\pi^* h + \eta \otimes \eta$ is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the group of the bundle S^1, the total space M is a smooth Sasakian manifold.*

Remark 10. The structure theorems discussed above show that there are two Kähler geometries naturally associated with every Sasakian manifold and we get the following diagram

$$\begin{tikzcd} \mathcal{C}(M) & M \arrow[l, hook'] \arrow[d, "\pi"] \\ & \mathcal{Z} \end{tikzcd} \tag{6}$$

The orbifold cohomology groups $H^p_{orb}(\mathcal{Z}, \mathbb{Z})$ were defined by Haefliger [Hae84]. In analogy with the smooth case a *Hodge orbifold* is then defined to be a compact Kähler orbifold whose Kähler class lies in $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. Alternatively, we can develop the concept of basic cohomology. This is useful in trying to extend the notion of

$\mathcal{Z}$ being Fano to the orbifold situation. This can be done in several ways. Here we will use the notion of basic Chern classes. Recall [Ton97] that a smooth p-form α on M is called *basic* if

$$\xi\rfloor\alpha = 0, \qquad \mathcal{L}_\xi\alpha = 0, \tag{7}$$

and we let Λ^p_B denote the sheaf of germs of basic p-forms on M, and by Ω^p_B the set of global sections of Λ^p_B on M. The sheaf Λ^p_B is a module over the ring, Λ^0_B, of germs of smooth basic functions on M. We let $C^\infty_B(M) = \Omega^0_B$ denote global sections of Λ^0_B, i.e. the ring of smooth basic functions on M. Since exterior differentiation preserves basic forms we get a de Rham complex

$$\cdots \longrightarrow \Omega^p_B \xrightarrow{\;d\;} \Omega^{p+1}_B \longrightarrow \cdots \tag{8}$$

whose cohomology $H^*_B(\mathcal{F}_\xi)$ is called the *basic cohomology* of $(M, \mathcal{F}_\xi)$. The basic cohomology ring $H^*_B(\mathcal{F}_\xi)$ is an invariant of the foliation $\mathcal{F}_\xi$ and hence, of the Sasakian structure on M. It is related to the ordinary de Rham cohomology $H^*(M,\mathbb{R})$ by the long exact sequence [Ton97]

$$\cdots \longrightarrow H^p_B(\mathcal{F}_\xi) \longrightarrow H^p(M,\mathbb{R}) \xrightarrow{\;j_p\;} H^{p-1}_B(\mathcal{F}_\xi) \xrightarrow{\;\delta\;} H^{p+1}_B(\mathcal{F}_\xi) \longrightarrow \cdots \tag{9}$$

where δ is the connecting homomorphism given by $\delta[\alpha] = [d\eta\wedge\alpha] = [d\eta]\cup[\alpha]$, and j_p is the composition of the map induced by $\xi\rfloor$ with the well known isomorphism $H^r(M,\mathbb{R}) \approx H^r(M,\mathbb{R})^{S^1}$ where $H^r(M,\mathbb{R})^{S^1}$ is the S^1-invariant cohomology defined from the S^1-invariant r-forms $\Omega^r(M)^{S^1}$. We also note that $d\eta$ is basic even though η is not. Next we exploit the fact that the transverse geometry is Kähler. Let $\mathcal{D}_\mathbb{C}$ denote the complexification of $\mathcal{D}$, and decompose it into its eigenspaces with respect to J, that is, $\mathcal{D}_\mathbb{C} = \mathcal{D}^{1,0}\oplus\mathcal{D}^{0,1}$. Similarly, we get a splitting of the complexification of the sheaf Λ^1_B of basic one forms on M, namely

$$\Lambda^1_B\otimes\mathbb{C} = \Lambda^{1,0}_B \oplus \Lambda^{0,1}_B.$$

We let $\mathcal{E}^{p,q}_B$ denote the sheaf of germs of basic forms of type (p,q), and we obtain a splitting

$$\Lambda^r_B\otimes\mathbb{C} = \bigoplus_{p+q=r}\mathcal{E}^{p,q}_B. \tag{10}$$

The basic cohomology groups $H^{p,q}_B(\mathcal{F}_\xi)$ are fundamental invariants of a Sasakian structure which enjoy many of the same properties as the ordinary Dolbeault cohomology of a Kähler structure.

Consider the complex vector bundle $\mathcal{D}$ on a Sasakian manifold (M,ξ,η,Φ,g). As such $\mathcal{D}$ has Chern classes $c_1(\mathcal{D}),\cdots,c_n(\mathcal{D})$ which can be computed by choosing a connection $\nabla^\mathcal{D}$ in $\mathcal{D}$ [Kob87]. Let us choose a local foliate unitary transverse frame $(X_1,\cdots,X_n)$, and denote by Ω^T the transverse curvature 2-form with respect to this frame. A simple calculation shows that Ω^T is a basic $(1,1)$-form. Since the curvature 2-form Ω^T has type $(1,1)$ it follows as in ordinary Chern-Weil theory that

Definition–Theorem 11. *The kth Chern class $c_k(\mathcal{D})$ of the complex vector bundle $\mathcal{D}$ is represented by the basic (k,k)-form γ_k determined by the formula*

$$\det(\mathbb{I}_n - \frac{1}{2\pi i}\Omega^T) = 1+\gamma_1+\cdots+\gamma_k.$$

Since γ_k is a closed basic (k,k)-form it represents an element in $H_B^{k,k}(\mathcal{F}_\xi) \subset H_B^{2k}(\mathcal{F}_\xi)$ that is called the basic kth Chern class and denoted by $c_k(\mathcal{F}_\xi)$.

We now concentrate on the first Chern classes $c_1(\mathcal{D})$ and $c_1(\mathcal{F}_\xi)$. We have

Definition 12. A Sasakian structure (ξ,η,Φ,g) is said to be *positive (negative)* if $c_1(\mathcal{F}_\xi)$ is represented by a positive (negative) definite $(1,1)$-form. If either of these two conditions is satisfied (ξ,η,Φ,g) is said to be *definite*, and otherwise (ξ,η,Φ,g) is called *indefinite*. (ξ,η,Φ,g) is said to be *null* if $c_1(\mathcal{F}_\xi)=0$.

In analogy with common terminology of smooth algebraic varieties we see that a positive Sasakian structure is a *transverse Fano structure*, while a null Sasakian structure is a *transverse Calabi-Yau structure*. The negative Sasakian case corresponds to the canonical bundle being ample; we refer to this as a *transverse canonical structure*.

Remark 13. Alternatively, a complex orbifold $\mathcal{Z}$ is Fano if its orbifold canonical bundle $K_{\mathcal{Z}^{orb}}$ is anti-ample. In the case $\mathcal{Z}$ is well-formed, that is when the orbifold singularities have codimension at least 2, the orbifold canonical bundle $K_{\mathcal{Z}^{orb}}$ can be identified with the ordinary canonical bundle. However, in the presence of codimension 1 singularities the orbifold canonical divisor **is not** the usual algebraic geometric canonical divisor, but is shifted off by the ramification divisors coming from the codimension one singularities [BGK03]. We shall give specific examples of this difference later.

4. The Einstein Condition

Definition 14. A Sasakian space (M,g) is *Sasakian-Einstein* if the metric g is also Einstein. For any 2n+1-dimensional Sasakian manifold $\mathrm{Ric}(X,\xi)=2n\eta(X)$ implying that any Sasakian-Einstein metric must have positive scalar curvature. Thus any complete Sasakian-Einstein manifold must have a finite fundamental group. Furthermore the metric cone on M $(\mathcal{C}(M),\bar{g}) = (\mathbb{R}_+\times M,\ dr^2+r^2g)$ is Kähler Ricci-flat (Calabi-Yau).

The following theorem [BG00] is an orbifold version of the famous Kobayashi bundle construction of Einstein metrics on bundles over positive Kähler-Einstein manifolds [Bes87, Kob56].

Theorem 15. *Let $(\mathcal{Z},h)$ be a compact Fano orbifold with $\pi_1^{orb}(\mathcal{Z})=0$ and Kähler-Einstein metric h. Let $\pi: M\longrightarrow\mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $\frac{c_1(\mathcal{Z})}{Ind(\mathcal{Z})}$. Suppose further that the local uniformizing groups of $\mathcal{Z}$ inject into S^1. Then with the metric $g=\pi^*h+\eta\otimes\eta$, M is a compact simply connected Sasakian-Einstein manifold.*

Here $\mathrm{Ind}(\mathcal{Z})$ is the *orbifold Fano index* [BG00] defined to be the largest positive integer such that $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$ defines a class in the orbifold cohomology group $H^2_{orb}(\mathcal{Z},\mathbb{Z})$. A very special class of Sasakian-Einstein spaces is naturally related to several quaternionic geometries.

Definition 16. Let (M,g) be a Riemannian manifold of dimension m. We say that (M,g) is 3-Sasakian if the metric cone $(\mathcal{C}(M),\bar{g}) = (\mathbb{R}_+\times\mathcal{S},\ dr^2+r^2g)$ on M is hyperkähler.

Remark 17. In the 3-Sasakian case there is an extra structure, i.e., the transverse geometry $\mathcal{O}$ of the 3-dimensional foliation which is quaternionic-Kähler. In this case, the transverse space $\mathcal{Z}$ is the twistor space of $\mathcal{O}$ and the natural map $\mathcal{Z} \longrightarrow \mathcal{O}$ is the orbifold twistor fibration [Sal82]. We get the following diagram which we denote by $\lozenge(M)$ and which extends the diagram in (6) [BGM93, BGM94]:

$$
\begin{tikzcd}
 & \boxed{\begin{array}{c}\textbf{Hyperkähler}\\ \textbf{Geometry}\end{array}} & \\
 & \mathcal{C}(M) \arrow[dl] \arrow[dr] \arrow[dd] & \\
\boxed{\begin{array}{c}\textbf{Twistor}\\ \textbf{Geometry}\end{array}}\ \mathcal{Z} \arrow[dr] & & M\ \boxed{\begin{array}{c}\textbf{3-Sasakian}\\ \textbf{Geometry}\end{array}} \arrow[ll] \arrow[dl] \\
 & \mathcal{O} & \\
 & \boxed{\begin{array}{c}\textbf{Quaternion Kähler}\\ \textbf{Geometry}\end{array}} &
\end{tikzcd}
\tag{11}
$$

Remark 18. The table below summarizes properties of cone and transverse geometries associated to various metric contact structures.

Cone Geometry of $\mathcal{C}(M)$	M	Transverse Geometry of $\mathcal{F}_\xi$
Symplectic	Contact	Symplectic
Kähler	Sasakian	Kähler
Kähler	positive Sasakian	Fano, $c_1(\mathcal{Z}) > 0$
Kähler	null Sasakian	Calabi-Yau, $c_1(\mathcal{Z}) = 0$
Kähler	negative Sasakian	canonical, $c_1(\mathcal{Z}) < 0$
Calabi-Yau	Sasakian-Einstein	Fano, Kähler-Einstein
Hyperkähler	3-Sasakian	$\mathbb{C}$-contact, Fano, Kähler-Einstein

5. Some Examples

Below we list some well-know constructions of Sasakian-Einstein and 3-Sasakian manifolds. We start with the latter.

Example 19. Examples of 3-Sasakian manifolds are numerous and they are easily constructed by way of the so called 3-Sasakian reduction [BGM94]. To begin with one starts with the canonical example of $\lozenge(M)$ where M is the round $(4n-1)$-dimensional sphere S^{4n-1} of constant sectional curvature 1.

$$
\begin{tikzcd}
 & \mathbb{C}^n \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathbb{P}^{n-1}_{\mathbb{C}} \arrow[dr] & & S^{4n-1}. \arrow[ll] \arrow[dl] \\
 & \mathbb{P}^{n-1}_{\mathbb{H}} &
\end{tikzcd}
\tag{12}
$$

Moreover, there is such a diamond diagram for any semisimple Lie group G and we get all homogeneous examples this way [BGM94], i.e.,

$$
\begin{tikzcd}
 & \mathcal{C}(G/L) \arrow[dl] \arrow[dr] \arrow[dd] & \\
\frac{G}{U(1)\cdot L} \arrow[dr] & & G/L. \arrow[ll] \arrow[dl] \\
 & \frac{G}{Sp(1)\cdot L} &
\end{tikzcd}
\tag{13}
$$

The spaces $\frac{G}{Sp(1)\cdot L}$ are the well-known Wolf spaces [Wol65] and they are the only know examples of smooth compact positive quaternion Kähler manifolds. A conjecture of LeBrun and Salamon [LS94] asserts that there are no other examples. This conjecture have been proved in the first three quaternionic dimensions by Hitchin [Hit74], Friedrich and Kurke [FK82], Poon and Salamon [PS91], and Herrera and Herrera [HH02a, HH02b].

Example 20. Now, one can start with any of the homogeneous diamonds $\Diamond(G/L)$ in 13. In principle, any subgroup $H \subset G$ of appropriate dimension leads to a reduction of $\Diamond(G/L)$ by symmetries of H. At various levels of the diamond such reductions are known as hyperkähler [HKLR87], 3-Sasakian [BGM94], and quaternionic Kähler quotients [GL88], respectively. In practice, it is not easy to assure that, say, the 3-Sasakian quotient of G/L by H be a smooth manifold. On the other hand, there are many cases when this happens. For instance, one can reduce the standard diagram 12 by an action of $T^k \subset T^n \subset Sp(n)$ k-dimensional torus. We get the reduction diagram

$$
\begin{tikzcd}
 & \mathbb{C}^n \arrow[dl] \arrow[dr] \arrow[dd] & & & & \mathcal{C}(\mathcal{S}(\Omega)) \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathbb{P}^{n-1}_{\mathbb{C}} \arrow[dr] & & S^{4n-1} \arrow[ll] \arrow[dl] & \boxed{\begin{matrix}\textbf{Reduction}\\ \Longrightarrow \\ \textbf{by k-torus}\end{matrix}} & \mathcal{Z}(\Omega) \arrow[dr] & & \mathcal{S}(\Omega). \arrow[ll] \arrow[dl] \\
 & \mathbb{P}^{n-1}_{\mathbb{H}} & & & & \mathcal{O}(\Omega) &
\end{tikzcd}
$$

The geometry of the 3-Sasakian reduced space $\mathcal{S}(\Omega)$ is completely determined by an integral $k \times n$ matrix Ω which for any $k \geq 1$ defines a homomorphism $f_\Omega : T^k \longrightarrow T^n \subset U(n) \subset Sp(n)$. The real dimension $\dim(\mathcal{S}(\Omega)) = 4(n-k)-1$. In the case $\dim(\mathcal{S}(\Omega)) = 7$, there are choices of Ω for any $k \geq 1$ which make $\mathcal{S}(\Omega)$ smooth. Since $b_2(\mathcal{S}(\Omega)) = k$ we conclude that in dimension 7 there exist Einstein manifolds with arbitrarily large second Betti number. These were the first such examples and they were constructed in [BGMR98]. Interestingly, this toric reduction does not give smooth manifolds with large second Betti numbers in dimensions greater than 7 [BGM98]. Nevertheless, one can obtain Sasakian-Einstein manifolds with arbitrary second Betti number in any odd dimension greater than seven by the join construction discussed in 31 below. Later Bielawski showed that in all allowed dimensions all toric examples must occur through the above procedure [Bie99].

Example 21. The first non-toric examples in dimension 11 and 15 were obtained by Boyer, Galicki, and Piccinni [BGP02]. These are toric quotients of the diamond digram $\Diamond(G/L)$ for $G = SO(n), L = SO(4) \times SO(n-4)$. Alternatively, these can be thought of as non-Abelian reductions of $\Diamond(S^{4n-1})$.

Example 22. Recently, the first non-toric examples in dimension 7 were obtained by Grove, Wilking, and Ziller [Zil]. They use an orbifold bundle construction with the examples of orbifold twistor space and self-dual Einstein metrics $\mathcal{Z}_k \longrightarrow \mathcal{O}_k$ discovered by Hitchin in 1992 [Hit95]. The self-dual Einstein metric on $\mathcal{O}_k$ is defined on $S^4 \setminus \mathbb{RP}^2$ and it has $\mathbb{Z}_k$ orbifold singularity along $\mathbb{RP}^2$. However, it turns out that the bundle $M_k \longrightarrow \mathcal{Z}_k$ is actually smooth. In particular, one can compute the integral cohomology ring of M_k. For odd k the 3-Sasakian manifold M_k is a rational homology 7-sphere with non-zero torsion depending on k. Hence, there exist infinitely many rational homology 7-spheres which have 3-Sasakian metrics

Let us turn our attention to complete examples of Sasakian-Einstein manifolds which are not 3-Sasakian.

Example 23. The standard example is that of complex Hopf fibration

$$\mathbb{C}^n \hookleftarrow S^{2n-1} \longrightarrow \mathbb{P}^{n-1}_{\mathbb{C}}. \tag{14}$$

Just as in 3-Sasakian case this example generalizes when one replaces the complex projective space with a generalized flag manifold. That is, consider any complex semi-simple Lie group G. A maximal solvable subgroup B of G is called a *Borel subgroup* and is unique up to conjugacy. Any $P \subset G$ containing B is called *parabolic.* It is known that any such generalized flag G/P admits a homogeneous Kähler-Einstein metric, and that any compact homogeneous simply connected Kähler manifold is a generalized flag manifold. Applying the construction of Theorem 15 gives all compact homogeneous Sasakian-Einstein metrics, in fact, all compact homogeneous Sasakian manifolds [BG00].

Example 24. The Kobayashi bundle construction also gives many inhomogeneous examples. These are all circle bundles over compact smooth Fano manifolds. For instance, in the case of surfaces all del Pezzo surfaces are classified and it is known which of them admit Kähler-Einstein metrics [Tia90, Tia99, Tia00].

Theorem 25. *The following del Pezzo surfaces admit Kähler-Einstein metrics:* $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}, \quad 3 \leq n \leq 8$. *Furthermore, the moduli space of K-E structures in each case is completely understood.*

As and immediate consequence, for 5-manifolds, we have the following result of Friedrich and Kath [FK90]

Theorem 26. *Let* $M_l = S^5 \# l(S^2 \times S^3)$.

1. *For each* $l = 0, 1, 3, 4$, *there is precisely one regular Sasakian-Einstein structure on* M_l.
2. *For each* $5 \leq l \leq 8$ *there is a* $2(l-4)$ *complex parameter family of inequivalent regular Sasakian-Einstein structures on* M_l.
3. *For* $l = 2$ *or* $l \geq 9$ *there are no regular Sasakian-Einstein structures on* M_l.

There are two del Pezzo surfaces which do not admit any K-E metrics due to theorem of Matsushima [Mat57]: the existence is obstructed by holomorphic vector fields. These are blow-ups of $\mathbb{CP}^2$ at one or two points.

Remark 27. A well known result of Martinet says that every orientable 3-manifold admits a contact structure. Furthermore, all Sasakian 3-manifolds have been classified [Bel00, Bel01, Gei97] and they are Seifert bundles over Riemann surfaces. In this case every compact Sasakian 3-manifold is either negative, null, or positive. In addition, if M is Sasakian-Einstein than it follows that the universal cover $\tilde{M}$ is isomorphic to the standard Sasakian-Einstein metric on S^3.

Example 28. [**Barden Manifolds**] Similarly one might try to classify all Sasakian manifolds in dimension 5. In the simply connected case there exists a classification result of all smooth 5-manifolds due to Smale [Sma62] and Barden [Bar65]. Extending Smale's theorem for spin manifolds Barden proves the following:

Theorem 29. *The class of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism. Furthermore, any such M is diffeomorphic to one of the spaces $M_{j;k_1,\ldots,k_s} = X_j \# M_{k_1} \# \cdots \# M_{k_s}$, where $-1 \leq j \leq \infty$, $s \geq 0$, $1 < k_1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$. A complete set of invariants is provided by $H_2(M,\mathbb{Z})$ and an additional diffeomorphism invariant $i(M) = j$ which depends only on the second Stiefel-Whitney class $w^2(M)$.*

In this article we will refer to a simply connected, closed, oriented, smooth, 5-manifold as a *Barden manifold*. The building blocks of Theorem 29 are given in the table below. They are listed with $H_2(M,\mathbb{Z})$ and Barden's $i(M)$ invariant.

M	$H_2(M,\mathbb{Z})$	$i(M)$
$X_{-1} = SU(3)/SO(3)$	$\mathbb{Z}_2$	-1
$X_n,\ n \geq 1$	$\mathbb{Z}_{2^n} \oplus \mathbb{Z}_{2^n}$	n
$X_\infty =$ non-trivial S^3 bundle over S^2	$\mathbb{Z}$	∞
$X_0 = S^5$	0	0
$M_\infty = S^3 \times S^2$	$\mathbb{Z}$	0
$M_n,\ n > 1$	$\mathbb{Z}_n \oplus \mathbb{Z}_n$	0

When M is spin $i(M) = j = 0$ as is $w^2(M) = 0$ and Barden's result is the extension of the well-known theorem of Smale for spin 5-manifolds. By an old theorem of Gray [Gra59] M admits an almost contact structure when $j = 0, \infty$ and by another result of Geiges M is in such a case necessarily contact [Gei91].

Question 30. It is natural to ask whether Barden manifolds admit Sasakian structures. Specifically, we would like to ask

1. Does every Barden manifold which is contact ($j = 0, \infty$) admit a Sasakian structure?
2. Does every Barden manifold which is spin ($j = 0$) admit positive (respectively negative) Sasakian structure? (Null Sasakian structures are obstructed. For example, Corollary 1.10 of [BGN03a] implies that S^5 and $S^2 \times S^3$ cannot admit null Sasakian structures).
3. Which of the Barden manifolds which are spin admit Sasakian-Einstein structures?

Remark 31. There is one more construction of non-regular Sasakian-Einstein manifolds which draws on examples of 3-Sasakian structures. In [BG00] the authors observed that the set of all Sasakian-Einstein manifolds has a monoid structure, i.e., for any two compact quasi-regular Sasakian-Einstein orbifolds M_1 and M_2 one can define $M_1 \star M_2$ which is automatically a compact Sasakian-Einstein orbifold of dimension $\dim(M_1)+\dim(M_2)-1$. This construction is an orbifold generalization of the well-known construction of Wang and Ziller in [WZ90] adapted to the Sasakian-Einstein setting. It turns out that a join of any non-regular Sasakian-Einstein manifold M_1 with a regular Sasakian-Einstein space M_2 (say, for example, $M_2 = S^3$) is automatically a compact smooth, non-regular Sasakian-Einstein manifold. The join construction produces new examples beginning in dimension 7. If, however, M_1

is 3-Sasakian, new examples begin in dimension 9. In addition, a completely new construction of inhomogeneous Sasakian-Einstein metrics have been considered by Gauntlett, Martelli, Sparks, and Waldram [GMSW04a, GMSW04b]. Their metrics are in fact very explicit and they were first obtained indirectly by considering general supersymmetric solutions in certain $D = 11$ supergravity theory.

6. Sasakian Geometry of Links

Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action given by $(z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n)$, where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w} \in (\mathbb{Z}^+)^{n+1}$, and we shall assume that $\gcd(w_0, \ldots, w_n) = 1$.

Definition 32. We say that f is a weighted homogeneous polynomial with weights $\mathbf{w}$ and of degree d if $f \in \mathbb{C}[z_0, \ldots, z_n]$ and satisfies

$$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n). \tag{15}$$

We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity.

Definition 33. The link of f is defined by

$$L_f = \{f = 0\} \cap S^{2n+1}, \tag{16}$$

where S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$.

Remark 34. L_f is endowed with a natural quasi-regular Sasakian structure [Tak78, YK84, BG01] inherited as a Sasakian submanifold of the sphere S^{2n+1} with its "weighted" Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which in the standard coordinates $\{z_j = x_j + iy_j\}_{j=0}^n$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is determined by

$$\eta_{\mathbf{w}} = \frac{\sum_{i=0}^n (x_i dy_i - y_i dx_i)}{\sum_{i=0}^n w_i (x_i^2 + y_i^2)}, \qquad \xi_{\mathbf{w}} = \sum_{i=0}^n w_i (x_i \partial_{y_i} - y_i \partial_{x_i}), \tag{17}$$

and the standard Sasakian structure (ξ, η, Φ, g) on S^{2n+1}.

The quotient of S^{2n+1} by the "weighted S^1-action" generated by the vector field $\xi_{\mathbf{w}}$ is the weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$, and we have a commutative diagram:

$$\begin{tikzcd} L_f \arrow[r] \arrow[d, "\pi"] & S^{2n+1}_{\mathbf{w}} \arrow[d] \\ \mathcal{Z}_f \arrow[r] & \mathbb{P}(\mathbf{w}), \end{tikzcd} \tag{18}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are orbifold Riemannian submersions. L_f is the total space of the principal S^1 V-bundle over the orbifold $\mathcal{Z}_f$. Alternatively we will sometimes denote $\mathcal{Z}_f$ as $X_d \subset \mathbb{P}(\mathbf{w})$ to indicate the weights and the degree of f. In such case we will also write $L_f = L(X_d \subset \mathbb{P}(\mathbf{w}))$.

Proposition 35. [BGK03] *The orbifold $\mathcal{Z}_f$ is Fano if and only if $d - \sum w_i < 0$.*

Example 36. At this point we return to the comments made in Remark 13. Consider links defined by

$$f_k(z_0,z_1,z_2)=z_0^{6k-1}+z_1^3+z_2^2. \tag{19}$$

The orbifold $\mathcal{Z}_{f_k}$ is a hypersurface X_d in $\mathbb{P}(6,2(6k-1),3(6k-1))$ of degree $d=6(6k-1)$. The corresponding link L_{f_k} is of Brieskorn-Pham type and will be denoted by $L(6k-1,3,2)$ (see (23)). All 3-dimensional Brieskorn-Pham links were classified by Milnor in [Mil75]. According to the Proposition 35, $L(6k-1,3,2)$ is positive only when $k=1$ and in all other cases it is negative. Indeed, $L(5,3,2)\simeq S^3/I^*$ is the famous Poincaré sphere, where $I^*\subset SU(2)$ is the binary isocahedral group. For $k>1$, the link $L(6k-1,3,2)$ is a homology sphere with infinite fundamental group. The complex orbifold $\mathcal{Z}_{f_k}$, for $k>1$ is not Fano. In particular, it cannot have an orbifold metric of **constant positive** curvature (though it has a natural metric of constant negative curvature). On the other hand, as an algebraic variety, for any k we must have $\mathcal{Z}_{f_k}\simeq\mathbb{P}^1$. This can be seen from the generalized genus formula. For any curve $X_d\subset\mathbb{P}(w_0,w_1,w_2)$ we have:

$$g(X_d)=\frac{1}{2}\Big(\frac{d^2}{w_0w_1w_2}-d\sum_{i<j}\frac{\gcd(w_i,w_j)}{w_iw_j}+\sum_i\frac{\gcd(d,w_i)}{w_i}-1\Big). \tag{20}$$

Hence, $\mathcal{Z}_{f_k}$ is certainly Fano as a smooth variety in the algebraic geometric sense, but it has codimension1 orbifold singularities and it **is not** Fano in the orbifold sense. Here the orbifold canonical class is not the usual algebraic geometric canonical class, but the codimension one orbifold ramification divisors are added in. By Milnor's classification $L(6k-1,3,2)$, for $(k>1)$, is the quotient of the universal cover $\widetilde{SL}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ by a co-compact discrete subgroup $\Gamma\subset\widetilde{SL}(2,\mathbb{R})$. Furthermore, $L(6k-1,3,2)$ has a finite covering by a manifold that is diffeomorphic to a non-trivial circle bundle over a Riemann surface of some genus $g>1$.

Now, recall the well-known construction of Milnor for isolated hypersurface singularities [Mil68, MO70]: there is a fibration of $(S^{2n+1}-L_f)\longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n\vee S^n\cdots\vee S^n$. The *Milnor number* μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $(w_0,\ldots,w_n)$ by the formula

$$\mu=\mu(L_f)=\prod_{i=0}^n\big(\frac{d}{w_i}-1\big) \tag{21}$$

The closure $\bar F$ of F has the same homotopy type as F and is a compact manifold with boundary precisely the link L_f. So the reduced homology of F and $\bar F$ is only non-zero in dimension n and $H_n(F,\mathbb{Z})\approx\mathbb{Z}^\mu$. Using the Wang sequence of the Milnor fibration together with Alexander-Poincare duality gives the exact sequence

$$0\rightarrow H_n(L_f,\mathbb{Z})\rightarrow H_n(F,\mathbb{Z})\xrightarrow{\mathbb{I}-h_*}H_n(F,\mathbb{Z})\rightarrow H_{n-1}(L_f,\mathbb{Z})\rightarrow 0 \tag{22}$$

where h_* is the *monodromy* map (or characteristic map) induced by the $S^1_{\mathbf{w}}$ action. From this we see that $H_n(L_f,\mathbb{Z})=\ker(\mathbb{I}-h_*)$ is a free Abelian group, and $H_{n-1}(L_f,\mathbb{Z})=\mathrm{Coker}(\mathbb{I}-h_*)$ which in general has torsion, but whose free part equals $\ker(\mathbb{I}-h_*)$. So the topology of L_f is encoded in the monodromy map h_*. There is a well-known algorithm due to Milnor and Orlik [MO70] for computing the free part of $H_{n-1}(L_f,\mathbb{Z})$ in terms of the characteristic polynomial

$\Delta(t) = \det(t\mathbb{I} - h_*)$, namely the Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$. First we mention an important immediate consequence of the exact sequence (22) which is due to Milnor:

Proposition 37. *The following hold:*

1. *L_f is a rational homology sphere if and only if $\Delta(1) \neq 0$.*
2. *L_f is a homology sphere if and only if $|\Delta(1)| = 1$.*
3. *If L_f is a rational homology sphere, then the order of $H_{n-1}(L_f, \mathbb{Z})$ equals $|\Delta(1)|$.*

Example 38. The following table lists some illustrating examples. All of the chosen links are Fano, but negative and null Sasakian structures can also be considered. We either explicitly identify the link with some smooth contact manifold or list non-vanishing homology groups. Σ^7_k and Σ^7_p indicate homotopy spheres where the differentiable structure depends on k and p.

$\mathcal{Z}_f$	L_f
$X_2 \subset \mathbb{P}(1,1,1,1)$	$S^2 \times S^3$
$X_3 \subset \mathbb{P}(1,1,1,1)$	$6\#(S^2 \times S^3)$
$X_4 \subset \mathbb{P}(1,1,1,2)$	$7\#(S^2 \times S^3)$
$X_6 \subset \mathbb{P}(1,1,2,3)$	$8\#(S^2 \times S^3)$
$X_{k+1} \subset \mathbb{P}(1,1,1,k)$	$k\#(S^2 \times S^3)$
$X_{10} \subset \mathbb{P}(1,2,3,5)$	$9\#(S^2 \times S^3)$
$X_{127} \subset \mathbb{P}(11,29,39,49)$	$2\#(S^2 \times S^3)$
$X_{256} \subset \mathbb{P}(13,35,81,128)$	$S^2 \times S^3$
$X_{3k} \subset \mathbb{P}(3,3,3,k)$, $k \neq 3n$	M_k
$X_{4k} \subset \mathbb{P}(4,4,4,4,k)$, $k \neq 2n$	$\lvert H_3(L_f,\mathbb{Z})\rvert = k^{21}$
$X_{6(6k-1)} \subset \mathbb{P}(6, 2(6k-1), 3(6k-1), 3(6k-1), 3(6k-1))$	$\Sigma_k \simeq S^7$
$X_{2(2p+1)} \subset \mathbb{P}(2, 2p+1, 2p+1, 2p+1, 2p+1, 2p+1)$	$\Sigma_p \simeq S^9$

In some of the above examples the homogeneous polynomial f can be chosen to contain no "mixed" monomial terms of the form $z_i^{c_i} z_j^{c_j}$. Such an f is called of Brieskorn-Pham type. In his famous work, in 1966 Brieskorn considered links $L(\mathbf{a})$ defined by

$$\sum_{i=0}^{n} |z_i^2| = 1, \qquad f_{\mathbf{a}}(\mathbf{z}) = z_0^{a_1} + \cdots + z_n^{a_n} = 0. \tag{23}$$

To the vector $\mathbf{a} = (a_0, \cdots, a_n) \in \mathbb{Z}_+^{n+1}$ one associates a graph $G(\mathbf{a})$ whose $n+1$ vertices are labeled by $a_0, \cdots, a_n$. Two vertices a_i and a_j are connected if and only if $\gcd(a_i, a_j) > 1$. Let C_{ev} denote the connected component of $G(\mathbf{a})$ determined by the even integers. Note that all even vertices belong to C_{ev}, but C_{ev} may contain odd vertices as well. Then we have the so-called **Brieskorn Graph Theorem** [Bri66]:

Theorem 39. *The following hold:*

1. *The link $L(\mathbf{a})$ is a rational homology sphere if and only if either $G(\mathbf{a})$ contains at least one isolated point, or C_{ev} has an odd number of vertices and for any distinct $a_i, a_j \in C_{ev}$, $\gcd(a_i, a_j) = 2$.*

(2) *The link $L(\mathbf{a})$ is an integral homology sphere if and only if either $G(\mathbf{a})$ contains at least two isolated points, or $G(\mathbf{a})$ contains one isolated point and C_{ev} has an odd number of vertices and $a_i, a_j \in C_{ev}$, implies $\gcd(a_i, a_j) = 2$ for any distinct i, j.*

Recall that by the seminal work of Milnor [Mil56], Kervaire and Milnor [KM63], and Smale [Sma61], for each $n \geq 5$, differentiable homotopy spheres of dimension n form an Abelian group Θ_n, where the group operation is connected sum. Θ_n has a subgroup bP_{n+1} consisting of those homotopy n-spheres which bound parallelizable manifolds V_{n+1}. Kervaire and Milnor proved that $bP_{2m+1} = 0$ for $m \geq 1$, $bP_{4m+2} = 0$, or $\mathbb{Z}_2$ and is $\mathbb{Z}_2$ if $4m+2 \neq 2^i - 2$ for any $i \geq 3$. The most interesting groups are bP_{4m} for $m \geq 2$. These are cyclic of order

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1}-1) \text{ numerator } \left(\frac{4B_m}{m}\right), \tag{24}$$

where B_m is the m-th Bernoulli number. Thus, for example $|bP_8| = 28, |bP_{12}| = 992, |bP_{16}| = 8128$ and $|bP_{20}| = 130,816$. In the first two cases these include all exotic spheres. The correspondence is given by

$$KM : \Sigma \mapsto \frac{1}{8}\tau(V_{4m}(\Sigma)) \mathrm{mod} |bP_{4m}|, \tag{25}$$

where $V_{4m}(\Sigma)$ is any parallelizable manifold bounding Σ and τ is its signature. Let Σ_i denote the exotic sphere with $KM(\Sigma_i) = i$. Now, the Brieskorn Graph Theorem tells us for which $\mathbf{a}$ the Brieskorn-Pham link $L(\mathbf{a})$ is a homotopy sphere. By (25) we need to be able to compute the signature to determine the diffeomorphism types of various links. We restrict our interest just to the case when $m = 2k+1$.

In this case, the diffeomorphism type of a homotopy sphere $L(\mathbf{a}) \in bP_{2m-2}$ is determined by the signature $\tau(M)$ of a parallelizable manifold M whose boundary is $\Sigma^{2m-3}_{\mathbf{a}}$. By the Milnor Fibration Theorem we can take M to be the Milnor fiber $M^{2m-2}_{\mathbf{a}}$ which, for links of isolated singularities coming from weighted homogeneous polynomials is diffeomorphic to the hypersurface $\{\mathbf{z} \in \mathbb{C}^m \mid f_{\mathbf{a}}(z_0, \cdots, z_{m-1}) = 1\}$.

Brieskorn shows that the signature of $M^{4k}(\mathbf{a})$ can be written combinatorially as

$$\begin{aligned} \tau(M^{4k}(\mathbf{a})) &= \#\Big\{\mathbf{x} \in \mathbb{Z}^{2k+1} \mid 0 < x_i < a_i \text{ and } 0 < \sum_{j=0}^{2k} \frac{x_i}{a_i} < 1 \mod 2\Big\} \\ &- \#\Big\{\mathbf{x} \in \mathbb{Z}^{2k+1} \mid 0 < x_i < a_i \text{ and } 1 < \sum_{j=0}^{2k} \frac{x_i}{a_i} < 2 \mod 2\Big\}. \end{aligned} \tag{26}$$

Using a formula of Eisenstein, Zagier (cf. [Hir71]) has rewritten this as:

$$\tau(M^{4k}(\mathbf{a})) = \frac{(-1)^k}{N} \sum_{j=0}^{N-1} \cot\frac{\pi(2j+1)}{2N} \cot\frac{\pi(2j+1)}{2a_1} \cdots \cot\frac{\pi(2j+1)}{2a_{2k+1}}, \tag{27}$$

where N is any common multiple of the a_i's.

Both formulas are quite well suited to computer use. A simple C code called `sig.c`, which for any m-tuple with $m = 2k+1 = 5, 7, 9$, computes the signature $\tau(\mathbf{a}) := \tau(M^{4k}_{\mathbf{a}})$ and the diffeomorphism type of the link using either of the above formulas can be found in [BGKT03].

Example 40. Let us consider the Brieskorn-Pham link $L(5,3,2,2,2)$. By Brieskorn Graph Theorem this is a homotopy 7-sphere. One can easily compute the signature using (2.14) to find out that $\tau(L(5,3,2,2,2)) = 8$. Hence $L(5,3,2,2,2) = \Sigma_1^7$ is an exotic 7-sphere and it is called Milnor generator (all others can be obtained from it by taking connected sums). It is interesting to note that one does not need a computer to find the signature of $L(6k-1,3,2,2,2)$. This was done in Brieskorn original paper [Bri66] where he used the combinatorial formula 26 to show that all the 28 diffeomorphism types of homotopy 7-spheres are realized by taking $k = 1, \cdots, 28$.

Question 41. Suppose that $\mathcal{Z}_f$ is Fano. How can one prove the existence of a Kähler-Einstein metric on $\mathcal{Z}_f$? When this can be done successfully we automatically get a Sasakian-Einstein metric on the link L_f.

7. Calabi Conjecture I

Recall that on a Kähler manifold the Ricci curvature 2-form ρ_ω of any Kähler metric represents the cohomology class $2\pi c_1(M)$. The well-known Calabi Conjecture is the question whether or not the converse is also true. To be more specific we begin with a couple of definitions

Definition 42. Let (M, J, g, ω_g) be a compact Kähler manifold. The Kähler cone of M

$$K(M) = \Big\{ [\omega] \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R}) \ \mid [\omega] = [\omega_h] \quad \text{for some Kähler metric } h \Big\}$$

is the set of all possible Kähler classes on M.

The global $i\partial\bar{\partial}$-lemma provides for a very simple description of the space of Kähler metrics $\mathcal{K}_{[\omega]}$. Suppose we have a Kähler metric g in a with Kähler class $[\omega_g] = [\omega] \in K(M)$. If $h \in \mathcal{K}_{[\omega]}$ is another Kähler metric then, up to a constant, there exists a global function $\phi \in C^\infty(M,\mathbb{R})$ such that $\omega_h - \omega_g = i\partial\bar{\partial}\phi$. We could fix the constant by requiring, for example, that $\int_M \phi\omega_g^n = 0$. Hence, we have

Corollary 43. *Let (M, J, g, ω_g) be a compact Kähler manifold with $[\omega_g] = [\omega] \in K(M)$. Then, relative to the metric g the space $\mathcal{K}_{[\omega]}$ of all Kähler metric in the same Kähler class can be described as*

$$\mathcal{K}_{[\omega]} = \{\phi \in C^\infty(M,\mathbb{R}) \mid \omega_h = \omega_g + i\partial\bar{\partial}\phi > 0, \quad \int_M \phi\omega_g^n = 0\},$$

where the 2-form $\omega_h > 0$ means that $\omega_h(X, JY)$ is a Hermitian metric on M.

We have the following theorem

Theorem 44. *Let (M, J, g, ω_g) be a compact Kähler manifold, $[\omega_g] = \omega \in K(M)$ the corresponding Kähler class and ρ_g the Ricci form. Consider any real $(1,1)$-form Ω on M such that $[\Omega] = 2\pi c_1(M)$. Then there exists a unique Kähler metric $h \in \mathcal{K}_{[\omega]}$ such that $\Omega = \rho_h$.*

The above statement is the celebrated Calabi Conjecture which was posed by Eugene Calabi in 1954. The conjecture in its full generality was eventually proved by Yau in 1978 [Yau78].

Let us reformulate the problem using the global $i\partial\bar{\partial}$-lemma. We start with a given Kähler metric g on M in the Kähler class $[\omega_g] = [\omega]$. Since ρ_g also represents $2\pi c_1(M)$ there exists a globally defined function $f \in C^\infty(M, \mathbb{R})$ such that

$$\rho_g - \Omega = i\partial\bar{\partial}f.$$

Appropriately, f may be called *a discrepancy potential function* for the Calabi problem and we could fix the constant by asking that $\int_M (e^f - 1)\omega_g = 0$.

Now, suppose that the desired solution of the problem is a metric $h \in \mathcal{K}_{[\omega]}$. We know that the Kähler form of h can be written as

$$\omega_h = \omega_g + i\partial\bar{\partial}\phi,$$

for some smooth function $\phi \in C^\infty(M, \mathbb{R})$. We normalize ϕ as in the previous corollary. Combining these two equations we see that

$$\rho_h - \rho_g = i\partial\bar{\partial}f.$$

If we define a smooth function $F \in C^\infty(M, \mathbb{R})$ relating the volume forms of the two metrics $\omega_h^n = e^F \omega_g^n$ then the left-hand side of the above equation takes the following form

$$i\partial\bar{\partial}F = \rho_h - \rho_g = i\partial\bar{\partial}f,$$

or simply $i\partial\bar{\partial}(F - f) = 0$. Hence, $F = f + c$. But since we normalized $\int_M (e^f - 1)\omega_g = 0$ we must have $c = 0$. Thus, $F = f$, or $\omega_h^n = e^f \omega_g^n$. We can now give two more equivalent formulations of the Calabi Problem.

Theorem 45. *Let (M, J, g, ω_g) be a compact Kähler manifold, $[\omega_g] = [\omega] \in K(M)$ the corresponding Kähler class and ρ_g the Ricci form. Consider any real $(1,1)$-form on M such that $[\Omega] = 2\pi c_1(M)$. Let $\rho_g - \Omega = i\partial\bar{\partial}f$, with $\int_M (e^f - 1)\omega_g = 0$.*

(1) *There exists a unique Kähler metric $h \in \mathcal{K}_{[\omega]}$ whose volume form ω_h^n equals $e^f \omega_g^n$.*
(2) *Let $(\mathcal{U}, z_1, \ldots, z_n)$ be a local complex chart on M with respect to which the metric $g = (g_{i\bar{j}})$. Then, up to a constant, there exists a unique smooth function ϕ in $\mathcal{K}_{[\omega]}$, which satisfies the following equation*

$$\frac{\det\Big(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\Big)}{\det(g_{i\bar{j}})} = e^f.$$

The equation in (2) is called the Monge-Ampère equation. Part (1) gives a very simple geometric characterization of the Calabi-Yau theorem. On a compact Kähler manifold one can always find a metric with arbitrarily prescribed volume form. The uniqueness part of this theorem was already proved by Calabi. This part involves only Maximum Principle. The existence proof uses the continuity method discussed briefly in Section 9 and it involves several difficult a priori estimates. These were found by Yau in 1978. We have the following:

Corollary 46. *Let (M, J, g, ω_g) be a compact Kähler manifold.*

(1) *If $c_1(M) > 0$ then M admits a Kähler metric of positive Ricci curvature.*
(2) *If $c_(M) = 0$ then M admits a unique Kähler Ricci-flat metric.*

It is "folklore" that the Calabi-Yau Conjecture is also true for compact orbifolds. In the context of Sasakian geometry with its characteristic foliation, the transverse space $\mathcal{Z}$ is typically a compact Kähler orbifold. In the context of foliations a **transverse Yau theorem** was proved by El Kacimi-Alaoui in 1990 [EKA90].

Theorem 47. *If $c_1(\mathcal{F}_\xi)$ is represented by a real basic $(1,1)$-form ρ^T, then it is the Ricci curvature form of a unique transverse Kähler form ω^T in the same basic cohomology class as $d\eta$.*

In the language of positive Sasakian manifolds this theorem provides the basis for proving [BGN03a]

Theorem 48. *Any positive Sasakian manifold (M,g) admits a Sasakian metric g' of positive Ricci curvature.*

There is a similar statement for negative and null Sasakian structures. This is studied in a forthcoming article [BGM04].

Corollary 49. *Let $L(f)$ be the link of an isolated hypersurface singularity where f is weighted homogeneous polynomial of weight $\mathbf{w}$ and degree d. If $\sum_i w_i - d > 0$ then $L(f)$ admits a Sasakian metric of positive Ricci curvature. In particular,*

(1) *$\#_k M_\infty = \#_k(S^2\times S^3) \simeq L(X_{k+1}\subset \mathbb{P}(1,1,1,k))$ admits Sasakian metric of positive Ricci curvature for all $k\geq 1$.*
(2) *The links $L(6k-1,3,2,\ldots,2)\simeq \Sigma_k^{4n+3}$ and $L(2p+1,2,\ldots,2)\simeq \Sigma_p^{4n+1}$ admit Sasakian metric of positive Ricci curvature for each n,k and p. Hence, all homotopy spheres that bound parallelizable manifolds admit metrics of positive Ricci curvature.*

Part (1) is the 5-dimensional case of a well known theorem of Sha and Yang [SY91]. Their theorem asserts the existence of positive Ricci curvature metrics on connected sums of product of spheres. Part (2) is in its final form a theorem of Wraith [Wra97]. Both papers rely on techniques of surgery theory. The proofs given in [BGN03a, BGN03c] are completely different being a consequence of the orbifold version of Yau's theorem.

8. Calabi Conjecture II – Kähler-Einstein Metrics

As pointed out in the previous section one of the consequences of the Yau's theorem is that a compact Kähler manifold with $c_1(M)=0$ must admit a Ricci-flat, hence, Einstein metric. More generally, we can consider existence of Kähler-Einstein metrics with arbitrary Einstein constant λ. By scaling we can assume that $\lambda = 0,\pm 1$. Specifically, let (M,J,g,ω_g) be a compact Käher manifold. We would like to know if one can always find a Kähler-Einstein metric $h\in\mathcal{K}_{[\omega_g]}$. Recall that on a Kähler-Einstein manifold $\rho_g=\lambda\omega_g$. This implies that $2\pi c_1(M)=\lambda[\omega_g]$. Now, if $c_1(M)>0$ we must have $\lambda=+1$ because $[\omega_g]$ is the Kähler class. Similarly, when $c_1(M)<0$ the only allowable sign of a Kähler-Einstein metric on M is $\lambda=-1$. Clearly, when $c_1(M)=0$ we must have $\lambda=0$ as $[\omega_g]\neq 0$. As we have already pointed out the $\lambda=0$ case follows from Yau's solution to the Calabi conjecture. For the reminder of this lecture we shall assume that $\lambda=\pm 1$.

Let (M,J,g,ω_g) be a Kähler manifold and $[\omega_g]=[\omega]\in K(M)$ the Kähler class. Let us reformulate the existence problem using the global $i\partial\bar\partial$-lemma. Suppose there exists an Einstein metric $h\in\mathcal{K}_{[\omega]}$. Starting with the original Kähler metric g on M we have a globally defined function $f\in C^\infty(M,\mathbb{R})$ such that

$$\rho_g-\lambda\omega_g = i\partial\bar\partial f. \tag{28}$$

As before we will call f *a discrepancy potential function.* We also fix the constant by asking that $\int_M (e^f - 1)\omega_g = 0$. Let $h \in \mathcal{K}_{[\omega]}$ be an Einstein metric for which $\rho_h = \lambda\omega_h$. Using the global $i\partial\bar{\partial}$-lemma once again we have a globally defined function $\phi \in C^\infty(M, \mathbb{R})$ such that $\omega_h - \omega_g = i\partial\bar{\partial}\phi$. We shall fix the constant in ϕ later. Using these two equations we easily get

$$\rho_g - \rho_h = i\partial\bar{\partial}(f - \lambda\phi).$$

Defining F so that $\omega_h^n = e^F\omega_g^n$ we can write this equation as

$$i\partial\bar{\partial}F = i\partial\bar{\partial}(f - \lambda\phi).$$

This implies that $F = f - \lambda\phi + c$. We have already fixed the constant in f so c depends only on the choice of ϕ. We can make $c = 0$ by choosing ϕ such that $\int_M (e^{f-\lambda\phi} - 1)\omega_g^n = 0$. Hence, we have the following

Proposition 50. *Let (M, J) be a compact Kähler manifold with $\lambda c_1(M) > 0$, where $\lambda = \pm 1$. Let $[\omega] \in K(M)$ be a Kähler class and g, h two Kähler metrics in $\mathcal{K}_{[\omega]}$ with Ricci forms ρ_g, ρ_h. Let $f, \phi \in C^\infty(M, \mathbb{R})$ be defined by $\rho_g - \lambda\omega_g = i\partial\bar{\partial}f$, $\omega_h - \omega_g = i\partial\bar{\partial}\phi$. Fix the relative constant of $f - \lambda\phi$ by setting $\int_M (e^{f-\lambda\phi} - 1)\omega_g = 0$. Then the metric h is Einstein with Einstein constant λ if and only if ϕ satisfies the following Monge-Ampère equation*

$$\omega_h^n = e^{f-\lambda\phi}\omega_g^n,$$

which in a local complex chart $(\mathcal{U}, z_1, \ldots, z_n)$ written as

$$\frac{\det\Big(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\Big)}{\det(g_{i\bar{j}})} = e^{f-\lambda\phi}.$$

Note that by setting $\lambda = 0$ we get the Monge-Ampère equation for the original Calabi problem. The character of the Monge-Ampère equation above very much depends on the choice of λ. The case of $\lambda = -1$ is actually the simplest as the necessary a priori C^0-estimates can be derived using the Maximum Principle. This was done by Aubin [Aub76] and independently by Yau [Yau78]. We have

Theorem 51. *Let (M, J, g, ω_g) be a compact Kähler manifold with $c_1(M) < 0$. Then there exists a unique Kähler metric $h \in \mathcal{K}_{[\omega_g]}$ such that $\rho_h = -\omega_h$.*

When $k = +1$ the problem is much harder. It has been known for quite some time that there are non-trivial obstructions to the existence of Kähler-Einstein metrics. Let $\mathfrak{h}(M)$ be the complex Lie algebra of all holomorphic vector fields on M. Matsushima [Mat57] proved that on a compact Kähler-Einstein manifold with $c_1(M) > 0$, $\mathfrak{h}(M)$ must be reductive, i.e. $\mathfrak{h}(M) = Z(\mathfrak{h}(M)) \oplus [\mathfrak{h}(M), \mathfrak{h}(M)]$ where $Z(\mathfrak{h}(M))$ denotes the center of $\mathfrak{h}(M)$. Now, suppose (M, g, J, ω) is a Kähler manifold and let f be the discrepancy potential defined by (28). Further, let $X \in \mathfrak{h}(M)$ and define

$$F(X) = \int_M X(f) d\mathrm{vol}_g. \tag{29}$$

At first glance $F(X)$ appears to depend on the Kähler metric. However, Futaki shows that this is not the case: $F(X)$ does not depend on the choice of the metric in $h \in \mathcal{K}_{[\omega_g]}$. Hence, $F : \mathfrak{h}(M) \longrightarrow \mathbb{C}$ is well-defined and it is called the *Futaki functional* or *character* [Fut83, Fut87, Fut88]. In particular, if an Einstein metric exists than we can choose f constant. Hence,

Corollary 52. *Suppose (M,g,J,ω) admits a Kähler-Einstein metric $h\in\mathcal{K}_{[\omega_g]}$. Then F must be identically zero.*

Remark 53. Futaki also showed that there are Fano manifolds for which $\mathfrak{h}(M)$ is reductive, but F is non-trivial [Fut88]. A folklore conjecture attributed to Calabi asserted that in the case $\mathfrak{h}(M)=0$ there are no obstructions to finding a Kähler-Einstein metric. This conjecture was disproved by Tian. First, Ding and Tian [DT92] constructed an example of an orbifold del Pezzo surface with $\mathfrak{h}(M)=0$ and no Kähler-Einstein metric. Later Tian found an example of a smooth Fano 3-fold with $\mathfrak{h}(M)=0$ and no Kähler-Einstein metric [Tia99]. In [Tia97] Tian shows that two different conditions are necessary for the existence of a Kähler-Einstein metric. One condition involves the generalized Futaki functional of every special degeneration of the manifold. The other condition is Mumford stability with respect to a certain polarization. Tian conjectures that the two conditions are equivalent and that they are also sufficient. This conjecture is still open. However, even if it is true neither of the conditions are easily checked for an arbitrary compact Fano manifold (orbifold). In principle, one should be able to compute generalized Futaki invariants for any Fano hypersurface $X_d\subset\mathbb{P}(\mathbf{w})$. (see, for example, Lu [Lu99], and Yotov [Yot99] for the computation of generalized Futaki invariants in the case of smooth complete intersections).

9. The Continuity Method and Kähler-Einstein Orbifolds

Let us briefly describe the main aspects of the continuity method. Let's say we are trying to show existence of Kähler-Einstein metric of positive sign. Here one tries to solve the Monge-Ampère equation

$$\frac{\det(g_{i\bar{j}}+\partial_i\bar{\partial}_{\bar{j}}\phi_t)}{\det(g_{i\bar{j}})}=e^{-t\phi_t+f},\quad g_{i\bar{j}}+\partial_i\bar{\partial}_{\bar{j}}\phi_t>0$$

for $t\in[0,1]$. Yau's Theorem tells us that this has a solution for $t=0$, and we try to solve this for $t=1$, where the metric will be Kähler-Einstein. The so called continuity method sets out to show that the interval where solutions exist is both open and closed. Openness follows from the Implicit Function Theorem, but there are well known obstructions to closedness. This problem has been studied most recently by Demailly and Kollár who work in the orbifold category [DK01]. Closedness is equivalent to the uniform boundedness of the integrals

$$\int_{\mathcal{Z}}e^{-\gamma t\phi_t}\omega_0^n$$

for any $\gamma\in(\frac{n}{n+1},1)$, where ω_0 is the Kähler form of h_0. This means that the *multiplier ideal sheaf* of Nadel [Nad90] $\mathcal{J}(\gamma\phi)=\mathcal{O}_{\mathcal{Z}}$ for all $\gamma\in(\frac{n}{n+1},1)$.

We will illustrate how the method works for links. The approach developed by Demailly and Kollár in [DK01] yields the following general theorem:

Theorem 54. *Let X^{orb} be a compact, n-dimensional orbifold such that $K^{-1}_{X^{orb}}$ is ample. The continuity method produces a Kähler-Einstein metric on X^{orb} if the* *following holds: There is a $\gamma > \frac{n}{n+1}$ such that for every $s \geq 1$ and for every holomorphic section $\tau_s \in H^0(X^{orb}, K^{-s}_{X^{orb}})$ the following integral is finite:*

$$\int |\tau_s|^{-\frac{2\gamma}{s}} \omega_0^n \ < \ +\infty.$$

In general, this condition is not hard to check. For hypersurfaces the situation is somewhat simpler and one gets

Corollary 55. *Let $\mathcal{Z}_f =: X_d \subset \mathbb{P}(w)$ be a hypersurface in $\mathbb{P}(w)$ given by the vanishing of the weighted homogeneous polynomial f of weight $\mathbf{w}$ and degree d. Let $Y_f := \{\{f = 0\} \subset \mathbb{C}^{n+1}\}$ so that $\mathcal{Z}_f = (Y_f \setminus \{0\})/\mathbb{C}^*_{\mathbf{w}}$. Assume $\mathcal{Z}_f$ is Fano, that is $d < \sum w_i$. The continuity method produces a Kähler-Einstein metric on $\mathcal{Z}_f$ if the following holds: There is a $\gamma > \frac{n}{n+1}$ such that for every weighted homogeneous polynomial g of weighted degree $s(\sum w_i - d))$, not identically zero on Y_f, the function*

$$|g|^{-\gamma/s} \quad \textit{is locally } L^2 \textit{ on } Y_f \setminus \{0\}.$$

Example 56. [**Log Del Pezzo Surfaces**] One can take $X_d \subset \mathbb{P}(w_0, w_1, w_2, w_3)$ with $I = \mathrm{Ind}(X_d) = \sum_i w_i - d > 0$. Assuming X_d has only isolated orbifold singularities one can classify all such log del Pezzo surfaces and check if the conditions of Corollary 55 are satisfied. This was done by Johnson and Kollár in [JK01b] when $I = 1$. In some cases the existence question was left open and more recently Araujo completely finished the analysis in [Ara02]. On the other hand, the similar question can be considered for an arbitrary index $I \geq 1$. This was done to a limited extend in [BGN03b]. That is, we enumerated all log del Pezzo surfaces for $2 < I \leq 10$ which can possibly admit Kähler-Einstein metrics as a consequence of the Corollary 55. Sometimes we were able to prove the existence, but unlike in the $I = 1$ situation, it has not been done for all the candidates. Furthermore, it remains to show that for $I > 10$ there are no examples of log del Pezzo surfaces satisfying condition of Corollary 55. As a result of this analysis we got Sasakian-Einstein structures on certain connected sums of $S^2 \times S^3$. The table below summarizes the results of [BGN03b, BGN02a, BG03b].

$\#_k(S^2 \times S^3)$	$\mathbf{N}$=$(n_0, n_1, \ldots, n_7, n_8)$	Example
$k = 1$	$(14 + 1, 0, 0, 0, 0, 0, 0, 0, 0)$	$X_{76} \subset \mathbb{P}(11, 13, 21, 38)$
$k = 2$	$(21, 2, 0, 0, 0, 0, 0, 0, 0)$	$X_{57} \subset \mathbb{P}(7, 8, 19, 25)$
$k = 3$	$(\aleph_0 + 1, 4, 2, 0, 0, 0, 0, 0, 0)$	$X_{64} \subset \mathbb{P}(7, 8, 19, 32)$
$k = 4$	$(2\aleph_0 + 1, 0, 1, 2, 0, 0, 0, 0, 0)$	$X_{20} \subset \mathbb{P}(3, 4, 5, 10)$
$k = 5$	$(2\aleph_0, 0 + 1, 0, 1, 1, 0, 0, 0, 0)$	$X_{28} \subset \mathbb{P}(3, 5, 7, 14)$
$k = 6$	$(2\aleph_0, 0, 0 + 1, 3, 0, 5, 0, 0, 0)$	$X_{18} \subset \mathbb{P}(2, 3, 5, 9)$
$k = 7$	$(0, 0, 0, 0 + 1, 0, \aleph_0, 0, 0, 0)$	$X_{8k+4} \subset \mathbb{P}(2, 2k + 1, 2k + 1, 4k + 1)$
$k = 8$	$(0, 0, 0, 0, 0 + 1, 0, 0, 0, 2)$	$X_{10} \subset \mathbb{P}(1, 2, 3, 5)$
$k = 9$	$(0, 0, 0, 0, 0, 0, 0, 0, 1)$	$X_{16} \subset \mathbb{P}(1, 3, 5, 8)$

For each $\#_k(S^2 \times S^3)$, with $1 \leq k \leq 9$ we list $\mathbf{N} = (n_0, n_1, n_2, \ldots, n_8)$, where n_i is the number of distinct families of links with complex dimension of deformation parameters equal to i. The largest family constructed this way had complex dimension 8 so that $n_i = 0$ when $i > 8$ for all $k = 1, \ldots, 9$. Also, the method produced no examples for $k > 9$. Furthermore, for instance, $n_0 = 2\aleph_0$ means that there are two distinct infinite sequence of examples which have no deformation parameters. We include regular examples in the count by writing "+1" where appropriate. In the third column we give an example with the largest moduli.

Remark 57. Example 56 forces an obvious question. Are there Sasakian-Einstein structures on $\#_k(S^2\times S^3)$ for arbitrary k? Recently Kollár [Kol04] has been able to answer this question in the affirmative. His method differs substantially from the one described here. The idea is to consider Seifert bundles over smooth surfaces, but with a non-trivial orbifold structure. Such a construction is more flexible in obtaining log del Pezzo surfaces with orbifold Kähler-Einstein metrics and more complicated topology. In particular, Kollár proves

Theorem 58. *For every integer $k\geq 6$ there are infinitely many complex $(k-1)$-dimensional families of Einstein metrics on $\#_k(S^2\times S^3)$.*

Combining this remarkable result with the links of hypersurfaces in Example 56, we get the following

Corollary 59. *Let M be any compact, smooth, simply-connected 5-manifold which is spin and has no torsion in $H_2(M,\mathbb{Z})$. Then M admits a Sasakian-Einstein metric.*

Example 60. [**Brieskorn-Pham Links**] Now, we consider Brieskorn–Pham links as defined in Equation (23). Let $Y(\mathbf{a}) := (\sum_{i=0}^n z_i^{a_i}=0)\subset\mathbb{C}^{n+1}$. One can easily see that $d=\mathrm{lcm}(a_i : i=0,\ldots,n)$ is the degree of $f_{\mathbf{a}}(\mathbf{z})$ and $w_i=d/a_i$ are the weights. The transverse space $\mathcal{Z}(\mathbf{a}) := (Y(\mathbf{a})\setminus\{0\})/\mathbb{C}^*$ is a Fano orbifold if and only if

$$1<\sum_{i=0}^n\frac{1}{a_i}.$$

More generally, we consider weighted homogeneous perturbations

$$Y(\mathbf{a},p) := (\sum_{i=0}^n z_i^{a_i}+p(z_0,\ldots,z_n)=0)\subset\mathbb{C}^{n+1},$$

where weighted degree$(p)=d$. The genericity condition we need, which is always satisfied by $p\equiv 0$ is: The intersections of $Y(\mathbf{a},p)$ with any number of hyperplanes $(z_i=0)$ are all smooth outside the origin.

The continuity methods produces the following sufficient conditions for the quotient $Y(\mathbf{a},p)/\mathbb{C}^*$ to admit a Kähler-Einstein metric [BGK03]:

Theorem 61. *Let $\mathcal{Z}(\mathbf{a},p)$ be the transverse space of a perturbed Brieskorn-Pham link $L(\mathbf{a},p)$. Let $C_i=\mathrm{lcm}(a_0,\ldots,\hat{a}_i,\ldots,a_n)$, $b_i=\gcd(C_i,a_i)$. Then $\mathcal{Z}(\mathbf{a},p)=Y(\mathbf{a},p)/\mathbb{C}^*$ is Fano and it has a Kähler-Einstein metric if*

(1) $1<\sum_{i=0}^n\frac{1}{a_i}$,
(2) $\sum_{i=0}^n\frac{1}{a_i}<1+\frac{n}{n-1}\min_i\{\frac{1}{a_i}\}$, *and*
(3) $\sum_{i=0}^n\frac{1}{a_i}<1+\frac{n}{n-1}\min_{i,j}\{\frac{1}{b_ib_j}\}$.

In this case the link $L(\mathbf{a},p)$ admits a Sasakian-Einstein metric with one-dimensional isometry group.

10. Sasakian-Einstein Structures on Brieskorn-Pham Links

In this section we will discuss some consequences of Theorem 61 of the previous section. We will investigate two separate cases: when $L(\mathbf{a})$ is a homotopy sphere and (2) when $L(\mathbf{a})$ is a rational homology sphere with non-vanishing torsion. If $L(\mathbf{a})$ is a homotopy sphere, for a fixed n, there are only finitely many examples of $\mathbf{a}$'s satisfying all three conditions of Theorem 61. However, the number of examples as well as the moduli grows doubly exponentially with each odd dimension. One can list all solution in dimensions 5 and 7 without difficulties. However, already in dimension 9 that task is too overwhelming. It is quite clear that many of our links will actually be diffeomorphic to standard spheres. Hence, let us begin with a remark concerning what is known about Einstein metrics on spheres in general.

Remark 62. Any standard sphere S^n, $n > 1$, admits a metric of constant positive sectional curvature. These canonical metrics are $SO(n+1)$-homogeneous and Einstein, i.e., the Ricci curvature tensor is a constant positive multiple of the metric. The spheres S^{4m+3}, $m > 1$ are known to have another $Sp(m+1)$-homogeneous Einstein metric discovered by Jensen [Jen73]. The metric is obtained from the "quaternionic Hopf fibration" $S^3 \to S^{4m+3} \to \mathbb{HP}^m$. Since both base and fiber are Einstein spaces with positive Einstein constant we obtain two Einstein metrics in the canonical variation. The second metric is also called "squashed sphere" metric in some physics literature. In addition, S^{15} has a third Spin(9)-invariant homogeneous Einstein metric discovered by Bourguignon and Karcher in 1978 [BK78]. The existence of such a metric has to do with the fact that S^{15}, in addition to fibering over $\mathbb{HP}^2$, also fibers over S^8 with fiber S^7. Thus the15-sphere admits 3 different homogeneous Einstein metrics. Ziller proved that these are the only homogeneous Einstein metrics on spheres [Zil82]. Böhm obtained infinite sequences of non-isometric Einstein metrics, of positive scalar curvature, on S^5, S^6, S^7, S^8, and S^9 [Böh98]. Böhm's metrics are of cohomogeneity one and they are not only the first inhomogeneous Einstein metrics on spheres but also the first non-canonical Einstein metrics on even-dimensional spheres.

Example 63. [**Sasakian-Einstein Metrics on S^5**] Consider $L(2,3,7,m)$. These are homotopy spheres as long as m is relatively prime to at least two of $2,3,7$. It is easy to see that $L(2,3,7,m)$ satisfies the condition of Theorem 61 if $5 \leq m \leq 41$ which gives 27 cases. The link $L(2,3,7,35)$ admits deformations, i.e., $C(u,v)$ is any sufficiently general homogeneous septic polynomial, then the link of

$$z_0^2 + x_1^3 + C(z_2, z_3^5)$$

also gives a Sasakian-Einstein metric on S^5. The relevant automorphism group of $\mathbb{C}^4$ is

$$(z_0, z_1, z_2, z_3) \mapsto (z_0, z_1, \alpha_2 z_2 + \beta z_3^5, \alpha_3 z_3).$$

Hence we get a $2(8-3) = 10$ real dimensional family of Sasakian-Einstein metrics on S^5. There are other examples, 68 in total, and we get [BGK03]

Theorem 64. *On S^5 there are at least 68 inequivalent families of Sasakian-Einstein metrics. Some of these families admit non-trivial continuous Sasakian-Einstein deformations. The biggest constructed family has has real dimension 10.*

Example 65. [**Sasakian-Einstein Metrics on Homotopy 7-Spheres**] Similarly $L(2,3,7,43,43\cdot 31)$ is the standard 7-sphere with a $2(43-2)=82$-dimensional family of Sasakian-Einstein metrics. One can do a computer search of all homotopy 7-spheres that satisfy the numerical conditions of Theorem 61. One finds 8610 such links. An additional computation of the Hirzebruch signature of the parallelizable manifold whose boundary is L shows that they are more or less evenly distributed among the 28 oriented diffeomorphism classes. This way we get [BGK03, BGKT03]

Theorem 66. *Let Σ^7_i, be a homotopy 7-sphere corresponding to the element $i\in bP_8\simeq \mathbb{Z}_{28}\simeq \Theta_7$ in the Kervaire-Milnor group. Σ^7_i admits at least n_i inequivalent families of Sasakian-Einstein metrics, where $(n_1,\ldots,n_{28})$ =*(376, 336, 260, 294, 231, 284, 322, 402, 317, 309, 252, 304, 258, 390,409, 352, 226, 260, 243, 309, 292, 452, 307, 298, 230, 307, 264, 353)*, giving a total of* 8610 *cases. In each oriented diffeomorphism class some of the families depend on a moduli. In particular, the standard 7-sphere Σ^7_{28} admits an 82-dimensional family of inequivalent Sasakian-Einstein metrics.*

Example 67 (**Sasakian-Einstein Metrics on Kervaire Spheres).** Let $\{c_i\}$ be an infinite sequence defined by the recursion relation

$$c_{k+1}=c_1\cdots c_k+1=c_k^2-c_k+1, \qquad c_1=2.$$

Consider sequences of the form $L(\mathbf{a})=L(2c_1,2c_2,\ldots,2c_{m-2},2,a_{m-1})$ where a_{m-1} is relatively prime to all the other a_is. Such $L(\mathbf{a})$ are rational homotopy spheres. The condition of Theorem 61 is satisfied if $2c_{m-2}<a_{m-1}<2c_{m-1}-2$. In particular, we can ask for a_{m-1} to be prime and estimate the number of primes in the range $(2c_{m-2},2c_{m-1}-2)$ which gives double exponential growth in m by the Prime Number Theorem. For odd m, $L(\mathbf{a})$ the standard sphere if one of the exponents of $\mathbf{a}$ equals to $\pm 1 \mod 8$ and it is the Kervaire sphere if one of the exponents equals $\pm 3 \mod 8$ [Bri66]. It is easy to check for all values of m that we get at least one solution of both types. Hence, we get [BGK03]

Proposition 68. *Theorem 61 yields a doubly exponential number of inequivalent Sasakian-Einstein metrics on both the standard and Kervaire spheres in every odd dimension $4m+1$.*

Example 69. $L(\mathbf{a})=L(2,3,7,43,1807,3263443,10650056950807,m)$ is just the standard 13-sphere for any suitably chosen m as bP_{14} is trivial. If we choose $m=(10650056950807-2)\cdot 10650056950807$ we get $2(10650056950807-2)$-dimensional family of deformations. By contrast the only Einstein metric on S^{13} known previously was the the canonical one.

All these examples point towards the following:

Conjecture 70. *All odd-dimensional spheres that bound parallazllizable manifolds admit Sasakian-Einstein metrics.*

The conjecture is true in dimension $4m+1$ by Proposition 68. It is also true in dimension 7. In addition, using computer programs we were able to verify that the conjecture holds in dimensions 11 and 15. Computational verification in arbitrary dimension $4m+3$ is not possible. On the other hand it does appear that Brieskorn-Pham links satisfying the conditions of Theorem 61 realize all oriented diffeomorphism types of homotopy spheres in every dimension.

Example 71. [**S-E Structures on Rational Homology Spheres**] Our final examples of Brieskorn-Pham links is that of $L(m,m,\ldots,m,k)$ with $\gcd(k,m)=1$. By Brieskorn Graph Theorem this is a rational homology sphere in every dimension. The conditions of Theorem 61 are satisfied as long as $k > m(m-1)$. The homology of $L(m,m,\ldots,m,k)$ contains torsion in $H_{m-1}(L,\mathbb{Z})$. It's order can be easily computed and it is $k^{b_{m-1}}$, where b_{m-1} is the $(m-1)^{th}$ Betti number of the null link $L(m,\ldots,m)$ which is regular circle bundle over Fermat hypersurface of degree m in $\mathbb{P}^{m-1}$. For example, with the appropriate restriction on k we have $|H_2(3,3,3,k)| = k^2$ and $|H_3(4,4,4,4,k)| = k^{21}$, a.s.o. In particular, we get [BG03a]

Proposition 72. *In each odd dimension greater than 3 there are infinitely many smooth, compact, simply-connected rational homology spheres admitting Sasakian-Einstein structures.*

Torsion group of each of the links $L(m,m,\ldots,m,k)$ can also be computed using an algorithm conjectured by Orlik [Or72] and proved by Randell in some special cases [Ran75]. In particular, Orlik's conjecture is true for all Brieskorn-Pham links and can be used to compute torsion of various examples discussed here [BG04]. Let us consider the 5-dimensional case in more detail. Using Theorem 61 and Orlik's algorithm we get the following list [OW75, Or72]

$L(\mathbf{a})$	Torsion
$L(3,3,3,k)$, $\gcd(k,3)=1, k>5$	$\mathbb{Z}_k \oplus \mathbb{Z}_k$
$L(2,4,4,k)$, $\gcd(k,2)=1$, $k>10$	$\mathbb{Z}_k \oplus \mathbb{Z}_k$
$L(2,3,6,k)$, $\gcd(k,6)=1$, $k>12$	$\mathbb{Z}_k \oplus \mathbb{Z}_k$

The three series above satisfy $\sum_{i=0}^{2}\frac{1}{a_i}=0$. In the case when $\sum_{i=0}^{2}\frac{1}{a_i}<0$ one can easily see that there are 16 more rational homology 5-spheres which satisfy inequalities of Theorem 61. An example of such a link is $L(3,4,4,4)$ whose 2-torsion equals $\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3$. Hence, $L(3,4,4,4)$ is diffeomorphic to $M_3\#M_3\#M_3$. For the torsion computation as well as the full list we refer interested readers to [BG04]. In particular, we get the following

Theorem 73. *The Barden manifold M_k admits Sasakian-Einstein structure for each $k>5$ prime to 3 and for each $k>10$ prime to 2.*

Example 74. Finally, note that the links in the last table have companions with non-trivial second Betti number and by Theorem 61 they too admit Sasakian-Einstein metrics. We list the relevant information in the table below:

$L(\mathbf{a})$	$b_2(L(\mathbf{a}))$
$L(3,3,3,3n)$, $n>2$	6
$L(2,4,4,2n)$, $\gcd(n,2)=1$, $n>5$	3
$L(2,4,4,4n)$, $n>2$	7
$L(2,3,6,2n)$, $\gcd(n,3)=1$, $n>12$	2
$L(2,3,6,3n)$, $\gcd(n,2)=1$, $n>12$	4
$L(2,3,6,6n)$, $n>4$	8

All the above links have 2-torsion equal to $\mathbb{Z}_n\oplus\mathbb{Z}_n$ which can be verified by Orlik's algorithm. In addition, just as in the case of rational homology 5-spheres, one can see that there are 16 exceptional cases of links which satisfy the inequalities of Theorem 61 and have non-vanishing second Betti number. An example of such a link is $L(2,4,6,10)$ which has $b_2 = 1$. Each line of the previous table gives infinite series of decomposable Barden manifolds of mixed type (i.e., having both a free part and a torsion in its second homology). For instance, we can rephrase first line as

Proposition 75. *The manifolds $6M_\infty \# M_n$ admit families of Sasakian-Einstein structures for any $n > 2$.*

Question 76. We conclude by asking some questions about Sasakian-Einstein structures on certain Barden manifolds:

(1) Does every Barden manifold M_k admit a Sasakian-Einstein structure? Is it possible that certain torsion in $H_2(M,\mathbb{Z})$ obstructs the existence of Sasakian-Einstein, or even positive Sasakian structures on M?
(2) Which Barden manifolds with $b_2(M) \neq 0$ and non-vanishing 2-torsion admit Sasakian-Einstein structures?

Remark 77. [**Einstein Metrics on Barden Manifolds**] The following table summarizes what we know about existence of Sasakian ($\mathcal{S}$), negative Sasakian ($\mathcal{S}_-$), null Sasakian ($\mathcal{S}_0$), positive Sasakian ($\mathcal{S}_+$), regular Sasakian-Einstein ($\mathcal{S}^r$) and non-regular Sasakian-Einstein ($\mathcal{S}^{nr}$) structures on some Barden manifolds. The last column (OE) indicates if an Einstein metric other than Sasakian-Einstein is known. Finally, "some k" means that we know existence of a given structure for some (possibly infinitely many) k's but we do not know if it exists for all k.

M	$\mathcal{S}$	$\mathcal{S}_-$	$\mathcal{S}_0$	$\mathcal{S}_+$	$\mathcal{SE}^r$	$\mathcal{SE}^{nr}$	OE
X_{-1}	?	no	no	no	no	no	yes
$X_n,\, n \geq 1$	?	no	no	no	no	no	?
X_∞	yes	no	no	no	no	no	yes
$X_0 \simeq S^5$	yes	yes	no	yes	yes	yes	yes
$M_\infty \simeq S^2 \times S^3$	yes	yes	no	yes	yes	yes	yes
$M_n, n \neq 3k$	yes	?	?	yes	no	$n > 5$	?
$M_n, n \neq 2k$	yes	?	?	yes	no	$n > 10$	?
$6M_\infty \# M_n$	yes	?	?	yes	no	$n > 2$	?
$kM_\infty, 1 < k$	yes	some k	some k	yes	$2 \neq k < 9$	yes	?

The non Sasakian-Einstein metric on Barden manifolds are the following: X_{-1} is a symmetric space and the metric is Einstein. $S^2 \times S^3$ is well known to have infinitely many inequivalent homogeneous Einstein metrics discovered by Wang and Ziller [WZ90]. S^5 and $S^2 \times S^3$ have infinitely many inequivalent Einstein metrics of cohomogeneity one discovered by Böhm [Böh98]. Finally X_∞ has infinitely many Einstein metrics recently constructed by several physicists [HSY04, LPP04].

References

[Ara02] C. Araujo, *Kähler-Einstein metrics for some quasi-smooth log del Pezzo surfaces*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4303–4312 (electronic). MR **2003g:**32039

[Aub76] T. Aubin, *Équations du type Monge-Ampère sur les variétés kähleriennes compactes*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 3, Aiii, A119–A121. MR 55 #6496

[Bar65] D. Barden, *Simply connected five-manifolds*, Ann. of Math. (2) **82** (1965), 365–385. MR 32 #1714

[Bel00] F. A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), no. 1, 1–40. MR **2002c:**32027

[Bel01] Florin Alexandru Belgun, *Normal CR structures on compact 3-manifolds*, Math. Z. **238** (2001), no. 3, 441–460. MR **2002k:**32065

[Bes87] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR **88f:**53087

[BG00] C. P. Boyer and K. Galicki, *On Sasakian-Einstein geometry*, Internat. J. Math. **11** (2000), no. 7, 873–909. MR **2001k:**53081

[BG01] ______, *New Einstein metrics in dimension five*, J. Differential Geom. **57** (2001), no. 3, 443–463. MR **2003b:**53047

[BG03a] C. P. Boyer and Galicki, *Einstein Metrics on Rational Homology Spheres*, arXiv:math.DG/0311355 (2003).

[BG03b] C. P. Boyer and K. Galicki, *New Einstein metrics on* $8\#(S^2 \times S^3)$, Differential Geom. Appl. **19** (2003), no. 2, 245–251. MR 2 002 662

[BG04] C. P. Boyer and Galicki, *Positive Ricci Curvature and Einstein Metrics on 5-Manifolds* , in preparation.

[BGK03] C. P. Boyer, K. Galicki, and J. Kollár, *Einstein Metrics on Spheres*, arXiv:math.DG/0309408 (2003).

[BGKT03] C. P. Boyer, K. Galicki, J. Kollár, and E. Thomas, *Einstein Metrics on Exotic Spheres in Dimensions 7,11, and 15*, arXiv:math.DG/0311293 (2003).

[BGM93] C. P. Boyer, K. Galicki, and B. M. Mann, *Quaternionic reduction and Einstein manifolds*, Comm. Anal. Geom. **1** (1993), no. 2, 229–279. MR **95c:**53056

[BGM94] ______, *The geometry and topology of* 3*-Sasakian manifolds*, J. Reine Angew. Math. **455** (1994), 183–220. MR **96e:**53057

[BGM98] ______, *A note on smooth toral reductions of spheres*, Manuscripta Math. **95** (1998), no. 2, 149–158. MR **99c:**53042

[BGM04] C. P. Boyer, K. Galicki, and P. Matzeu, *On Eta-Einstein Sasakian Geometry*, to appear (2004).

[BGMR98] C. P. Boyer, K. Galicki, B. M. Mann, and E. G. Rees, *Compact* 3*-Sasakian* 7*-manifolds with arbitrary second Betti number*, Invent. Math. **131** (1998), no. 2, 321–344. MR **99b:**53066

[BGN02a] C. P. Boyer, K. Galicki, and M. Nakamaye, *Sasakian-Einstein structures on* $9\#(S^2 \times S^3)$, Trans. Amer. Math. Soc. **354** (2002), no. 8, 2983–2996 (electronic). MR **2003g:**53061

[BGN02b] Ch. P. Boyer, K. Galicki, and M. Nakamaye, *Einstein metrics on rational homology 7-spheres*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 5, 1569–1584. MR **2003j:**53056

[BGN03a] C. P. Boyer, K. Galicki, and M. Nakamaye, *On positive Sasakian geometry*, Geom. Dedicata **101** (2003), 93–102. MR 2 017 897

[BGN03b] ______, *On the geometry of Sasakian-Einstein 5-manifolds*, Math. Ann. **325** (2003), no. 3, 485–524. MR **2004b:**53061

[BGN03c] ______, *Sasakian geometry, homotopy spheres and positive Ricci curvature*, Topology **42** (2003), no. 5, 981–1002. MR 1 978 045

[BGP02] C. P. Boyer, K. Galicki, and P. Piccinni, *3-Sasakian geometry, nilpotent orbits, and exceptional quotients*, Ann. Global Anal. Geom. **21** (2002), no. 1, 85–110. MR **2003d:**53076

[Bie99] R. Bielawski, *Complete hyper-Kähler* $4n$*-manifolds with a local tri-Hamiltonian* $\mathbf{R}^n$*-action*, Math. Ann. **314** (1999), no. 3, 505–528. MR **2001c:**53058

[BK78] J-P. Bourguignon and H. Karcher, *Curvature operators: pinching estimates and geometric examples*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 1, 71–92. MR 58 #12829

[Böh98] Ch. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, Invent. Math. **134** (1998), no. 1, 145–176. MR **99i:**53046

[Bri66] E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1–14. MR 34 #6788

[DK01] J-P. Demailly and J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 525–556. MR **2002e:**32032

[DT92] W. Y. Ding and G. Tian, *Kähler-Einstein metrics and the generalized Futaki invariant*, Invent. Math. **110** (1992), no. 2, 315–335. MR **93m:**53039

[EKA90] A. El Kacimi-Alaoui, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. **73** (1990), no. 1, 57–106. MR **91f:**58089

[FK82] Th. Friedrich and H. Kurke, *Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature*, Math. Nachr. **106** (1982), 271–299. MR **84b:**53043

[FK90] Th. Friedrich and I. Kath, *Compact* 5*-dimensional Riemannian manifolds with parallel spinors*, Math. Nachr. **147** (1990), 161–165. MR **92m:**53080

[Fut83] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), no. 3, 437–443. MR **84j:**53072

[Fut87] ———, *On a character of the automorphism group of a compact complex manifold*, Invent. Math. **87** (1987), no. 3, 655–660. MR **88e:**32048

[Fut88] ———, *Kähler-Einstein metrics and integral invariants*, Lecture Notes in Mathematics, vol. 1314, Springer-Verlag, Berlin, 1988. MR **90a:**53053

[Gei91] H. Geiges, *Contact structures on* 1*-connected* 5*-manifolds*, Mathematika **38** (1991), no. 2, 303–311 (1992). MR **93e:**57042

[Gei97] ———, *Normal contact structures on* 3*-manifolds*, Tohoku Math. J. (2) **49** (1997), no. 3, 415–422. MR **98h:**53046

[GL88] K. Galicki and H. B. Lawson, Jr., *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), no. 1, 1–21. MR **89m:**53075

[GMSW04a] J. P. Gauntlett, D. Martelli, J. F. Sparks, and D. Waldram, *Sasaki-Einstein Metrics on* $S^2 \times S^3$, arXiv:hep-th/0403002 (2004).

[GMSW04b] J. P. Gauntlett, D. Martelli, J. F. Sparks, and D. Waldram, *A New Infinite Class of Sasaki-Einstein Manifolds*, arXiv:hep-th/0403038 (2004).

[Gra59] J. W. Gray, *Some global properties of contact structures*, Ann. of Math. (2) **69** (1959), 421–450. MR 22 #3016

[Hae84] A. Haefliger, *Groupoïdes d'holonomie et classifiants*, Astérisque (1984), no. 116, 70–97, Transversal structure of foliations (Toulouse, 1982). MR **86c:**57026a

[HH02a] H. Herrera and R. Herrera, *Classification of positive quaternion-Kähler 12-manifolds*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 1, 43–46. MR **2003e:**53056

[HH02b] ———, *$\hat{A}$-genus on non-spin manifolds with S^1 actions and the classification of positive quaternion-Kähler 12-manifolds*, J. Differential Geom. **61** (2002), no. 3, 341–364. MR 1 979 364

[Hir71] F. Hirzebruch, *Pontrjagin classes of rational homology manifolds and the signature of some affine hypersurfaces*, Proceedings of Liverpool Singularities Symposium, II (1969/1970) (Berlin), Springer, 1971, pp. 207–212. Lecture Notes in Math., Vol. 209. MR 49 #1525

[Hit74] N. J. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55. MR 50 #11332

[Hit95] ———, *Poncelet polygons and the Painlevé equations*, Geometry and analysis (Bombay, 1992), Tata Inst. Fund. Res., Bombay, 1995, pp. 151–185. MR **97d:**32042

[HKLR87] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), no. 4, 535–589. MR **88g:**53048

[HSY04] Y. Hashimoto, M. Sakaguchi, and Y. Yasui, *New infinite series of einstein metrics on sphere bundles from ads black holes*, arXiv: hep-th/0402199 (2004), 1–19.

[Jen73] G. R. Jensen, *Einstein metrics on principal fibre bundles*, J. Differential Geometry **8** (1973), 599–614. MR 50 #5694

[JK01a] J. M. Johnson and J. Kollár, *Fano hypersurfaces in weighted projective 4-spaces*, Experiment. Math. **10** (2001), no. 1, 151–158. MR **2002a:**14048

[JK01b] ———, *Kähler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 1, 69–79. MR **2002b:**32041

[KM63] M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537. MR 26 #5584

[Kob56] S. Kobayashi, *Principal fibre bundles with the* 1*-dimensional toroidal group*, Tôhoku Math. J. (2) **8** (1956), 29–45. MR 18,328a

[Kob87] ———, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, vol. 15, Princeton University Press, Princeton, NJ, 1987, Kanô Memorial Lectures, 5. MR **89e:**53100

[Kol04] J. Kollár, *Einstein Metrics on connected sums of* $S^2 \times S^3$, arXiv:math.DG/0402141 (2004).

[LPP04] H. Lu, D. N. Page, and C.N. Pope, *New inhomogeneous einstein metrics on sphere bundles over einstein-kähler manifolds*, arXiv: hep-th/0403079 (2004), 1–14.

[LS94] C. LeBrun and S. Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. **118** (1994), no. 1, 109–132. MR **95k:**53059

[Lu99] Z. Lu, *On the Futaki invariants of complete intersections*, Duke Math. J. **100** (1999), no. 2, 359–372. MR **2000h:**32033

[Mat57] Y. Matsushima, *Sur les espaces homogènes kählériens d'un groupe de Lie réductif*, Nagoya Math. J. **11** (1957), 53–60. MR 19,315c

[Mil56] J. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Ann. of Math. (2) **64** (1956), 399–405. MR 18,498d

[Mil68] ———, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., 1968. MR 39 #969

[Mil75] ———, *On the* 3*-dimensional Brieskorn manifolds* $M(p,q,r)$, Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox), Princeton Univ. Press, Princeton, N. J., 1975, pp. 175–225. Ann. of Math. Studies, No. 84. MR 54 #6169

[MO70] J. Milnor and P. Orlik, *Isolated singularities defined by weighted homogeneous polynomials*, Topology **9** (1970), 385–393. MR 45 #2757

[Nad90] A. M. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), no. 3, 549–596. MR **92d:**32038

[Or72] P. Orlik, *On the homology of weighted homogeneous manifolds*, Proceedings of the Second Conference on Compact Transformation Groups (Univ. Massachusetts, Amherst, Mass., 1971), Part I (Berlin), Springer, 1972, pp. 260–269. Lecture Notes in Math., Vol. 298.

[OW75] P. Orlik and P. Wagreich, *Seifert* n*-manifolds*, Invent. Math. **28** (1975), 137–159. MR 50 #13596

[PS91] Y. S. Poon and S. M. Salamon, *Quaternionic Kähler* 8*-manifolds with positive scalar curvature*, J. Differential Geom. **33** (1991), no. 2, 363–378. MR **92b:**53071

[Ran75] R. C. Randell, *The homology of generalized Brieskorn manifolds*, Topology **14** (1975), no. 4, 347–355. MR 54 #1270

[Sal82] S. M. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), no. 1, 143–171. MR **83k:**53054

[Siu88] Y. T. Siu, *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. of Math. (2) **127** (1988), no. 3, 585–627. MR **89e:**58032

[Sma61] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406. MR 25 #580

[Sma62] ———, *On the structure of* 5*-manifolds*, Ann. of Math. (2) **75** (1962), 38–46. MR 25 #4544

[SY91] J.-P. Sha and D. Yang, *Positive Ricci curvature on the connected sums of* $S^n \times S^m$, J. Differential Geom. **33** (1991), no. 1, 127–137. MR **92f:**53048

[Tak78] T. Takahashi, *Deformations of Sasakian structures and its application to the Brieskorn manifolds*, Tôhoku Math. J. (2) **30** (1978), no. 1, 37–43. MR **81e:**53024

[Tia87a] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with* $C_1(M) > 0$, Invent. Math. **89** (1987), no. 2, 225–246. MR **88e:**53069

[Tia87b] ———, *On Kähler-Einstein metrics on certain Kähler manifolds with* $C_1(M) > 0$, Invent. Math. **89** (1987), no. 2, 225–246. MR **88e:**53069

[Tia90] ———, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990), no. 1, 101–172. MR **91d:**32042

[Tia97] ———, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), no. 1, 1–37. MR **99e:**53065

[Tia99] ———, *Kähler-Einstein manifolds of positive scalar curvature*, Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999, pp. 67–82. MR **2001m:**32052

[Tia00] ______, *Canonical metrics in Kähler geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2000, Notes taken by Meike Akveld. MR **2001j:**32024

[Ton97] P. Tondeur, *Geometry of foliations*, Monographs in Mathematics, vol. 90, Birkhäuser Verlag, Basel, 1997. MR **98d:**53037

[TY87] G. Tian and S.-T. Yau, *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$*, Comm. Math. Phys. **112** (1987), no. 1, 175–203. MR **88k:**32070

[Wol65] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. **14** (1965), 1033–1047. MR 32 #3020

[Wra97] D. Wraith, *Exotic spheres with positive Ricci curvature*, J. Differential Geom. **45** (1997), no. 3, 638–649. MR **98i:**53058

[WZ90] M. Y. Wang and W. Ziller, *Einstein metrics on principal torus bundles*, J. Differential Geom. **31** (1990), no. 1, 215–248. MR **91f:**53041

[Yau78] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411. MR **81d:**53045

[YK84] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, vol. 3, World Scientific Publishing Co., Singapore, 1984. MR **86g:**53001

[Yot99] M. Yotov, *On generalized futaki invariant*, arXiv: math.AG/9907055 (1999), 1–23.

[Zil] W. Ziller, private communication.

[Zil82] ______, *Homogeneous Einstein metrics on spheres and projective spaces*, Math. Ann. **259** (1982), no. 3, 351–358. MR **84h:**53062

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