Volume 95 (1998), 149-158 `manuscripta mathematica`

A Note on Smooth Toral Reductions of Spheres

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In previous work [BGMR1, BGMR2] the authors and E. Rees, using 3-Sasakian toral reductions of the standard $(4n-1)$-dimensional sphere, constructed compact 3-Sasakian 7-manifolds with arbitrary second Betti number. These were the first examples of smooth compact Einstein manifolds of positive scalar curvature with arbitrary second Betti number. This toral reduction procedure actually constructs more general families of 3-Sasakian $(4n-1)$-dimensional orbifolds $\mathcal{S}(\Omega)$. However, to obtain $(4n-1)$-manifolds with $b_2>1$ one should answer two separate questions: (1) What are the orbifold Betti numbers of $\mathcal{S}(\Omega)$? and (2) When is $\mathcal{S}(\Omega)$ smooth?

In [BGMR1] sufficient conditions for guarenteeing smoothness for general toral reductions were given, and one easily sees that these conditions are also necessary. This is formulated in Proposition 1.3 below. However, the Betti number computations were carried out only for dimension seven. In this case the we gave a $2k$-parameter family of smooth 3-Sasakian 7-manifolds with second Betti number k. Contemporaneously, R. Bielawski [Bi] computed Betti numbers of all these toral examples; more precisely, he showed that

$$b_{2i}=\binom{k+i-1}{i}$$

for $i\leq n-k-1$ for any 3-Sasakian orbifold obtained by 3-Sasakian reduction of S^{4n-1} by a k-dimensional torus. Since the odd Betti numbers b_{2i+1} of any 3-Sasakian manifold are known to vanish [GS] for $i\leq n-k-1$ (for orbifolds this follows from results of [BG]), Bielawski's results and Poincaré duality would determine the rational homology of any 3-Sasakian manifold obtained from toral reduction. However, there remains the question as to when these higher dimensional orbifolds can be smooth manifolds. In this note we give mod 2 obstructions to smoothness which show, somewhat surprisingly, that if the reduced 3-Sasakian orbifold $\mathcal{S}$ is a smooth manifold of dimension greater than 7 then $b_2\leq 4$, and if the dimension of the reduced 3-Sasakian manifold is greater than 15 then $b_2\leq 1$. That is, we prove

Theorem A: *Let $\mathcal{S}$ be a 3-Sasakian manifold obtained by 3-Sasakian reduction of S^{4n-1} by a torus T^k. If $k>1$ then $\dim\mathcal{S}=7,11,15$. In addition, if $k>4$ then $\dim\mathcal{S}=7$.*

During the preparation of this work the authors were supported by an NSF grant.

After proving Theorem A we show that these bounds are sharp in that there exist infinite families of manifold examples in all cases not eliminated by Theorem A. This also shows that any additional mod p obstructions to smoothness for p an odd prime give no further constraints on the Betti numbers.

§1. Proof of Theorem A.

Recall from [BGMR1] that any k-torus action on S^{4n-1} that preserves the 3-Sasakian structure on the sphere is given by a quaternionic representation of the k-torus T^k on a quaternionic vector space $\mathbb{H}^n$. This action can be described by left multiplication on a quaternionic column vector $\mathbf{u} = (u_1, \cdots, u_n)^t$ by a diagonal matrix $h_\Omega(\tau_1, .., \tau_k) \in T^n \subset Sp(n)$

1.1 $$h_\Omega(\tau_1, .., \tau_k) = \begin{pmatrix} \prod_{i=1}^k \tau_i^{a_1^i} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \prod_{i=1}^k \tau_i^{a_n^i} \end{pmatrix},$$

where $(\tau_1, .., \tau_k) \in S^1 \times \cdots \times S^1 = T^k$ are the complex coordinates on T^k, and $a_j^i \in \mathbb{Z}$. This representation gives rise to the integral weight matrix

1.2 $$\Omega = \begin{pmatrix} a_1^1 & a_2^1 & \ldots & a_k^1 & \ldots & a_n^1 \\ a_1^2 & a_2^2 & \ldots & a_k^2 & \ldots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\ a_1^k & a_2^k & \ldots & a_k^k & \ldots & a_n^k \end{pmatrix}.$$

Two such weight matrices $\Omega, \Omega' \in \mathcal{M}_{k\times n}(\mathbb{Z})$ are equivalent if there are $A \in GL(k, \mathbb{Z})$ and $w \in \mathcal{W}(Sp(n))$ such that $\Omega' = A\Omega w$. Equivalent weight matrices give rise to isomorphic 3-Sasakian quotients $\mu_{\Omega'}^{-1}(0)/T^k(\Omega') \simeq \mu_\Omega^{-1}(0)/T^k(\Omega)$. The question of whether or not the converse is true is more subtle and will be addressed elsewhere. The conditions on Ω that guarantee "nice" behavior of the quotient are given by

PROPOSITION 1.3: [BGMR1] *Let $\mathcal{S} = \mu_\Omega^{-1}(0)/T^k(\Omega)$ be a 3-Sasakian quotient space. Then $\mathcal{S}$ is an orbifold if the k by k minor determinants*

$$\Delta_{\alpha_1\ldots\alpha_k} = \begin{vmatrix} a_{\alpha_1}^1 & \ldots & a_{\alpha_k}^1 \\ \vdots & & \vdots \\ a_{\alpha_1}^k & \ldots & a_{\alpha_k}^k \end{vmatrix} \neq 0$$

for all sequences $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$. Moreover, $\mathcal{S}$ is a smooth manifold if and only if, in addition, we have

$$\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\alpha_k}) = g$$

for all sequences $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n$*, where* g *denotes the* k^{th} *determinantal divisor of* Ω.

An $\Omega \in \mathcal{M}_{k\times n}(\mathbb{Z})$ that satisfies the two conditions of Proposition 1.3 was called *admissible* in [BGMR1]. Recall also that if $g > 1$ then the T^k action is not effective, and there is a normal subgroup Γ of order g such that the factor group $T^k/\Gamma \simeq T^k$ acts effectively. Thus, we may assume without loss of generality that $g = 1$, *i.e.*, Ω is in *reduced* form. The subset of reduced admissible matrices in $\mathcal{M}_{k\times n}(\mathbb{Z})$ is denoted by $\mathcal{A}_{k\times n}(\mathbb{Z})$. It is invariant under the action of $GL(k,\mathbb{Z}) \times \mathcal{W}(Sp(n))$. Furthermore, the unordered set $\{|\Delta_s|\}$ of absolute values of all k by k minor determinants is an invariant of $\mathcal{A}_{k\times n}(\mathbb{Z})$.

Let $\Omega \in \mathcal{A}_{k\times n}(\mathbb{Z})$. Since Ω is reduced there is a k by k minor determinant that is odd. By permuting columns if necessary this minor can be taken to be the first k columns. Now consider the mod 2 reduction $\mathcal{M}_{k\times n}(\mathbb{Z}) \longrightarrow \mathcal{M}_{k\times n}(\mathbb{Z}_2)$. We have the following commutative diagram

1.4
$$\begin{array}{ccc} GL(k,\mathbb{Z}) \times \mathcal{M}_{k\times n}(\mathbb{Z}) & \longrightarrow & \mathcal{M}_{k\times n}(\mathbb{Z}) \\ \big\downarrow & & \big\downarrow \\ GL(k,\mathbb{Z}_2) \times \mathcal{M}_{k\times n}(\mathbb{Z}_2) & \longrightarrow & \mathcal{M}_{k\times n}(\mathbb{Z}_2). \end{array}$$

Let $\tilde{\Omega} \in \mathcal{A}_{k\times n}(\mathbb{Z}_2)$ denote the mod 2 reduction of $\Omega \in \mathcal{A}_{k\times n}(\mathbb{Z})$. Since the first k by k minor determinant of Ω is odd, the mod 2 reduction of this minor in $\tilde{\Omega}$ is invertible. Thus, we can use the $GL(k,\mathbb{Z}_2)$ action to put $\tilde{\Omega}$ in the form

1.5
$$\tilde{\Omega} = \begin{pmatrix} 1 & 0 & \dots & 0 & a^1_{k+1} & \dots & a^1_n \\ 0 & 1 & \dots & 0 & a^2_{k+1} & \dots & a^2_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 1 & a^k_{k+1} & \dots & a^k_n \end{pmatrix}$$

with $a^i_j \in \mathbb{Z}_2$.

Lemma 1.6: *The set* $\mathcal{A}_{k\times n}(\mathbb{Z})$ *is empty for* $n > k+2$ *and* $k > 4$.

Proof: We analyze the second admissibility condition. Since Ω is reduced the second condition of Proposition 1.3 implies

1.7
$$(\tilde{\Delta}_{\alpha_2\cdots\alpha_{k+1}}, \ldots, \tilde{\Delta}_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \ldots, \tilde{\Delta}_{\alpha_1\cdots\alpha_k}) \neq (0,\cdots,0)$$

for all sequences $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n$, where $\tilde{\Delta}$ denotes the mod 2 reduction of Δ. Thus, to check admissibility we choose $k+1$ columns and look at all k by k minor determinants mod 2. The first set to choose is the first k columns and any one of the remaining $n-k$ columns. But since $\tilde{\Delta}_{1\cdots k} = 1$, the second condition of Proposition 1.3 is identically satisfied. Next choose any $k-1$ of the first k columns and two of the remaining $n-k$ columns. The minor determinants $\tilde{\Delta}_{1\cdots\hat{j}\cdots kl}$, where $\hat{j}$ indicates the j^{th}

column is deleted, have the form $\tilde{\Delta}_{1\cdots\hat{j}\cdots kl} = a_l^j$, while those with two of the first k-columns deleted have the form

1.8
$$\tilde{\Delta}_{1\cdots\hat{\imath}\cdots\hat{j}\cdots klm} = \begin{vmatrix} a_l^i & a_m^i \\ a_l^j & a_m^j \end{vmatrix}.$$

Thus the admissibility condition becomes

1.9
$$(a_l^j, a_m^j, \begin{vmatrix} a_l^1 & a_m^1 \\ a_l^j & a_m^j \end{vmatrix}, \cdots, \begin{vmatrix} a_l^j & a_m^j \\ a_l^k & a_m^k \end{vmatrix}) \neq (0, \cdots, 0)$$

for all $j = 1, \cdots k$, where, of course, the term $\begin{vmatrix} a_l^j & a_m^j \\ a_l^j & a_m^j \end{vmatrix}$ is deleted. This condition is satisfied if and only if

1.10
$$(a_l^j, a_m^j) \neq (0, 0)$$

for all $j = 1, \cdots, k$, and $k + 1 \leq l < m \leq n$.

Next consider sequences of minors formed from $k - 2$ of the first k columns and 3 of the remaining $n - k$ columns (Here we use the fact that $n - k > 2$). There are two types of minors, those of the form of (6) above, and

1.11
$$\tilde{\Delta}_{1\cdots\hat{\imath}\cdots\hat{j}\cdots\hat{l}\cdots mr} = \begin{vmatrix} a_l^i & a_m^i & a_r^i \\ a_l^j & a_m^j & a_r^j \\ a_l^k & a_m^k & a_r^k \end{vmatrix}.$$

Then the second admissibility condition implies

1.12
$$(\begin{vmatrix} a_l^i & a_m^i \\ a_l^j & a_m^j \end{vmatrix}, \begin{vmatrix} a_l^i & a_r^i \\ a_l^j & a_r^j \end{vmatrix}, \begin{vmatrix} a_m^i & a_r^i \\ a_m^j & a_r^j \end{vmatrix}, \begin{vmatrix} a_l^i & a_m^i & a_r^i \\ a_l^j & a_m^j & a_r^j \\ a_l^1 & a_m^1 & a_r^1 \end{vmatrix}, \cdots, \begin{vmatrix} a_l^i & a_m^i & a_r^i \\ a_l^j & a_m^j & a_r^j \\ a_l^k & a_m^k & a_r^k \end{vmatrix}) \neq (0, \cdots, 0),$$

where the terms in the 3 by 3 determinants with last row equal to the i^{th} or j^{th} row are deleted. Now this condition implies that for all pairs of triples (a_l^i, a_m^i, a_r^i) of an admissible $\tilde{\Omega}$ we must have

(1.13)
$$\begin{pmatrix} a_l^i & a_m^i & a_r^i \\ a_l^j & a_m^j & a_r^j \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

On the other hand 1.10 says:

(i) We cannot have two 0's in any row of such a triple.

Now suppose that there were two 0's in a column of a pair of triples. Then by column permutation the pair of triples must have the form

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

But then one easily sees that 1.12 does not hold. Thus,

(ii) We cannot have two 0's in any column of any pair of triples.

Hence, it follows that, up to column and row permutations, that any four triples of the last $n-k$ columns of an admissible $\tilde{\Omega}$ must have the form

1.14
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

So we see that we cannot add another row without violating the above conditions. It follows that $k \leq 4$. ∎

We can also use our analysis to prove:

LEMMA 1.15: *Let $\Omega \in \mathcal{A}_{k\times n}(\mathbb{Z})$ and let $k > 1$, then $n-k \leq 4$.*

PROOF: By Lemma 1.6 we need only check this for $k = 2,3,4$. We consider the case $k=2$ as the cases $k=3,4$ are similar. As before, let $\tilde{\Omega}$ denote the mod 2 reduction of Ω. Then by 1.13, and rules (i) and (ii) above up to $GL(2,\mathbb{Z})$ transformations and column permutations any $\Omega \in \mathcal{A}_{2\times 6}(\mathbb{Z})$ will have a reduced matrix of the form

$$\tilde{\Omega} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

One cannot add another column to this matrix since 1.10 implies that this must be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, but this is forbidden by 1.13. ∎

Lemmas 1.6 and 1.15 now imply Theorem A. Notice that one could replace $\mathbb{Z}_2$ by $\mathbb{Z}_p$ for any prime p in Lemma 1.6 and carry out a similar analysis to obtain "mod p" obstructions to smoothness. However, it is not surprising that the $p=2$ bound is the sharpest, and as we show in the next sections there are no further Betti number bounds.

§2. Some 11 Dimensional Examples.

In [BGM] we constructed 3-Sasakian manifolds obtained by toral reduction for $k=1$ for every n and in [BGMR1] we constructed 3-Sasakian 7-manifolds with arbitrary k. Thus, to show that Theorem A is sharp we need only construct 11 and 15 dimensional manifold examples when $k=2,3,4$. These last constructions give explicit examples of non-regular 3-Sasakian manifolds for which the Betti number relations given in [GS] for regular 3-Sasakian manifolds fail as Bielawski [Bi] showed that the regular [GS] relations can hold for a 3-Sasakian toral quotient if and only if $k=1$.

We begin by assuming that dim $\mathcal{S} = 11$ and $k = 4$. We take Ω to be in the special form

2.1
$$\Omega = \Big(\mathbb{I}_4 \; A\Big),$$

where $\mathbb{I}_4$ denotes the 4 by 4 identity matrix and $A \in \mathcal{M}_{4\times 3}(\mathbb{Z})$. For matrices of this form the first admissibility condition implies that all the entries (a^i_j) of the matrix A are non vanishing and that all minor determinants are non vanishing. The second admissibility condition amounts to

2.2a
$$\gcd(a^j_l, a^j_m) = 1$$

and

2.2b
$$\gcd\Big(\begin{vmatrix} a^i_l & a^i_m \\ a^j_l & a^j_m \end{vmatrix}, \begin{vmatrix} a^i_l & a^i_r \\ a^j_l & a^j_r \end{vmatrix}, \begin{vmatrix} a^i_m & a^i_r \\ a^j_m & a^j_r \end{vmatrix}, \begin{vmatrix} a^i_l & a^i_m & a^i_r \\ a^j_l & a^j_m & a^j_r \\ a^1_l & a^1_m & a^1_r \end{vmatrix}, \cdots, \begin{vmatrix} a^i_l & a^i_m & a^i_r \\ a^j_l & a^j_m & a^j_r \\ a^k_l & a^k_m & a^k_r \end{vmatrix}\Big) = 1$$

for all $1 \le i < j \le 4$, and $1 \le l < m < r \le 3$.

Condition 2.2b is equivalent to the condition

2.2c
$$\gcd\Big(\begin{vmatrix} a^i_l & a^i_m \\ a^j_l & a^j_m \end{vmatrix}, \begin{vmatrix} a^i_l & a^i_r \\ a^j_l & a^j_r \end{vmatrix}, \begin{vmatrix} a^i_m & a^i_r \\ a^j_m & a^j_r \end{vmatrix}\Big) = 1$$

for all $1 \le i < j \le 4$, and $1 \le l < m < r \le 3$. There are six equations of the form 2.2c. Moreover, it follows from the analysis of the previous section that up to row and column permutations the mod 2 reduction $\tilde{A}$ of A must take the form of 1.14. Thus, three of the six two by two minor determinants of A must be even, while the other three are odd. In addition, three of the six equations of the form 2.2c contain precisely one even two by two minor, and the remaining three equations have all odd two by two minors. Moreover, precisely one of the four three by three minor determinants is even and this occurs in the three equations with all odd two by two minor determinants. The three equations with an even two by two minor determinant can be automatically satisfied by choosing each of the three even two by two minors to have determinant that is a power of 2. This leaves only three equations of the form 2.2c to satisfy.

These equations are still somewhat complicated in their full generality, so we shall look for a 3 parameter family of solutions with A taken in the form

2.3
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1+2^l \\ 1 & 16 & 1+2^m \\ -1 & 3 & 2c \end{pmatrix},$$

where $l, m \in \mathbb{Z}^+$, and $c \in \mathbb{Z}$. We denote the quotient space by $\mathcal{S}(c, l, m)$. This space will be a 3-Sasakian orbifold as long as the determinant

2.4
$$\Delta = 2(31c + 6 + 19 \cdot 2^{l-1} - 7 \cdot 2^{m-1}) \neq 0.$$

An example of a singular quotient that is not an orbifold is $\mathcal{S}(1,1,4)$. Henceforth, we shall assume that 2.4 is always satisfied. The three equations of the form 2.2c that are not automatically satisfied are

2.5
$$\begin{aligned} \gcd(7, 4c+2^l+1, 2c-3\cdot 2^l-3) &= 1, \\ \gcd(19, 2c+2^m+1, 32c-3\cdot 2^l-3) &= 1, \\ \gcd(31, 2^{m+1}-2^l+1, 2^m-2^{l+4}-15) &= 1. \end{aligned}$$

One can check that these equations hold if and only if $7, 19$ and 31 do not divide Δ. Thus, we have

PROPOSITION 2.6: *The* 3*-Sasakian quotients* $\mathcal{S}(c,l,m)$ *are smooth manifolds of dimension* 11 *if and only if* 3 *does not divide* c, *and neither* $7, 19$ *nor* 31 *divide* Δ.

COROLLARY 2.7: *There are smooth* 3*-Sasakian* 11*-manifolds* $\mathcal{S}(c,l,m)$ *with second Betti number equal to* 4 *for an infinite number of values of the parameters* c, l, m.

PROOF: It is routine to verify that the three parameter infinite family given by

2.8
$$\begin{aligned} c &\equiv 14 \pmod{21}, \\ l &\not\equiv 1 \pmod{5}, \\ m &\not\equiv \alpha(c) \pmod{18}, \end{aligned}$$

where $2^{\alpha(c)} = 22(31c+6) \pmod{18}$ satisfies the hypothesis of Proposition 2.6. Notice that as 2 is a primitive root of 19 the equation defining $\alpha(c)$ has a unique solution $\pmod{18}$ for each value of c. ∎

COROLLARY 2.9: *The quotient spaces* $\mathcal{S}(3i,l,m)$ *and* $\mathcal{S}(1,1,m)$ *are never smooth for any values of* i, l, m.

COROLLARY 2.10: *There are smooth* 3*-Sasakian* 11*-manifolds with second Betti number equal to* 2 *and* 3 *depending on an infinite number of parameters.*

PROOF: Let A be any 4×3 matrix of the form in equation 2.3 such that A also satisfies equation 2.8. Let $A(3)$ and $A(2)$ be obtained from A by omitting any row and two rows, respectively. Then $\Omega(2) = (\,\mathbb{I}_2 \quad A(2)\,)$ and $\Omega(3) = (\,\mathbb{I}_3 \quad A(3)\,)$ are 11-dimensional smooth examples with $k=2$ and 3, respectively. ∎

REMARK 2.11: We do not claim that the infinite number of 3-Sasakian manifolds constructed in Corollaries 2.7 and 2.10 are actually inequivalent as 3-Sasakian manifolds, although this is quite probably the case. They are, however, inequivalent as toral reductions which follows from the fact that at least one k by k minor determinant depends linearly on $c, 2^l$, and 2^m.

§3. Some 15 Dimensional Examples.

Again we begin by considering the highest possible second Betti number and hence assume that $\dim \mathcal{S} = 15$ and $k = 4$. In this case the matrix A of equation 2.1 becomes a 4 by 4 matrix. We choose A to be the matrix obtained by adding a 4th column to the matrix 2.3 so as to make our computations as simple as possible. Some of the computations leading to our examples stated below were done with the aid of MAPLE symbolic manipulation program. Thus, we set

3.1
$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1+2^l & -1 \\ 1 & 16 & 1+2^m & 3 \\ -1 & 3 & 2c & -1 \end{pmatrix}.$$

According to the reduction theorem, assuming that all minor determinants are non vanishing we get 3-Sasakian orbifolds $\mathcal{S}(c,l,m)$ of dimension 15 and second Betti number $b_2 = 4$. We need to check the conditions for smoothness of $\mathcal{S}(c,l,m)$. Now one easily sees that all the equations of the form 2.2a are satisfied if 3 does not divide c and m is even.

There are 24 equations of type 2.2c, and four equations of the form

3.2
$$\gcd(M_{i1}, \cdots, M_{i4}, \det A) = 1,$$

where M_{ij} denotes ij minor determinant of A. Of the 24 equations of type 2.2c, 14 are satisfied automatically. Three are precisely equations 2.5, and three of the remaining seven equations contain only one parameter each. These first three equations are

3.3
$$\begin{aligned} \gcd(5, 2^l - 1, 3 + 2^{l+1}) &= 1, \\ \gcd(29, 2^{m+1} - 1, 2^m - 15) &= 1, \\ \gcd(7, |2c - 3|, |4c + 1|) &= 1. \end{aligned}$$

These equations can be solved using elementary number theory. First, since m is even the second equation is automatically satisfied. The first equation holds if and only if 4 does not divide l, and the third holds if and only if $c \not\equiv 5 (mod\ 7)$. The remaining four equations of type 2.2c are:

3.4
$$\begin{aligned} \gcd(3, 4c + 2^l + 1, 2c - 2^l - 1) &= 1 \\ \gcd(7, 2^{m+1} - 2^l + 1, 3 \cdot 2^l + 2^m + 4) &= 1 \\ \gcd(19, 2^m - 2^{l+4} - 15, 3 \cdot 2^l + 2^m + 4) &= 1 \\ \gcd(25, 32c - 3 \cdot 2^m - 3, 6c + 2^m + 1) &= 1. \end{aligned}$$

The four equations given by 3.2 are determined by the determinant of A

3.5
$$\det A = 19 \cdot 2^m - 63 - 148c - 11 \cdot 2^l,$$

and the matrix of minor determinants (M_{ij}) is

$$\begin{pmatrix} 2^{m+1}+27-38c+25\cdot 2^l & -3\cdot 2^m-5-14c-2^{l+1} & -71 & 62c-7\cdot 2^m+12+19\cdot 2^l \\ -7\cdot 2^m+18+58c & 2^m-2c-1 & 11 & 30c-2^{m+2}+15 \\ -7\cdot 2^l-9+6c & 2^l+10c+4 & 19 & -2c-2^{l+2}+3 \\ -29\cdot 2^l-45+3\cdot 2^m & 5\cdot 2^m+2^l-1 & 74 & -15\cdot 2^l-2^m+15 \end{pmatrix}.$$

THEOREM 3.6: *Suppose that all entries and all minor determinants of A are also non vanishing. Then the 3-Sasakian orbifolds $\mathcal{S}(c,l,m)$ of dimension 15 with $b_2=4$ determined by $(\,\mathbb{I}_4\quad A\,)$ are smooth manifolds if and only if the following conditions hold*

(1) $c\not\equiv 0 \pmod 3$.

(2) m *is even.*

(3) $l\not\equiv 0 \pmod 4$.

(4) $c\not\equiv 5 \pmod 7$.

(5) $11, 19, 37, 71$ *do not divide* $\det A$.

(6) Conditions 3.4 hold.

PROOF: One only needs to check that if for a fixed row of the matrix (M_{ij}) the number M_{i3} divides any entry of the same row it divides all entries of that row. ∎

COROLLARY 3.7: *For an infinite number of values of the parameters c,l,m there are smooth 3-Sasakian 15-manifolds $\mathcal{S}(c,l,m)$ with second Betti number equal to 4.*

PROOF: It is straightforward to verify that the infinite family given by

3.8
$$\begin{aligned} c &= 2, \\ l &= 1, \\ n &\equiv 21 \pmod{90}, \end{aligned}$$

where $m=2n$ satisfies the conditions of Theorem 3.6. It is helpful to use the Law of Quadratic Reciprocity and the Chinese Remainder Theorem when checking conditions (5) and (6) of Theorem 3.6. ∎

COROLLARY 3.9: *For an infinite number of parameter values c,l,m there are smooth 3-Sasakian 15-manifolds $\mathcal{S}(c,l,m)$ with second Betti number equal to 2 and 3.*

PROOF: Let A be any 4×4 matrix of the form in equation 3.1 such that A also satisfies equation 3.8. Let $A(3)$ and $A(2)$ be obtained from A by omitting any row and two rows, respectively. Then $\Omega(2) = (\,\mathbb{I}_2 \quad A(2)\,)$ and $\Omega(3) = (\,\mathbb{I}_3 \quad A(3)\,)$ are 15-dimensional smooth examples with $k = 2$ and 3, respectively. ∎

We note that Remark 2.11 applies equally well to the 15 dimensional examples. Finally, it is intriguing to ponder the question of whether there are any 3-Sasakian manifolds at all that fall outside of the bounds given by Theorem A. A first step towards understanding this would be to prove a Delzant type theorem (See [Gui] or [BD]) saying that any compact 3-Sasakian manifold of dimension $4n-1$ with a T^n action preserving the 3-Sasakian structure must be one of our toral reductions. This is currently under investigation.

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December 1996