Rational Homology 5-Spheres with Positive Ricci Curvature

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Abstract: We prove that for every integer $k>1$ there is a simply connected rational homology 5-sphere M^5_k with spin such that $H_2(M^5_k,\mathbb{Z})$ has order k^2, and M^5_k admits a Riemannian metric of positive Ricci curvature. Moreover, if the prime number decomposition of k has the form $k=p_1\cdots p_r$ for distinct primes p_i then M^5_k is uniquely determined.

Introduction

Recently, Stephan Stolz brought to our attention the fact that preciously little is known about the existence of metrics of positive Ricci curvature on simply connected 5-manifolds in the presence of torsion in H_2. Indeed up until now there appears to be only one known non-trivial example of a rational homology 5-sphere admitting metrics of positive Ricci curvature. This is somewhat surprising in light of the fact that it is precisely in dimension 5 that there is a diffeomorphism classification of compact simply connected 5-manifolds. In 1965 Barden proved the following remarkable theorem:

Theorem [Bar]: *The class of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism. Furthermore, any such M is diffeomorphic to one of the spaces $M_{j;k_1,\ldots,k_s} = X_j\#M_{k_1}\#\cdots\#M_{k_s}$, where $-1\leq j\leq\infty$, $s\geq 0$, $1<k_1$ and k_i divides k_{i+1} or $k_{i+1}=\infty$. A complete set of invariants is provided by $H_2(M,\mathbb{Z})$ and $i(M)$.*

It is understood that $s=0$ means that no M_{k_i} occurs. The diffeomorphism invariant $i(M)$ depends only on the second Stiefel-Whitney class $w^2(M)$. When M is spin $i(M)=0$ as is $w^2(M)=0$. Otherwise $i(M)\neq 0$. Barden's result is the extension of the similar theorem of Smale for spin manifolds. In fact, in Barden's notation $M_{j;k_1,\ldots,k_s}$ is spin if and only if $j=0$ in which case $X_0=S^5$ and $M_{0;k_1,\ldots,k_s}=M_{k_1}\#\cdots\#M_{k_s}$ which is precisely Smale's list [Sm]. Recall that here, $X_{-1}=SU(3)/SO(3)$ is the well-known symmetric space, X_∞ is the non-trivial S^3 bundle over S^2. For all other values of j, X_j is a rational homology sphere determined by its 2-torsion which is $H_2(X_j,\mathbb{Z})=\mathbb{Z}_{2^j}\oplus\mathbb{Z}_{2^j}$ and the value of $i(X_j)=j$. All X_j other than X_0 are non-spin, as their second Stiefel-Whitney class $w^2(X_j)\neq 0$. Now, M_k, $1<k<\infty$ is a rational homology sphere with $H_2(M_k,\mathbb{Z})=\mathbb{Z}_k\oplus\mathbb{Z}_k$ and vanishing $i(M_k)=0$. Finally, $M_\infty=S^2\times S^3$ is the trivial S^3 bundle over S^2. All of the pieces in the above theorem are indecomposable with the exception of $X_1=X_{-1}\#X_{-1}$. Note that for $1<j<\infty$, M_{2^j} and X_j have the same $H_2(M,\mathbb{Z})$ but $i(M_{2^j})=0$ while $i(X_j)=j$.

Regarding the question of positive Ricci curvature metrics on simply connected 5-manifolds, there is the well-known result of Sha and Yang [SY] which, in dimension 5, implies that any k-fold connected sum of $S^2\times S^3$ admits a metric of positive Ricci curvature (an alternative proof using methods similar to this note was given in [BGN2]). Using

During the preparation of this work the authors were partially supported by NSF grant DMS-9970904.

Barden's notation, there exists positive Ricci curvature metrics on $M = M_\infty \# \stackrel{k}{\cdots} \# M_\infty$ for any $k \geq 1$. Now, $X_0 \simeq S^5$ and the symmetric metric has constant curvature. Furthermore, X_∞ admits a metric of positive Ricci curvature because of its bundle structure [Na]. Remarkably, when the torsion in $H_2(M^5, \mathbb{Z})$ is non-trivial we know of just one example, namely X_{-1} where $H_2(X_{-1}, \mathbb{Z}) = \mathbb{Z}_2$. However, our knowledge of this rational homology sphere is due to the special circumstance that it is the symmetric space $SU(3)/SO(3)$ which is known to admit a metric of positive Ricci curvature. Indeed, the standard metric on this compact symmetric space happens to be a positive Einstein metric [Bes]. On the other hand there are no known obstructions to the existence of positive Ricci curvature metrics on any of the manifolds in Barden's classification. This poses a natural

Question: *Which smooth simply connected closed 5-manifolds admit metrics of positive Ricci curvature?*

We do not propose to answer this question here in its full generality. Our purpose is to prove the existence of metrics with positive Ricci curvature on an infinite number of the indecomposable building blocks in Barden's classification theorem by giving infinite families of simply connected rational homology 5-spheres $M^5_{k_i}$ admitting metrics with positive Ricci curvature. Our techniques use contact Riemannian geometry, more specifically Sasakian geometry. Thus, our examples are complementary to the homogeneous manifold $X_{-1} = SU(3)/SO(3)$ whose second and third Stiefel-Whitney classes are both non-vanishing, and so it does not even admit an almost contact structure. In fact our method will not apply to any M with $i(M) \neq 0$. Our examples are necessarily spin and, in this note, they are rational homology spheres. By Smale's theorem [Sm] any rational homology 5-sphere M^5 with vanishing second Stiefel-Whitney class w_2 can be written uniquely as (for our purposes it is more convenient to rephrase Smale's result in terms of elementary divisors instead of invariant factors as he does):

$$M^5 = M^5_{p_1^{s_1}} \# \cdots \# M^5_{p_r^{s_r}} \tag{1}$$

for some positive integers $r, s_1, \cdots, s_r$ where the p_i's are (not necessarily distinct) primes, and $H_2(M^5_{p_i^{s_i}}, \mathbb{Z}) = \mathbb{Z}_{p_i^{s_i}} \oplus \mathbb{Z}_{p_i^{s_i}}$. A result of Geiges [Gei] says that any simply connected rational homology 5-sphere that admits a contact structure must be spin and thus a Smale manifold of the type given by equation (1). In [Gei] it was shown that a simply connected 5-manifold admits a contact structure if and only if its integral third Stiefel-Whitney class vanishes. Hence, from Barden's theorem, the $M_{j;k_1,\cdots,k_s}$ admits a contact structure if and only if $j = 0$ or 1.

Our main result is the following existence theorem:

Theorem A: *For every integer $k > 1$, there exists a simply connected rational homology 5-sphere M^5_k such that $H_2(M^5_k, \mathbb{Z})$ has order k^2, $i(M^5_k) = 0$, and M^5_k admits a Sasakian metric with positive Ricci curvature.*

Our construction gives all possible orders of H_2, but it does not pin down the group precisely in all cases. However, the form of H_2 given by Smale's theorem says that the elementary divisors of H_2 must occur in pairs. Thus, when all of the primes p_i in equation (1) are distinct and all of the s_i's equal 1, the order of H_2 uniquely determines the manifold. For example, if $|H_2(M^5, \mathbb{Z})| = 36$, the elementary divisors must be $\{2, 2, 3, 3\}$. This determines M^5 to be $M^5_2 \# M^5_3$. However, if for example, $|H_2(M^5, \mathbb{Z})| = 64$ the elementary divisors can be $\{2^3, 2^3\}, \{2, 2, 2^2, 2^2\}$, or $\{2, 2, 2, 2, 2, 2\}$, giving the possibilities for M^5 as $M^5_{2^3}, M^5_2 \# M^5_{2^2}$, or $M^5_2 \# M^5_2 \# M^5_2$. In this case we are unable to determine which manifolds occur. Thus, we stop short of proving that all simply connected rational homology 5-spheres with $i(M) = 0$ admit metrics with positive Ricci curvature, although we certainly believe this to be the case.

However, in cases when the order determines the group we have the following corollary

COROLLARY B: *For every positive integer r and every list of distinct primes $p_1, \cdots, p_r$, the manifolds*

$$M^5 = M^5_{p_1} \# \cdots \# M^5_{p_r}$$

admits Sasakian metrics with positive Ricci curvature.

In the absence of known obstructions the question asked above is an intriguing one. Since we are using methods of contact Riemannian geometry to prove Theorem A we do not have any new ideas how to obtain such metrics on manifolds with non-vanishing $i(M)$ such as for, for examples, the rational homology 5-spheres X_j, $1 \leq j < \infty$. Now, X_∞ is actually a contact, even a Sasakian manifold, but it is not a positive Sasakian manifold as positivity implies spin [BGN2] and X_∞ is not spin. As already mentioned, it follows from a theorem of Nash [Na] that X_∞ does have a metric of positive Ricci curvature, but it cannot be Sasakian.

On the other hand, we believe that in the spin case one should be able to extend Theorem A to many other cases. First it is reasonable that actually all spin rational homology 5-spheres can be realized as links with positive Sasakian structure. The main problem in proving this is that we do not know how to compute the torsion group in all cases when it is not determined by its order. One could also try to obtain examples of links M when $H_2(M, \mathbb{Z})$ has both free part and torsion. Regarding this there is a conjectured algorithm due to Orlik [Or3] which has been verified in certain special cases, and we plan to address this question in the future. Another approach would be to use surgery, assuming that one has the existence of positive Ricci curvature metrics on the indecomposable pieces M_k. However, apparently the only known method, e.g. [SY], entails finding 2-spheres in M_k such that the metric in a neighborhood of the S^2 is in standard form [St], and this appears to be obstructed by the 2-torsion.

1. Positive Sasakian Geometry

Let (M, J) is a compact complex manifold and g a Kähler metric on M, with Kähler form M. Suppose that ρ' is a real, closed $(1,1)$-form on M with $[\rho'] = 2\pi c_1(M)$. Then there exists a unique Kähler metric g' on M with Kähler form ω', such that $[\omega] = [\omega'] \in H^2(M, \mathbb{R})$, and the Ricci form of g' is ρ'. The above statement is the celebrated Calabi Conjecture which was posed by Eugene Calabi in 1954. The conjecture in its full generality was eventually proved by Yau in 1976. In the Fano case when $c_1(M) > 0$, i.e., when the first Chern class can be represented by a positive-definite real, closed $(1,1)$-form ρ' on M, the conjecture implies that the Kähler form of M can be represented by a metric of positive Ricci curvature. The key idea behind the proof of Theorem A is based on a more general Calabi Problem when M is not necessarily a smooth manifold but rather a V-manifold or an orbifold [DK, Joy]. In the contex of foliations one actually can prove a "transverse Yau theorem" and this was done by El Kacimi-Alaoui in 1990 [ElK]. In [BGN2] we adapted this to a very special case of Sasakian foliations.

Recall [Bl, YK] that a Sasakian structure on a manifold M of dimension $2n+1$ is a metric contact structure (ξ, η, Φ, g) such that the Reeb vector field ξ is a Killing field and whose underlying almost CR structure is integrable. Briefly, let $(M, \mathcal{D})$ be a contact manifold, and choose a 1-form η so that $\eta \wedge (d\eta)^n \neq 0$ and $\mathcal{D} = \ker\ \eta$. The pair $(\mathcal{D}, \omega)$, where ω is the restriction of $d\eta$ to $\mathcal{D}$ gives $\mathcal{D}$ the structure of a symplectic vector bundle.

Choose an almost complex structure J on $\mathcal{D}$ that is compatible with ω, that is J is a smooth section of the endomorphism bundle End $\mathcal{D}$ that satisfies

$$J^2 = -\mathbb{I}, \qquad d\eta(JX, JY) = d\eta(X, Y), \qquad d\eta(X, JX) > 0 \tag{1.1}$$

for any smooth sections X, Y of $\mathcal{D}$. Notice that J defines a Riemannian metric $g_{\mathcal{D}}$ on $\mathcal{D}$ by setting $g_{\mathcal{D}}(X, Y) = d\eta(X, JY)$. One easily checks that $g_{\mathcal{D}}$ satisfies the compatibility condition $g_{\mathcal{D}}(JX, JY) = g_{\mathcal{D}}(X, Y)$. Now we can extend J to an endomorphism Φ on all of TM by putting $\Phi = J$ on $\mathcal{D}$ and $\Phi\xi = 0$. Likewise we can extend the metric $g_{\mathcal{D}}$ on $\mathcal{D}$ to a Riemannian metric g on M by setting

$$g = g_{\mathcal{D}} + \eta \otimes \eta. \tag{1.2}$$

The quadruple (ξ, η, Φ, g) is called a *metric contact structure* on M. If in addition ξ is a Killing vector field and the almost complex structure J on $\mathcal{D}$ is integrable the underlying almost contact structure is said to be *normal* and (ξ, η, Φ, g) is called a *Sasakian structure*. The fiduciary examples of compact Sasakian manifolds are the odd dimensional spheres S^{2n+1} with the standard contact structure and standard round metric g.

Every Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ has a 1-dimensional foliation $\mathcal{F}_\xi$ associated to it, defined by the flow of the Reeb vector field ξ and called the *characteristic foliation*. Associated with this foliation are important invariants, namely, the basic cohomology groups $H^p_B(\mathcal{F}_\xi)$, (cf. [Ton]) and in particular one can consider the *basic first Chern class* $c_1(\mathcal{F}_\xi)$ [ElK, BGN2] as an element in $H^2_B(\mathcal{F}_\xi)$. These are not only invariants of the Sasakian structure, but of the entire deformation class of Sasakian structures. Notice that on a compact Sasakian manifold $H^2_B(\mathcal{F}_\xi) \neq 0$ since $[d\eta]_B$ is a non-vanishing class.

DEFINITION 1.3: *A Sasakian manifold M is said to be positive if its basic first Chern class $c_1(\mathcal{F}_\xi)$ can be represented by a basic positive definite $(1,1)$-form.*

As in [BGN2, BGN4] we consider deformation classes $\mathfrak{F}(\mathcal{F}_\xi)$ of Sasakian structures that have the same characteristic foliation. Recall that two Sasakian structures $\mathcal{S} = (\xi, \eta, \Phi, g)$ and $\mathcal{S}' = (\xi', \eta', \Phi', g')$ in $\mathfrak{F}(\mathcal{F}_\xi)$ on a smooth manifold M are said to be *a-homologous* if there is an $a \in \mathbb{R}^+$ such that $\xi' = a^{-1}\xi$ and $[d\eta']_B = a[d\eta]_B$. On a rational homology sphere every $\mathcal{S} \in \mathfrak{F}(\mathcal{F}_\xi)$ belongs to precisely one of two a-homology classes corresponding to a given Sasakian structure or its conjugate. In [BGN2] we proved, using El-Kacimi-Alaoui's [ElK] "transverse Yau Theorem",

THEOREM 1.4 [BGN2]: *Let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a positive Sasakian structure on a compact manifold M of dimension $2n+1$. Then M admits a Sasakian structure $\mathcal{S}' = (\xi', \eta', \Phi', g')$ with positive Ricci curvature a-homologous to $\mathcal{S}$ for some $a > 0$.*

Theorem 1.4 says that to prove the existence of a Sasakian metric with positive Ricci curvature it suffices to prove the existence of positive Sasakian structures. In the next section we shall discuss how to construct positive Sasakian structures on homotopy 5-spheres, and prove Theorem A of the Introduction.

2. The Construction

Our 5-manifolds are constructed as k-fold branched covers of S^5 branched over certain Seifert manifolds that are in turn S^1 orbifold V-bundles over a compact Riemann surface of genus g. Our construction is similar to that in [Sav]. Let $f_3(z_1, z_2, z_3)$ be a weighted homogeneous polynomial of an isolated hypersurface singularity in $\mathbb{C}^3$ with weights $\mathbf{w} =$

(w_1, w_2, w_3) and degree d. The link $L_{\mathbf{w}}$ defined by $L_{\mathbf{w}} = \{f_3 = 0\} \cap S^5$ is a Seifert fibration over an algebraic curve $C_{\mathbf{w}}$ in the weighted projective space $\mathbb{P}(\mathbf{w})$. Let $g = g(\mathbf{w})$ denote the genus of the curve $C_{\mathbf{w}}$. Then,

PROPOSITION 2.1: *Let L_f denote the link of the weighted homogeneous polynomial*

$$f = z_0^k + f_3(z_1, z_2, z_3)$$

with weights $(d, k\mathbf{w})$ where k is an integer > 1 where f_3 is a weighted homogeneous polynomial of degree d with weights $\mathbf{w} = (w_1, w_2, w_3)$ as above. Suppose further that $\gcd(d, k) = 1$. *Then the link L_f is a smooth simply connected rational homology 5-sphere such that the order of $H_2(L_f, \mathbb{Z})$ is k^{2g}. Furthermore, L_f admits Sasakian metrics with positive Ricci curvature.*

PROOF: Let us briefly recall the construction of the Alexander (characteristic) polynomial $\Delta_3(t)$ in [MO] associated to a 3-dimensional link L_{f_3}. It is the characteristic polynomial of the monodromy map $\mathbb{I} - h_* : H_2(F, \mathbb{Z}) \longrightarrow H_2(F, \mathbb{Z})$ induced by the $S^1_{\mathbf{w}}$ action on the Milnor fibre F. Thus, $\Delta_3(t) = \det(t\mathbb{I} - h_*)$. Now both F and its closure $\bar{F}$ are homotopy equivalent to a bouquet of 3-spheres, and the boundary of $\bar{F}$ is the link L_{f_3}. Now L_{f_3} is connected and its Betti numbers $b_1(L_{f_3}) = b_2(L_{f_3})$ equal the number of factors of $(t-1)$ in $\Delta_3(t)$. Now since the curve $C_{\mathbf{w}}$ is algebraic, $b_1(L_{f_3}) = 2g$ where g is the genus of $C_{\mathbf{w}}$. Following Milnor and Orlik we let Λ_j denote the divisor of $t^j - 1$ in the group ring $\mathbb{Z}[\mathbb{C}^*]$. Then the divisor of $\Delta_3(t)$ is given by

$$\text{div } \Delta_3 = \prod_{i=1}^{n} (\frac{\Lambda_{u_i}}{v_i} - 1) \tag{2.2}$$

where we write $\frac{d}{w_i} = \frac{u_i}{v_i}$ in irreducible form. Using the relations $\Lambda_a \Lambda_b = \gcd(a, b)\Lambda_{lcm(a,b)}$, equation equation 1 takes the form $\sum a_j \Lambda_j - 1$, where $a_j \in \mathbb{Z}$ and the sum is taken over the set of all least common multiples of all combinations of the $u_1, \cdots, u_n$. The Alexander polynomial is then given by

$$\Delta_3(t) = (t-1)^{-1} \prod_j (t^j - 1)^{a_j}, \tag{2.3}$$

and

$$b_1(L_{f_3}) \;=\; 2g \;=\; \sum_j a_j - 1. \tag{2.4}$$

Now we compute the divisor div Δ of the Alexander polynomial $\Delta_4(t)$ for f. We have

$$\begin{aligned} \text{div } \Delta \;&=\; (\Lambda_k - 1)\text{div } \Delta_f \;=\; (\Lambda_k - 1)\big(\sum a_j \Lambda_j - 1\big) \\ &=\; \sum_j \gcd(k, j) a_j \Lambda_{\text{lcm}(k,j)} - \sum_j a_j \Lambda_j - \Lambda_k + 1. \end{aligned}$$

Since the j's run through all the least common multiples of the set $\{u_1, \cdots, u_n\}$ and $\gcd(k, u_i) = 1$ for all i, we see that for all j, $\gcd(k, j) = 1$. This implies

$$b_2(L_f) \;=\; \sum_j a_j - \sum_j a_j - 1 + 1 \;=\; 0.$$

Thus, L_f is a rational homology sphere. Next we compute the Alexander polynomial for L_f.

$$\Delta_4(t) \;=\; \frac{(t-1)}{(t^k-1)} \prod_j \frac{(t^{kj}-1)^{a_j}}{(t^j-1)^{a_j}}$$

$$(2.5) \qquad = \; (t^{k-1} + \cdots + t + 1)^{-1} \prod_j \Big(\frac{t^{kj-1} + \cdots + t + 1}{t^{j-1} + \cdots + t + 1}\Big)^{a_j}.$$

This gives

$$\Delta_4(1) \;=\; k^{-1} \prod_j \Big(\frac{kj}{j}\Big)^{a_j} \;=\; k^{\Sigma_j a_j - 1} \;=\; k^{2g}.$$

So by [MO] the order of $H_2(L_f, \mathbb{Z})$ is $\Delta_4(1) = k^{2g}$.

To finish the proof it suffices by Theorem 1.4 to show that the induced Sasakian structure on L_f is positive, i.e. that the basic first Chern class $c_1(\mathcal{F}_\xi) \in H^2_B(\mathcal{F}_\xi)$ is positive. Now from our previous work [BG1-2,BGN1-4] L_f is the total space of a V-bundle over a Kähler orbifold $\mathcal{Z}_f$. Moreover, $c_1(\mathcal{F}_\xi)$ is just $c_1(\mathcal{Z}_f)$ pulled back to L_f. Thus, it is enough to prove that $\mathcal{Z}_f$ is Fano. This follows from Lemma 2.6 below which is a special case of Lemma 3.12 of [BGN4].

Lemma 2.6: *As algebraic varieties $\mathcal{Z}_f$ is isomorphic to the weighted projective space $\mathbb{P}(\mathbf{w})$. Hence, its Fano index is $|\mathbf{w}| = \sum_i w_i > 0$.*

This concludes the proof of Proposition 2.1. ∎

Next we want to show that every order of H_2 can be realized. First we show that f_3 realizes curves of any genus. In fact there is a formula due to Orlik and Wagreich [OW] for the genus of the curve $C_\mathbf{w}$ which generalizes the well known genus formula for curves in $\mathbb{P}^2$ (See also the books [Dim, Or1]). It is

$$2.7 \qquad g(C_\mathbf{w}) = \frac{1}{2}\Big(\frac{d^2}{w_1w_2w_3} - d\sum_{i<j}\frac{\gcd(w_i, w_j)}{w_iw_j} + \sum_i \frac{\gcd(d, w_i)}{w_i} - 1\Big).$$

We are interested in the case $g > 0$ which implies $w_1 + w_2 + w_3 \leq d$ [Or2]. It is easy to see that there are quasi-smooth weighted homogeneous polynomials f_3 with arbitrary genus, but we claim that genus one will suffice to realize all rational homology spheres of the form given in our main theorem.

Proposition 2.8: *For every integer $k > 1$, there exists a rational homology 5-sphere M^5_k whose second homology group $H_2(M^5_k, \mathbb{Z})$ has order k^2 and that can be realized as the link L_f of a weighted homogeneous polynomial f given in Proposition 2.1 where f_3 cuts out* *a projective curve of genus one. Furthermore if k has the form $k = p_1 \cdots p_r$ for distinct primes p_i, the manifold M^5_k is uniquely determined up to diffeomorphism.*

Proof: By Smale's classification theorem [Sm] and Proposition 2.1 it suffices to exhibit for each integer $k > 1$ an infinite family of weighted homogeneous polynomials f_3 of prime degree p with $g(C_{\mathbf{w}}) = 1$. For then for a given k we can choose p such that $\gcd(k, p) = 1$. The infinite family of polynomials is given by

$$f_p(z_1, z_2, z_3) = z_1^p + z_2^2 z_3 + z_3^2 z_1$$

with weights $\mathbf{w} = (1, \frac{p+1}{4}, \frac{p-1}{2})$ and degree p where p is a prime of the form $p = 4l - 1$. It is well known that there are an infinite number of such primes. The genus formula then gives

$$\begin{aligned} g =& \frac{1}{2}\Big(\frac{8p^2}{p^2-1} - p(\frac{4}{p+1} + \frac{2}{p-1} + \frac{8}{p^2-1}) + 1 + \frac{4}{p+1} + \frac{2}{p-1} - 1\Big) \\ =& \frac{1}{2}\frac{1}{p^2-1}(2p^2 - 6p + 6p - 2) = 1, \end{aligned}$$

where we have used the fact that $\gcd(\frac{p+1}{4}, \frac{p-1}{2}) = \gcd(l, 2l-1) = 1$, and this proves the first statement. The second statement follows from the classification of finite Abelian groups and Smale's classification theorem [Sm]. ∎

Remarks 2.7:

(1) In the case that $k = p_1 \cdots p_r$ for distinct primes p_i, the links cannot be realized using curves of higher genus $(g > 1)$.

(2) It is still an open question as to whether all simply connected rational homology 5-spheres can be realized by our methods, and if so how does one distinguish the different elementary divisors. The curves of higher genus should play a role here.

(3) For each $k > 1$ there are an infinite number of p's that satisfy $\gcd(k, p) = 1$. This gives rise to an infinite number of Sasakian deformation classes $\mathfrak{F}(\mathcal{F}_\xi)$ each with Sasakian metrics of positive Ricci curvature. It is also quite plausable that the different deformation classes belong to distinct underlying contact structures, but we have not proven this last statement.

Acknowledgments: The authors would like to thank Stephan Stolz and Wolfgang Ziller for fruitful discussions. The second author would also like to than Max-Planck-Institut in Bonn for hospitality and support.

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February 2002