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Duality and Yang-Mills Fields on Quaternionic Kähler Manifolds

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Abstract: The concept of a self-dual connection on a four dimensional Riemannian manifold is generalized to the 4n-dimensional case of any quaternionic Kähler manifold. The generalized self-dual connections are minima of a modified Yang-Mills functional. It is shown that our definitions give a correct framework for a mapping theory of quaternionic Kähler manifolds. The mapping theory is closely related to the construction of Yang-Mills fields on such manifolds. We also discuss some monopole-like equations.

1. Introduction

A quaternionic Kähler manifold is a Riemannian manifold whose holonomy group can be reduced to a subgroup of $Sp(n)\cdot Sp(1),\ n>1$ [1, 2]. By definition, such manifold has dimension $4n$. As demonstrated by Salamon [2, 3] it can be also viewed as a higher dimensional analogy of the anti-self-dual Einstein 4-manifold. The bundle of 2-forms on a quaternionic Kähler manifold M has the following irreducible decomposition as representation of $Sp(n)\cdot Sp(1)$

$$\Lambda^2 T^*M = S^2\mathbb{H}\oplus S^2\mathbb{E}\oplus\Big(S^2\mathbb{H}\oplus S^2\mathbb{E}\Big)^{\perp}, \tag{1.1}$$

where $\mathbb{H}$ and $\mathbb{E}$ are vector bundles associated to the standard representations of $Sp(n)$ and $Sp(1)$ respectively. This decomposition resembles the decomposition of Λ^2T^*M into the direct sum of self-dual and anti-self-dual 2-forms when M is four dimensional. Just as in the four dimensional case we are able to interpret the decomposition (1.1) in terms of the Hodge $*$-operator.

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If the curvature of a connection ∇ is in either the $S^2\mathbb{H}$ or the $S^2\mathbb{E}$ part of (1.1) then ∇ is a minimum of the Yang-Mills functional and if the curvature is in the orthogonal complement of $S^2\mathbb{H} \oplus S^2\mathbb{E}$ then ∇ is most likely a saddle point. We have found that the Yang-Mills functional can be modified so that whenever the curvature of ∇ is in one and only one component of (1.1) the connection is its minimum.

We demonstrate that our definitions are compatible with the description of Yang-Mills fields on 4-manifolds and that it gives a correct framework for mapping theory of quaternionic Kähler manifolds. On the other hand, when the energy functional is interpreted as a classical Lagrangian, our quaternionic mapping theory yields many new examples of quantum field theories with $SU(2)$ (or $SO(3)$) gauge symmetry and composite gauge fields: 4-dimensional sigma models. We show that some fundamental properties of the well-known 4-dimensional σ-models on the quaternionic projective spaces are shared by such models on arbitrary quaternionic Kähler manifolds. Finally, we demonstrate that our formalism provides a global picture for the generalized monopole equation of Pedersen and Poon [4].

2. Duality

Let M be a $4n$-dimensional Riemannian manifold whose holonomy group is contained in $Sp(n)\cdot Sp(1) \subset SO(4n)$. Then the cotangent bundle of M can be identified with

$$T^*M = \mathbb{E} \otimes \mathbb{H},$$

where $\mathbb{E}$ and $\mathbb{H}$ are the standard representations of $Sp(n)$ and $Sp(1)$ respectively. Then $S^2\mathbb{H}$ is a real rank 3 subbundle of $\mathrm{End}TM$. Locally, at each $x \in M$, $S^2\mathbb{H}$ has a basis $\{I, J, K\}$ satisfying

$$I^2 = J^2 = -1\!\!1, \qquad IJ = -JI = K. \tag{2.1}$$

The metric g on M is compatible with the bundle $S^2\mathbb{H}$ in the sense that for each $A \in S^2\mathbb{H}_x$, g is Hermitian with respect to A, *i.e.*, $g(AX, AY) = g(X, Y)$ for all $X, Y \in T_xM$. One can use the metric to define an isomorphism

$$\mathrm{End}TM \cong T^*M \otimes T^*M$$

under which $S^2\mathbb{H}$ is isometrically embedded in Λ^2T^*M. Explicitly, any element $A \in S^2\mathbb{H}_x$ is mapped into ω_A by

$$\omega_A(X, Y) = g(AX, Y), \qquad X, Y \in T_xM.$$

Let $\{\omega_1, \omega_2, \omega_3\}$ be a local orthogonal frame of $S^2\mathbb{H} \subset \Lambda^2T^*M$. For convenience of further computations let us normalize $\{\omega_1, \omega_2, \omega_3\}$ to have length $2n$ and then define

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3. \tag{2.2}$$

This Ω is a globally defined, non-degenerate 4-form on M and it is parallel. It is usually called the fundamental 4-form or the quaternionic structure on M as its parallelism determines reduction of the structure group on M. The condition $\nabla\Omega = 0$ can be used to define quaternionic Kähler geometry in dimension bigger than 4. In dimension 4 we shall say that M is quaternionic Kähler if it is self-dual and Einstein. The parallelism of Ω immediately implies that $d\Omega = 0$. Recently, Swann [5] showed that the converse is also true provided $\dim M \geq 12$.

Pointwisely, Ω can be described as follows. At any point $x \in M$, $T^*_x M = \mathbb{E}_x \otimes \mathbb{H}_x$, where $\mathbb{E}_x$ is the $2n$-dimensional complex representation of $Sp(n)$ and $\mathbb{H}_x$ is the 2-dimensional complex representation of $Sp(1)$. Let ω_E and ω_H be the symplectic forms on $\mathbb{E}_x$ and $\mathbb{H}_x$ respectively and j_E and j the quaternionic structures. Then the metric g on $T^*_x M$ can be expressed as

$$g = \omega_E \otimes \omega_H. \tag{2.3}$$

Let $\{e^j,\ j_E e^j \quad : j = 1, ..., n\}$ be a symplectic basis on $\mathbb{E}_x$ and $\{h, jh\}$ a symplectic basis on $\mathbb{H}_x$. We define

$$\begin{aligned}
\omega_0^j &\doteq \frac{1}{\sqrt{2}}(e^j \otimes h + j_E e^j \otimes jh) \\
\omega_1^j &\doteq \frac{i}{\sqrt{2}}(e^j \otimes h - j_E e^j \otimes jh) \\
\omega_2^j &\doteq \frac{1}{\sqrt{2}}(j_E e^j \otimes h - e^j \otimes jh) \\
\omega_3^j &\doteq \frac{i}{\sqrt{2}}(j_E e^j \otimes h + e^j \otimes jh).
\end{aligned} \tag{2.4}$$

Now $\{\omega_0^j,\ \omega_1^j,\ \omega_2^j,\ \omega_3^j,\quad j = 1, ..., n\}$ forms an orthonormal basis on $T^*_x M$. Let

$$\omega_1 \doteq \sum_{j=1}^{n} (\omega_0^j \wedge \omega_1^j + \omega_2^j \wedge \omega_3^j)$$

$$\omega_2 \doteq \sum_{j=1}^{n} (\omega_0^j \wedge \omega_2^j - \omega_1^j \wedge \omega_3^j)$$

$$\omega_3 \doteq \sum_{j=1}^{n} (\omega_0^j \wedge \omega_3^j + \omega_1^j \wedge \omega_2^j). \tag{2.5}$$

Then $\{\omega_1, \omega_2, \omega_3\}$ forms an orthogonal basis on $S^2\mathbb{H}_x$. We shall choose Ω as in (2.2). The orthogonal basis for $S^2\mathbb{E}_x$ can be written as

$$\begin{cases}
\Sigma_0^{ij} \doteq (\omega_0^i \wedge \omega_0^j + \omega_1^i \wedge \omega_1^j) + (\omega_2^i \wedge \omega_2^j + \omega_3^i \wedge \omega_3^j), \ 1 \leq i < j \leq n, \\
\Sigma_1^{ij} \doteq (\omega_0^i \wedge \omega_1^j - \omega_2^i \wedge \omega_3^j) + (\omega_0^j \wedge \omega_1^i - \omega_2^j \wedge \omega_3^i), \ 1 \leq i \leq j \leq n, \\
\Sigma_2^{ij} \doteq (\omega_0^i \wedge \omega_2^j + \omega_1^i \wedge \omega_3^j) + (\omega_0^j \wedge \omega_2^i + \omega_1^j \wedge \omega_3^i), \ 1 \leq i \leq j \leq n, \\
\Sigma_3^{ij} \doteq (\omega_0^i \wedge \omega_3^j - \omega_1^i \wedge \omega_2^j) + (\omega_0^j \wedge \omega_3^i - \omega_1^j \wedge \omega_2^i), \ 1 \leq i \leq j \leq n.
\end{cases} \tag{2.6}$$

Σ_0^{ij} give $n(n-1)/2$ basis elements and Σ_A^{ij}, $A=1,2,3$, give $n(n+1)/2$ basis elements respectively. One can easily check that

$$\mathrm{vol}(M) = \frac{1}{(2n+1)!}\Omega^n \tag{2.7}$$

$$\mathrm{vol}(M) = \frac{1}{12n(2n+1)}\Omega \wedge *\Omega, \tag{2.8}$$

where $\mathrm{vol}(M)$ is the volume form of M and "$*$" is the Hodge $*$-operator. As a consequence we have

$$*\Omega = \frac{6}{(2n-1)!}\Omega^{n-1}. \tag{2.9}$$

Note that all these equations are valid even when n is equal to 1.

DEFINITION 2.1: *A 2-form ω on M is c-self-dual if*

$$*\omega = c\omega \wedge \Omega^{n-1}. \tag{2.10}$$

When $n=1$ then $c^2=1$, because $*^2=1$, and the above equation is reduced to the conformally invariant self-dual or anti-self-dual equations on a 4-dimensional oriented Riemannian manifold. Notice that the above definition depends on the choice of both the fundamental 4-form Ω and the constant c. In dimension higher than 4, as we shall now see, there are three different constants c that give non-trivial solutions to (2.10). Similar equations were studied in [6].

THEOREM 2.2: *Let ω be a non-zero c-self-dual 2-form. Then $c=c_i, i=1,2,3$, where*

$$c_1 = \frac{6n}{(2n+1)!}, \quad c_2 = \frac{-1}{(2n-1)!}, \quad c_3 = \frac{3}{(2n-1)!}. \tag{2.11}$$

Moreover, when $c=c_1$ then $\omega \in S^2\mathbb{H}$, when $c=c_2$ then $\omega \in S^2\mathbb{E}$, and when $c=c_3$ then ω is in the orthogonal complement of $S^2\mathbb{H} \oplus S^2\mathbb{E}$ in Λ^2T^*M.

PROOF: As the basis for $S^2\mathbb{H}$ is given in (2.5) and the basis for $S^2\mathbb{E}$ in (2.6) the proof is an easy exercise in linear algebra. Therefore we only spell out the constraints on the coefficients of the 2-form ω. Using the orthonormal basis $\{\omega_0^j,\ \omega_1^j,\ \omega_2^j,\ \omega_3^j:\ \ j=1,...,n\}$ any 2-form ω can be written as

$$\omega = \sum_{i,j,\alpha,\beta} \omega_{\binom{i}{\alpha}\binom{j}{\beta}} \omega_\alpha^i \wedge \omega_\beta^j.$$

Then $*\omega = c_1\omega \wedge \Omega^{n-1}$ if and only if

$$\begin{cases} \omega_{\binom{i}{0}\binom{i}{1}} = \omega_{\binom{i}{2}\binom{i}{3}} = \omega_{\binom{j}{0}\binom{j}{1}} = \omega_{\binom{j}{2}\binom{j}{3}} \\ \omega_{\binom{i}{0}\binom{i}{2}} = -\omega_{\binom{i}{1}\binom{i}{3}} = \omega_{\binom{j}{0}\binom{j}{2}} = -\omega_{\binom{j}{1}\binom{j}{3}} \\ \omega_{\binom{i}{0}\binom{i}{3}} = \omega_{\binom{i}{1}\binom{i}{2}} = \omega_{\binom{j}{0}\binom{j}{3}} = \omega_{\binom{j}{1}\binom{j}{2}} \end{cases} \tag{2.12}$$

for all i,j

$$\omega_{\binom{i}{0}\binom{j}{0}} = \omega_{\binom{i}{1}\binom{j}{1}} = \omega_{\binom{i}{0}\binom{j}{2}} = \omega_{\binom{i}{3}\binom{j}{3}} = 0 \quad \forall i,j \tag{2.13}$$

and

$$\omega_{\binom{i}{\alpha}\binom{j}{\beta}} = 0 \qquad \forall i \neq j \ \ \forall \alpha, \beta. \tag{2.14}$$

Similarly, $*\omega = c_2\omega \wedge \Omega^{n-1}$ if and only if

$$\begin{cases} \omega_{\binom{i}{0}\binom{j}{1}} = -\omega_{\binom{i}{2}\binom{j}{3}}, \omega_{\binom{i}{0}\binom{j}{2}} = \omega_{\binom{i}{1}\binom{j}{3}}, \omega_{\binom{i}{0}\binom{j}{3}} = -\omega_{\binom{i}{1}\binom{j}{2}} \quad \forall i,j \\ \omega_{\binom{i}{\alpha}\binom{j}{\alpha}} = \omega_{\binom{i}{\beta}\binom{j}{\beta}}, \ \forall i,j,\alpha,\beta \\ \omega_{\binom{i}{\alpha}\binom{j}{\beta}} = \omega_{\binom{j}{\alpha}\binom{i}{\beta}}, \ \forall i,j,\alpha,\beta,\alpha\neq\beta. \end{cases} \tag{2.15}$$

Finally, $*\omega = c_3\omega \wedge \Omega^{n-1}$ if and only if

$$\begin{cases} \sum_{i=1}^{n} \omega_{\binom{i}{0}\binom{i}{1}} = \sum_{i=1}^{n} \omega_{\binom{i}{0}\binom{i}{2}} = \sum_{i=1}^{n} \omega_{\binom{i}{0}\binom{i}{3}} = 0 \\ \sum_{\alpha=0}^{3} \omega_{\binom{i}{\alpha}\binom{j}{\alpha}} = 0 \ \ \forall \ i,j \\ \omega_{\binom{i}{0}\binom{j}{1}} + \omega_{\binom{j}{0}\binom{i}{1}} = \omega_{\binom{i}{2}\binom{j}{3}} + \omega_{\binom{j}{2}\binom{i}{3}} \quad \forall i,j \\ \omega_{\binom{i}{0}\binom{j}{2}} + \omega_{\binom{j}{0}\binom{i}{2}} = -\left(\omega_{\binom{i}{1}\binom{j}{3}} + \omega_{\binom{j}{1}\binom{i}{3}}\right) \quad \forall i,j \\ \omega_{\binom{i}{0}\binom{j}{3}} + \omega_{\binom{j}{0}\binom{i}{3}} = \omega_{\binom{i}{1}\binom{j}{2}} + \omega_{\binom{j}{1}\binom{i}{2}} \quad \forall i,j. \ \blacksquare \end{cases} \tag{2.16}$$

DEFINITION 2.3: *Let P be a principal bundle on M with connection ∇. This connection is c-self-dual if its curvature 2-form is c-self-dual.*

DEFINITION 2.4: *For any real constant c , a generalized "Yang-Mills" functional on the space of connections on P is defined by*

$$YM_c(\nabla) \doteq \frac{1}{2}\int_M [\|F\|^2 + c^2\|F\wedge\Omega^{n-1}\|^2]\mathrm{vol}(M), \tag{2.17}$$

where F is the curvature of the connection.

$YM_c(\nabla)$ has the following Euler-Lagrange equations

$$d*F + c^2\left(d*(F\wedge\Omega^{n-1})\right)\wedge\Omega^{n-1} = 0. \tag{2.18}$$

Notice that

$$
\begin{aligned}
0 &\leq \| *F - cF \wedge \Omega^{n-1} \|^2 \\
&= \| *F\|^2 - 2\langle *F, cF \wedge \Omega^{n-1}\rangle + c^2 \|F \wedge \Omega^{n-1}\|^2 \\
&= \|F\|^2 - 2c(\mathrm{tr}F \wedge F) \wedge \Omega^{n-1} + c^2 \|F \wedge \Omega^{n-1}\|^2 \\
&= \|F\|^2 - 16c\pi^2 p_1(P) \wedge \Omega^{n-1} + c^2 \|F \wedge \Omega^{n-1}\|^2
\end{aligned}
$$

or

$$c(8\pi^2)p_1(P) \wedge \Omega^{n-1} \leq \frac{1}{2}[\|F\|^2 + c^2\|F \wedge \Omega^{n-1}\|^2],$$

where $p_1(P)$ is the first Pontrjagin class of the bundle P on M. Hence, after integrating over M, we get

$$8\pi^2 c \int_M p_1(P) \wedge \Omega^{n-1} \mathrm{vol}(M) \leq YM_c(\nabla). \tag{2.19}$$

The equality holds if and only if

$$*F = cF \wedge \Omega^{n-1},$$

i.e., if F is c-self-dual. In such case we shall call the connection ∇ itself a c-self-dual-connection. As $p_1(P)$ is a topological invariant of the bundle P, we define

$$Q(P) \doteq 8\pi^2 \int_M p_1(P) \wedge \Omega^{n-1} \mathrm{vol}(M) \tag{2.20}$$

and call it a topological charge of the bundle P. We have just demonstrated that

PROPOSITION 2.5: *Any c-self-dual connection is a minimum of the Yang-Mills energy functional $YM_c(\nabla)$.*

The following result is due to [7].

PROPOSITION 2.6: *Any c-self-dual connection is an extremum of the Yang-Mills energy functional $YM(\nabla)$. Moreover, c_1- and c_2-self-dual connections are minimizing.*

PROOF: Suppose ∇ is a c-self-dual connection. Then

$$d * F = cd * (F \wedge \Omega^{n-1}) = 0$$

as $dF = d\Omega = 0$. Hence, $d * F = 0$ or ∇ is a Yang-Mills connection.

Let us write $F(\nabla) \in \Lambda^2 T^*M$ as

$$F(\nabla) = F_1 + F_2 + F_3,$$

where $F_1 \in S^2\mathbb{H}$, $F_2 \in S^2\mathbb{E}$, and $F_3 \in (S^2\mathbb{H} \oplus S^2\mathbb{E})^\perp$. Then

$$YM(\nabla) = \frac{1}{2}\int_M \Big(\|F_1\|^2 + \|F_2\|^2 + \|F_3\|^2\Big)\mathrm{vol}(M)$$

because (1.1) is an orthogonal decomposition with respect to the usual norm $\|\cdot\|$ on Λ^2T^*M. Notice that the topological charge of P can be written in terms of the components of $F(\nabla)$

$$Q(P)=\int_M \mathrm{tr}(F\wedge F)\wedge\Omega^{n-1}\mathrm{vol}(M)=\int_M\Big(\frac{1}{c_1}\|F_1\|^2+\frac{1}{c_2}\|F_2\|^2+\frac{1}{c_3}\|F_3\|^2\Big)\mathrm{vol}(M).$$

Hence, we can write $YM(\nabla)$ as

$$\begin{aligned}2YM(\nabla)&=c_1Q(P)+\int_M\Big((1-\frac{c_1}{c_2})\|F_2\|^2+(1-\frac{c_1}{c_3}\|F_3\|^2\Big)\mathrm{vol}(M)=\\&=c_1Q(P)+\int_M\Big((1+\tfrac{3}{2n+1})\|F_2\|^2+(1-\tfrac{1}{2n+1})\|F_3\|^2\Big)\mathrm{vol}(M)\end{aligned}\tag{2.21}$$

or

$$\begin{aligned}2YM(\nabla)&=c_2Q(P)+\int_M\Big((1-\frac{c_2}{c_1})\|F_1\|^2+(1-\frac{c_2}{c_3})\|F_3\|^2\Big)\mathrm{vol}(M)=\\&=c_2Q(P)+\int_M\Big((1+\tfrac{2n+1}{3})\|F_1\|^2+\tfrac{4}{3}\|F_3\|^2\Big)\mathrm{vol}(M)\end{aligned}\tag{2.22}$$

or

$$\begin{aligned}2YM(\nabla)&=c_3Q(P)+\int_M\Big((1-\frac{c_3}{c_1})\|F_1\|^2+(1-\frac{c_3}{c_2})\|F_2\|^2\Big)\mathrm{vol}(M)=\\&=c_3Q(P)+\int_M\Big((-2n)\|F_1\|^2+4\|F_2\|^2\Big)\mathrm{vol}(M)\end{aligned}\tag{2.23}$$

It follows now from (2.21), (2.22) and Theorem 2.2 that c_1- and c_2-self-dual connections are minima of $YM(\nabla)$. ∎

We do not know of any examples of c_3-self-dual connections but (2.23) seems to indicate that, if they exist, they will be unstable.

3. Quaternionic maps and sigma models

In this chapter we introduce a new concept of quaternionic maps. We shall do it in such a way that it generalizes the theory of holomorphic mappings between Kähler manifolds. On the other hand we shall see that it is also very natural in studying instantons on 4-manifolds and 4-dimensional σ-models with composite $SU(2)$ (or $SO(3)$) gauge fields and Yang-Mills fields on quaternionic Kähler manifolds.

It is well-known that, if one defines a quaternionic Kähler submanifold to be a submanifold with a quaternionic structure given by restrictions, then it is automatically a totally geodesic submanifold [4]. We shall therefore not insist that the whole quaternionic structure be preserved by such mappings. Instead we adopt a weaker definition.

DEFINITION 3.1: *Let M, N be quaternionic Kähler manifolds. A map f from M to N is called quaternionic if $f^*S^2\mathbb{H}_N \subset S^2\mathbb{H}_M$.*

The following theorem is in an obvious analogy to the well-known result stating that holomorphic maps between Kähler manifolds are energy minimizing.

THEOREM 3.2: *On the space of differentiable mappings between two compact oriented quaternionic Kähler manifolds M and N define the following functional*

$$E(f) \doteq \frac{1}{2}\sum_{i=1}^{3}\int_M \Big(\|f^*\omega_i\|^2 + c^2\|f^*\omega_i \wedge \Omega^{m-1}\|^2\Big)\mathrm{vol}(M), \tag{3.1}$$

where $c = c_1 = \frac{6m}{(2m+1)!}$, $4m = \dim M$, and

$$Q(f) \doteq \int_M f^*\Omega_N \wedge \Omega_M^{m-1}. \tag{3.2}$$

Then $cQ(f) \leq E(f)$ and equality holds if and only if the map f is quaternionic.

PROOF: Let Ω_M, Ω_N be the fundamental 4-forms on M and N respectively. Once they are fixed $Q(f)$ is a homotopy invariant. As usual, we shall call it the degree or the topological charge of f.

Let $\{\omega_1, \omega_2, \omega_3\}$ be a local orthogonal frame on $S^2\mathbb{H}_N$ such that

$$\Omega_N = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.$$

We have to show that $E(f)$ is well-defined. If $\omega_i = \sum_j \phi_{ij}\mu_j$ is an $SO(3)$-rotation of the frame field on $S^2\mathbb{H}_N$ then pointwisely

$$f^*\omega_i = \sum_{i=1}^{3}(\phi_{ij})f^*\mu_j.$$

Furthermore,

$$\sum_{i=1}^{3}\|f^*\omega_i\|^2 = \sum_{i=1}^{3} f^*\omega_i \wedge *f^*\omega_i = \sum_{i=1}^{3}\sum_{j,k}\Big(\phi_{ij}f^*\mu_j\Big) \wedge *\Big(\phi_{ik}f^*\mu_k\Big) =$$

$$= \sum_{i=1}^{3}\sum_{j,k}\Big(\phi_{ij}\phi_{ik}\Big)\Big(f^*\mu_j \wedge *f^*\mu_k\Big) = \sum_{i=1}^{3}\sum_{j,k}\Big(\phi_{ji}^{-1}\phi_{ik}\Big)\Big(f^*\mu_j \wedge *f^*\mu_k\Big) =$$

$$= \sum_{j,k} \delta_{jk} f^*\mu_j \wedge *f^*\mu_k = \sum_{j=1}^{3} f^*\mu_j \wedge *f^*\mu_j = \sum_{j=1}^{3} \|f^*\mu_j\|^2.$$

Similarly,

$$\sum_{i=1}^{3} \|f^*\omega_i \wedge \Omega^{m-1}\|^2 = \sum_{i=1}^{3} \Big(f^*\omega_i \wedge \Omega^{m-1}\Big) \wedge * \Big(f^*\omega_i \wedge \Omega^{m-1}\Big) =$$

$$= \sum_{i=1}^{3} \sum_{j,k} \Big(\phi_{ij}\phi_{ik}\Big) \Big(f^*\mu_j \wedge \Omega^{m-1}\Big) \wedge * \Big(f^*\mu_k \wedge \Omega^{m-1}\Big) =$$

$$= \sum_{j=1}^{3} \Big(f^*\mu_j \wedge \Omega^{m-1}\Big) \wedge * \Big(f^*\mu_j \wedge \Omega^{m-1}\Big) = \sum_{j=1}^{3} \|f^*\mu_j \wedge \Omega^{m-1}\|^2.$$

Hence, $E(f)$ is independent of the choice of any normalized frame on $S^2\mathbb{H}_N$ and therefore well defined. Now the inequality $cQ(f) \le E(f)$ follows from

$$0 \le \| * f^*\omega_i - cf^*\omega_i \wedge \Omega^{m-1}\|^2$$

which can be written as

$$c\Big\langle *f^*\omega_i,\ f^*\omega_i \wedge \Omega^{m-1}\Big\rangle \le \frac{1}{2}\Big(\|f^*\omega_i\|^2 + c^2\|f^*\omega_i \wedge \Omega^{m-1}\|^2\Big). \tag{3.3}$$

Since

$$\Big\langle *f^*\omega_i,\ f^*\omega_i \wedge \Omega^{m-1}\Big\rangle = f^*\omega_i \wedge f^*\omega_i \wedge \Omega^{m-1}$$

and

$$f^*\Omega_N = \sum_{i=1}^{3} f^*\omega_i \wedge f^*\omega_i,$$

the inequality $cQ(f) \le E(f)$ is simply obtained by summation of (3.3) over i and integration over M.

Finally, when $c = \frac{6m}{(2m+1)!}$, the assertion that $cQ(f) = E(f)$ is equivalent to the requirement that

$$*f^*\omega = cf^*\omega \wedge \Omega_M^{m-1}$$

holds for all $\omega \in S^2\mathbb{H}_N$, or that $f^*\omega \in S^2\mathbb{H}_M$ by Theorem 2.2., *i.e.*, f is quaternionic. ∎

EXAMPLE 3.3: If $\dim M = 4$, $S^2\mathbb{H}_M \cong \Lambda^2_+$. As the Hodge $*$-operator is conformally invariant, any orientation preserving conformal automorphism is a quaternionic map in our sense.

In [8] Atiyah gave a geometric construction for all basic $SU(2)$-instantons, *i.e.,* anti-self-dual Yang-Mills fields on the Euclidean 4-sphere with topological charge -1, as follows: The Euclidean 4-sphere is viewed as the quaternionic projective line $\mathbb{HP}^1$. The tautological bundle is the bundle $\mathbb{H}$ with charge -1. The natural connection ∇ of $\mathbb{H}$ is anti-self-dual. Let f be an orientation preserving conformal automorphism which is not an isometry. Then $f^*\nabla$, the pull-back connection on $f^*\mathbb{H}$, is a new anti-self-dual connection.

EXAMPLE 3.4: The above example can be easily generalized as follows: The quaternionic projective space $\mathbb{HP}^n$ has a tautological bundle $\mathbb{H}$. By definition, any element of $GL^+(n+1,\mathbb{H})$ is an orientation preserving quaternionic linear map. In other words, if $f \in GL^+(n+1,\mathbb{H})$ is considered as an automorphism of $\mathbb{HP}^n$, then $f^*\mathbb{H}$ is isomorphic to $\mathbb{H}$. It follows that $f^*S^2\mathbb{H} \equiv S^2\mathbb{H}$ and hence f is a quaternionic map. As the natural connection ∇ on $\mathbb{H}$ is c_1-self-dual, so is $f^*\nabla$. Besides, as long as f is not an isometry, $f^*\nabla$ is not gauge equivalent to ∇. We do not know if these are all c_1-self-dual connections on $\mathbb{HP}^n$.

EXAMPLE 3.5: Another well-known example of a mapping which in our language is quaternionic is a general $SU(2)$-instanton over 4-sphere with the topological charge k [8, 9]. The $S^2\mathbb{H}$ bundle on the quaternionic projective space $\mathbb{HP}^k$ has a canonical $Sp(1)$-connection and all instantons over S^4 are induced by an appropriate choice of $f: S^4 \longrightarrow \mathbb{HP}^k$. In fact f can be described explicitly as follows: If $\mathbf{u} \in \mathbb{HP}^k$ is a local (Fubini-Study) quaternionic coordinate on the quaternionic projective space and $x \in S^4$ is a local quaternionic coordinate on the 4-sphere identified with the quaternionic projective line $\mathbb{HP}^1$ then

$$\mathbf{u}(x) = \left[\vec{\lambda} \cdot (\mathbb{B} - x\mathbb{1})\right]^\dagger \tag{3.4}$$

where $\vec{\lambda} = (\lambda_1, ..., \lambda_k)$ is a quaternionic row vector, $\mathbf{u}$ is a quaternionic column vector, $\mathbb{B}$ is a symmetric quaternionic $n \times n$ matrix, $\dagger$ denotes quaternionic conjugation and transposition, and $(\vec{\lambda}, \mathbb{B})$ are subject to the following two conditions

$$\begin{cases} \mathrm{Im}(\mathbb{B}^\dagger\mathbb{B} + \vec{\lambda}^\dagger\vec{\lambda}) = 0 \\ \left(\forall x \in \mathbb{HP}^1 \quad (\mathbb{B} - x\mathbb{1})\vec{\xi} = 0, \quad \vec{\lambda}\cdot\vec{\xi} = 0 \quad \text{where} \quad \vec{\xi} \in \mathbb{H}^k\right) \quad \Rightarrow \quad \vec{\xi} = 0. \end{cases} \tag{3.5}$$

In the same way k-instantons over the complex projective plane can be generated by quaternionic maps from $bbc\mathbb{P}^2 \longrightarrow \mathbb{HP}^{2k}$ [10, 11].

The energy functional (3.1) may also be interpreted as an $SO(3)$ locally gauge invariant lagrangian of an interesting class of nonlinear field theories called σ-models. In particular, if $\dim M = 4$, one can think of M as a physical, possibly curved, space-time and $f(x)$, $x \in M$, becomes an N-valued classical field with the action functional given by $E(f)$. $E(f)$ is manifestly invariant with respect to the global coordinate transformation on M (diffeomorphisms of M) as well as it is gauge invariant under the following gauge transformations

$$\left(f^*\omega_i\right)_x \longrightarrow \sum_j \Phi_{ij}(x)\left(f^*\omega_j\right)_x, \tag{3.6}$$

where $\Phi_{ij}(x)$ is a local $SO(3)$ transformation and $\Big(f^*\omega_i\Big)$ is the curvature 2-form of a gauge field A_j on M defined as follows

$$d\Big(f^*\omega_i\Big) = \sum_{j,k} \epsilon_{ijk} A_j \wedge f^*\omega_k. \tag{3.7}$$

The gauge potential 1-form on A_j transforms in the usual way

$$\delta\Big(\epsilon_{ijk} A_k\Big) = -d\Phi_{ij}(x). \tag{3.8}$$

$A_j(f)$ depends on the choice of $f(x)$, *i.e.,* it is a composite gauge field. If $N = \mathbb{HP}^n$ and $\mathbf{u} \in \mathbb{HP}^n$ as before then

$$A(\mathbf{u}) = -\tfrac{1}{2}\frac{\mathbf{u}^\dagger \cdot d\mathbf{u} - d\mathbf{u}^\dagger \cdot \mathbf{u}}{1 + \mathbf{u}^\dagger \cdot \mathbf{u}} = iA_1 + jA_2 + kA_3.$$

This particular example was introduced and extensively studied by Gürsey and Tze [12]. Here we see that many interesting global and local properties of $\mathbb{HP}^n$-model are common for a large class of field theoretical models based on $E(f)$. All of them have duality equations built in and all possess global topological invariants.

4. Generalized Bogomolny equations

In this section we discuss some special solutions of the c-self-duality equations. If $M = \mathbb{R}^4 \ni (x_0, x_1, x_2, x_3)$ and P is a principal bundle over M then one can study x_0-invariant solutions to the usual self-dual equations. They are called time invariant instantons or monopoles. In our case, let $M = \mathbb{R}^{4n} \simeq \mathbb{R}^4 \otimes \mathbb{R}^n \ni \{x^i_\alpha\}^{i=1,\ldots,n}_{\alpha=0,\ldots,3}$, P be a principal bundle over M, and let $YM_c(\nabla)$ be our Yang-Mills functional. In an obvious analogy to the 4-dimensional case we can study x^i_0 invariant c-self-dual connections on M or "c-monopoles" on $\mathbb{R}^3 \otimes \mathbb{R}^n$. Let us start with the following observation.

PROPOSITION 4.1: *Let $M = \mathbb{R}^4 \otimes \mathbb{R}^n$ be the $4n$-dimensional Euclidean flat space with global linear coordinates x^i_α, $\alpha = 0, 1, 2, 3$; $i = 1, ..., n$. For any $(x_1, ..., x_n)$ in $\mathbb{R}^n$ we define*

$$p:\ \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \otimes \mathbb{R}^n$$

by

$$(x_0, x_1, x_2, x_3) \ \longrightarrow\ x^i_\alpha = x_\alpha x^i. \tag{4.1}$$

Suppose P is a principal bundle over M with connection ∇ and curvature F. Then $p^*\nabla$ is an anti-self-dual connection on p^*P if

$$*F = -\frac{1}{(2n-1)!} F \wedge \Omega^{n-1}, \tag{4.2}$$

i.e., F is c_2-self-dual.

PROOF: In the x^i_α-coordinates dx^i_α is exactly the 1-form ω^i_α of (2.4). Now a 2-form ω satisfies the equation

$$*F = -\frac{1}{(2n-1)!} F \wedge \Omega^{n-1}$$

if and only if

$$F = -\frac{1}{6} * (F \wedge *\Omega). \tag{4.3}$$

Using Theorem 2.2 we get the following solutions

$$\begin{cases} F_{\binom{i}{0}\binom{j}{1}} = -F_{\binom{i}{2}\binom{j}{3}}, F_{\binom{i}{0}\binom{j}{2}} = F_{\binom{i}{1}\binom{j}{3}}, F_{\binom{i}{0}\binom{j}{3}} = -F_{\binom{i}{1}\binom{j}{2}} \quad \forall i,j \\ F_{\binom{i}{\alpha}\binom{j}{\alpha}} = F_{\binom{i}{\beta}\binom{j}{\beta}}, \quad \forall i,j,\alpha,\beta \\ F_{\binom{i}{\alpha}\binom{j}{\beta}} = F_{\binom{j}{\alpha}\binom{i}{\beta}}, \quad \forall i,j,\alpha,\beta,\alpha \neq \beta. \end{cases} \tag{4.4}$$

Let us denote the components of p^*F by $F_{\alpha\beta}$. As a consequence of the chain rule we get

$$F_{\alpha\beta} = \sum_{i,j} x^i x^j F_{\binom{i}{\alpha}\binom{j}{\beta}} \tag{4.5}$$

and therefore

$$F_{01} = \sum_{i,j} x^i x^j F_{\binom{i}{0}\binom{j}{1}} = -F_{23}$$

$$F_{02} = \sum_{i,j} x^i x^j F_{\binom{i}{0}\binom{j}{2}} = F_{13}$$

$$F_{03} = \sum_{i,j} x^i x^j F_{\binom{i}{0}\binom{j}{3}} = -F_{12}. \tag{4.6}$$

In other words, $p^*\nabla$ is an anti-self-dual connection. ∎

Recently, Pedersen and Poon used twistorial approach to find a generalization of the Bogomolny equations [5]. They introduced Yang-Mills-Higgs equations on $\mathbb{R}^3 \otimes \mathbb{R}^n$. If one considers monopoles on $\mathbb{R}^3$ as time invariant instantons on $\mathbb{R}^4$ the following simple geometric description of generalized monopoles comes with no surprise.

PROPOSITION 4.2: *Let x^i_μ, $\mu = 0,1,2,3$; $i = 1,...,n$ be a global linear coordinate on $\mathbb{R}^4 \otimes \mathbb{R}^n$ and let*

$$p: \quad \mathbb{R}^4 \otimes \mathbb{R}^n \longrightarrow \mathbb{R}^3 \otimes \mathbb{R}^n$$

be a projection

$$(x^i_0, x^i_1, x^i_2, x^i_3) \longrightarrow (x^i_1, x^i_2, x^i_3).$$

If (∇, Φ^i) is a generalized monopole then

$$\nabla' \doteq p^* \nabla + \sum_i \Phi^i dx_0^i \tag{4.7}$$

is a c_2-self-dual connection. Conversely, any c_2-self-dual connection that is x_0^i-invariant determines a solution of the generalized monopole equation.

PROOF: The curvature F' of the connection ∇' is given by

$$F' = p^* F + \sum_j (\nabla \Phi^j) \wedge dx_0^j + \frac{1}{2} \sum_{i<j} [\Phi^i, \Phi^j] dx_0^i \wedge dx_0^j, \tag{4.8}$$

where F is the curvature 2-form of ∇. Now, using equations (4.4), we get

$$\begin{cases} \nabla_{\binom{j}{1}} \Phi^i = F_{\binom{i}{2}\binom{j}{3}}, \nabla_{\binom{j}{2}} \Phi^i = -F_{\binom{i}{1}\binom{j}{3}}, \nabla_{\binom{j}{3}} \Phi^i = F_{\binom{i}{1}\binom{j}{2}} \quad \forall i, j \\ F_{\binom{i}{\alpha}\binom{j}{\alpha}} = \frac{1}{2} [\Phi^i, \Phi^j], \ \forall i, j; \ \alpha = 1, 2, 3 \\ \nabla_{\binom{i}{\alpha}} \Phi^j = \nabla_{\binom{j}{\alpha}} \Phi^i \quad \forall i, j; \ \alpha = 1, 2, 3 \end{cases} \tag{4.9}$$

which can be written as

$$\begin{cases} F_{\binom{i}{\alpha}\binom{j}{\beta}} = \sum_\gamma \epsilon_{\alpha\beta\gamma} \nabla_{\binom{i}{\gamma}} \Phi^j + \frac{1}{2} \delta_{\alpha\beta} [\Phi^i, \Phi^j] \quad \forall i, j; \ \forall \alpha, \beta = 1, 2, 3 \\ \nabla_{\binom{i}{\alpha}} \Phi^j = \nabla_{\binom{j}{\alpha}} \Phi^i \quad \forall i, j; \ \alpha = 1, 2, 3. \end{cases} \tag{4.10}$$

The converse is obvious. ∎

We can also obtain "monopole" analogues of c-self duality equations in the c_1 and c_3 cases. The first one is not interesting, however, because it yields n decoupled self-dual Bogomolny equations. In the second case we can explicitly write down the set of equations

$$\begin{cases} F_{\binom{i}{\alpha}\binom{j}{\beta}} + F_{\binom{j}{\alpha}\binom{i}{\beta}} = \sum_\gamma \epsilon_{\alpha\beta\gamma} \left(\nabla_{\binom{i}{\gamma}} \Phi^j + \nabla_{\binom{j}{\gamma}} \Phi^i \right), \quad \forall i, j; \ \forall \alpha, \beta \\ \sum_{i=1}^n \nabla_{\binom{i}{\alpha}} \Phi^i = 0 \ \ \forall \alpha, \qquad [\Phi^i, \Phi^j] = -\sum_{\alpha=1}^3 F_{\binom{i}{\alpha}\binom{j}{\alpha}} \quad \forall i, j. \end{cases} \tag{4.11}$$

For $n = 1$ these are just the usual Bogomolny equations with the reversed orientation. We do not known any non-trivial solutions of (4.11) for $n > 1$ at the moment. Finally, let us remark that we could introduce additional invariance and reduce the c-self-duality equation to $2n$ dimensions, assuming that the c-self-dual equations on $\mathbb{R}^4 \otimes \mathbb{R}^n$ be both x_0^i and x_1^i-invariant. Then we obtain an analogue of the well-known vortex equation of the two-dimensional Yang-Mills-Higgs theory. Again c_2 case is the most natural generalization and we shall address this problem in a future work.

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