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3-Sasakian Manifolds

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Introduction

In 1960 Sasaki [Sas1] introduced a geometric structure related to an almost contact structure on a smooth manifold. This structure, which became known as a Sasakian structure, was studied extensively in the 1960's by an entire school of Japanese geometers (See [YK] and references therein). In 1970 Kuo [Ku] refined this notion and introduced manifolds with Sasakian 3-structures. The same year Kuo and Tachibana, Tachibana and Yu, and Tanno [KuTach,TachYu,Tan] published foundational papers discussing Sasakian 3-structures and these structures were then vigorously studied by many Japanese mathematicians from 1970-1975. This intense analysis culminated with an important paper of Konishi [Kon] which shows the existence of a Sasakian 3-structure on a certain principal $SO(3)$ bundle over any quaternionic Kähler manifold of positive scalar curvature.

Earlier on, in 1973, Ishihara [I2] had shown that if the distribution formed by the three Killing vector fields which define the Sasakian 3-structure is regular then the space of leaves is a quaternionic Kähler manifold. This fact led Ishihara to his foundational work on quaternionic Kähler manifolds [I1]. Ishihara's and Konishi's observation that quaternionic Kähler and 3-Sasakian geometries are related is fundamental and we will return to this point at the end of section two below.

It is notable that in this early period the only examples of manifolds with a Sasakian 3-structure appearing in the literature were those of constant curvature, namely the spheres S^{4k-1}, the real projective spaces $\mathbb{RP}^{4k-1}$, and spherical space forms in dimension three [Sas2]. Even though Konishi's result mentioned above combined with the earlier work of Wolf [Wo] on the classification of homogeneous quaternionic Kähler manifolds of positive scalar curvature give many new homogeneous examples, no further work on manifolds with a Sasakian 3-structure seems to have been done until very recently [BGM1,FK].

The purpose of this note is to announce some of our recent results about geometry of Sasakian 3-structures. Full details and proofs of the results stated below can be found in [BGM1,BGM2,BGM3,BGM4]. To begin we recall two fundamental definitions.

Definition A: *Let $(\mathcal{S},g)$ be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. Then $(\mathcal{S},g)$ has a Sasakian structure if there exists a Killing vector field*

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ξ of unit length on $\mathcal{S}$ so that the tensor field Φ of type $(1,1)$, defined by

$$(i) \qquad \Phi = \nabla\xi$$

satisfies the condition

$$(ii) \qquad (\nabla_X\Phi)(Y) = \eta(Y)X - g(X,Y)\xi$$

for any pair of vector fields X and Y on $\mathcal{S}$. Here η denotes the 1-form dual to ξ with respect to g; i.e., $g(Y,\xi) = \eta(Y)$ for any vector field Y, and satisfies the dual equation to (i); namely,

$$(iii) \qquad (\nabla_X\eta)(Y) = g(Y, \Phi X).$$

We write (Φ,ξ,η) to denote the specific Sasakian structure on $(\mathcal{S},g)$ and will refer to $\mathcal{S}$ with such a structure as a Sasakian manifold.

DEFINITION B: *Let $(\mathcal{S},g)$ be a Riemannian manifold that admits three distinct Sasakian structures $\{\Phi^a,\xi^a,\eta^a\}_{a=1,2,3}$ such that*

$$g(\xi^a,\xi^b) = \delta^{ab} \qquad \text{and} \qquad [\xi^a,\xi^b] = 2\epsilon^{abc}\xi^c$$

for $a,b,c = 1,2,3$. Then $(\mathcal{S},g)$ is a 3-Sasakian manifold with Sasakian 3-structure $(\mathcal{S},g,\xi^a)$.

It follows directly from the definition that every 3-Sasakian manifold admits a local action of either $Sp(1)$ or $SO(3)$ as local isometries and, if the vector fields ξ^a are complete, then these are global isometries. We refer to this action as the *standard* $Sp(1)$ action on $\mathcal{S}$. In the remainder of this note we shall assume that the vector fields ξ^a are complete. It is well-known that every 3-Sasakian manifold $(\mathcal{S},g,\xi^a)$ has dimension $4n+3$ and defines a Riemannian foliation $(\mathcal{S},\mathcal{F})$ of codimension $4n$ with totally geodesic leaves of constant curvature 1 [IKon,KuTach]. Furthermore, $(\mathcal{S},g,\xi^a)$ is an Einstein manifold [Ka].

1. The Structure Theorem

Our first main theorem generalizes the result of Ishihara [I2] which say that if the space of leaves $\mathcal{S}/\mathcal{F}$ is a manifold then it has a canonical quaternionic Kähler structure. In addition, we show that every 3-Sasakian manifold is of positive scalar curvature and that it admits a second non-isometric Einstein metric. However, this second Einstein metric does not have a compatible Sasakian 3-structure. More precisely we prove

THEOREM C: [BGM2] *Let $(\mathcal{S},g,\xi^a)$ be a 3-Sasakian manifold of dimension $4n+3$ such that the Killing vector fields ξ^a are complete for $a = 1,2,3$. Then*

(i) $(\mathcal{S},g,\xi^a)$ is an Einstein manifold of positive scalar curvature equal to $2(2n+1)(4n+3)$.

(ii) $\mathcal{S}$ admits a second Einstein metric g' of positive scalar curvature which is not homothetic to g.

(iii) The metric g is bundle-like with respect to the foliation $\mathcal{F}$.

(iv) Each leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ is a 3-dimensional homogeneous spherical space form.

(v) The space of leaves $\mathcal{S}/\mathcal{F}$ is a quaternionic Kähler orbifold of dimension $4n$ with positive scalar curvature equal to $16n(n+2)$.

Hence, every complete 3-Sasakian manifold is compact with finite fundamental group and diameter less than or equal to π.

Notice that the space of leaves described in part (v) is not necessarily a smooth manifold but rather is a V-manifold, now commonly referred to as an orbifold, originally studied by Satake [Sat]. As pointed out above there is always at least one 3-Sasakian manifold associated with every quaternionic Kähler manifold of positive scalar curvature [Kon]. However, as shown in [BGM1], 3-Sasakian manifolds are much more plentiful than quaternionic Kähler manifolds of positive scalar curvature. In all but five quaternionic dimensions there are only 3 explicitly known examples of compact quaternionic Kähler manifolds of positive scalar curvature (in dimension 1 there are only two such examples whereas in dimensions 7, 10, 16, and 28 there are four). Moreover, all of these examples are symmetric spaces and can be found in Wolf's classification [Wo]. It is also known that in quaternionic dimensions 1 and 2 there are no others [H], [PS].

By contrast, a 3-Sasakian manifold must be of real dimension $4k+3$ and in each such allowable dimension we have constructed infinitely many distinct compact simply-connected 3-Sasakian manifolds [BGM1,BGM2]. Moreover, these examples range through infinitely many distinct homotopy types in every dimension.

Thus, Theorem C explains why, in order to understand the connection between quaternionic Kähler and 3-Sasakian geometry, as first observed by Ishihara and Konishi, one must consider the larger category of quaternionic Kähler *orbifolds*.

2. The Embedding Theorem

Our next theorem shows that every 3-Sasakian manifold embeds naturally in a hyperkähler manifold, which generalizes Swann's associated bundle [Sw] to the orbifold category. In particular we have

THEOREM D: *Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a complete 3-Sasakian manifold. Then the product manifold $M = \mathcal{S}\times\mathbb{R}^+$ with the cone metric $g_M = dr^2 + r^2 g_{\mathcal{S}}$ is hyperkähler so that there is a commutative diagram of (orbifold) fibrations*

$$\begin{tikzcd} & \mathcal{S}\times\mathbb{R}^+ \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathcal{Z} \arrow[dr] & & \mathcal{S}. \arrow[dl] \\ & \mathcal{O} & \end{tikzcd}$$

Here $\mathcal{O}$ is a quaternionic Kähler orbifold appearing in part (v) of Theorem 3, $\mathcal{Z}$ is the twistor space of $\mathcal{O}$ with its Kähler-Einstein orbifold metric, and $\mathbb{R}^+\times\mathcal{S}$ is an orbifold generalization of Swann's associated quaternionic bundle.

An application of Theorem D to hypercomplex structures is given in section four below.

3. Some Classification Theorems

We obtain the following classification theorem of all 3-Sasakian homogeneous spaces; that is, 3-Sasakian manifolds with transitive action of the group of automorphisms of the

Sasakian 3-structure. Combining Wolf's [Wo] classification with the results of Ishihara [I2], Tanno [Tan], and Theorem C above we prove the following theorem.

THEOREM E: [BGM2] *Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian homogeneous space. Then $\mathcal{S}$ is precisely one of the following homogeneous spaces:*

$$\frac{Sp(n)}{Sp(n-1)} \simeq S^{4n-1}, \qquad \frac{Sp(n)}{Sp(n-1)\times \mathbb{Z}_2} \simeq \mathbb{RP}^{4n-1},$$

$$\frac{SU(m)}{S\big(U(m-2)\times U(1)\big)}, \qquad \frac{SO(k)}{SO(k-4)\times Sp(1)},$$

$$\frac{G_2}{Sp(1)}, \qquad \frac{F_4}{Sp(3)}, \qquad \frac{E_6}{SU(6)}, \qquad \frac{E_7}{\mathrm{Spin}(12)}, \qquad \frac{E_8}{E_7}.$$

Here $n \geq 1$, $Sp(0)$ denotes the identity group, $m \geq 3$, and $k \geq 7$. Furthermore, the fiber F over the quaternionic Kähler base space is $Sp(1)$ if and only if $(\mathcal{S}, g, \xi^a)$ is simply connected with constant curvature; that is, when $\mathcal{S} = S^{4n-1}$. In all other cases $F = SO(3)$.

The metrics on all these cosets spaces are Einstein and they were considered in this context in Besse [Bes]. However, with the exception of the constant curvature case, these are not the normal homogeneous metrics, as they are not naturally reductive and thus are not obtained from the bi-invariant metric on G by Riemannian submersion. Inhomogeneous 3-Sasakian geometries were constructed and studied in [BGM2,BGM3] as indicated in section four below.

Combing the result of Ishihara [I2] with the results of Hitchin who classified all 4-dimensional compact quaternionic Kähler manifolds of positive scalar curvature [H], and Poon and Salamon who extended this classification to dimension 8 [PS], we obtain the following classification of all fibered Riemannian spaces with Sasakian 3-structure.

THEOREM F: [BGM1] *Let $\pi : \mathcal{S} \rightarrow \mathcal{S}/\mathcal{F}$ be a complete principal Riemannian fibration with Sasakian 3-structure. Then,*

1. *If $\mathcal{S}$ has dimension 7, then $\mathcal{S}$ is either S^7, $\mathbb{RP}^7$, or $SU(3)/U(1)$.*
2. *If $\mathcal{S}$ has dimension 11, then $\mathcal{S}$ is either S^{11}, $\mathbb{RP}^{11}$, $SU(4)/S(U(2)\times U(1))$, or $G_2/SU(2)$.*

In particular, every such fibered Riemannian manifold of dimension 7 or 11 with Sasakian 3-structure is homogeneous.

4. Applications to Other Geometries

We have already seen that results about 3-Sasakian manifolds are related to results about quaternionic Kähler orbifolds. We conclude this announcement by indicating how our results on 3-Sasakian manifolds can be used to deduce results in other areas of differential geometry. To begin we give some non-homogeneous examples of 3-Sasakian manifolds that are not covered by the Classification Theorem E.

DEFINITION G: [BGM2] *Let $n \geq 3$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n$ be an n-tuple of non-decreasing, pairwise relatively prime, positive integers. Let $\mathcal{S}(\mathbf{p})$ be the left-right quotient of the unitary group $U(n)$ by $U(1) \times U(n-2) \subset U(n)^2 = U(n)_L \times U(n)_R$ where the* *action is given by the formula*

$$\mathbb{W} \xrightarrow{(\tau,\mathbb{B})} \begin{pmatrix} \tau^{p_1} & & \\ & \ddots & \\ & & \tau^{p_n} \end{pmatrix} \mathbb{W} \begin{pmatrix} \mathbb{I}_2 & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix}.$$

Here $\mathbb{W} \in U(n)$ *and* $(\tau, \mathbb{B}) \in U(1) \times U(n-2)$.

THEOREM H: [BGM2] *Let* $n \geq 3$ *and* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n_+$ *be an* n*-tuple of non-decreasing, pairwise relatively prime, positive integers. Then* $\mathcal{S}(\mathbf{p})$ *is a compact, simply connected,* $(4n-5)$*-dimensional 3-Sasakian manifold. Furthermore,* $\mathcal{S}(\mathbf{p})$ *is inhomogeneous as long as* $\mathbf{p} \neq (1, \ldots, 1)$.

Combining Theorem H with a homology calculation given in [BGM2] we have

COROLLARY I: [BGM2] *There are infinitely many non-homotopy equivalent compact simply-connected inhomogeneous 3-Sasakian manifolds in dimension* $4n-5$ *for every* $n \geq 3$.

Following Eschenburg we call a manifold *strongly inhomogeneous* if it is not homotopy equivalent to any homogeneous Riemannian manifold. The next corollary follows directly from a theorem of Eschenburg [E] and the proof of Corollary I.

COROLLARY J: [BGM2] *There are infinitely many homotopically distinct 7-dimensional strongly inhomogeneous compact simply connected 3-Sasakian manifolds.*

To our knowledge, the families in Corollary J are the only known examples of strongly inhomogeneous compact 3-Sasakian manifolds. Combining Theorem C with Kashiwada's fundamental observation [Ka] that every 3-Sasakian manifold is Einstein we can replace the phrase "3-Sasakian manifold" by the phrase "Einstein manifold of positive scalar curvature" in both Corollaries I and J [BGM3].

Finally, Theorem D can be used to give a generalization of the standard Hopf surface construction which we can then use to construct many new compact hypercomplex manifolds. Consider the manifold $\mathcal{S} \times S^1$ obtained from $\mathcal{S} \times \mathbb{R}^+$ as the quotient by the multiplicative action of $\mathbb{Z}$ on $\mathbb{R}^+$ generated by $r \mapsto ar$ where $a \neq 1$ is a fixed positive real number.

COROLLARY K: [BGM2] *Let* $\mathcal{S}$ *be a complete 3-Sasakian manifold, then the compact manifold* $\mathcal{S} \times S^1$ *constructed above has a naturally induced hypercomplex structure. In fact, the product metric is locally conformally hyperkähler.*

More generally we have

THEOREM L: [BGM4] *Let* $\mathcal{S}$ *be a 3-Sasakian manifold and* P *any circle bundle over* $\mathcal{S}$*. Then*

(i) P *is almost hypercomplex.*

(ii) P *is hypercomplex if and only if the nowhere vanishing vertical vector field on* P *which generates the circle action is compatible with the Sasakian 3-structure.*

The technical definition of compatibility in part (ii) of Theorem L is given in [BGM4]. Moreover, we have

THEOREM M: [BGM4] *Let* $H(\mathcal{S})$ *be a compatible hypercomplex circle bundle over any complete 3-Sasakian manifold* $\mathcal{S}$*. Then*

(i) $H(\mathcal{S})$ *does not admit any Kähler metric.*

(ii) All the Chern numbers of $H(\mathcal{S})$ *are zero.*

(iii) There is a compatible hyperhermitian metric on $H(\mathcal{S})$ whose isometry group contains a copy of $U(2)$.

We conclude this note by pointing out that Theorems L and M are not vacuous as we have infinite families of examples in every quaternionic dimension.

THEOREM N: [BGM4] *Let $n \geq 3$, $\mathbf{p}$ be an n-tuple of non-decreasing, pairwise relatively prime, positive integers, and k a non-negative integer. Then there exist compatible hypercomplex circle bundles $H(\mathcal{S}(\mathbf{p}); k)$ over the complete 3-Sasakian manifold $\mathcal{S}(\mathbf{p})$ given in Definition G.*

The hypercomplex manifolds in Theorem N are constructed so that $\pi_1(H(\mathcal{S}(\mathbf{p}); k)) = \mathbb{Z}_k$. In addition, the integral cohomology ring of $H(\mathcal{S}(\mathbf{p}); k)$ is computed in [BGM4].

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