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New Einstein Metrics in Dimension Five

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Abstract: The purpose of this note is to prove the existence of new Sasakian-Einstein metrics on $S^2\times S^3$ and on $(S^2\times S^3)\#(S^2\times S^3)$. These give the first known examples of non-regular Sasakian-Einstein 5-manifolds. Our method involves describing the Sasakian-Einstein structures as links of certain isolated hypersurface singularities, and makes use of the recent work of Demailly and Kollár who obtained new examples of Kähler-Einstein del Pezzo surfaces with quotient singularities.

0. Introduction

Since any three dimensional Einstein manifold has constant curvature, the essential study of Sasakian-Einstein manifolds begins in dimension five. Moreover, since a complete Sasakian-Einstein manifold is necessarily spin with finite fundamental group, Smale's classification [Sm] of simply connected compact 5-manifolds with spin applies. If there is no torsion in H_2 then Smale's theorem says that any such 5-manifold is diffeomorphic to $S^5\#_k(S^2\times S^3)$ for some k. In the classification of Sasakian-Einstein manifolds it is judicious to distinguish between regular Sasakian-Einstein manifolds and non-regular ones. The compact simply connected five dimensional manifolds admitting a regular Sasakian-Einstein structure have been classified by Friedrich and Kath [FK], and it follows from the classification of smooth del Pezzo surfaces admitting Kähler-Einstein metrics due to Tian and Yau [TY]. These 5-manifolds are S^5 and $\#_k(S^2\times S^3)$ for $k=1,3,\cdots,8$. For $k=3,\cdots,8$ they are are circle bundles over $\mathbb{P}^2$ blown up at k generic points, whereas for $k=1$ this is the homogeneous Stiefel manifold $V_{4,2}(\mathbb{R})$ which is a circle bundle over $\mathbb{P}^1\times\mathbb{P}^1$. Notice that $k=2$ is missing from this list as are the circle bundles over $\mathbb{P}^2$ blown-up at one or two points. The reason for this is Matsushima's well-known obstruction to the existence of Kähler-Einstein metrics when the complex automorphism group is not reductive. Thus, any Sasakian-Einstein structure on the connected sum of two copies of $S^2\times S^3$ must be non-regular. In this note we prove the existence of such a non-regular Sasakian-Einstein metric thus filling this "$k=2$ gap". It is an interesting question as to whether Sasakian-Einstein structures exist on $\#_k(S^2\times S^3)$ for $k>8$. Of course if such structures exist they must be non-regular.

We also prove the existence of two inhomogeneous non-regular Sasakian-Einstein metrics on $S^2\times S^3$. Recently it has been shown that the manifold $S^2\times S^3$ admits a quite a few Einstein metrics. First Wang and Ziller [WZ] proved the existence of a countable number of homogeneous Einstein metrics on $S^2\times S^3$. More recently Böhm [Böh] showed the existence of a countable number of cohomogeneity one Einstein metrics on $S^2\times S^3$. Our new Sasakian-Einstein metrics are also inhomogeneous and they are not isometric to any of the Böhm's examples. Explicitly, we prove

Theorem A: *There exists a non-regular Sasakian-Einstein metric on* $(S^2\times S^3)\#(S^2\times S^3)$.

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THEOREM B: *There exist two inequivalent inhomogeneous Sasakian-Einstein metrics on $S^2 \times S^3$. These metrics are inequivalent as Riemannian metrics to the inhomogeneous metrics of Böhm. Hence, $S^2 \times S^3$ admits at least three distinct Sasakian-Einstein metrics.*

A Sasakian structure on a manifold defines several interesting objects. It defines a one-dimensional foliation, a CR-structure, and a contact structure. Indeed, combining all three of these, it defines a Pfaffian structure with a transverse Kähler geometry. Now in each case above the Sasakian-Einstein structure is unique within the CR-structure. Thus, on $S^2 \times S^3$ we have three distinct CR-structures. It is interesting to ask the question as to whether the three Sasakian-Einstein structures on $S^2 \times S^3$ belong to distinct contact structures. This is a more subtle question as contact geometry has no local invariants. Perhaps there is a connection between certain link invariants and contact invariants as suggested by Arnold [Arn].

Our method of proofs of Theorems A and B is to consider the Sasakian geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. We then make use of a recent result of Demailly and Kollár [DK] proving the existence of Kähler-Einstein metrics on certain del Pezzo orbifolds given as hypersurfaces in certain weighted projective spaces. The links which can then be represented as the total space of V-bundles over these orbifolds admit Sasakian-Einstein metrics. We then use a well-known algorithm of Milnor and Orlik [MO] to compute the characteristic polynomials of the monodromy maps associated to the links. This allows us to determine the second Betti number of the link. Then using a method of Randell [Ran] we can show that the links have no torsion, and apply Smale's classification theorem.

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1. The Sasakian Geometry of Links of Weighted Homogeneous Polynomials

In this section we discuss the Sasakian geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action given by $(z_0,\ldots,z_n) \mapsto (\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n)$, where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w} \in (\mathbb{Z}^+)^{n+1}$, and we shall assume that $\gcd(w_0,\ldots,w_n) = 1$. Let f be a quasi-homogeneous polynomial, that is $f \in \mathbb{C}[z_0,\ldots,z_n]$ and satisfies

1.1 $$f(\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n) = \lambda^d f(z_0,\ldots,z_n),$$

where $d \in \mathbb{Z}^+$ is the degree of f. We are interested in the *weighted affine cone* C_f defined by the equation $f(z_0,\ldots,z_n) = 0$. We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity, in fact the only singularity, of f. Then the link L_f defined by

1.2 $$L_f = C_f \cap S^{2n+1},$$

where

$$S^{2n+1} = \{(z_0,\ldots,z_n) \in \mathbb{C}^{n+1} | \sum_{j=0}^{n} |z_j|^2 = 1\}$$

is the unit sphere in $\mathbb{C}^{n+1}$, is a smooth manifold of dimension $2n-1$. Furthermore, it is well-known [Mil] that the link L_f is $(n-2)$-connected.

On S^{2n+1} there is a well-known [YK] "weighted" Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which in the standard coordinates $\{z_j = x_j + iy_j\}_{j=0}^n$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is determined by

$$\eta_{\mathbf{w}} = \frac{\sum_{i=0}^n (x_i dy_i - y_i dx_i)}{\sum_{i=0}^n w_i(x_i^2 + y_i^2)}, \qquad \xi_{\mathbf{w}} = \sum_{i=0}^n w_i(x_i \partial_{y_i} - y_i \partial_{x_i}),$$

and the standard Sasakian structure (ξ, η, Φ, g) on S^{2n+1}. Explicitly, we have

1.3
$$\begin{aligned} \Phi_{\mathbf{w}} &= \Phi - \Phi\xi_w \otimes \eta_w \\ g_{\mathbf{w}} &= \frac{1}{\eta(\xi_{\mathbf{w}})}[g - \eta_{\mathbf{w}} \otimes \xi_{\mathbf{w}} \rfloor g - \xi_{\mathbf{w}} \rfloor g \otimes \eta_{\mathbf{w}} + g(\xi_{\mathbf{w}}, \xi_{\mathbf{w}})\eta_{\mathbf{w}} \otimes \eta_{\mathbf{w}}] + \eta_{\mathbf{w}} \otimes \eta_{\mathbf{w}}. \end{aligned}$$

Now, by equation 1.1, the $\mathbb{C}^*(\mathbf{w})$ action on $\mathbb{C}^{n+1}$ restricts to an action on C_f, and the associated S^1 action restricts to an action on both S^{2n+1} and L_f. It follows that $\xi_{\mathbf{w}}$ is tangent to the submanifold L_f and, by abuse of notation, we shall denote by $\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}}$ the corresponding tensor fields on both S^{2n+1} and L_f. Now $\Phi_{\mathbf{w}}$ coincides with Φ on the contact subbundle $\mathcal{D}$ on S^{2n+1} which defines an integrable almost complex structure on $\mathcal{D}$. Moreover, since f is a holomorphic function on $\mathbb{C}^{n+1}$ the Cauchy-Riemann equations imply that for any smooth section X of $\mathcal{D}$ we have $\Phi_{\mathbf{w}} X(f) = 0$. Thus, L_f is an invariant submanifold of S^{2n+1} with its weighted Sasakian structure. We have arrived at a theorem given by Takahashi [Tak, YK] in the case of Brieskorn-Pham links and we have seen that Takahashi's proof easily generalizes to the case of arbitrary weighted homogeneous hypersurface singularities.

THEOREM 1.4: *The quadruple $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ gives L_f a quasi-regular Sasakian structure.*

Actually as with Kähler structures there are many Sasakian structures on a given Sasakian manifold. In fact there are many Sasakian structures which have ξ as its characteristic vector field. To see this let (ξ, η, Φ, g) be a Sasakian structure on a smooth manifold (orbifold) M, and consider a deformation of this structure by adding to η a continuous one parameter family of 1-forms ζ_t that are basic with respect to the characteristic foliation. We require that the 1-form $\eta_t = \eta + \zeta_t$ satisfy the conditions

1.5
$$\eta_0 = \eta, \qquad \zeta_0 = 0, \qquad \eta_t \wedge (d\eta_t)^n \neq 0 \ \ \forall \ \ t \in [0, 1].$$

This last non-degeneracy condition implies that η_t is a contact form on M for all $t \in [0, 1]$. Then by Gray's Stability Theorem [MS] η_t belongs to the same contact structure as η. Moreover, since ζ_t is basic ξ is the Reeb (characteristic) vector field associated to η_t for all t. Now let us define

1.6
$$\begin{aligned} \Phi_t &= \Phi - \xi \otimes \zeta_t \circ \Phi \\ g_t &= g + d\zeta_t \circ (\Phi \otimes \mathrm{id}) + \zeta_t \otimes \eta + \eta \otimes \zeta_t + \zeta_t \otimes \zeta_t. \end{aligned}$$

Note that it is not at all clear from this definition that g_t is a Riemannian metric, but we shall check this below. We have

THEOREM 1.7: *Let (M, ξ, η, Φ, g) be a Sasakian manifold. Then for all $t \in [0, 1]$ and every basic 1-form ζ_t such that $d\zeta_t$ is of type $(1, 1)$ and such that 1.5 holds $(\xi, \eta_t, \Phi_t, g_t)$ defines a Sasakian structure on M belonging to the same underlying contact structure as η.*

PROOF: The conditions of 1.5 guarantee that $(\xi, \eta_t, \Phi_t, g_t)$ defines a Pfaffian structure, i.e. a contact structure with a fixed contact 1-form. We need to check that it is a metric contact structure and that it is normal. It is easy to check that the metric g_t of 1.6 can be rewritten as

1.8 $$g_t = d\eta_t \circ (\Phi_t \otimes \mathrm{id}) + \eta_t \otimes \eta_t.$$

It follows from the fact $d\eta_t$ is type $(1,1)$ on the contact bundle $\mathcal{D}_t = \ker\, \eta_t$ that g_t is a symmetric bilinear form and then a straightforward computation checks the compatibility condition
$$g_t(\Phi_t X, \Phi_t Y) = g_t(X,Y) - \eta_t(X)\eta_t(Y).$$
The positive definiteness of g_t follows from the positive definiteness of g and the non-degeneracy condition in 1.5. Moreover, one easily checks the identity $\Phi_t^2 = -\mathrm{id} + \xi \otimes \eta_t$. Next we check normality which amounts to checking two conditions, that the almost CR structure defined by Φ_t on the contact bundle $\mathcal{D}_t$ is integrable, and that ξ is a Killing vector field for the metric g_t. The last condition is equivalent to vanishing of the Lie derivative $\mathcal{L}_\xi \Phi_t$ for all t which follows immediately from the first of equations 1.6 and the facts that it holds for $t = 0$ and that ζ_t is basic. Integrability follows from the fact that the almost CR structure defined by Φ on $\mathcal{D}$ is integrable, and that the first of equations 1.6 is just the projection of the image of Φ onto $\mathcal{D}_t$. ∎

In general these structures are inequivalent and the moduli space of Sasakian structures having the same characteristic vector field is infinite dimensional. Indeed since the link of a hypersurface is determined by the $S^1(\mathbf{w})$ action we formulate the following:

DEFINITION 1.8: *A Sasakian structure (ξ, η, Φ, g) on L_f is said to be compatible with the link L_f if ξ is a generator of the S^1 action on L_f. We say that ξ is the standard generator if $\xi = \xi_{\mathbf{w}}$, where $\mathbf{w}$ is the weight vector of L_f satisfying* $\gcd(w_0, \ldots, w_n) = 1$.

For every compatible Sasakian structure there is one with a standard generator, and hereafter we shall always choose the standard generator for a compatible Sasakian structure unless otherwise stated. We are interested in the following question:

PROBLEM 1.9: *Given a link L_f with a given Sasakian structure (ξ, η, Φ, g), when can we find a 1-form ζ such that the deformed structure $(\xi, \eta + \zeta, \Phi', g')$ is Sasakian-Einstein ?*

This is a Sasakian version of the Calabi problem for the link L_f which is discussed further in [BGN1]. Here we use the fact that the leaf space of the characteristic foliation of a Sasakian structure of a link L_f is a compact Kähler orbifold together with recent results of Demailly and Kollár [DK] on the existence of Kähler-Einstein orbifold metrics on certain singular del Pezzo surfaces to construct the Sasakian-Einstein metrics on the corresponding link.

2. Kähler-Einstein Orbifolds and Sasakian-Einstein Manifolds

Given a sequence $\mathbf{w} = (w_0, \ldots, w_n)$ of positive integers one can form the graded polynomial ring $S(\mathbf{w}) = \mathbb{C}[z_0, \ldots, z_n]$, where z_i has grading or *weight* w_i. The weighted projective space [BR, Del, Dol, Fle] $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$ is defined to be the scheme $\mathrm{Proj}(S(\mathbf{w}))$. It is the quotient space $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*(\mathbf{w})$, where $\mathbb{C}^*(\mathbf{w})$ is the weighted action defined in section 1. Clearly, $\mathbb{P}(\mathbf{w})$ is also the quotient of the weighted Sasakian sphere $S^{2n+1}_{\mathbf{w}} = (S^{2n+1}, \xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ by the weighted circle action $S^1(\mathbf{w})$ generated by $\xi_{\mathbf{w}}$. As such $\mathbb{P}(\mathbf{w})$ is also a compact complex orbifold with an induced Kähler structure. At times it will be important to distinguish between $\mathbb{P}(\mathbf{w})$ as a complex orbifold and $\mathbb{P}(\mathbf{w})$ as an algebraic variety.

Now the cone C_f in $\mathbb{C}^{n+1}$ cuts out a hypersurface $\mathcal{Z}_f$ of $\mathbb{P}(\mathbf{w})$ which is also a compact orbifold with an induced Kähler structure $\omega_{\mathbf{w}}$. So there is a commutative diagram

2.1
$$\begin{tikzcd} L_f \arrow[r] \arrow[d, "\pi"] & S^{2n+1}_{\mathbf{w}} \arrow[d] \\ \mathcal{Z}_f \arrow[r] & \mathbb{P}(\mathbf{w}), \end{tikzcd}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are orbifold Riemannian submersions. Furthermore, by the inversion theorem of [BG1] L_f is the total space of the principal S^1 V-bundle over the orbifold $\mathcal{Z}_f$ whose first Chern class is $[\omega_{\mathbf{w}}] \in H^2_{orb}(\mathcal{Z}_f, \mathbb{Z})$, and $\eta_{\mathbf{w}}$ is the connection in this V-bundle whose curvature is $\pi^*\omega_{\mathbf{w}}$. For further discussion of the orbifold cohomology groups we refer the reader to [Hae] and [BG1].

Now $H^2_{orb}(\mathcal{Z}_f, \mathbb{Z}) \otimes \mathbb{Q} \approx H^2(\mathcal{Z}_f, \mathbb{Q})$, so $[\omega_{\mathbf{w}}]$ defines a rational class in the usual cohomology. Thus, by Baily's [Bai] projective embedding theorem for orbifolds, $\mathcal{Z}_f$ is a projective algebraic variety with at most quotient singularities. It follows that as an algebraic variety $\mathcal{Z}_f$ is normal. As with $\mathbb{P}(\mathbf{w})$ it is important to distinguish between $\mathcal{Z}_f$ as an orbifold and $\mathcal{Z}_f$ as an algebraic variety. For example the singular loci may differ. There are examples where the orbifold singular locus $\Sigma^{orb}(\mathcal{Z}_f)$ has codimension one over $\mathbb{C}$, whereas $\Sigma^{alg}(\mathcal{Z}_f)$ always has codimension ≥ 2 by normality. Indeed there are examples with $\Sigma^{orb}(\mathcal{Z}_f) \neq \emptyset$, but $\mathcal{Z}_f$ is smooth as an algebraic variety. In these cases an orbifold metric is NOT a Riemannian metric in the usual sense.

We are interested in finding Sasakian-Einstein structures $(\xi_{\mathbf{w}}, \eta'_{\mathbf{w}}, \Phi'_{\mathbf{w}}, g'_{\mathbf{w}})$ on L_f with the same Sasakian vector field $\xi_{\mathbf{w}}$ as the original Sasakian structure, that is Sasakian-Einstein structures that are compatible with the link L_f. Since a Sasakian-Einstein metric necessarily has positive Ricci curvature, we see that a necessary condition for a Sasakian-Einstein metric g to exist on L_f is that there be an orbifold metric h on $\mathcal{Z}_f$ with positive Ricci form ρ_h such that $g = \pi^*h + \eta \otimes \eta$. The positive definiteness of ρ_h follows from the relation

2.2
$$\pi^*\mathrm{Ric}_h = \mathrm{Ric}_g|_{\mathcal{D}\times\mathcal{D}} + 2g|_{\mathcal{D}\times\mathcal{D}}.$$

But since ρ_h represents the first Chern class $c_1(K^{-1})$ of the anti-canonical line V-bundle K^{-1} we see that $\mathcal{Z}_f$ must be a Fano orbifold, i.e., some power of K^{-1} is invertible and ample. Here K is the canonical line V-bundle of Baily [Bai]. By the previous section and the well-known theory of Sasakian-Einstein structures [BG1] we have

THEOREM 2.3: *The link L_f has a compatible Sasakian-Einstein structure if and only if the Fano orbifold $\mathcal{Z}_f$ admits a compatible Kähler-Einstein orbifold metric of scalar curvature $4n(n+1)$.*

Of course, if we find a Kähler-Einstein metric of positive scalar curvature on the orbifold $\mathcal{Z}_f$, we can always rescale the metric so that the scalar curvature in $4n(n+1)$.

We want conditions that guarantee that the hypersurfaces be Fano, but first we restrict somewhat the hypersurfaces that we treat.

DEFINITION 2.3 [Fle]:

(1) A weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$ is said to be well-formed if

$$\gcd(w_0, \ldots, \hat{w}_i, \ldots, w_n) = 1 \text{ for all } i = 1, \ldots, n.$$

(2) *A hypersurface in a weighted projective space* $\mathbb{P}(w_0,\ldots,w_n)$ *is said to be well-formed if in addition it contains no codimension 2 singular stratum of* $\mathbb{P}(w_0,\ldots,w_n)$,

where the hat means delete that element.

In (1) of definition 2.3, we prefer the terminology introduced in [BR] for an equivalent condition which seems to have first appeared in [Dol]. So if a weighted projective space satisfies (1) of definition 2.3 we say that $\mathbf{w}$ is *normalized.* In [Fle] it is shown that (2) of definition 2.3 can be formulated solely in terms of the weighted homogeneous polynomial f. That is, a hypersurface in $\mathbb{P}(\mathbf{w})$ defined by a weighted homogeneous polynomial f is well-formed if and only if $\mathbf{w}$ is normalized and

2.4 $$\gcd(w_0,\ldots,\hat{w}_i,\ldots,\hat{w}_j,\ldots,w_n) \mid d \qquad \text{for all distinct } i,j=0,\ldots,n.$$

In this case we shall also say that the weighted homogeneous polynomial f defining the hypersurface is well-formed. The following proposition is essentially an exercise in [Kol]:

Proposition 2.5: *Let* $f_{\mathbf{w}}$ *be a well-formed weighted homogeneous polynomial with weights* $\mathbf{w}$ *and degree* d*. Then the hypersurface* $\mathcal{Z}_f$ *in* $\mathbb{P}(w_0,\ldots,w_n)$ *is Fano if and only if the following condition holds*

$(*)$ $d<|\mathbf{w}|=\sum_{i=0}^n w_i.$

Thus if $\mathbf{w}$ *is normalized and satisfies 2.4, a necessary condition for the link* L_f *to admit a compatible Sasakian-Einstein metric is that* $(*)$ *be satisfied.*

Proof: Following [Mo,BR] in any weighted projective space $\mathbb{P}(\mathbf{w})$ we define the Mori-singular locus as follows: Let $m=\mathrm{lcm}(w_0,\ldots,w_m)$ and consider for each prime divisor p of m the subscheme S_p cut out by the ideal I_p generated by those indeterminates z_i such that $p \not| w_i$, and define the Mori-singular locus by

$$S(\mathbf{w})=\cup_{p|m}S_p,$$

and the Mori-regular locus by

$$\mathbb{P}^0(\mathbf{w})=\mathbb{P}(\mathbf{w})-S(\mathbf{w}).$$

Now $\mathbb{P}^0(\mathbf{w})$ is contained in the smooth locus of $\mathbb{P}(\mathbf{w})$ but can be a proper subset of it. Thus, $S(\mathbf{w})$ contains the singular locus $\Sigma(\mathbf{w})$ of $\mathbb{P}(\mathbf{w})$ but in general is larger. However, when $\mathbf{w}$ is normalized the local uniformizing groups of the orbifold $\mathbb{P}(\mathbf{w})$ contain no quasi-reflections as in [Fle]. It follows in this case that the singular locus $\Sigma(\mathbf{w})$ coincides with the Mori-singular locus $S(\mathbf{w})$, and $\mathbb{P}^0(\mathbf{w})$ is precisely the smooth locus of $\mathbb{P}(\mathbf{w})$.

Now the sheaves $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(n)$ for $n\in\mathbb{Z}$ are not in general invertible, but they are reflexive [BR]. Moreover, the Mori-regular locus $\mathbb{P}^0(\mathbf{w})$ is the largest open subset U of $\mathbb{P}(\mathbf{w})$ on which

(1) $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(1)|U$ is invertible, and

(2) the natural map $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(1)^{\otimes n}|U \longrightarrow \mathcal{O}_{\mathbb{P}(\mathbf{w})}(n)|U$ is an isomorphism for each $n\in\mathbb{Z}^+$ [Mo].

Furthermore, if $\mathbf{w}$ is normalized $\mathbb{P}^0(\mathbf{w})$ is characterized by condition (1) alone. As an orbifold $\mathbb{P}(\mathbf{w})$ has a canonical V-bundle $K_{\mathbb{P}(\mathbf{w})}$ whose associated sheaf, the dualizing sheaf

$\kappa_{\mathbb{P}(\mathbf{w})}$, is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(-|\mathbf{w}|)$. Generally the adjunction formula does not hold, but if $\mathbf{w}$ is normalized we have

2.6 $$\kappa_{\mathcal{Z}_f} \cong \mathcal{O}_{\mathcal{Z}_f}(d-|\mathbf{w}|).$$

These sheaves are invertible on $\mathbb{P}^0(\mathbf{w})\cap\mathcal{Z}_f$ and we have

$$\kappa^{-1}_{\mathcal{Z}_f\cap\mathbb{P}^0(\mathbf{w})} \cong \mathcal{O}_{\mathcal{Z}_f\cap\mathbb{P}^0(\mathbf{w})}(|\mathbf{w}|-d).$$

Moreover, since $\mathcal{Z}_f$ is well-formed, $\mathcal{Z}_f\cap S(\mathbf{w})=\mathcal{Z}_f\cap\Sigma(\mathbf{w})$ has codimension ≥ 2 in $\mathcal{Z}_f$. It follows by a standard result (cf. [KMM] Lemma 0-1-10) that

$$\kappa^{-1}_{\mathcal{Z}_f}=\iota_*(\kappa^{-1}_{\mathcal{Z}_f\cap\mathbb{P}^0(\mathbf{w})}), \qquad \mathcal{O}_{\mathcal{Z}_f}(|\mathbf{w}|-d)=\iota_*(\mathcal{O}_{\mathcal{Z}_f\cap\mathbb{P}^0(\mathbf{w})}(|\mathbf{w}|-d)),$$

where $\iota:\mathcal{Z}_f\cap\mathbb{P}^0(\mathbf{w})\longrightarrow\mathcal{Z}_f$ is the natural inclusion. We then have

2.7 $$\kappa^{-1}_{\mathcal{Z}_f} \cong \mathcal{O}_{\mathcal{Z}_f}(|\mathbf{w}|-d).$$

This proves the result. ∎

There are further obstructions to the existence of a Sasakian-Einstein structure on L_f, namely those coming from the well-known failure of the zeroth order a priori estimate for solving the Monge-Ampere equation for the Calabi problem on $\mathcal{Z}_f$. Now the link L_f admits a compatible Sasakian-Einstein structure if and only if $\mathcal{Z}_f$ has a Kähler-Einstein structure ω' in the same cohomology class as $\omega_{\mathbf{w}}$. So the well-known obstructions in the Kähler-Einstein case, such as the non-reductiveness of the connected component of the complex automorphism group, or the vanishing of the Futaki invariant become obstructions to the existence of Sasakian-Einstein metrics. The problem of solving the Calabi problem in the positive case has drawn much attention in the last decade [Sui, Ti1, Ti2, Ti3, TY, DT, Nad] but still remains open. Here it suffices to consider three examples given recently by Demailly and Kollar [DK].

3. The Demailly-Kollár Examples

In a recent preprint Demailly and Kollár [DK] give a new derivation of Nadel's existence criteria for positive Kähler-Einstein metrics [Nad] which is valid for orbifolds. Furthermore their method of implementing Nadel's theorem does not depend on the existence of a large finite group of symmetries, but rather uses intersection inequalities. In this sense it is complementary to the work of Nadel [Nad] and others [Siu, Ti1-Ti4]. As an application of their method Demailly and Kollár [DK] construct three new del Pezzo orbifolds which admit Kähler-Einstein metrics. Explicitly, they prove

THEOREM 3.1 [DK]: *Let $\mathcal{Z}_f$ be a del Pezzo orbifold, and let f be given by one of the following three quasi-homogeneous polynomials of degree d whose zero set $\mathcal{Z}_f$ is a surface in the weighted projective space $\mathbb{P}(w_0,w_1,w_2,w_3)$:*

(1) $f=z_0^5z_1+z_0z_2^3+z_1^4+z_3^3$ *with* $\mathbf{w}=(9,15,17,20)$ *and* $d=60$.

(2) $f=z_0^{17}z_2+z_0z_1^5+z_1z_2^3+z_3^2$ *with* $\mathbf{w}=(11,49,69,128)$ *and* $d=256$.

(3) $f=z_0^{17}z_1+z_0z_2^3+z_1^5z_2+z_3^2$ *with* $\mathbf{w}=(13,35,81,128)$ *and* $d=256$.

Then $\mathcal{Z}_f$ admits a Kähler-Einstein orbifold metric.

We denote these del Pezzo surfaces by $\mathcal{Z}_{60}, \mathcal{Z}^{(1)}_{256}, \mathcal{Z}^{(2)}_{256}$, respectively. Actually more is true. By a result of Bando and Mabuchi [BM] (which also holds in the case of orbifolds) the Kähler-Einstein metric is unique up to complex automorphisms. Let us denote the corresponding links L_f of these del Pezzo surfaces by $L_{60}, L^{(1)}_{256}, L^{(1)}_{256}$, respectively. Due to the form of the polynomials as $f = g + z_3^m$, there is a well-known description of both $\mathcal{Z}_f$ and L_f in terms of branched covers (cf. [DuKa]). We briefly describe this geometry here. $\mathcal{Z}_{60}$ is a 3-fold cover of $\mathbb{P}(9, 15, 17)$ branched over the curve $C_{60} = \{z_0^5z_1 + z_0z_2^3 + z_1^4 = 0\}$. Similarly, the surfaces $\mathcal{Z}^{(i)}_{256}$ are 2-fold covers of $\mathbb{P}(11, 49, 69)$ and $\mathbb{P}(13, 35, 81)$, respectively, branched over the curves $C^{(1)}_{256} = \{z_0^{17}z_2 + z_0z_1^5 + z_1z_2^3 = 0\}$, and $C^{(2)}_{256} = \{z_0^{17}z_1 + z_0z_2^3 + z_1^5z_2 = 0\}$, respectively. In section 5 we show that in each case the genus of these curves is zero, so they are all $\mathbb{P}^1$'s (there are no quotient singularities in dimension one). Similarly, L_{60} is a 3-fold cyclic cover of S^5 branched over the Seifert manifold $K(4, 3, 5; IV)$, and $L^{(i)}_{256}$ are double covers of S^5 branched over the Seifert manifolds $K(17, 3, 5; V)$ and $K(17, 5, 3; V)$ for $i = 1, 2$, respectively. The notation $K(a_0, a_1, a_2; \cdot)$ is that of Orlik [Or1]. These Seifert manifolds are quite complicated. For example, they all have infinite fundamental group, but have finite abelianization H_1, and they all are Seifert fibrations over the Riemann sphere. As we shall see shortly the links $L^{(1)}_{256}$ and $L^{(2)}_{256}$ have the same characteristic polynomial although their Sasakian structures are distinct since their leaf holonomy groups are different. In particular they can be distinguished by their orders [BG1]. One easily finds by analyzing the orbifold singularity structure that $\mathrm{ord}(L^{(1)}_{256}) = 37191 = 3\cdot 7^2\cdot 11\cdot 23$, while $\mathrm{ord}(L^{(2)}_{256}) = 36855 = 3^4\cdot 5\cdot 7\cdot 13$. Now combining the Demailly-Kollár Theorem with Theorem 2.3 gives

THEOREM 3.2: *The simply connected 5-manifolds $L_{60}, L^{(i)}_{256}$ admit compatible Sasakian-Einstein metrics. Furthermore, these Sasakian-Einstein metrics are unique within the CR structure up to a CR automorphism.*

4. Link Invariants and the Milnor Fibration

Recall the well-known construction of Milnor [Mil] for isolated hypersurface singularities: There is a fibration of $(S^{2n+1} - L_f)\longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n \vee S^n \cdots \vee S^n$. The *Milnor number* μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $(w_0, \ldots, w_n)$ by the formula [MO]

4.1
$$\mu = \mu(L_f) = \prod_{i=0}^{n}\Big(\frac{d}{w_i} - 1\Big).$$

One immediately has

PROPOSITION 4.2: *The Milnor numbers of the simply connected Sasakian-Einstein 5-manifolds $L_{60}, L^{(1)}_{256}, L^{(2)}_{256}$ are*

$$\mu(L_{60}) = 86, \qquad \mu(L^{(i)}_{256}) = 255.$$

The closure $\bar{F}$ of F has the same homotopy type as F and is a compact manifold with boundary precisely the link L_f. So the reduced homology of F and $\bar{F}$ is only non-zero in dimension n and $H_n(F,\mathbb{Z}) \approx \mathbb{Z}^\mu$. Using the Wang sequence of the Milnor fibration together with Alexander-Poincare duality gives the exact sequence [Mil]

4.3 $$0 \longrightarrow H_n(L_f,\mathbb{Z}) \longrightarrow H_n(F,\mathbb{Z}) \xrightarrow{\mathbb{I}-h_*} H_n(F,\mathbb{Z}) \longrightarrow H_{n-1}(L_f,\mathbb{Z}) \longrightarrow 0,$$

where h_* is the *monodromy* map (or characteristic map) induced by the $S^1_{\mathbf{w}}$ action. From this we see that $H_n(L_f,\mathbb{Z}) = \ker(\mathbb{I}-h_*)$ is a free Abelian group, and $H_{n-1}(L_f,\mathbb{Z}) = \mathrm{coker}(\mathbb{I}-h_*)$ which in general has torsion, but whose free part equals $\ker(\mathbb{I}-h_*)$. So the topology of L_f is encoded in the monodromy map h_*. There is a well-known algorithm due to Milnor and Orlik [MO] for computing the free part of $H_{n-1}(L_f,\mathbb{Z})$ in terms of the characteristic polynomial $\Delta(t) = \det(t\mathbb{I}-h_*)$, namely the Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$. We now compute the characteristic polynomials $\Delta(t)$ for our examples.

Proposition 4.4: *The characteristic polynomials $\Delta(t)$ of the simply connected Sasakian-Einstein 5-manifolds $L_{60}, L^{(1)}_{256}, L^{(2)}_{256}$ are given by*

$$\begin{aligned}\Delta(t) = \quad &(t-1)^2(t^{14}+t^{13}+\cdots+1)(t^{15}+1)(t^{30}+1)(t^4+t^3+t^2+t+1)\\ &\times(t^5+1)(t^{10}+1)(t^4-t^2+1)(t^2-t+1)\end{aligned}$$

for L_{60} and

$$\Delta(t) = (t-1)(t^2+1)(t^4+1)(t^8+1)(t^{16}+1)(t^{32}+1)(t^{64}+1)(t^{128}+1)$$

for $L^{(1)}_{256}$ and $L^{(2)}_{256}$.

Proof: The Milnor and Orlik [MO] algorithm for computing the characteristic polynomial of the monodromy operator for weighted homogeneous polynomials is as follows: First associate to any monic polynomial F with roots $\alpha_1,\ldots,\alpha_k \in \mathbb{C}^*$ its divisor

$$\mathrm{div}\ F = <\alpha_1> + \cdots + <\alpha_k>$$

as an element of the integral ring $\mathbb{Z}[\mathbb{C}^*]$ and let $\Lambda_n = \mathrm{div}\ (t^n-1)$. The rational weights w_i' used in [MO] are related to our integer weights w_i by $w_i' = \frac{d}{w_i}$, and we write the $w_i' = \frac{u_i}{v_i}$ in irreducible form. So for the surface of degree 60 we have

$$w_0' = \frac{20}{3}, \quad w_1' = 4, \quad w_2' = \frac{60}{17}, \quad w_3' = 3.$$

Then by Theorem 4 of [MO] the divisor of the characteristic polynomial is

$$\mathrm{div}\ \Delta(t) = (\frac{\Lambda_{20}}{3}-1)(\Lambda_4-1)(\frac{\Lambda_{60}}{17}-1)(\Lambda_3-1).$$

Using the relations $\Lambda_a\Lambda_b = \gcd(a,b)\Lambda_{lcm(a,b)}$ we find in this case

4.5 $$\mathrm{div}\ \Delta(t) = \Lambda_{60}+\Lambda_{20}+\Lambda_{12}-\Lambda_4-\Lambda_3+1.$$

Then $\Delta(t)$ is obtained from the formula

$$\text{div } \Delta(t) = 1 - \sum \frac{s_j}{j} \Lambda_j$$

yielding

$$\Delta(t) = \frac{(t-1)}{\prod (t^j - 1)^{s_j/j}}.$$

We see from 4.5 that the only nonzero s_j's are

$$s_{60} = -60, \quad s_{20} = -20, \quad s_{12} = -12, \quad s_4 = 4, \quad s_3 = 3,$$

and $\Delta(t)$ becomes

$$\Delta(t) = \frac{(t-1)(t^{60}-1)(t^{20}-1)(t^{12}-1)}{(t^4-1)(t^3-1)}$$

which we see easily reduces the the expression given.

Similar computations for the two surfaces of degree 256 show that

4.6 $$\text{div } \Delta(t) = \Lambda_{256} - \Lambda_2 + 1,$$

giving the characteristic polynomial noted above. ∎

5. The Topology of L_f

Since the multiplicity of the root 1 in $\Delta(t)$ is precisely the second Betti number of L_f we have an immediate corollary of Proposition 4.4:

Corollary 5.1: *The second Betti numbers of the 5-manifolds $L_{60}, L_{256}^{(i)}$ are $b_2(L_{60}) = 2$, and $b_2(L_{256}^{(i)}) = 1$ for $i = 1, 2$.*

One can give another proof of this corollary by making use of a formula due to Steenbrink [Dim]. One considers the Milnor algebra

5.2 $$M(f) = \frac{\mathbb{C}[z_0, z_1, z_2, z_3]}{\theta(f)},$$

where $\theta(f)$ is the Jacobi ideal. $M(f)$ has a natural $\mathbb{Z}$ grading

$$M(f) = \oplus_{i \geq 0} M(f)_i,$$

and Steenbrink's formula determines the primitive Hodge numbers

5.3 $$h_0^{i,n-i-1} = \dim_{\mathbb{C}} M(f)_{(i+1)d-w}$$

of the projective surface $f = 0$ in the weighted projective space $\mathbb{P}(\mathbf{w})$. Then $b_2(L_f) = h_0^{1,1}$ can be computed by finding a basis of residue classes in $M(f)_{2d-|\mathbf{w}|}$.

Steenbrink's formula can also be used to compute the Hirzebruch signature τ of the orbifolds $\mathcal{Z}_f$ as well as the genus of the curves C_{60} and $C_{256}^{(i)}$ discussed at the end of the last section. We have

PROPOSITION 5.4: *The Fano orbifold $\mathcal{Z}_{60}$ has $\tau=-1$ while the orbifolds $\mathcal{Z}_{256}^{(i)}$ have $\tau=0$. The curves C_{60} and $C_{256}^{(i)}$ are all isomorphic to the projective line $\mathbb{P}^1$.*

PROOF: The formula for the signature is [Dim]

$$\tau = 1 + 2\mathrm{dim}_{\mathbb{C}} M(f)_{d-|\mathbf{w}|} - \mathrm{dim}_{\mathbb{C}} M(f)_{2d-|\mathbf{w}|},$$

whereas the genus of the curves is given by

$$g = h_0^{0,1} = \mathrm{dim}_{\mathbb{C}} M(f_0)_{d-|\mathbf{w}|},$$

where f_0 is a weighted homogeneous polynomial in z_0, z_1, z_2 which is related to f by $f = f_0 + z_3^{a_3}$. Now $d-|\mathbf{w}| = -1$ for all the surfaces and $\mathrm{dim}_{\mathbb{C}} M(f)_{2d-|\mathbf{w}|} = h_0^{1,1} = b_2(L_f)$ which is 2 for $\mathcal{Z}_{60}$ and 1 for $\mathcal{Z}_{256}^{(i)}$. Now for the curve C_{60} we have $d-|\mathbf{w}| = 19$, and one easily sees that $g = \mathrm{dim}_{\mathbb{C}} M(f_0)_{19} = 0$. For the curves $C_{256}^{(i)}$ we have $d-|\mathbf{w}| = 127$, so $g = \mathrm{dim}_{\mathbb{C}} M(f_0)_{127}$. For example, for $C_{256}^{(1)}$ one easily checks that there are no monomials of the form $z_0^a z_1^b z_2^c$ such that $11a+49b+69c = 127$. In both cases we find $g=0$. ∎

Next we turn to the calculation of the torsion in $H_2(L_f,\mathbb{Z})$. Actually we show that it is zero. We follow a method due to Randell [Ran] for computing the torsion of generalized Brieskorn manifolds (complete intersections of Brieskorn's) where he verified a conjecture due to Orlik [Or2] for this class of manifolds. Randell's methods apply to our more general weighted homogeneous polynomials case. For any $x \in L_f$ let Γ_x denote the isotropy subgroup of the $S^1(\mathbf{w})$-action on L_f and by $|\Gamma_x|$ its order. Following [Ran] for each prime p we define the *p-singular set* by

5.5 $$S_p = \{x \in L_f |\ p \text{ divides } |\Gamma_x|\}.$$

Then we have

LEMMA 5.6: *Suppose that for each prime p the p-singular set S_p is contained in a submanifold S of codimension 4 in L_f. Then the isomorphism holds:*

5.7 $$Tor(H_{n-1}(L_f,\mathbb{Z})) \approx Tor(H_{n-3}(S,\mathbb{Z})).$$

In particular, if the corresponding projective hypersurface $\mathcal{Z}_f$ is well-formed, the isomorphism 5.7 holds.

PROOF: The first statement is due to Randell [Ran]. To prove the second statement we notice that the set $\cup_p S_p$, where the union is over all primes, is precisely the subset of L_f, where the leaf holonomy groups are non-trivial and its projection to $\mathcal{Z}_f$ is just the orbifold singular locus $\Sigma^{orb}(\mathcal{Z}_f)$. Now $\mathcal{Z}_f \subset \mathbb{P}(\mathbf{w})$ and $\mathbf{w}$ is normalized, so the fact that $\mathcal{Z}_f$ is well-formed says that the orbifold singular locus $\Sigma^{orb}(\mathcal{Z}_f)$ has complex codimension at least two in $\mathcal{Z}_f$; hence, for each prime p, S_p has real codimension at least four in L_f. ∎

For the case at hand, $n=3$ so Randell's lemma says that $H_2(L_f,\mathbb{Z})$ is torsion free. Thus, we have arrived at:

LEMMA 5.8: *Let $L_f \subset \mathbb{C}^4$ be the link of an isolated singularity defined by a weighted homogeneous polynomial f in four complex variables. Suppose further that the del Pezzo surface $\mathcal{Z}_f \subset \mathbb{P}(\mathbf{w})$ is well-formed, then Tor $(H_2(L_f, \mathbb{Z})) = 0$.*

Now a well-known theorem of Smale [Sm] says that any simply connected compact 5-manifold which is spin, and whose second homology group is torsion free, is diffeomorphic to $S^5 \#_k (S^2 \times S^3)$ for some non-negative integer k. Furthermore, it is known [BG2,Mor] that any simply connected Sasakian-Einstein manifold is spin. Combining this with the development above gives

THEOREM 5.9: *Let L_f be the link associated to a well-formed weighted homogeneous polynomial f in four complex variables. Suppose also that L_f admits a Sasakian-Einstein metric. Then L_f is diffeomorphic to $S^5 \#_k (S^2 \times S^3)$, where k is the multiplicity of the root 1 of the characteristic polynomial $\Delta(t)$ of L_f.*

Let us now consider the links $L_{60}, L_{256}^{(i)}$ of Theorem 3.2 which admit Sasakian-Einstein metrics.

THEOREM 5.10: *The link L_{60} is diffeomorphic to $(S^2 \times S^3) \# (S^2 \times S^3)$ while the links $L_{256}^{(i)}$ for $i = 1, 2$ are diffeomorphic to the Stiefel manifold $S^2 \times S^3$. In particular, $S^2 \times S^3$ admits 3 distinct Sasakian-Einstein structures.*

PROOF: This follows from Smale's Theorem [Sm], Theorems 3.2, 5.9, and Corollary 5.1 as soon as we check that the weighted homogeneous polynomials are well-formed. But this follows easily from the definition and equation 2.4. ∎

6. Proofs of the Main Theorems and Further Discussion

Theorem A of the Introduction now follows immediately from Theorems 3.2 and 5.10. Similarly for Theorem B these two theorems give the existence of Sasakian-Einstein metrics on $S^2 \times S^3$. Furthermore, since the Sasakian structures are non-regular the metrics are inhomogeneous. The statement that the Sasakian-Einstein structures are inequivalent follows from the fact that their characteristic foliations are inequivalent (indeed, they have different orders). It remains to show that our Sasakian-Einstein metrics are inequivalent as Riemannian metrics to any of the metrics of Böhm. To see this we notice that for all of Böhm's metrics on $S^2 \times S^3$, the connected component of the isometry group $\mathfrak{I}_0(g)$ is $SO(3) \times SO(3)$ (see [Wa] Theorem 2.19). But a Theorem of Tanno [Tan] says that the connected component of the group of Sasakian automorphisms, $\mathfrak{A}_0(g)$ coincides with the connected component of the group of isometries $\mathfrak{I}_0(g)$, and $\mathfrak{A}_0(g)$ has the form $\mathfrak{G} \times S^1$, where the S^1 is generated by the Sasakian vector field ξ. Thus, our metrics are not isometric to the any of Böhm's metrics. ∎

Smale's Theorem actually says more than we have mentioned. It says that the torsion in H_2 must be of the form

5.11 $$\mathrm{Tor}(H_2(M, \mathbb{Z})) \approx \bigoplus (\mathbb{Z}_{q_i} \oplus \mathbb{Z}_{q_i}).$$

Thus we have

THEOREM 6.1: *Let M be a complete simply connected Sasakian-Einstein 5-manifold. Then the torsion group Tor$(H_2(M, \mathbb{Z}))$ must be of the form 5.11. In particular, the number of torsion generators must be even.*

It is a very interesting question whether the form of the torsion actually provides an obstruction to the existence of a Sasakian-Einstein structure or whether it is always satisfied. The general form of Randell's proof suggests that the torsion may always vanish, but we do not yet have a proof of this even in the case of hypersurfaces defined by weighted homogeneous polynomials when we drop the well-formed assumption. If there were an honest obstruction, it would provide a new type of obstruction to the existence of positive Kähler-Einstein metrics on del Pezzo orbifolds.

Finally we mention that many new Sasakian-Einstein structures can be constructed in higher dimension by applying the join construction introduced in [BG1] to our new examples. For example,

PROPOSITION 6.2: *The joins $S^3 \star L_{60}$ and $S^3 \star L^{(i)}_{256}$ are all smooth Sasakian-Einstein 7-manifolds. $S^3 \star L_{60}$ has the rational cohomology type of $S^2 \times \big((S^2 \times S^3)\#(S^2 \times S^3)\big)$, while $S^3 \star L^{(i)}_{256}$ have the rational cohomology type of $S^3 \star S^3 \star S^3$.*

PROOF: By Proposition 4.6 of [BG1] these joins are smooth when the Fano indices of the links are one. But it follows from the proof of Proposition 2.5 that $\kappa^{-1}(L_f) \approx \mathcal{O}(|\mathbf{w}| - d)$ and in all three cases we have $|\mathbf{w}| - d = 1$. The rational cohomology types can easily be seen from Theorem 5.22 of [BG1]. ∎

Many other examples can be worked out along the lines of [BG1]. However, what is perhaps a more interesting question is, for example, whether $S^3 \star L^{(1)}_{256}, S^3 \star L^{(2)}_{256}$ and $S^3 \star S^3 \star S^3$ have the same integral cohomology type, and if they do, are they homeomorphic (diffeomorphic)? We plan to study these types of questions in the future.

NOTE ADDED: The ideas of this paper have been developed much further in [BGN1,BGN2] where the existence of infinite families of Sasakian-Einstein metrics on the k-fold connected sum $k\#(S^2 \times S^3)$ is proven for $k = 2, \cdots, 9$. The moduli of such structures is also discussed.

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