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Non-zero Scalar Curvature Generalizations of the ALE Hyperkähler Metrics

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Abstract: We describe a quaternionic quotient construction of families of self-dual Einstein metrics with positive scalar curvature on compact Riemannian orbifolds (V-manifolds). The metrics are the positive scalar curvature quaternionic analogues of the ALE gravitational instantons constructed by Kronheimer.

1. Introduction

Self-dual and Einstein 4-manifolds are of particular interest to both mathematicians and physicists. Many new examples of such geometries were discovered through the development of the Euclidean approach to quantum gravity. There the zero-temperature vacuum state of the gravitational field can be thought of as a Ricci-flat metric with certain asymptotic behavior. This leads to, so-called, asymptotically locally Euclidean (ALE) Ricci-flat manifolds. A special case of such geometries, which is now completely understood, are the hyperkähler ALE instantons. These are Riemannian 4-manifolds with $SU(2)$ holonomy group for which some neighborhood of infinity has a finite covering $\tilde{U}$ diffeomorphic to the complement of the unit ball in $\mathbb{R}^4$. If x_i is the natural coordinate on $\mathbb{R}^4$ then the metric $g_{ij} = \delta_{ij} + h_{ij}$ on $\tilde{U}$ tends to the standard Euclidean metric with $\partial^p(h_{ij}) = O(r^{-4-p})$, where r is the proper distance. The fall-off conditions given here are these of [1]. (Weaker conditions can often be found in some mathematical physics literature referring to the ALE metrics, see for example [2]).

The first example of the hyperkähler ALE metric was constructed by Eguchi and Hanson [3]. It describes the Kähler Ricci-flat metric on the cotangent bundle of the complex

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projective line. At the same time Calabi [4] gave a description of hyperkähler metrics on $T^*\mathbb{CP}(n)$ and the Eguchi-Hanson metric is precisely the Calabi metric on $T^*\mathbb{CP}(1)$. More examples were given later by Gibbons and Hawking [5]. They were called multi-Eguchi-Hanson metrics. In the independent work of Hitchin [6] the multi-Eguchi-Hanson metrics were obtained as metrics on the minimal resolution of the singularity of $\mathbb{C}^2/\mathbb{Z}_k$. In the same work Hitchin conjectured the existence of such metrics on the minimal resolution of bbc^2/Γ where $\Gamma \subset SU(2)$ is any discrete subgroup. This conjecture was finally proven by Kronheimer [1, 7]. He used the quotient techniques of Hitchin et al. [8] to describe these metrics explicitly and showed that any ALE hyperkähler manifold is isometric to a member of one of the families obtained in his construction.

The ALE hyperkähler manifolds are examples of self-dual Einstein manifolds for which classification results were obtained. In general case our understanding of such geometries is far from satisfactory. Very little is known about non-compact case. If the manifold is compact, we have the well-known result of Hitchin [9] which says that, if the scalar curvature κ of a compact self-dual Einstein manifold M is non-negative, then

(i) M is isometric to S^4 or $\mathbb{CP}^2$ with their canonical metrics ($\kappa > 1$),

(ii) M is either flat or its universal covering is a $K3$ surface with the Calabi-Yau metric ($\kappa = 0$).

Again, not much is known if M is compact and has negative scalar curvature, and the only examples here are compact quotients of the hyperbolic 4-ball.

If M is not a manifold but rather a V-manifold, or a Riemannian orbifold, then the Hitchin's result no longer applies. Examples of infinitely many non-symmetric self-dual Einstein metrics with positive scalar curvature on compact orbifolds were constructed by Galicki and Lawson [10]. All of them are metrics on the weighted projective spaces $\mathbb{CP}^2_{p,q,s}$. However, even with the powerful technique of the quaternionic quotient, it is not easy to find new examples of such metrics with only orbifold singularities. In this paper we show that the orbifolds of Galicki and Lawson are special cases of a more general construction. For any ALE family of hyperkähler spaces $M(\Gamma, \xi)$ we obtain a new family of compact self-dual Einstein orbifolds with positive scalar curvature $\mathcal{O}(\Gamma, \xi, b)$. Just as in the case of the Kronheimer's construction [1, 7], the metrics are given only implicitly as quaternionic quotients of some quaternionic projective space. Their explicit calculation would involve solving a large set of quadratic constraints. However, in principle, our orbifolds provide a large family of local self-dual and Einstein metrics with positive cosmological constant.

The paper is organized as follows: In Section 2 we review the necessary facts about the geometry of the hyperkähler and the quaternionic Kähler quotients. In Section 3 we describe our construction of $\mathcal{O}(\Gamma, \xi, b)$. In Section 4 we discuss the quaternionic associated bundle of $\mathcal{O}(\Gamma, \xi, b)$ and its twistor space. Finally, in Section 5 we describe some simple examples.

2. Hyperkähler and Quaternionic Kähler Quotients

We begin by recalling basic definitions of hyperkähler and quaternionic Kähler geometries. We also briefly review the quotient constructions of [8] and [10].

Let $(M, <\cdot,\cdot>)$ be a hyperkähler manifold, *i.e.*, M is a $4n$-dimensional Riemannian manifold with three parallel complex structures $J^i \in \mathrm{End}(TM), i = 1, 2, 3$ where

$$J^i \circ J^k = -\delta^{ik} + \epsilon^{ikj} J^j. \tag{2.1}$$

Let the metric $<\cdot,\cdot>$ be Hermitian with respect to all three complex structures. We can define three symplectic forms $\omega^i(X, Y) = (J^i X, Y), \quad X, Y \in \Gamma(TM)$. Let $G \times M \longrightarrow M$ be a compact action on M by isometries commuting with all three complex structures. We call such isometries hyperkählerian. Let $\mathfrak{g}$ be the Lie algebra of G. Then there is a hyperkähler moment map

$$\mu = \mu^1 i + \mu^2 j + \mu^3 k : \ M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathfrak{sp}(1), \tag{2.2}$$

defined as

$$i_V \omega^i =< d\mu^i, V >, \tag{2.3}$$

where $V \in \mathfrak{g} \otimes \Gamma(TM)$.

Let ξ be an invariant element of $\mathfrak{g}^* \otimes_{\mathbb{R}} \mathfrak{sp}(1)$ under the coadjoint action $Ad^* \otimes id$ of G. Now, suppose $\widehat{M}(\xi) = \mu^{-1}(\xi)/G$ is a smooth Riemannian manifold. Consider the inclusion and projection maps

$$M \stackrel{i}{\hookleftarrow} \mu^{-1}(\xi) \stackrel{\pi}{\to} \widehat{M}(\xi). \tag{2.4}$$

Theorem [8] 2.1: *If $\widehat{M}(\xi)$ is a manifold, then its induced Riemannian metric is hyperkählerian.*

The above construction was recently used by Kronheimer to obtain families of the hyperkähler ALE spaces for all discrete subgroups Γ of $SU(2)$ (cf. [1, 7]). We describe his construction using quaternionic notation and leaving out details. We refer the reader to the original articles.

Let $M = \mathbb{H}^{|\Gamma|}$ be the $|\Gamma|$-dimensional quaternionic vector space, where $|\Gamma|$ is the order of Γ. For every Γ, one can uniquely define a Lie group $G(\Gamma) \subset U(|\Gamma|)/U(1)$ and its representation in $Sp(|\Gamma|)$

$$G(\Gamma) \ni g \quad \longrightarrow \quad A(g) \in Sp(|\Gamma|). \tag{2.5}$$

The group $G(\Gamma)$ acts on $\mathbb{H}^{|\Gamma|} \ni \mathbf{w}$ by hyperkähler isometries as

$$g \cdot \mathbf{w} = A(g)\mathbf{w}. \tag{2.6}$$

The momentum mapping $\mu: \mathbb{H}^{|\Gamma|} \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathfrak{sp}(1)$ is defined as follows

$$< \mu(\mathbf{w}), X >= {}^t\overline{\mathbf{w}} dA(X)\mathbf{w}, \tag{2.7}$$

where $A(g) = \exp(dA(X))$, $X \in \mathfrak{g}$, and $<\cdot,\cdot>$ is the natural pairing on $\mathfrak{g}^* \times \mathfrak{g}$. One obtains the Kronheimer's ALE spaces $M(\Gamma, \xi)$ as the quotient of $\mu^{-1}(\xi) \equiv \{\mathbf{w} \in \mathbb{H}^{|\Gamma|} : \mu(\mathbf{w}) = \xi\}$ by $G(\Gamma)$. The quotient is a smooth Riemannian manifold if and only if ξ is an element of a "good set" (see [1]). The set of all invariant elements of $\mathfrak{g}^* \otimes_{\mathbb{R}} \mathfrak{sp}(1)$ is given by $\mathfrak{t}^* \otimes_{\mathbb{R}} \mathfrak{sp}(1)$, where $\mathfrak{t}$ is the Lie algebra of the center of $G(\Gamma)$. As $\mathfrak{t}$ is identified with the Cartan subalgebra of one of root systems A_n, D_n, E_6, E_7, E_8 associated to the set of all irreducible representations of Γ, the "good set" consists of $\xi \in \mathfrak{t}^* \otimes_{\mathbb{R}} \mathfrak{sp}(1)$ whose components are regular.

In the next chapter we will demonstrate that one can generalize this construction to obtain non-zero scalar curvature quaternionic analogues of all the hyperkähler ALE spaces. First, let us recall the quaternionic reduction of Galicki and Lawson [10].

Let N be a $4n$-dimensional quaternionic manifold. There is a quaternionic structure on N, *i.e.,* a rank 3 vector bundle $\mathcal{V} \subset \mathrm{End}(TM)$ of endomorphisms in a local frame described by J^1, J^2, J^3 satisfying relations (1.1). Let g be a Riemannian metric on N, Hermitian with respect to J^1, J^2, J^3. Then we can isomorphically identify the bundle $\mathcal{V}$ with a subbundle of $\Lambda^2 T^* M$ spanned by $\omega^1, \omega^2, \omega^3$. Now $\Omega = \omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 + \omega^3 \wedge \omega^3$ is a globally defined 4-form on N. If $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection of g, then the holonomy of N reduces to a subgroup of $Sp(\frac{n}{4}) \cdot Sp(1)$ and N is a quaternionic Kähler manifold.

Let G act on N by quaternionic isometries, *i.e.,* preserving Ω, and let the scalar curvature κ of N be non-zero. As demonstrated in [10], for any vector field V_X of the G-action on N generated by $X \in \mathcal{G}$ there exists a unique section f_X of the bundle $\mathcal{V}$ defined as

$$\nabla f_X = \sum_{\alpha=1,2,3} \omega_\alpha(V_X) \otimes \omega_\alpha. \tag{2.8}$$

We define a "zero level" set as

$$N \supset f^{-1}(0) \equiv \{y \in N : \quad f_X(y) = 0, \ X \in \mathcal{G}\}. \tag{2.9}$$

THEOREM [10] 2.2: *If the quotient $f^{-1}(0)/G$ is a smooth manifold, then its induced metric is quaternionic Kähler.*

As in the previous case, both the metric and the quaternionic structure are induced by the inclusion and projection maps. In the case when the action of G on $f^{-1}(0)$ is not free but only locally free, the quotient yields a quaternionic Kähler orbifold.

Using this theorem one can obtain families of compact 4-dimensional orbifolds $\mathcal{O}_{p,q}(1)$, $q, p \in \mathbb{Z}$, $q \le p$, $(q,p) = 1$, with self-dual Einstein metrics of positive scalar curvature.

Let $N = \mathbb{HP}^n \ni [u_o, \mathbf{u}]$ and let $G = U(1)$ act on N as follows:

$$[u_0, \mathbf{u}] \longrightarrow [e^{ipt}u_0, e^{iqt}\mathbf{u}]. \tag{2.10}$$

The zero level set for this action is

$$f^{-1}(0) = \{[u_0, \mathbf{u}] \in \mathbb{HP}^n : \ \overline{u}_0 ipu_0 +^t \overline{\mathbf{u}} iq\mathbf{u} = 0\} \tag{2.11}$$

and

$$\mathcal{O}_{p,q}(n) = f^{-1}(0)/U(1).$$

The singular structure of $\mathcal{O}_{p,q}(n)$ was described in [10, 11]. $\mathcal{O}_{p,q}(1)$ is a 4-dimensi- onal compact orbifold with positive scalar curvature self-dual Einstein metric at all regular points. The metric is not locally symmetric and $\mathcal{O}_{p,q}(1)$ is smoothly equivalent to $\mathbb{CP}^2_{2q,p+q,p+q}$ for $(p+q)$ odd and $\mathbb{CP}^2_{q,\frac{p+q}{2},\frac{p+q}{2}}$ for $(p+q)$ even. The orbifold $\mathcal{O}_{p,1}(1)$ is an analogue of the Eguchi-Hanson hyperkähler metric which is $\Gamma = \mathbb{Z}_2$ case in the Kronheimer's construction. In Section 3 we will see that this is a special situation. There are orbifold analogues of all other hyperkähler ALE spaces.

3. Quotient Construction of Orbifolds

Consider the quaternionic projective space $\mathbb{P}(\mathbb{H}\times\mathbb{H}^{|\Gamma|})$, where Γ is a discrete subgroup of $SU(2)$. Let the Kronheimer group $G(\Gamma)$ act on $\mathbb{P}(\mathbb{H}\times\mathbb{H}^{|\Gamma|})$ as follows

$$G(\Gamma) \ni g \quad : \quad g\cdot[u_0, \mathbf{u}] \equiv [b(g)u_0, A(g)\mathbf{u}] \tag{3.1}$$

where $A : \ G(\Gamma) \longrightarrow Sp(|\Gamma|)$ is the representation of $G(\Gamma)$ in $Sp(|\Gamma|)$ as in (2.5) and $b : \ G(\Gamma) \longrightarrow Sp(1)$ is a group homomorphism. The above action is well-defined on $\mathbb{P}(\mathbb{H}\times\mathbb{H}^{|\Gamma|})$ for any choice of the homomorphism b. The zero section level for this action can be described by the following constraints

$$\mathbb{HP}^{|\Gamma|} \supset f^{-1}(0) = \{[u_0, \mathbf{u}] \in \mathbb{HP}^{|\Gamma|} : \ \overline{u}_0 db(X)u_0 + \ {}^t\overline{\mathbf{u}} dA(X)\mathbf{u} = 0, X \in \mathfrak{g}\}, \tag{3.2}$$

where $\mathfrak{g}$ is the Lie algebra of $G(\Gamma)$.

Lemma 3.1: *Let $db(X) = -\xi(X), (db = -\xi), \ \ \xi \in \mathfrak{g}^* \otimes \mathfrak{sp}(1)$. The action (3.1) on $f^{-1}(0)$ is then locally free if ξ is in the Kronheimer's "good set".*

Proof: The vector field on $f^{-1}(0)$ associated to $X \in \mathfrak{g}$ vanishes if there exists $\lambda \in \mathbb{H}^*$ such that

$$(db(X)u_0, dA(X)\mathbf{u}) = (u_0, \mathbf{u})\lambda. \tag{3.3}$$

Take the Hermitian product of the both sides with $(u_0, \mathbf{u})$

$$\overline{u}_0 db(X)u_0 + \ {}^t\overline{\mathbf{u}} dA(X)\mathbf{u} = (|u_0|^2 + |\mathbf{u}|^2)\lambda. \tag{3.4}$$

However, the left-hand side in (3.4) vanishes on $f^{-1}(0)$ and therefore $\lambda = 0$ as $|u_0|^2 + |\mathbf{u}|^2 > 0$. Hence, the vector field V_X vanishes on $f^{-1}(0) \in \mathbb{HP}^{|\Gamma|}$ if and only if $(db(X)u_0, dA(X)\mathbf{u}) = (0,0)$.

We need to consider two cases:

Case 1. Let $\mathcal{U}_0 \in \mathbb{HP}^{|\Gamma|}$ be an open set such that $u_0 \neq 0$. Since $u_0 \neq 0$ then $db(X) = 0$. Let $\mathbf{w} = \mathbf{u}u_0^{-1}$. In terms of $\mathbf{w}$ we can write

$$f^{-1}(0) \cap \mathcal{U}_0 = \{[1, \mathbf{w}] : \ {}^t\overline{\mathbf{w}} dA \mathbf{w} = \xi\}. \tag{3.5}$$

But if ξ is in the "good set" then the action of $A(g)$ on $\mathbf{w}$ is free and therefore its vector field $dA(X)\mathbf{w}$ is nowhere zero. Hence $dA(X)\mathbf{u}$ cannot vanish on $f^{-1}(0) \cap \mathcal{U}_0$.

Case 2. Let $\mathcal{U}_1 = \mathbb{HP}^{|\Gamma|} \setminus \mathcal{U}_0$. Then

$$f^{-1}(0) \cap \mathcal{U}_1 = \{[0, \mathbf{u}] \in \mathbb{HP}^{|\Gamma|} : \quad {}^t\overline{\mathbf{u}} dA \mathbf{u} = 0\}. \tag{3.6}$$

However, as

$$\{\mathbf{u} \in \mathbb{H}^{|\Gamma|} : \quad {}^t\overline{\mathbf{u}} dA \mathbf{u} = 0\}/G(\Gamma) \simeq \mathbb{C}^2/\Gamma$$

$dA\mathbf{u} \neq 0$ unless $\mathbf{u} = 0$. This is because the action $A(g)\mathbf{u}$, $\mathbf{u} \in \mathbb{H}^{|\Gamma|}$ is free on $\{\mathbf{u} \in \mathbb{H}^{|\Gamma|} : {}^t\overline{\mathbf{u}} dA \mathbf{u} = 0\}$ away from $\mathbf{u} = 0$ (see [1]).

Since $0 = \mathbf{u} \notin f^{-1}(0) \cap \mathcal{U}_1$, $dA(X)\mathbf{u} \neq 0$ there and the action of $G(\Gamma)$ is locally free on $f^{-1}(0) \cap \mathcal{U}_1$. ∎

It follows now from the Theorem 2.2 that

THEOREM 3.2: *If $db(X) = -\xi(X)$ where $b : G(\Gamma) \longrightarrow Sp(1)$ is a homomorphism and ξ is the Kronheimer's "good set" then $\mathcal{O}(\Gamma, \xi, b) \equiv f^{-1}(0)/G(\Gamma)$ is a compact 4-dimensional orbifold with self-dual Einstein metric of positive scalar curvature at all regular parts.*

The condition $db = -\xi$ clearly puts further restrictions on ξ. We observe that, according to the result of Salamon [12], the twistor space $\mathcal{Z}(\Gamma, \xi, b)$ of $\mathcal{O}(\Gamma, \xi, b)$ is a compact Kähler-Einstein orbifold of complex dimension 3 and of positive scalar curvature. We will discuss the quotient construction of $\mathcal{Z}(\Gamma, \xi, b)$ in the next chapter.

Let us briefly describe the relationship between the Kronheimer's ALE spaces and our orbifolds $\mathcal{O}(\Gamma, \xi, b)$. One can consider two different actions of the group $G(\Gamma)$ on $f^{-1}(0) \cap \mathcal{U}_0 \subset \mathbb{H}^{|\Gamma|}$:

$$\rho : G(\Gamma) \times f^{-1}(0) \ni (g, \mathbf{w}) \longrightarrow A(g)\mathbf{w} \in f^{-1}(0),$$

$$\rho' : G(\Gamma) \times f^{-1}(0) \ni (g, \mathbf{w}) \longrightarrow A(g)\mathbf{w}b(g)^{-1} \in f^{-1}(0). \tag{3.7}$$

Then the the Kronheimer's ALE space is obtained as the quotient of $f^{-1}(0) \cap \mathcal{U}_0$ by the first action and our orbifold $\mathcal{O}(\Gamma, \xi, b)$ with one point removed ($u_0 = 0$) is the quotient of the same space by the second action. We have

$$M(\Gamma, \xi) \; \stackrel{\rho}{\longleftarrow} \; f^{-1}(0) \cap \mathcal{U}_0 \; \stackrel{\rho'}{\longrightarrow} \; \mathcal{O}(\Gamma, \xi, b) \setminus \{pt.\}. \tag{3.8}$$

Let us point out that, although topologically both $M(\Gamma, \xi)$ and $\mathcal{O}(\Gamma, \xi, b)$ are quotients of the same space by ρ and ρ', the metrics on $f^{-1}(0) \cap \mathcal{U}_0$ are different in those two constructions. In the first case we must take the flat metric on $\mathbb{H}^{|\Gamma|}$ and restrict it to $f^{-1}(0) \cap \mathcal{U}_0$. In the second case we take the Fubini-Study metric on $\mathbb{H}^{|\Gamma|} \simeq \mathbb{HP}^{|\Gamma|} \setminus \{pt.\}$.

If we replace $\mathbb{HP}^{|\Gamma|}$ by the quaternionic hyperbolic space $\mathbb{HH}^{|\Gamma|} = \mathbb{P}(\mathbb{H}^{|\Gamma|,1})$ we get 4-dimensional orbifold Einstein metrics with negative scalar curvature. In this case the construction can sometimes lead to complete examples [13].

4. The Associated Bundle of $\mathcal{O}(\Gamma, \xi, b)$ and Its Twistor Space

Notice that our extension of the action of $G(\Gamma)$ on $\mathbb{P}(\mathbb{H}^{|\Gamma|+1})$ can be lifted to a hyperkähler action on the quaternionic vector space $\mathbb{H}^{|\Gamma|+1}$ as follows

$$G(\Gamma) \times \mathbb{H}^{|\Gamma|+1} \longrightarrow \mathbb{H}^{|\Gamma|+1}$$

$$g \cdot (u_0, \mathbf{u}) \equiv (b(g)u_0, A(g)\mathbf{u}), \tag{4.1}$$

with A and b as before. The momentum map for this action $\widetilde{\mu}: \; \mathbb{H}^{|\Gamma|+1} \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathfrak{sp}(1)$ reads

$$< \widetilde{\mu}(u_0, \mathbf{u}), X >= \overline{u}_0 db(X) u_0 + {}^t\overline{\mathbf{u}} dA(X)\mathbf{u}, \quad X \in \mathfrak{g}. \tag{4.2}$$

Lemma 4.1: *Let $db = -\xi \in \mathfrak{g}^* \otimes_{\mathbb{R}} \mathfrak{sp}(1)$. Then $G(\Gamma)$ acts freely on $\tilde{\mu}^{-1}(0) \cap \widetilde{\mathcal{U}_0}$ where $\widetilde{\mathcal{U}_0} \equiv \{(u_0, \mathbf{u}) \in \mathbb{H}^{|\Gamma|+1} \; : \; u_0 \neq 0\}$ if ξ is in the "good set".*

Proof: Since

$$\tilde{\mu}^{-1}(0) = \{(u_0, \mathbf{u}) \in \mathbb{H}^{|\Gamma|+1} \; : \; \overline{u}_0 db(X) u_0 + {}^t\overline{\mathbf{u}} dA(X)\mathbf{u} = 0, \quad X \in \mathfrak{g}\}. \tag{4.3}$$

Let $\mathbf{w} = \mathbf{u} u_0^{-1}$ on $\tilde{\mu}^{-1}(0) \cap \widetilde{\mathcal{U}_0}$. In the $\mathbf{w}$-coordinates

$$\tilde{\mu}^{-1}(0) \cap \widetilde{\mathcal{U}_0} = \{(u_0, \mathbf{w}) \in \mathbb{H}^{|\Gamma|+1} \; : \; u_0 \neq 0, \quad -\xi + {}^t\overline{\mathbf{w}} dA \mathbf{w} = 0\}. \tag{4.4}$$

Suppose

$$(b(g)u_0, A(g)\mathbf{u}) = (u_0, \mathbf{u}).$$

On $\tilde{\mu}^{-1}(0) \cap \widetilde{\mathcal{U}_0}$ we then must have $b(g) \equiv id$. Since $G(\Gamma)$ acts on $(u_0, \mathbf{w}) \in \tilde{\mu}^{-1}(0) \cap \widetilde{\mathcal{U}_0}$ as follows

$$(u_0, \mathbf{w}) \longrightarrow (b(g)u_0, A(g)\mathbf{w} b(g)^{-1}), \tag{4.5}$$

the condition that $b(g) = id$ implies that $A(g)\mathbf{w} = \mathbf{w}$. But $A(g)$ acts freely on $\mathbf{w}$ if $\mathbf{w} \in \{\mathbf{w} \in \mathbb{H}^{|\Gamma|} : \ \xi = {}^t\overline{\mathbf{w}}dA\mathbf{w}\}$ and ξ is in the "good set". ∎

Let $\widetilde{\mathcal{U}_1} = \mathbb{H}^{|\Gamma|+1} \setminus \widetilde{\mathcal{U}_0} \simeq \mathbb{H}^{|\Gamma|}$. Then $G(\Gamma)$ acts on $\widetilde{\mathcal{U}_1}$

$$(\tilde{\mu}^{-1}(0) \cap \widetilde{\mathcal{U}_1})/G(\Gamma) \simeq \mathbb{C}^2/\Gamma. \tag{4.6}$$

It is easy to see that $G(\Gamma)$ act freely on $(\mathbb{H}^{|\Gamma|+1} \setminus \{0\}) \cap \tilde{\mu}^{-1}(0)$. Hence

$$\widehat{M}(\Gamma,\xi) = \Big((\mathbb{H}^{|\Gamma|+1} \setminus \{0\}) \cap \tilde{\mu}^{-1}(0)\Big)/G(\Gamma)$$

is a smooth 8-dimensional hyperkähler manifold. It follows from the above remark and the work of Swann [14] that we have the following fiberation

$$\widehat{M}(\Gamma,\xi,b) \stackrel{\mathbb{H}/\mathbb{Z}_2*}{\longrightarrow} \mathcal{O}(\Gamma,\xi,b), \tag{4.7}$$

where $\widehat{M}(\Gamma,\xi,b)$ is the quaternionic associated bundle of $\mathcal{O}(\Gamma,\xi,b)$. $\mathcal{O}(\Gamma,\xi,b)$ can also be interpreted as a certain $Sp(1)$ quotient of its associated bundle where $Sp(1)$ acts by isometries rotating hyperkähler 2-forms [15, 16]. (Or the $\mathbb{H}^*/\mathbb{Z}_2$ quotient, where $\mathbb{H}^*$ acts by the quaternionic multiplication from the left). The manifold $\widehat{M}(\Gamma,\xi,b)$ represents a very special case of the hyperkähler geometry. It admits an isometric $Sp(1)$ action rotating the hyperkähler structure. Moreover, $\widehat{M}(\Gamma,\xi,b)$ has a hyperkähler potential [14,15], *i.e.,* there exists a function ν on $\widehat{M}(\Gamma,\xi,b)$ such that its hyperkähler metric g is given by $g = \nabla^2\nu$. Other examples of hyperkähler manifolds with such properties are furnished by the instanton moduli spaces and the nilpotent adjoint orbits of complex Lie groups [15].

The twistor space of $\mathcal{O}(\Gamma,\xi,b)$ can be expressed as a Kähler quotient of $\widehat{M}(\Gamma,\xi,b)$ by the action of any circle subgroup $U(1) \subset Sp(1)$ with respect to the Kähler structure that is preserved by this particular $U(1)$. By the theorem of Salamon [12], $\mathcal{Z}(\Gamma,\xi,b)$ carries a Kähler-Eintein metric of positive scalar curvature. Hence, the Kähler quotient of $\widehat{M}(\Gamma,\xi,b)$ by any $U(1) \subset Sp(1)$ is not just Kähler. It is an example of the Kähler-Einstein quotient, *i.e.,* it is a Kähler quotient of a Kähler-Einstein manifold with the property that the reduced manifold is Einstein. We can describe all these quotients in the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbb{H}^{|\Gamma|+1} \setminus \{0\} & & & \stackrel{G(\Gamma)}{\Longrightarrow} & & & \widehat{M}(\Gamma,\xi,b) \\
 & \searrow & & & & \swarrow & \\
\downarrow \ \mathbb{H}^* & & \mathbb{CP}^{2|\Gamma|+1} & \stackrel{G(\Gamma)\otimes\mathbb{C}}{\Longrightarrow} & \mathcal{Z}(\Gamma,\xi,b) & & \downarrow \mathbb{H}^*/\mathbb{Z}_2 \\
 & \swarrow & & & & \searrow & \\
\mathbb{P}(\mathbb{H}^{|\Gamma|+1}) & & & \stackrel{G(\Gamma)}{\Longrightarrow} & & & \mathcal{O}(\Gamma,\xi,b)
\end{array} \tag{4.8}$$

The horizontal double arrows represent the hyperkähler quotient by $G(\Gamma)$, the Kähler quotient by $G(\Gamma) \otimes \mathbb{C}$, and the quaternionic Kähler quotient by $G(\Gamma)$ respectively. The maps $\mathbb{H}^{|\Gamma|+1} \setminus \{0\} \xrightarrow{\mathbb{C}^*} \mathbb{CP}^{2|\Gamma|+1}$ and $\widehat{M}(\Gamma,\xi,b) \xrightarrow{\mathbb{C}^*/\mathbb{Z}_2} \mathcal{Z}(\Gamma,\xi,b)$ are the Kähler-Einstein quotients. Finally, $\mathbb{CP}^{2|\Gamma|+1} \xrightarrow{\pi} \mathbb{P}(\mathbb{H}^{|\Gamma|+1})$ and $\mathcal{Z}(\Gamma,\xi,b) \xrightarrow{\pi} \mathcal{O}(\Gamma,\xi,b)$ are the twistor fiberations.

5. Some Examples

In the following section we consider some simple examples.

EXAMPLE 5.1: Let $\Gamma = \mathbb{Z}_2$. Then $G(\Gamma) = U(1)$ and $\mathbb{H}^{|\Gamma|} = \mathbb{H}^2$. Let $g = e^{it} \in U(1)$. Then $A : U(1) \longrightarrow Sp(2)$ is given by the diagonal action

$$A(g) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix} \tag{5.1}$$

or

$$\mathbb{H}^2 \ni (u_1, u_2) \xrightarrow{t} (e^{it}u_1, e^{it}u_2).$$

The hyperkähler moment map reads

$$\mu(u_1, u_2) = \overline{u}_1 i u_1 + \overline{u}_2 i u_2. \tag{5.2}$$

Now, it is easy to see that

$$\mu^{-1}(\xi)/U(1) = T^*\mathbb{CP}(1).$$

Consider a group homomorphism $b : U(1) \longrightarrow Sp(1)$. As $U(1)$ is Abelian any such homomorphism is given by a number p and

$$g = e^{it}, \qquad b(g) = e^{\rho p t}, \tag{5.3}$$

where $\overline{\rho} = -\rho \in \mathfrak{sp}(1)$. By rotating the quaternionic structure on u_0 we can always choose $\rho = i$ and $p \in \mathbb{Z}_+$. Consider the action of $U(1)$ on $\mathbb{HP}^2 \in [u_0, u_1, u_2]$ in homogeneous coordinates

$$g \cdot [u_0, u_1, u_2] = [e^{ipt}u_0, e^{it}u_1, e^{it}u_2]. \tag{5.4}$$

The quaternionic moment map is then

$$\mu([u_0, u_1, u_2]) = \overline{u}_1 i u_1 + \overline{u}_2 i u_2 + p\overline{u}_0 i u_0. \tag{5.5}$$

Comparing with (2.11) shows that the quotient

$$\mu^{-1}(0)/U(1) = \mathcal{O}_{p,1}(1).$$

The diagram (3.8) in this case gives

$$\begin{array}{ccc} \mathbb{H}^3 \setminus \{0\} & \longrightarrow & \widehat{M}(\mathbb{Z}_2, p) \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathbb{H}^2) & \longrightarrow & \mathcal{O}(\mathbb{Z}_2, p) \simeq \mathcal{O}_{p,1}(1) \end{array} \tag{5.6}$$

Here $\widehat{M}(\mathbb{Z}_2, p)$ is the associated bundle of the orbifold $\mathcal{O}_{p,1}(1)$. For $p = 1$ $\mathcal{O}_{1,1}(1) = \mathbb{CP}^2$ and its associated bundle is the singular limit of $T^*\mathbb{CP}^2$.

EXAMPLE 5.2: Let $\Gamma = \mathbb{Z}_3$ and $G(\Gamma) = U(1) \times U(1)$. If $g = (e^{is}, e^{it}) \in U(1) \times U(1)$ we have

$$A(g) = \begin{pmatrix} e^{is} & 0 & 0 \\ 0 & e^{i(t-s)} & 0 \\ 0 & 0 & e^{-it} \end{pmatrix} \in Sp(3). \tag{5.7}$$

Thus, the Kronheimer's construction gives the following action of $U(1) \times U(1)$ on $\mathbb{H}^3$:

$$g\cdot(u_1, u_2, u_3) = (e^{is}u_1, e^{i(t-s)}u_2, e^{-it}u_3)$$

and the hyperkähler momentum maps is

$$\mu(u_1, u_2, u_3) = \begin{pmatrix} \overline{u}_1 i u_1 - \overline{u}_2 i u_2 \\ \overline{u}_2 i u_2 - \overline{u}_3 i u_3 \end{pmatrix} \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathfrak{sp}(1). \tag{5.8}$$

Now

$$\mu^{-1}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} / U(1) \times U(1) = M(\mathbb{Z}_3, \xi), \tag{5.9}$$

where $\xi \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathfrak{sp}(1)$ is in the "good set", is the two-center multi-Eguchi-Hanson metric. The element ξ in the "good set" means here that $\xi_1 \neq -\xi_2$. Now we consider a homomorphism $b : U(1) \times U(1) \longrightarrow Sp(1)$

$$b(e^{is}, e^{it}) = e^{ias+ibt} \in Sp(1), \quad a, b \in \mathbb{Z}. \tag{5.10}$$

so that the action of $U(1) \times U(1)$ extends to $\mathbb{HP}^3 \ni [u_0, u_1, u_2, u_3]$ as

$$g\cdot[u_0, u_1, u_2, u_3] = [e^{i(as+bt)}u_0, e^{is}u_1, e^{i(t-s)}u_2, e^{-it}u_3] \tag{5.11}$$

and the quaternionic moment maps is

$$\mu([u_0, u_1, u_2, u_3]) = \begin{pmatrix} \overline{u}_1 i u_1 - \overline{u}_2 i u_2 + a\overline{u}_0 i u_0 \\ \overline{u}_2 i u_2 - \overline{u}_3 i u_3 + b\overline{u}_0 i u_0 \end{pmatrix}. \tag{5.12}$$

Consider the $Sp(1)$-invariant zero section $\mu^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $\mathbb{HP}^3$. For any $a, b \in \mathbb{Z} \setminus \{0\}$, $a \neq -b$ the action of $U(1) \times U(1)$ is locally free on $\mu^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{HP}^3$. Hence

$$\mathcal{O}(\mathbb{Z}_3; a, b) = \mu^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} / U(1) \times U(1) \tag{5.13}$$

is a compact 4-dimensional orbifold with self-dual Einstein metric and positive scalar curvature.

Let us analyze the singular structure of $\mathcal{O}(\mathbb{Z}_3; a, b)$ in the simple case when $a = b = 1$. We write $\mathbb{HP}^3 = \cap_{i=0}^{3}\mathcal{U}_i$ where $\mathcal{U}_i = \{[u_0, u_1, u_2, u_3] \in \mathbb{HP}^3 : \quad u_i \neq 0\}$. The action of $U(1) \times U(1)$ is

$$[u_0, u_1, u_2, u_3] \xrightarrow{(s,t)} [e^{i(s+t)}u_0, e^{is}u_1, e^{i(t-s)}u_2, e^{-it}u_3] \tag{5.14}$$

and $\mu^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is described by the following constraints

$$\begin{aligned} &\overline{u}_1 i u_1 - \overline{u}_2 i u_2 + \overline{u}_0 i u_0 = 0 \\ &\overline{u}_2 i u_2 - \overline{u}_3 i u_3 + \overline{u}_0 i u_0 = 0. \end{aligned} \tag{5.15}$$

On $\mathcal{U}_0$ we introduce the non-homogeneous coordinates $w_i = u_i u_0^{-1}, \quad i = 1, 2, 3$ and then

$$(w_1, w_2, w_3) \xrightarrow{(s,t)} (e^{is}w_1 e^{-i(s+t)}, e^{i(t-s)}w_2 e^{-i(s+t)}, e^{-it}w_3 e^{-i(s+t)}) \tag{5.16}$$

$$\begin{aligned} &\overline{w}_1 i w_1 - \overline{w}_2 i w_2 = -i \\ &\overline{w}_2 i w_2 - \overline{w}_3 i w_3 = -i. \end{aligned} \tag{5.17}$$

Let us write $w_i = x_i + j y_i, \quad i = 1, 2, 3$ where $(x_i, y_i) \in bbc^3 \times \mathbb{C}^3 \simeq \mathbb{H}^3$. As

$$\overline{w}_i i w_i = i(|x_i|^2 - |y_i|^2) - 2ji(x_i y_i)$$

we can rewrite both (5.16) and (5.17)

$$(x_1, y_1, x_2, y_2, x_3, y_3) \xrightarrow{(s,t)} (e^{-it}x_1, e^{-i(2s+t)}y_1, e^{-i2s}x_2, e^{-i2t}y_2, e^{-i(s+2t)}x_3, e^{-is}y_3), \tag{5.18}$$

$$\begin{aligned} &|x_2|^2 - |y_2|^2 - (|x_1|^2 - |y_1|^2) = 1 \\ &|x_3|^2 - |y_3|^2 - (|x_2|^2 - |y_2|^2) = 1 \\ &x_1 y_1 = x_2 y_2 = x_3 y_3. \end{aligned} \tag{5.19}$$

One can easily see that there are three tori which are fixed on $\mathcal{U}_0$ by discrete subgroups of $U(1) \times U(1)$ namely

$$T^2_{_1} = \{(x_i, y_i) \in \mathbb{C}^3 \times \mathbb{C}^3 : x_1 = x_2 = x_3 = y_3 = 0, \quad 2|y_2|^2 = |y_1|^2 = 2\}$$

$$T^2_{_2} = \{(x_i, y_i) \in \mathbb{C}^3 \times \mathbb{C}^3 : x_1 = y_1 = y_2 = y_3 = 0, \quad |x_2|^2 = |x_3|^2 = 1\}$$
$$T^2_{_3} = \{(x_i, y_i) \in \mathbb{C}^3 \times \mathbb{C}^3 : x_1 = x_2 = y_2 = y_3 = 0, \quad |y_1|^2 = |x_3|^2 = 1\}$$

correspondingly by

$$\Gamma_1 = \mathbb{Z}_4 = \{(s, t) \in U(1) \times U(1) : \quad (s, t) = (\frac{k\pi}{2}, k\pi)\}$$

$$\Gamma_2 = \mathbb{Z}_4 = \{(s,t) \in U(1) \times U(1) : \quad (s,t) = (k\pi, \frac{k\pi}{2})\}$$

$$\Gamma_3 = \mathbb{Z}_3 = \{(s,t) \in U(1) \times U(1) : \quad (s,t) = (\frac{k\pi}{3}, \frac{k\pi}{3})\}$$

The tori T_i^2 are single orbits of the $U(1) \times U(1)$ action and they project to three isolated singular points on the quotient. Using methods similar to these of [10], it is easy to see that in the neighborhood of the singular points our orbifold looks like $\mathbb{C}^2/\mathbb{Z}_4, \mathbb{C}^2/\mathbb{Z}_4, \mathbb{C}^2/\mathbb{Z}_3$ respectively.

There is one more singular point on $\mathcal{O}(\mathbb{Z}_3; 1, 1)$. One can verify it by repeating the similar calculations for $u_0 = 0$. The singularity is easily seen to be $\mathbb{C}^2/\Gamma = \mathbb{C}^2/\mathbb{Z}_3$. This singular point is common for all orbifolds $\mathcal{O}(\mathbb{Z}_3; a, b), \quad a \neq -b$. Clearly $\mathcal{O}(\mathbb{Z}_3; 1, 1)$ is not equivalent to any weighted complex projective 2-space as it has 4 isolated orbifold points.

Similar analysis can be carried out for other quotients. However, the geometry and the singular structure of $\mathcal{O}(\Gamma, \xi, b)$ very much depend on the choice of the homomorphism b. We do not know how to describe the geometry of our orbifolds for all Γ's and all choices of b. In the scalar curvature going to zero limit locally our self-dual Einstein orbifold metrics give the hyperkähler ALE metrics of Kronheimer. It would be interesting to know if $\mathcal{O}(\Gamma, \xi, b)$ are the only possible generalizations of the hyperkähler ALE instantons with this property.

Recently Joyce [17] has shown that one can construct a family of self-dual metrics on the connected sum $k\mathbb{CP}^2$ as a quaternionic (but not quaternionic Kähler) quotient of $\mathbb{HP}^{k+1}$ by an action of $U(1)^k$. In fact, our orbifold $\mathcal{O}(\mathbb{Z}_{k+1}, \xi, b)$ is a quaternionic Kähler quotient of $\mathbb{HP}^{k+1}$ by $G(\mathbb{Z}_k) = U(1)^k$. The metric on $\mathcal{O}(\mathbb{Z}_{k+1}, \xi, b)$ is not only self-dual but also Einstein and, as a consequence of Hitchin's theorem [4], we necessarily must have orbifold singularities in the quotient. Joyce uses a different notion of the moment map and therefore his quotients give smooth self-dual metrics. It would be interesting to investigate the relationship between the Joyce's construction of self-dual metrics on $k\mathbb{CP}^2$ and our orbifolds $\mathcal{O}(\mathbb{Z}_{k+1}, \xi, b)$. We plan to address some of these questions in our future work.

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