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A Note on Toric Contact Geometry

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Abstract: After observing that the well-known convexity theorems of symplectic geometry also hold for compact contact manifolds with an effective torus action whose Reeb vector field corresponds to an element of the Lie algebra of the torus, we use this fact together with a recent symplectic orbifold version of Delzant's theorem due to Lerman and Tolman [LT] to show that every such compact toric contact manifold can be obtained by a contact reduction from an odd dimensional sphere.

1. Introduction

The main purpose of this note is to prove a Delzant-type Theorem [Del] which says that every compact toric contact manifold whose Reeb vector field corresponds to an element of the Lie algebra of the torus (a condition we call *of Reeb type*) can be obtained by contact reduction from an odd dimensional sphere. This result makes use of the symplectic orbifold version of Delzant's Theorem by Lerman and Tolman [LT] thereby showing the close relationship between the two. Toric contact geometry has been previously considered by Banyaga and Molino [BM1,BM2]. They show that there are two cases: (1) The action of the torus is regular in which case the image of the moment map is a sphere. Thus, for $n \geq 3$ the original contact manifold is the product $T^{n+1} \times S^n$. (2) The action is singular in which case the image generates a closed convex polytope. Then Banyaga and Molino give a Delzant-type theorem by showing that this polytope determines the toric contact structure up to isomorphism. But they do not prove that every such manifold can be obtained from reduction. Indeed, this may not be true. However, if one makes the added assumption that the Reeb vector field corresponds to an element of the Lie algebra of the torus, such a result is true as we shall show. While [BM1,BM2] do consider this case they do not prove such a result since their proofs are entirely different from the usual proofs of the Delzant theorems in symplectic geometry [Del, Gui, LT] which make use of reduction.

We begin by reviewing some well known facts about contact manifolds and symplectic cones. We then discuss the moment maps associated to toral actions. In particular, we show that the well-known convexity theorem of Atiyah [At], Guillemin and Sternberg [GS] hold for compact contact manifolds of Reeb type, a result given previously in the toric case by Banyaga and Molino [BM1,BM2]. In contact geometry we need to fix a contact 1-form, that is a Pfaffian structure, within the contact structure. Doing so we show that a torus that acts effectively on a compact contact manifold and preserves the Pfaffian form gives

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rise to a moment map whose image is a convex polytope lying in a certain hyperplane, which we call the *characteristic hyperplane*, in the dual of the Lie algebra of the torus. If one changes the Pfaffian form by a function that is invariant under the torus, the resulting polytope then differs from the former by a change of scale. Indeed, it is always possible to choose the Pfaffian form so that the polytope is rational. Following Lerman and Tolman we also give a theorem relating the geometry of labeled polytopes to the geometry of the contact manifold with a fixed Pfaffian structure with its torus action and characteristic foliation.

Our motivation for this study is that of obtaining explicit positive Einstein metrics using methods of contact geometry that the authors have recently developed [BGM, BGMR, BG1-2]. In a forthcoming work we investigate precisely which toric contact manifolds admit Sasakian-Einstein metrics.

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2. Symplectic Cones and Contact Manifolds

DEFINITION 2.1 *A contact structure on a manifold M of a manifold of dimension $2n+1$ is a subbundle $\mathcal{D}$, called the contact distribution, of the tangent bundle TM that is maximally non-integrable.*

When M is orientable, which we shall always assume, the contact distribution can be given as the kernel of a 1-form η that satisfies the nondegeneracy condition

2.2 $$\eta\wedge (d\eta)^n\neq 0$$

and $d\eta$ is a symplectic form on the subbundle $\mathcal{D}$. It should be realized, however, that the contact structure is defined not by η, but by an equivalence class of such 1-forms. Explicitly, two such 1-forms η,η' are equivalent if there exists a nowhere vanishing function f on M such that $\eta'=f\eta$. For us it will often be convenient to fix a contact form η in the equivalence class. In [LM] this is referred to as a *Pfaffian structure*.

DEFINITION 2.3: *A symplectic cone is a cone $C(M)=M\times\mathbb{R}^+$ with a symplectic structure ω with a one parameter group ρ_t of homotheties of ω whose infinitesimal generator is a vector field on $\mathbb{R}^+$.*

Definition 2.3 is referred to as a symplectic Liouville structure in [LM]. We denote by r the coordinate on $\mathbb{R}^+$ in which case the infinitesimal generator of the homothety group is the Liouville vector field $\Psi=r\partial_r$. Define the one form $\tilde{\eta}$ on $C(M)$ by $\tilde{\eta}=\Psi\rfloor\omega$. As Ψ is an infinitesimal generator of homotheties of ω it follows that ω is the exact 2-form $\omega=d\tilde{\eta}$. Let us identify M with $M\times\{1\}$, and define $\eta=\tilde{\eta}|_{M\times\{1\}}$. Then we see that $\tilde{\eta}=r\eta$, and this gives ω as

2.4 $$\omega=dr\wedge\eta+rd\eta.$$

The condition that $\omega^{n+1}\neq 0$ implies that η gives M the structure of a contact manifold with a fixed 1-form, *i.e.,* 2.2 is satisfied. Conversely, one can easily reverse this procedure:

given a contact manifold with a fixed 1-form (M,η), defining ω on $C(M)$ by 2.4 gives the cone $C(M)$ a symplectic structure with homotheties. In [Eli, MS] $(C(M),\omega)$ is called the *symplectization* of M while it is called the *symplectification* of M in [LM]. The terminology is apparently due to Arnold. We have arrived at the well-known result

PROPOSITION 2.5: *An orientable manifold (orbifold) is a contact manifold (orbifold) if and only if the cone $C(M)=M\times\mathbb{R}^+$ is a symplectic cone.*

Now a contact manifold with a fixed 1-form η has associated to it a unique vector field called the *characteristic* or Reeb vector field ξ defined by the conditions

2.6 $$\eta(\xi)=1, \qquad \text{and} \qquad \xi\rfloor d\eta=0.$$

Clearly ξ is nowhere vanishing on M and thus induces a foliation $\mathcal{F}^\xi$ of M called the *characteristic foliation.* It should be mentioned that both ξ and its characteristic foliation depend on the choice of 1-form η, that is, they are invariants of the Pfaffian structure, but not the contact structure. Conversely, a characteristic vector field uniquely determines the 1-form η within the contact structure $\mathcal{D}$.

3. Contact Transformations

Let M be an orientable contact manifold, and let $\mathfrak{C}(M,\mathcal{D})$ denote the group of contact transformations, *i.e.,* the subgroup of the group $\mathrm{Diff}(M)$ of diffeomophisms of M that leaves the contact distribution $\mathcal{D}$ invariant. If we fix a contact form η such that $\mathcal{D}=\ker\eta$, then $\mathfrak{C}(M,\mathcal{D})$ can be characterized as the subgroup of diffeomorphisms $\phi: M\longrightarrow M$ that satisfy $\phi^*\eta=f\eta$ for some nowhere vanishing $f\in C^\infty(M)$. With the 1-form η fixed we are interested in the subgroup $\mathfrak{C}(M,\eta)$ of *strict contact transformations* defined by the condition $\phi^*\eta=\eta$. Likewise, we denote the "Lie algebras" of $\mathfrak{C}(M,\mathcal{D})$ and $\mathfrak{C}(M,\eta)$ by $\mathfrak{c}(M,\mathcal{D})$ and $\mathfrak{c}(M,\eta)$, respectively. They can be characterized as follows:

$$\mathfrak{c}(M,\mathcal{D})=\{X\in\Gamma(M)|\mathcal{L}_X\eta=g\eta \text{ for some } g\in C^\infty(M)\},$$

$$\mathfrak{c}(M,\eta)=\{X\in\Gamma(M)|\mathcal{L}_X\eta=0\},$$

where $\Gamma(M)$ denotes the Lie algebra of smooth vector fields on M. Clearly, $\mathfrak{c}(M,\eta)$ is a Lie subalgebra of $\mathfrak{c}(M,\mathcal{D})$. In contrast to the symplectic case, every infinitesimal contact transformation is Hamiltonian. More explicitly,

PROPOSITION 3.1: [LM] *The contact 1-form η induces a Lie algebra isomorphism Φ between the Lie algebra $\mathfrak{c}(M,\mathcal{D})$ of infinitesimal contact transformations and the Lie algebra $C^\infty(M)$ of smooth functions with the Jacobi bracket defined by*

$$\Phi(X)=\eta(X).$$

Furthermore, under this isomorphism the subalgebra $\mathfrak{c}(M,\eta)$ is isomorphic to the subalgebra $C^\infty(M)^\xi$ of functions in $C^\infty(M)$ that are invariant under the flow generated by the Reeb vector field ξ. In particular, $\xi=\Phi^{-1}(1)$.

The function $\eta(X)$ is known as a *contact Hamiltonian function.*

Similarly, on $C(M)$ we let $\mathfrak{S}(C(M),\omega)$ denote the group of symplectomorphisms of $(C(M),\omega)$, and $\mathfrak{S}_0(C(M),\omega)$ the subgroup of $\mathfrak{S}(C(M),\omega)$ that commutes with homotheties, *i.e.,* the automorphism group of the symplectic Liouville structure. The corresponding Lie algebras are denoted by $\mathfrak{s}(C(M),\omega)$ and $\mathfrak{s}_0(C(M),\omega)$, respectively. Given a vector field $X\in\mathfrak{s}_0(C(M),\omega)$, the fact that X commutes with the Liouville vector field Ψ implies that X is projectable to a vector field X_M on M and one easily sees that [LM]

PROPOSITION 3.2: *The maps $X\mapsto X_M\mapsto\eta(X_M)$ induce Lie algebra isomorphisms $\mathfrak{s}_0(C(M),\omega)\approx\mathfrak{c}(M,\eta)\approx C^\infty(M)^\xi$. Furthermore, ξ is in the center of $\mathfrak{c}(M,\eta)$.*

4. Convexity and the Moment Map

We now consider the moment map construction for $C(M)$ and M. Let $\mathfrak{G}$ be a Lie group acting on the symplectic cone $(C(M),\omega)$ which leaves invariant the symplectic form ω and which commutes with the homothety group. In particular, $\mathfrak{G}$ is a subgroup of the group $\mathfrak{S}_0(C(M),\omega)$ and gives rise to a moment map

4.1 $$\tilde{\mu}:C(M)\longrightarrow\mathfrak{g}^*,$$

where $\mathfrak{g}$ denotes the Lie algebra of $\mathfrak{G}$ and $\mathfrak{g}^*$ its dual. Explicitly $\tilde{\mu}$ is defined by

4.2 $$d<\tilde{\mu},\tau>=-X^\tau\rfloor\omega,$$

where X^τ denotes the vector field on $C(M)$ corresponding to $\tau\in\mathfrak{g}$. For simplicity we denote the function $<\tilde{\mu},\tau>$ by $\tilde{\mu}^\tau$. An easy computation then shows that up to an additive constant

4.3 $$\tilde{\mu}^\tau=\tilde{\eta}(X^\tau).$$

With this choice $\tilde{\mu}$ is clearly homogeneous. Indeed, under homotheties $r\mapsto e^t r$ we have

$$\omega\mapsto e^t\omega,\qquad\tilde{\eta}\mapsto e^t\tilde{\eta},\qquad\tilde{\mu}\mapsto e^t\tilde{\mu}.$$

Now consider the case when the Lie group $\mathfrak{G}$ is a torus $\mathfrak{T}$ with Lie algebra $\mathfrak{t}$. Then there is the following convexity theorem [deMT]:

THEOREM 4.4 [deMT]: *Let $(C(M),\omega)$ be a symplectic cone with M compact, and let $\mathfrak{T}\subset\mathfrak{S}_0(C(M),\omega)$ be a torus group. Assume further that there is an element $\tau\in\mathfrak{t}$ such that $<\mu,\tau>>0$. Then the image $\mu(C(M))$ is a convex polyhedral cone.*

Now let us return to the case of an arbitrary Lie group $\mathfrak{G}\subset\mathfrak{S}_0(C(M),\omega)$. Again we identify M with $M\times\{1\}$. Since $\mathfrak{G}$ commutes with homotheties we get an induced action of $\mathfrak{G}$ on M, and by Proposition 3.2 we can identify $\mathfrak{G}$ with a Lie subgroup of $\mathfrak{C}(M,\eta)$. This gives a moment map by restriction, viz.

$$\mu:M\longrightarrow\mathfrak{g}^*,$$

4.5 $$\mu=\tilde{\mu}|_{M\times\{1\}},\qquad\mu^\tau=\eta(X^\tau).$$

Such a moment map was noticed previously by Geiges [Gei] and in [BGM] within the context of 3-Sasakian geometry.

We wish to consider the special case when the Lie group $\mathfrak{G}$ is an $n+1$-dimensional torus $\mathfrak{T}^{n+1}$. Let $\mathfrak{t}_{n+1}$ denote the Lie algebra of $\mathfrak{T}^{n+1}$, and let $\{e_i\}_{i=0}^n$ denote the standard basis for $\mathfrak{t}_{n+1} \approx \mathbb{R}^{n+1}$. Corresponding to each basis element e_i there is a vector field X^{e_i} which for convenience we denote by H_i. We shall assume that the Reeb vector field ξ corresponds to an element ς, which we call the *characteristic vector*, in the Lie algebra $\mathfrak{t}_{n+1}$. Hence, the Reeb vector field is almost periodic. In this case we say that the torus action is *of Reeb type*. Let $\{e_i^*\}_{i=0}^n$ denote the dual basis of $\mathfrak{t}_{n+1}^*$. Then we can write the corresponding characteristic vector ς and moment map μ as

4.6
$$\varsigma = \sum_{i=0}^n a_i e_i, \qquad \mu = \sum_{i=0}^n \mu_i e_i^*$$

for some $a_i \in \mathbb{R}$. Then the moment map μ maps ς to the hyperplane given by

4.7
$$< \mu, \varsigma >= \sum_{i=0}^n \mu_i a_i = 1.$$

We call this hyperplane *the characteristic hyperplane*. The nondegeneracy of η implies that the plane defined by 4.7 is actually a hyperplane, *i.e.,* its codimension is one. Now there is a commutative diagram

4.8
$$\begin{tikzcd} C(M) \arrow[dr, "\tilde{\mu}"] & \\ M \arrow[u] \arrow[r, "\mu"] & \mathfrak{t}_{n+1}^*, \end{tikzcd}$$

where the vertical arrow is the natural inclusion M into $C(M)$ as $M \times \{1\}$. Thus, intersecting the characteristic hyperplane 4.6 with the convex cone of Theorem 4.4 we arrive at:

THEOREM 4.9: *Let (M,η) be a compact contact manifold with a fixed contact form η, and let μ be the moment map of a torus $\mathfrak{T}$ acting effectively on M which preserves the contact form η. Suppose also that the Reeb vector field ξ corresponds to an element of the Lie algebra $\mathfrak{t}$ of $\mathfrak{T}$, so that the torus action is of Reeb type. Then the image $\mu(M)$ is a convex compact polytope lying in the characteristic hyperplane.*

This theorem was proved earlier by a different method by Banyaga and Molino [BM1,BM2], and they also give an example due to R. Lutz which shows that the assumption on the Reeb vector field is necessary.

We are mainly interested in rational polytopes. Let $\ell \subset \mathfrak{t}$ denote the lattice of circle subgroups of $\mathfrak{T}$. We recall some definitions for convex polytopes [Zie, Gui, LT]

DEFINITION 4.10: *A facet is a codimension one face. A convex polytope of dimension n is called simple if there are precisely n facets meeting at each vertex. A convex polytope*

$\Delta \subset \mathfrak{t}^*$ *is rational if, for some* $\lambda_i \in \mathbb{R}$, *there are* $y_i \in \ell$ *such that*

$$\Delta = \bigcap_{i=1}^{N} \{\alpha \in \mathfrak{t}^* \mid \ < \alpha, y_i > \leq \lambda_i\},$$

where N *is the number of facets of* Δ.

We are now ready for

THEOREM 4.11: *Under the hypothesis of Theorem 4.9, the polytope* $\mu(M) \subset \mathfrak{t}^*$ *is simple of dimension* $\dim \mathfrak{t} - 1$, *and it is rational if and only if the characteristic vector* ς *lies in the lattice* $\ell \subset \mathfrak{t}$ *of circle subgroups of* $\mathfrak{T}$.

PROOF: The nondegeneracy of η implies that the dimension of the polytope Δ is dim $\mathfrak{t} - 1$, and the fact that Δ is the intersection of a convex polyhedral cone and a hyperplane implies that Δ is simple. To verify the rationality condition consider the characteristic foliation $\mathcal{F}^\xi$. The space of leaves $\mathcal{Z}$ is a compact orbifold if and only if ς is in the lattice ℓ. So by the quasi-regular version [Tho] of the Boothby-Wang fibration, this orbifold has a symplectic structure. Moreover, since $\mathfrak{t}$ is Abelian and contains ς, there is a Lie algebra $\hat{\mathfrak{t}}$ acting on $\mathcal{Z}$ by infinitesimal symplectomorphisms which fits into the exact sequence

$$0 \longrightarrow \{\varsigma\} \longrightarrow \mathfrak{t} \xrightarrow{\varpi} \hat{\mathfrak{t}} \longrightarrow 0,$$

where $\{\varsigma\}$ denotes the Lie algebra generated by ς. Consider the dual sequence

$$0 \longrightarrow \hat{\mathfrak{t}}^* \xrightarrow{\varpi^*} \mathfrak{t}^* \longrightarrow \{\varsigma\}^* \longrightarrow 0. \tag{4.12}$$

By a theorem of Lerman and Tolman [LT] there is a moment map $\hat{\mu} : \mathcal{Z} \longrightarrow \hat{\mathfrak{t}}^*$ whose image in $\hat{\mathfrak{t}}^*$ is a rational convex polytope $\hat{\Delta}$. Furthermore, if $\pi : M \longrightarrow \mathcal{Z}$ denotes the orbifold V-bundle map and ω the symplectic form on $\mathcal{Z}$ we have that $\pi^*\omega = d\eta$. It follows that $\varpi^*\pi^*\hat{\mu}$ differs from μ by a constant $c \in \mathfrak{t}^*$, that is

$$\mu = \varpi^*\pi^*\hat{\mu} + c. \tag{4.13}$$

Moreover, since μ maps into the characteristic hyperplane c satisfies $< c, \varsigma >= 1$. Let Δ denote the polytope $\mu(M)$. From Definition 4.10 we see that the polytope Δ is rational if and only if we can choose the y_i to lie in ℓ. But since $\hat{\Delta}$ is rational and $\alpha \in \Delta$ if and only if $\alpha - c \in \hat{\Delta}$, there are $y_i \in \ell$ such that $\hat{\Delta}$ is determined by $< \hat{\alpha}, y_i > \leq \hat{\lambda}_i$. Thus, Δ is given by the equation

$$\bigcap_{i=1}^{N} \{\alpha \in \mathfrak{t}^* \mid \ < \alpha, y_i > \ \leq \ \lambda_i = \hat{\lambda}_i + < c, y_i >\}$$

which is clearly rational. ∎

For quasi-regular contact Pfaffian structures equation 4.13 gives the fundamental relation between our contact moment map and the moment map for compact symplectic orbifolds described by Lerman and Tolman. Moreover, in [LT] it is shown that the polytope $\hat{\Delta}$ has labels associated with each facet. The outward normal vector y_i to the $i^{\rm th}$ facet $\hat{f}_i$ of $\hat{\Delta}$ lies in ℓ, so there is a positive integer m_i and a primitive vector $p_i\in \ell$ such that $y_i=m_ip_i$. Thus, by associating m_i to the $i^{\rm th}$ facet f_i for each $i=1,\ldots,N$ we obtain $\hat{\Delta}$ as a labeled polytope. But from the discussion above the outward normals y_i to the $i^{\rm th}$ facet f_i of Δ coincide with the outward normals to $\hat{f}_i$ of $\hat{\Delta}$, so that Δ is also a labeled polytope with the integers m_i associated to the $i^{\rm th}$ facet f_i. Thus an immediate consequence of 4.13 and Lemma 6.6 of [LT] is

THEOREM 4.14: *Let (M,η) be a compact contact manifold with a fixed quasi-regular contact form η with an effective action of a torus $\mathfrak{T}$ of Reeb type that leaves the 1-form η invariant, For every point $x\in M$ let $F(x)$ denote the set of open facets of Δ whose closure contains $\mu(x)$ and let m_i and p_i denote the labels and primitive outward normal vectors to the $i^{\rm th}$ facet. Then the Lie algebra $\mathfrak{h}_x$ of the isotropy subgroup $\mathfrak{H}_x$ of $\mathfrak{T}$ at x is the linear span of the vectors p_i for all i such that the $i^{\rm th}$ open facet f_i lies in $F(x)$. In particular, if $\mu(x)$ is a vertex of the polytope Δ, then $\mathfrak{H}_x$ is isomorphic to the factor group $\mathfrak{T}/S^1(\xi)$ where $S^1(\xi)$ denotes the circle subgroup generated by the characteristic vector field ξ. Furthermore, Δ is the convex hull of its vertices.*

Let $\ell_x\in\mathfrak{h}_x$ denote the lattice of circle subgroups of $\mathfrak{H}_x$ and let $\hat{\ell}_x$ denote the sublattice of ℓ generated by the vectors $\{m_ip_i\}_{f_i\in F(x)}$. Then the leaf holonomy group at x of the characteristic foliation $\mathcal{F}^\xi$ is isomorphic to $\ell_x/\hat{\ell}_x$. In particular, (M,η) is regular if and only if the set $\{m_ip_i\}_{f_i\in F(x)}$ generates ℓ_x for all $x\in M$. (In this case $m_i=1\ \ \forall i$).

We now consider varying the contact form η within the contact structure. Let $\eta'=f\eta$ where f is a nowhere vanishing function on M, and let ξ' be the Reeb vector field associated to η'. Write $\xi'=\xi+\rho$. Suppose further that $X\in\mathfrak{c}(M,\eta)$. Then we have the following elementary lemma whose proof we leave to the reader:

LEMMA 4.15: *The following hold:*

(i) $f=\dfrac{1}{\eta(\rho)+1}$.

(ii) $\rho\rfloor d\eta=-d(\eta(\rho))+\xi(\eta(\rho))\eta$.

(iii) $\mathcal{L}_\rho\eta=\mathcal{L}_{\xi'}\eta=\xi(\eta(\rho))\eta$, *i.e., both ξ' and ρ are infinitesimal contact transformations.*

(iv) $\xi',\rho\in\mathfrak{c}(M,\eta)$ *if and only if* $f\in C^\infty(M)^\xi$.

(v) $X\in\mathfrak{c}(M,\eta')$ *if and only if* $Xf=0$.

We return to the case when (M,η) is a contact manifold with an $(n+1)$-torus $\mathfrak{T}^{n+1}$ acting as strict contact transformations whose characteristic vector ς lies in the Lie algebra $\mathfrak{t}_{n+1}$. The moment map for this torus action is given by 4.6 with $\mu_i=\eta(H_i)$. If $\eta'=f\eta$ is another 1-form in same contact structure where f is invariant under the torus action, then this action preserves η' as well, and the corresponding moment map μ' satisfies $\mu'=f\mu$.

Let Δ be the polytope associated with η. Then the polytope Δ' associated to η' has the same dimension as Δ with the same number of facets and the same number of vertices. The size of the faces and labels, however, depend on the contact form η, that is on the Pfaffian structure, and the labels are defined only when η is quasi-regular. But in the projective space $\mathbb{RP}^{n+1}$ the lines through the origin in $\mathfrak{t}_{n+1}$ that intersect the lattice ℓ of circle subgroups are dense, so from the point of view of the contact structure we can always perturb the Reeb vector field and contact form so that the characteristic foliation is quasi-regular, and by Theorem 4.11 so that the polytope is rational. We are ready for

Definition 4.16: *A contact manifold (orbifold) $(M,\mathcal{D})$ of dimension $2n+1$ is called a toric contact manifold (orbifold) (written as the triple $(M,\mathcal{D},\mathfrak{T})$) if there are a 1-form η that represents the contact structure $\mathcal{D}$, and an effective action of an $(n+1)$-dimensional torus $\mathfrak{T}$ on M that preserves the contact form η. If in addition the Reeb vector field associated to η corresponds to an element of the Lie algebra $\mathfrak{t}$ of $\mathfrak{T}$, we say that $(M,\mathcal{D},\mathfrak{T})$ is a toric contact manifold of Reeb type.*

If we wish to fix a contact form η we write $(M,\eta,\mathfrak{T})$ for a toric contact manifold instead of $(M,\mathcal{D},\mathfrak{T})$. Our discussion above proves

Proposition 4.17: *Let $(M,\mathcal{D},\mathfrak{T})$ be a compact toric contact orbifold of Reeb type, then $\mathcal{D}$ can be represented by a quasi-regular contact form, and hence, by a rational polytope.*

The fiduciary examples of compact contact manifolds are the odd dimensional spheres S^{2n+1}.

Example 4.18: S^{2n+1} with the standard contact structure. This is the contact structure induced from the standard symplectic structure on $\mathbb{R}^{2n+2}$ given in Cartesian coordinates $(x_0,y_0,\ldots,x_n,y_n)\in\mathbb{R}^{2n+2}$ by

$$\omega = 2\sum_{i=0}^n dx_i\wedge dy_i.$$

In this case the standard contact form η and the standard characteristic vector field ξ are given by

$$\eta=\frac{1}{r}\sum_{i=0}^n (x_i dy_i - y_i dx_i), \qquad \xi=\sum_{i=0}^n (x_i\partial_{y_i} - y_i\partial_{x_i}),$$

where $r=\sum_{i=0}^n(x_i^2+y_i^2)$ (not the usual r). The maximal torus $\mathfrak{T}^{n+1}$ is generated by the vector fields $H_i = x_i\partial_{y_i} - y_i\partial_{x_i}$ for $i=0,\ldots,n$, so (S^{2n+1},η) is a toric contact manifold. The moment map is easily seen to be

$$\mu(x_0,y_0,\ldots,x_n,y_n)=\frac{1}{r}\sum_{i=0}^n (x_i^2+y_i^2)e_i^*.$$

Letting $r_0,\ldots,r_n$ denote the coordinates for $\mathfrak{t}_{n+1}^*$ we see that the characteristic hyperplane is just

$$r_0+\cdots+r_n=1.$$

Thus, the image $\mu(S^{2n+1})$ is just the standard n-simplex with r_i's as barycentric coordinates.

Now we can consider non-standard characteristic vector fields and Pfaffian forms within the standard contact structure on S^{2n+1}. These are deformations [YK] of the standard form depending on $n+1$ positive real parameters $(a_0,\ldots,a_n)\in(\mathbb{R}^+)^{n+1}$. In this case the 1-form η and characteristic vector fields are given by

$$\eta_{\mathbf{a}}=\frac{\sum_{i=0}^n(x_idy_i-y_idx_i)}{\sum_{i=0}^n a_i(x_i^2+y_i^2)}\qquad \xi_{\mathbf{a}}=\sum_{i=0}^n a_i(x_i\partial_{y_i}-y_i\partial_{x_i}),$$

so that

$$\eta_{\mathbf{a}}=\left(\frac{r}{\sum_{i=0}^n a_i(x_i^2+y_i^2)}\right)\eta.$$

The characteristic hyperplane is

$$\sum_{i=0}^n a_ir_i=1,$$

so the polytope is given by the "weighted" n-simplex determined by this and

$$0\leq r_i\leq\frac{1}{a_i}.$$

The special case where $a_i=a$ for all $i=0,\ldots,n$ is just the dilated standard n-simplex

$$a(r_0+\cdots+r_n)=1,\qquad 0\leq r_i\leq\frac{1}{a}.$$

5. A Delzant Theorem for Toric Contact Manifolds of Reeb type

We begin by considering contact reduction [BGM, Gei]. Let $(\tilde{M},\tilde{\eta})$ be a compact contact manifold with a fixed quasi-regular contact form $\tilde{\eta}$. Suppose also that a compact Lie group $\mathfrak{G}$ acts on $\tilde{M}$ preserving the contact form $\tilde{\eta}$ and let $\mu:\tilde{M}\longrightarrow\mathfrak{g}^*$ denote the corresponding moment map. Then if $\mathfrak{G}$ acts freely on the zero set $\mu^{-1}(0)\overset{\iota}{\hookrightarrow}\tilde{M}$, the quotient $M=\mu^{-1}(0)/\mathfrak{G}$ is a compact contact manifold with a unique fixed 1-form η satisfying $\iota^*\tilde{\eta}=p^*\eta$ where $p:\mu^{-1}(0)\longrightarrow M$ denotes the natural projection.

We shall prove

THEOREM 5.1: *Let $(M,\eta,\mathfrak{T})$ be a compact toric contact manifold of Reeb type and a fixed quasi-regular contact form η. Then (M,η) is isomorphic to the reduction by a torus of a sphere S^{2N-1} with its standard contact structure and with a fixed 1-form $\eta_{\mathbf{a}}$.*

PROOF: Now M is compact of dimension, say, $2n+1$, and since η is quasi-regular, the space of leaves $\mathcal{Z}$ of the characteristic foliation $\mathcal{F}$ is a compact symplectic orbifold of dimension $2n$. Furthermore, since M is toric, so is $\mathcal{Z}$, that is, there is an n dimensional torus $\mathfrak{T}^n$ preserving the symplectic structure ω on $\mathcal{Z}$. Furthermore, by [BG2] ω represents an integral class in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ and M is the total space of the principal S^1 V-bundle $M\xrightarrow{\pi}\mathcal{Z}$ whose first Chern class is represented by $d\eta=\pi^*\omega$.

Now by Theorem 8.1 of Lerman and Tolman [LT] $(\mathcal{Z},\omega)$ is isomorphic to the symplectic reduction $(\mathbb{C}^N,\omega_0)$ with the standard symplectic structure by a torus $\mathfrak{T}^{N-n}$of dimension $N-n$. If $\mu_{N-n}:\mathbb{C}^N\longrightarrow\mathfrak{t}^*_{N-n}$ denotes the moment map for the $\mathfrak{T}^{N-n}$ action, then $(\mathcal{Z},\omega)$ is isomorphic to $(\mu^{-1}_{N-n}(\lambda)/\mathfrak{T}^{N-n},\hat{\omega})$ where λ is a regular value of μ_{N-n} and $\hat{\omega}$ is the unique symplectic 2-form induced by reduction. Let $\hat{\phi}$ denote the above isomorphism. It follows that the cohomology class of $\hat{\omega}=(\hat{\phi}^{-1})^*\omega$ is integral in $H^2_{orb}(\mu^{-1}_{N-n}(\lambda)/\mathfrak{T}^{N-n},\mathbb{Z})$. By the orbifold version of the Boothby-Wang theorem there is an S^1 V-bundle $\pi:P\longrightarrow\mu^{-1}_{N-n}(\lambda)/\mathfrak{T}^{N-n}$, a connection form $\hat{\eta}$ on P such that $d\hat{\eta}=\hat{\pi}^*\hat{\omega}$. Thus there is an S^1-equivariant V-bundle map

5.2
$$\begin{array}{ccc} M & \xrightarrow{\phi} & P \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\hat{\pi}} \\ \mathcal{Z} & \xrightarrow{\hat{\phi}} & \mu^{-1}_{N-n}(\lambda)/\mathfrak{T}^{N-n} \end{array}$$

such that $\phi^*d\hat{\eta}=d\eta$. Thus, $\phi^*\hat{\eta}$ and η differ by a closed 1-form. But the space of closed 1-forms on M is path-connected, so one can find a one parameter family of connections having the same curvature that connect $\phi^*\hat{\eta}$ to η. So by Gray's stability theorem [MS] $\phi^*\hat{\eta}$ and η define the same contact structure. Thus, we can choose $\phi^*\hat{\eta}=\eta$. Moreover, by equivariance the characteristic vector ς of the contact manifold $(P,\hat{\eta})$ lies in the Lie algebra $\mathfrak{t}_{N-n}$, so we can split off the circle that it generates and write $\mathfrak{T}^{N-n}=S^1_\varsigma\times\mathfrak{T}^{N-n-1}$, where $\mathfrak{T}^{N-n-1}$ is an $(N-n-1)$-dimensional torus. It follows that $P=\mu^{-1}_{N-n}(\lambda)/\mathfrak{T}^{N-n-1}$. Hereafter, we identify (M,η) with $(\mu^{-1}_{N-n}(\lambda)/\mathfrak{T}^{N-n-1},\hat{\eta})$. Now $\mu^{-1}_{N-n}(\lambda)$ is a torus bundle over a compact manifold, so it is a compact manifold which by construction is an intersection of $N-n$ real quadrics in $\mathbb{C}^N$. It follows that there is a component of the moment map μ_{N-n} which takes the form $\sum_i a_i|z_i|^2$ with $a_i>0$ for all i. Let $\mathbf{a}$ denote the vector in $\mathbb{R}^N$ whose $i^{\rm th}$ component is a_i, and consider the ellipsoid $\Sigma_{\mathbf{a}}=\{\sum_i a_i|z_i|^2=1\}\cong S^{2N-1}$. Then there is a $\mathfrak{T}^{N-n}$-moment map $\nu_{\mathbf{a}}:\Sigma_{\mathbf{a}}\longrightarrow t^*_{N-n}$ such that $\nu_{\mathbf{a}}^{-1}(0)=\mu^{-1}_{N-n}(\lambda)$. Furthermore, letting $\eta_0=\sum_i(x_idy_i-y_idx_i)$ we see that $d\eta_0=\omega_0$ on $\mathbb{C}^N$ and that $\eta_0|_{\Sigma_{\mathbf{a}}}=\eta_{\mathbf{a}}|_{\Sigma_{\mathbf{a}}}$, where $\eta_{\mathbf{a}}$ is the deformed 1-form of Example 4.18. Thus, letting $p:\mu^{-1}_{N-n}(\lambda)\longrightarrow M$ denote the natural submersion, and $\iota:\mu^{-1}_{N-n}(\lambda)\longrightarrow\Sigma_{\mathbf{a}}$ the natural inclusion, we see that $p^*\eta=\iota^*\eta_{\mathbf{a}}$, so (M,η) is obtained from $(\Sigma_{\mathbf{a}},\eta_{\mathbf{a}})$ by contact reduction. ∎

Now Theorem 9.1 of [LT] says that every symplectic toric orbifold possesses an invariant complex structure which is compatible with its symplectic form. This means that every symplectic toric orbifold is actually Kähler and we can combine this fact with our results to get the following:

THEOREM 5.3: *Every compact toric contact manifold of Reeb type admits a compatible invariant Sasakian structure.*

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