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Compact 3-Sasakian 7-Manifolds with Arbitrary Second Betti Number

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Introduction

Amongst all Riemannian geometries the class of Einstein metrics stands out as perhaps the most natural and interesting [Bes]. Even so there are still many open questions about the relation between the topology of a compact manifold and the possible existence of Einstein metrics. In dimensions bigger than four almost nothing seems to be known in general. Yet, Einstein metrics on compact manifolds are relatively rare and they usually appear as part of additional geometric structure which makes their study tractable. In recent years the first three authors [BGM1,BGM2] have studied a class of Riemannian manifolds known as 3-Sasakian manifolds which have proven to be a remarkable source of compact Einstein manifolds of positive scalar curvature. In view of this work several seemingly unrelated questions regarding the possible breakdown of finiteness and Betti number bounds come to mind:

Question 1: Do there exist 3-Sasakian manifolds with second Betti number greater than one?

Question 2: Are there singular Fano varieties with arbitrarily large Picard number?

Question 3: Are there compact Einstein manifolds of positive scalar curvature with arbitrarily large total Betti number?

Question 4: Are there compact Einstein manifolds of positive scalar curvature which do not admit metrics of nonnegative sectional curvature?

These four questions are related and it is the purpose of this paper to give an affirmative answer to all of these questions by explicit construction. The first question is of interest in light of the recent bound of LeBrun [Le,LeSa] that for quaternionic Kähler manifolds of positive scalar curvature we must have $b_2 \leq 1$. It then follows [GS] that for regular 3-Sasakian manifolds $b_2 \leq 1$ also. The methods used to prove this bound relate to question 2 through twistor geometry. Indeed the bound follows from a result of Wiśniewski

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[Wi] about Fano manifolds of large index. Question 2 is of interest in its own right owing to the result of Mori and Mukai [MM] which says that for smooth Fano 3-folds $b_2 \leq 10$. That an affirmative answer to question 3 implies an affirmative answer to question 4 is a well known result of Gromov [Gro] that says that if a compact manifold admits a Riemannian metric of nonnegative sectional curvature then the total Betti number is bounded by a constant depending only on dimension. In fact, more is true when there is a bound on the diameter [Gro]. As the total Betti number grows some sectional curvatures must become more and more negative. The main result of our paper is:

THEOREM A: *Let k be a positive integer, and let $(\mathbf{a},\mathbf{b}) \in (\mathbb{Z}^*)^k \oplus (\mathbb{Z}^*)^k$ whose components (a^i, b^i) are pairs of relatively prime integers for $i = 1, \cdots, k$ that satisfy the condition that if for some pair i, j $a^i = \pm a^j$ or $b^i = \pm b^j$ then we must have $b^i \neq \pm b^j$ or $a^i \neq \pm a^j$, respectively. Then there exist 3-Sasakian manifolds $\mathcal{S}(\mathbf{a},\mathbf{b})$ of dimension 7 depending on $(\mathbf{a},\mathbf{b})$ such that $b_2(\mathcal{S}(\mathbf{a},\mathbf{b})) = k$. In particular, there exist simply connected Einstein 7-manifolds of positive scalar curvature with arbitrary second Betti number.*

Actually we construct a family of 3-Sasakian orbifolds depending on $\frac{k(k+5)}{2}$ integers, but there appears to be no general algorithm for determining when these orbifolds are smooth manifolds. It should also be emphasized that the existence of the manifolds $\mathcal{S}(\mathbf{a},\mathbf{b})$ is determined constructively, and that they are generalizations of the 3-Sasakian manifolds described in example 4.6 of [BGM1].

There are several important corollaries of Theorem A. The first follows immediately from Theorem A and Gromov's Theorem [Gro].

COROLLARY B: *There are infinitely many compact simply-connected Einstein 7-manifolds of positive scalar curvature, namely the $\mathcal{S}(\mathbf{a},\mathbf{b})$ of Theorem A, that do not admit metrics of nonnegative sectional curvature. Furthermore, for any negative real number κ there are infinitely many 3-Sasakian manifolds $\mathcal{S}(\mathbf{a},\mathbf{b})$ which do not admit metrics whose sectional curvatures are all greater than or equal to κ.*

Actually question 4 with the condition "Einstein manifold of positive scalar curvature" replaced by "nonnegative Ricci curvature" was problem 5 of Yau's famous problem section of the 1979-80 Princeton Seminar [Y]. This question was answered affirmatively in 1989 by Sha and Yang [SY], but to the best of the authors' knowledge our construction gives the first examples for Einstein manifolds of positive scalar curvature.

Another result relating to the breakdown of finiteness comes from a theorem of Anderson [An] which implies there is finite number of diffeomorphism types of Einstein manifolds of positive scalar curvature with a lower bound on the injectivity radius. Thus using our examples in [BGM2,BGM3] or those of Theorem A give:

COROLLARY C: *There are infinitely many 3-Sasakian 7-manifolds with arbitrarily small injectivity radius.*

It is interesting to compare the examples of Theorem A with our previous examples [BGM2] where in particular we constructed infinitely many 3-Sasakian manifolds that are homotopy inequivalent and which admit metrics of positive sectional curvature, but which by Anderson's theorem have arbitrarily small injectivity radii. On the other hand by Gromov's theorem, Anderson's theorem, and Theorem A, there are infinitely many 3-Sasakian manifolds with arbitrarily small injectivity radii which cannot admit any metric of nonnegative sectional curvature.

Our next corollary is a partial classification result. It follows immediately from Theorem A and results of [GS].

Corollary D: *In dimension seven there exist 3-Sasakian manifolds with every allowable rational homology type.*

It is known [BGM1,BGM2] that every 3-Sasakian manifold has two distinct homothety classes of Einstein metrics only one of which is 3-Sasakian. Furthermore, in dimension 7 both of these metrics have weak G_2 holonomy [GS,FKMS]. Thus, we have

Corollary E: *There exist 7-manifolds with arbitrary second Betti number having metrics of weak G_2 holonomy.*

In [BG] it was shown that the twistor space of any 3-Sasakian manifold has the structure of a $\mathbb{Q}$-factorial Fano variety, and this provides the link with question 2. Thus, results of [BG] and Theorem A give:

Corollary F: *There exist $\mathbb{Q}$-factorial contact Fano 3-folds X with $b_2(X) = l$ for any positive integer l.*

There is a well known relationship [BGM1,BG] between 3-Sasakian geometry on the one hand and both quaternionic Kähler geometry of positive scalar curvature and Fano contact geometry on the other (Here $l = k+1$ for the l in Corollary F and k in Theorem A). But in general this relationship involves Riemannian metrics with orbifold singularities for both the quaternionic Kähler and Fano geometries. It is the existence of these singularities that cause the violation of finiteness. This is most easily seen in the Fano contact case where Wiśniewski's theorem fails in the orbifold category since both the contact divisor and the anticanonical divisor are now $\mathbb{Q}$-divisors, and the singularity index can be arbitrarily high.

It is interesting to view our results from the perspective of holonomy. In the case of an Einstein n-manifold M with positive scalar curvature and irreducible special holonomy (*i.e.,* M is not locally symmetric and the restricted holonomy group is a proper subgroup of $SO(n)$ given by the Berger's classification theorem [Bes]), there are only two possibilities; M is Kähler-Einstein of positive scalar curvature, or M is quaternionic Kähler of positive scalar curvature. In both cases there is a finiteness theorem. The first is the boundedness of deformation types of Fano manifolds by Kollár, Miyaoka, and Mori [KMM], and the second is LeBrun's Finiteness theorem [Le] mentioned above. This contrasts with 3-Sasakian manifolds whose holonomy is always the full special orthogonal group $SO(4n-1)$ [GS].

Another known case where finiteness fails are the homogeneous manifolds of Aloff and Wallach [AlWa]. Moreover, they admit Einstein metrics of positive scalar curvature [Wa]. However, since they are homogeneous, b_2 is bounded (in fact, $b_2 = 1$ here). In our case we have:

COROLLARY G: *If the second Betti number* $b_2(\mathcal{S}(\mathbf{a},\mathbf{b})) = k > 3$, *the* 3*-Sasakian manifolds* $\mathcal{S}(\mathbf{a},\mathbf{b})$ *(of Theorem A) are not homotopy equivalent to any homogeneous space.*

Another question related to those mentioned above is whether there exist compact quaternionic Kähler orbifolds $\mathcal{O}$ of positive scalar curvature with arbitrary second Betti number. In dimension four, these spaces are compact, self-dual, Einstein orbifolds. To each 3-Sasakian manifold $\mathcal{S}(\Omega)$ there is a naturally associated quaternionic Kähler orbifold $\mathcal{O}(\Omega)$ [BGM1, BGM2].

THEOREM H: *Let* $\mathcal{O}(\mathbf{a},\mathbf{b})$ *be the compact, self-dual, Einstein orbifold associated to the 7-dimensional* 3*-Sasakian manifold* $\mathcal{S}(\mathbf{a},\mathbf{b})$ *given in Theorem A. Then*

$$b_2(\mathcal{O}(\mathbf{a},\mathbf{b})) = b_2(\mathcal{S}(\mathbf{a},\mathbf{b})) = k.$$

Hence, there are compact, self-dual, Einstein orbifolds of positive scalar curvature with arbitrary second Betti number.

We begin our constructions in section one by reviewing the 3-Sasakian reduction [BGM2] of the standard sphere S^{4n-1} by a k-torus action. Next, in section two, we determine necessary and sufficient conditions for the k-torus to act freely on the zero level set of the 3-Sasakian moment map. Moreover, we determine normal forms for these toral action.

Since the k-torus is a subtorus of the maximal torus in the group $Sp(n)$ of 3-Sasakian isometries, the reduced 3-Sasakian manifold $\mathcal{S}$ we construct admits an $(n-k)$-torus of 3-Sasakian isometries. Combining these isometries with the $Sp(1)$ group of 3-Sasakian isometries generated by the 3-Sasakian vector fields we obtain a larger group of 3-Sasakian isometries on our quotients. In particular, in the case when $n = k+2$, the resulting 3-Sasakian 7-manifolds $\mathcal{S}$ each have a five dimensional isometry group. Taking the quotient of $\mathcal{S}$ by this isometry group gives rise to a 2-dimensional quotient space with a cell structure represented by a 2-disc whose boundary is a $(k+2)$-gon. This in turn gives a natural stratification of $\mathcal{S}$ where the fibres are tori crossed with 3-dimensional lens spaces. This stratification is described in section three.

Then, in section four we use a Leray spectral sequence to determine the rational homology of $\mathcal{S}$ and thus prove Theorem A. Finally, Theorem H follows from a slightly modified version of the stratification argument used to prove Theorem A.

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§1. 3-Sasakian Reductions

In this section we review the 3-Sasakian reduction procedure [BGM2] which constructs new 3-Sasakian manifolds from a given one. The key to this construction is the quaternionic quotient described in [BGM1]. With each quaternionic Kähler space $\mathcal{O}$, in a canonical way, one associates three bundles: the twistor bundle $\mathcal{Z}$, the Swann bundle $\mathcal{U}$, and the Konishi bundle $\mathcal{S}$. The twistor bundle $\mathcal{Z}$ carries a complex contact structure and a Kähler-Einstein metric. The Swann bundle $\mathcal{U}$ is a hyperkähler manifold with a certain isometric $SU(2)$ action. Finally, the Konishi bundle $\mathcal{S}$ is a 3-Sasakian manifold. (In case $\mathcal{O}$ is a compact orbifold all the bundles exist as V-bundles.) The quaternionic Kähler reduction of [GL] has a canonical lift to three other reductions: the twistor space or *contact* reduction of Hitchin [Hi2], the hyperkähler reduction of [HKLR] and the 3-Sasakian reduction given in [BGM2] and reviewed in this section. In contrast to the better known quaternionic Kähler and hyperkähler quotients, it should be mentioned that this last reduction is effective in producing new compact and smooth manifolds. There is not a single known example of a hyperkähler reduction which leads to a compact manifold and in the quaternionic Kähler case such examples are extremely rare.

To begin let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a 3-Sasakian manifold and let $I_0(\mathcal{S}, g_{\mathcal{S}})$ denote the connected component of the group of 3-Sasakian isometries. By the embedding theorem [BGM2], $M = \mathcal{S} \times \mathbb{R}^+$ is a hyperkähler manifold with respect to the cone metric g_M. The group $I_0(\mathcal{S}, g_{\mathcal{S}})$ extends to a group $I_0(M, g_M) \cong I_0(\mathcal{S}, g_{\mathcal{S}})$ of isometries on M by defining each element to act trivially on $\mathbb{R}^+$. Furthermore, it follows that these isometries $I_0(M, g_M)$ preserve the hyperkähler structure on M. Recall from [HKLR] that any subgroup $G \subset I_0(M, g_M)$ gives rise to a hyperkähler moment map $\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$, where $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^*$ is its dual. Thus, we can define a 3-Sasakian moment map

1.1
$$\mu_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

by restriction $\mu_{\mathcal{S}} = \mu \,|\, \mathcal{S}$. We denote the components of $\mu_{\mathcal{S}}$ with respect to the standard basis of $\mathbb{R}^3$, which we have identified with the purely imaginary quaternions, by $\mu^a_{\mathcal{S}}$. Recall that ordinarily moment maps determined by Abelian group actions (in particular, those associated to 1-parameter groups) are only specified up to an arbitrary constant. This is not the case for 3-Sasakian moment maps since we require that the group $Sp(1)$ generated by the Sasakian vector fields ξ^a acts on the level sets of $\mu_{\mathcal{S}}$. In fact, there is a unique 3-Sasakian moment map $\mu_{\mathcal{S}}$ such that the zero set $\mu_{\mathcal{S}}^{-1}(0)$ is invariant under the group $Sp(1)$ generated by the vector fields ξ^a, and this moment map is given by the simple expression

1.2
$$< \mu^a_{\mathcal{S}}, \tau > = \frac{1}{2}\eta^a(X^\tau),$$

where $\tau \in \mathfrak{g}$ has associated vector field X^τ, and η^a denotes the 1-form dual to the vector field ξ^a. We now have:

REDUCTION THEOREM 1.3 [BGM2]: *Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a 3-Sasakian manifold with a compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries. Let $\mu_{\mathcal{S}}$ be the corresponding 3-Sasakian moment map and assume both that 0 is a regular value of $\mu_{\mathcal{S}}$ and that G acts freely on the submanifold $\mu_{\mathcal{S}}^{-1}(0)$. Furthermore, let*

$$\iota : \mu_{\mathcal{S}}^{-1}(0) \longrightarrow \mathcal{S}$$

and

$$\pi : \mu_{\mathcal{S}}^{-1}(0) \longrightarrow \mu_{\mathcal{S}}^{-1}(0)/G$$

denote the corresponding embedding and submersion. Then

$$(\check{\mathcal{S}} = \mu_{\mathcal{S}}^{-1}(0)/G, \check{g}_{\mathcal{S}}, \check{\xi}^a)$$

is a smooth 3-Sasakian manifold of dimension $4(n - \dim \mathfrak{g}) - 1$ with metric $\check{g}_{\mathcal{S}}$ and Sasakian vector fields $\check{\xi}^a$ determined uniquely by the two conditions

$$\iota^* g_{\mathcal{S}} \;=\; \pi^* \check{g}_{\mathcal{S}}$$

and

$$\pi_*(\xi^a \,|\, \mu_{\mathcal{S}}^{-1}(0)) \;=\; \check{\xi}^a.$$

In the next section we apply this theorem with $\mathcal{S} = S^{4n-1}$ and $G = T^k$ the k dimensional torus. We also need the following result concerning 3-Sasakian isometries.

PROPOSITION 1.4 [BGM2]: *Assume that the hypothesis of Theorem 1.3 holds. In addition assume that $(\mathcal{S}, g_{\mathcal{S}})$ is complete and hence compact. Let $C(G) \subset I_0(\mathcal{S}, g_{\mathcal{S}})$ denote the centralizer of G in $I_0(\mathcal{S}, g_{\mathcal{S}})$ and let $C_0(G)$ denote the subgroup of $C(G)$ given by the connected component of the identity. Then $C_0(G)$ acts on the submanifold $\mu_{\mathcal{S}}^{-1}(0)$ as isometries with respect to the restricted metric $\iota^* g_{\mathcal{S}}$ and the 3-Sasakian isometry group $I_0(\check{\mathcal{S}}, \check{g}_{\mathcal{S}})$ of the quotient $(\check{\mathcal{S}}, \check{g}_{\mathcal{S}})$ contains an isomorphic copy of $C_0(G)$.*

By abuse of notation we denote this isomorphic copy by $C_0(G)$. In the case at hand $G = T^k$, $C_0(T^k) = T^n$ the maximal torus of $Sp(n)$. Notice that $C_0(T^k)$ does not act effectively on the quotient. The normalizer of the maximal torus, namely, the Weyl group $\mathcal{W}(Sp(n))$ will also play a role when discussing equivalence between quotients in the next section.

We end this section with an example. The diagram below describes all the *homogeneous* circle quotients [Bat] (see [BGM2] for details and further examples):

1.5
$$\begin{tikzcd}
& \mathbb{H}^n \setminus \{0\} \arrow[ld, "\mathbb{R}^+"'] \arrow[rd, "\mathbb{C}^*"] \arrow[dd, "\mathbb{H}^*"] \arrow[rrr, "S^1"] & & & \mathcal{N} \arrow[ld, "t(\mathbb{C})"'] \arrow[rd, "t(\mathbb{R})"] \arrow[dd, "t(\mathbb{H})"] & \\
S^{4n-1} \arrow[rd] & & \mathbb{CP}^{2n-1} \arrow[ld] \arrow[r, "S^1"] & \mathcal{Z} \arrow[rd] & & \mathcal{S}, \arrow[ld] \\
& \mathbb{HP}^{n-1} \arrow[rrr, "S^1"] & & & \mathrm{Gr}_2(\mathbb{C}^n) &
\end{tikzcd}$$

where $\mathcal{S} = \frac{U(n)}{U(n-2)\times U(1)}$ and $\mathcal{Z}$ is the complex flag manifold $F_{n,2,1} = \frac{U(n)}{U(n-2)\times U(1)\times U(1)}$.

In the next section we describe a quotient construction of a large family of inhomogeneous examples by considering k-dimensional torus actions on spheres. On the one hand these generalize the inhomogeneous examples $\mathcal{S}(\mathbf{p})$ given in [BGM2] since $\mathcal{S}(\mathbf{p})$ is obtained by considering the most general 1-torus action. On the other hand the general toral quotients are very different as they are never homogeneous, whereas $\mathcal{S}(\mathbf{p})$ is actually homogeneous for $\mathbf{p} = \mathbf{1}$.

§2. Torus Actions on the 3-Sasakian Sphere

Let $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{H}^n$, be the quaternionic coordinates on the n-dimensional quaternionic vector space $\mathbb{H}^n$, equipped with the flat metric. Consider the unit sphere

$$S^{4n-1} = \{\mathbf{u} \in \mathbb{H}^n \quad | \quad \sum_{\alpha=1}^{n} \overline{u}_\alpha u_\alpha = 1\},$$

with its canonical metric g_{can} obtained from the flat metric by the inclusion $S^{4n-1} \subset \mathbb{H}^n$. This sphere has two natural 3-Sasakian structures determined by whether one treats $\mathbb{H}^n$ as a right or left quaternionic vector space. We choose the left module structure on $\mathbb{H}^n$ and this chooses the right 3-Sasakian vector fields ξ^a_r. The subgroup of the isometry group $O(n)$ of the (S^{4n-1}, g_{can}) that normalizes this structure is $Sp(n)\cdot Sp(1) = \big(Sp(n)\times Sp(1)\big)/\mathbb{Z}_2$, where the $Sp(1)$ is the group generated by the 3-Sasakian vector fields ξ^a_r. The group $Sp(n)\cdot Sp(1)$ acts on the sphere as:

$$Sp(n)\times Sp(1)\times S^{4n-1} \to S^{4n-1}$$

2.1
$$\big((\mathbb{A}, \sigma); \mathbf{u}\big) \longrightarrow \mathbb{A}\mathbf{u}\sigma^{-1},$$

where $\mathbb{A} \in Sp(n)$ is the quaternionic $n\times n$ matrix of the quaternionic representation of $Sp(n)$, and $\sigma \in Sp(1)$ is a unit quaternion. As the diagonal $\mathbb{Z}_2$ acts trivially we see that $Sp(n)\cdot Sp(1)$ acts on the sphere by isometries. The group $Sp(n)$ is the subgroup of the full isometry group which commutes with the 3-Sasakian $Sp(1)$ action, *i.e.,* we have $I_0(S^{4n-1}) = Sp(n)$. In this paper we shall consider the maximal torus $T^n \subset Sp(n)$ and its subgroups acting on S^{4n-1}.

Every quaternionic representation of a k-torus T^k on $\mathbb{H}^n$ can be described by a diagonal matrix of the form

2.2
$$\begin{pmatrix} \prod_{i=1}^k \tau_i^{a_1^i} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \prod_{i=1}^k \tau_i^{a_n^i} \end{pmatrix},$$

where $(\tau_1, .., \tau_k) \in S^1 \times \cdots \times S^1 = T^k$ are the complex coordinates on T^k, and $a_j^i \in \mathbb{Z}$. In turn this representation defines a $k \times n$ integral *weight* matrix

2.3
$$\Omega = \begin{pmatrix} a_1^1 & a_2^1 & \dots & a_k^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & \dots & a_k^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ a_1^k & a_2^k & \dots & a_k^k & \dots & a_n^k \end{pmatrix}.$$

Let $\{e_j\}_{j=1}^k$ denote the standard basis for $\mathfrak{t}_k^* \simeq \mathbb{R}^k$. Then the 3-Sasakian moment map [BGM2] $\mu_\Omega : S^{4n-1} \longrightarrow \mathfrak{t}_k^* \otimes \mathbb{R}^3$ defined by the action with Ω given by 2.3 is $\mu_\Omega = \sum_j \mu_\Omega^j e_j$ with

2.4
$$\mu_\Omega^j(\mathbf{u}) = \sum_\alpha \bar{u}_\alpha i a_\alpha^j u_\alpha.$$

We choose the complex structure in $\mathbb{H}^n$ determined by i and write the quaternionic coordinates in terms of complex coordinates $u_\alpha = z_\alpha + w_\alpha j$. The moment map then takes the form

2.4′
$$\mu_\Omega^j(\mathbf{z}, \mathbf{w}) = i \sum_\alpha a_\alpha^j (|z_\alpha|^2 - |w_\alpha|^2) + 2k \sum_\alpha a_\alpha^j \bar{w}_\alpha z_\alpha$$

where $k = ij$. We are interested in the zero set of this moment map. Under the appropriate conditions, 0 is a regular value of μ_Ω, so that $N(\Omega) = \mu_\Omega^{-1}(0)$ is a smooth compact submanifold of S^{4n-1} of codimension $3k$.

Assuming that $N(\Omega)$ is a smooth submanifold, we are interested in conditions that guarantee that the T^k-action is free on $N(\Omega)$. Actually this condition is a bit too strong, since the action may not be effective. So what we want is a free action after quotienting to an effective action. However, by considering an appropriate notion of equivalence, we shall show that any 'good' action will be equivalent to an effective one. Let us assume that no row has all zeroes since this would correspond to a T^{k-1} action, and consider the $\binom{n}{k}$ minor determinants

2.5 $$\Delta_{\alpha_1\dots\alpha_k} = \det\begin{pmatrix} a^1_{\alpha_1} & \dots & a^1_{\alpha_k} \\ \vdots & & \vdots \\ a^k_{\alpha_1} & \dots & a^k_{\alpha_k}\end{pmatrix}$$

obtained by deleting $n-k$ columns of Ω.

Definition 2.6: *Let $\Omega \in \mathcal{M}_{k\times n}(\mathbb{Z})$ satisfy*

(1) $\Delta_{\alpha_1\cdots\cdots\alpha_k} \neq 0, \forall \quad 1 \leq \alpha_1 < \cdots < \alpha_k \leq n.$

Suppose that condition (1) is satisfied and let g be the kth determinantal divisor, i.e. the gcd of all the k by k minor determinants $\Delta_{\alpha_1\cdots\cdots\alpha_k}$. Then Ω is said to be admissible if in addition we have

(2) $\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, \dots, \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \dots, \Delta_{\alpha_1\cdots\alpha_k}) = g$ *for all sequences* $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n.$

Next we discuss the notion of equivalent T^k-actions on S^{4n-1} and obtain a normal form for admissible weight matrices. We are free to change bases of the Lie algebra $\mathfrak{t}_k$. This can be done by the group of unimodular matrices $GL(k,\mathbb{Z})$. Moreover, if we fix a maximal torus T^n of $Sp(n)$, its normalizer, the Weyl group $\mathcal{W}(Sp(n)) \simeq \Sigma_n \rtimes (\mathbb{Z}_2)^n$, preserves the 3-Sasakian structure on S^{4n-1} and intertwines the T^k actions. Thus, there is an induced action of $GL(k,\mathbb{Z}) \times \mathcal{W}(Sp(n))$ on the set of weight matrices $\mathcal{M}_{k\times n}(\mathbb{Z})$. The group $GL(k,\mathbb{Z})$ acts on $\mathcal{M}_{k\times n}(\mathbb{Z})$ by matrix multiplication from the left, and the Weyl group $\mathcal{W}(Sp(n))$ acts by permutation and overall sign changes of the columns. Actually we want a slightly stronger notion of equivalence than that described above. If the ith row of Ω has a gcd d_i greater than one, then by reparameterizing the one-parameter subgroup $\tau_i' = \tau_i^{d_i}$ we obtain $\tau_i^{a^j_\alpha} = (\tau_i')^{b^i_\alpha}$ where $\gcd\{b^i_\alpha\}_\alpha = 1$. So the action obtained by using the matrix whose ith row is divided by its gcd d_i is the same as the original action. This is related to the kth determinantal divisor g. We say that a matrix Ω satisfying condition (1) of definition 2.6 is in *reduced* form (or simply *reduced*) if $g=1$. The following lemma says that it is sufficient to consider matrices in reduced form.

Lemma 2.7: *Every weight matrix Ω satisfying condition (1) of definition 2.6 is equivalent to a matrix in reduced form.*

Proof: By standard theory there are $U \in GL(k,\mathbb{Z})$ and $V \in GL(n,\mathbb{Z})$ such that

$$\Omega = USV \quad \text{where} \quad S = \begin{pmatrix} s_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & s_k & 0 & \dots & 0 \end{pmatrix}.$$

Here $s_1, \cdots, s_k$ are the invariant factors of the matrix Ω satisfying $s_i | s_{i+1}$ and $s_1 s_2 \cdots s_k = g$. The matrix SV is a k by n matrix with the gcd of the ith row equal to s_i. Furthermore, the kth determinantal divisor of the k by n matrix V' obtained by deleting the last $n-k$ rows of V is one, and so is in reduced form. ∎

Henceforth, we shall only consider matrices in reduced form.

Definition 2.8: *Let $\mathcal{A}_{k\times n}(\mathbb{Z})$ denote the subset of reduced admissible matrices in $\mathcal{M}_{k\times n}(\mathbb{Z})$.*

The subset $\mathcal{A}_{k\times n}(\mathbb{Z})$ is invariant under the action of $GL(k, \mathbb{Z}) \times \mathcal{W}(Sp(n))$, and the unordered set $\{|\Delta_s|\}$ of the absolute values of all k by k minor determinants are invariants of this subset. Let $m = \min_s\{|\Delta_s|\}$, and let Δ_m realize this minimum. Then by transforming Ω by an element of $\mathcal{W}(Sp(n))$ if necessary, we can arrange that the first k columns of Ω has minor determinant Δ_m.

Recall (See e.g. [AW]) that a Hermite normal form is a normal form for k by n matrices over principal ideal domains R under multiplication by $GL(k, R)$ from the left. We are working with the classical case $R = \mathbb{Z}$. For any nonzero integer a, we let $P(a) = \{0, 1, \cdots, |a|-1\}$ denote the set of residues modulo $|a|$. In our case of the $GL(k, \mathbb{Z})$-invariant subset $\mathcal{A}_{k\times n}(\mathbb{Z})$ the Hermite normal form takes the form:

2.9
$$\Omega = \begin{pmatrix} a_1^1 & a_2^1 & \dots & a_k^1 & \dots & a_n^1 \\ 0 & a_2^2 & \dots & a_k^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_k^k & \dots & a_n^k \end{pmatrix}$$

with $a_i^i > 0$, and if $j < i$ then $a_i^j \in P(a_i^i)$. From the standard theory we have

Proposition 2.10: *Every $\Omega \in \mathcal{A}_{k\times n}(\mathbb{Z})$ is equivalent by left multiplication by an element of $GL(k, \mathbb{Z})$ to a matrix of the form 2.9.*

The Hermite normal form is unique when transforming by $GL(k, \mathbb{Z})$, but in general there may be several Hermite normal forms when transforming by the larger group $GL(k, \mathbb{Z}) \times \mathcal{W}(Sp(n))$. For example, the weight matrices

2.11
$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

are inequivalent under both $GL(2, \mathbb{Z})$ and $\mathcal{W}(Sp(4))$ separately, but are equivalent under the product.

As described above we always choose a normal form such that the absolute value of the determinant of the first k by k minor is minimal. Thus, the $GL(k, \mathbb{Z}) \times \mathcal{W}(Sp(n))$-invariant subset $\mathcal{A}_{k\times n}(\mathbb{Z})$ decomposes as a disjoint union of invariant subsets labelled by minima $|\Delta_m|$, viz.

2.12 $$\mathcal{A}_{k\times n}(\mathbb{Z}) = \bigsqcup_{m=1}^{\infty} \mathcal{A}_{k\times n}(m;\mathbb{Z}).$$

As the subsets $\mathcal{A}_{k\times n}(m;\mathbb{Z})$ are somewhat intractable for $m > 1$, we focus our attention on the case $m = 1$. In this case the Hermite normal form gives

PROPOSITION 2.13: *Let Ω lie in the invariant subset $\mathcal{A}_{k\times n}(1;\mathbb{Z})$, then Ω is equivalent to a weight matrix of the form*

$$\Omega = \begin{pmatrix} 1 & 0 & \dots & 0 & a^1_{k+1} & \dots & a^1_n \\ 0 & 1 & \dots & 0 & a^2_{k+1} & \dots & a^2_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 1 & a^k_{k+1} & \dots & a^k_n \end{pmatrix},$$

where the all the entries of the last $n-k$ columns are non vanishing. Furthermore, if Δ_1 is unique, then this normal form is unique up to permutations and sign changes of the last $n-k$ columns.

Our interest in admissible weight matrices is the following:

THEOREM 2.14: *Let $\Omega \in \mathcal{A}_{k\times n}(\mathbb{Z})$, that is Ω is a reduced admissible weight matrix. Then the quotient space $N(\Omega)/T^k$ is a smooth 3-Sasakian manifold of dimension $4(n-k)-1$.*

The proof of this theorem follows directly from the Reduction Theorem 1.3 and the following lemma.

LEMMA 2.15: *Let $\Omega \in \mathcal{M}_{k\times n}(\mathbb{Z})$ satisfy condition (1) of definition 2.6, then*

(i) The isotropy subgroup $\Gamma \subset T^k$ of any point of $N(\Omega)$ is discrete.

(ii) 0 is a regular value of the moment map μ_Ω.

(iii) If in addition Ω is in reduced form and satisfies condition (2) of 2.6, then T^k acts freely on $N(\Omega)$.

PROOF: The 3-Sasakian version of a standard fact of symplectic reduction says that the image of the differential $D\mu_\Omega(p)$ of the moment map at $p \in N(\Omega)$ is $\mathfrak{g}_p^\perp \otimes \mathbb{R}^3$ where $\mathfrak{g}_p$ is the Lie algebra of the isotropy subgroup at p. So (i) and (ii) are equivalent. We prove (i). At most $n-k-1$ quaternionic coordinates u_j can vanish on $N(\Omega)$ for if $n-k$ of the u_j's were to vanish the moment map equation would become in matrix notation

$$\Omega_k \cdot \begin{pmatrix} \bar{u}_{\alpha_1} i u_{\alpha_1} \\ \vdots \\ \bar{u}_{\alpha_k} i u_{\alpha_k} \end{pmatrix} = 0$$

for some indices $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$ and k by k minor Ω_k. But by (1) of definition 2.6 Ω_k is invertible implying $\bar{u}_{\alpha_j} i u_{\alpha_j} = 0$ for $j = 1, \cdots k$. Then equation $2.4'$ implies that all the coordinates u_j vanish. But this is impossible since $N(\Omega) \subset S^{4n-1}$. Thus, there are $k+1$ quaternionic coordinates that do not vanish. Let $\{\alpha_j\}_{j=1}^{k+1}$ denote the indices of the non-vanishing coordinates. The equations that determine a fixed point of $N(\Omega)$ are

2.16 $$\prod_i \tau_i^{a_\alpha^i} = 1 \quad \text{for each } \alpha = \alpha_j,\ j = 1, \cdots k+1.$$

This system of $k+1$ equations in k unknowns τ_i can be linearized by taking logarithms. Writing $\tau_j = e^{2\pi i t_j}$ 2.16 becomes

2.17 $$\sum_i a_\alpha^i t_i = r_\alpha$$

for each $\alpha = \alpha_j$ and some integers r_{α_j}. Since all minor determinants are non vanishing this system has only a discrete set of solutions. Hence, the isotropy groups are discrete proving (i).

Now suppose that Ω is in reduced form and satisfies condition (2) of definition 2.6, *i.e.*, $\Omega \in \mathcal{A}_{k\times n}(\mathbb{Z})$. From above at most $n-k-1$ of the quaternionic coordinates can vanish, so there must be at least $k+1$ non-vanishing coordinates u_{α_j}. The equations determining the fixed points of T^k are Eq. 2.16 which must hold for all α such that $u_\alpha \neq 0$. Again let Ω_k be any k by k minor of Ω and let C_k denote the cofactor matrix of Ω_k. Then inverting 2.17 gives

$$\Delta_k t_i = (C_k \cdot \mathbf{r})_i,$$

where Δ_k is the determinant of Ω_k and $\mathbf{r} = (r_1, \cdots, r_n) \in \mathbb{Z}^n$. Since the matrix C_k is integer valued this implies

2.18 $$\tau_i^{\Delta_k} = 1$$

for all minor determinants Δ_k. But since the gcd of all minor determinants is 1, this implies $\tau_i = 1$ for all $i = 1, \cdots k$. This proves the lemma, and thus Theorem 2.14. ∎

Remark 2.19: If condition (1) of definition 2.6 holds, but condition (2) fails, then the quotient $N(\Omega)/T^k$ will be a 3-Sasakian orbifold, but not a smooth manifold.

In general Theorem 2.14 is not yet an existence theorem, since $\mathcal{A}_{k\times n}(\mathbb{Z})$ could be empty. Indeed, we shall show in a forthcoming work that for each k there is an upper bound on n in order that admissible weight matrices exist. However, in this work we are only concerned with the case $n = k+2$ when the 3-Sasakian quotient is a 7-manifold. So we now restrict ourselves to this case. Our next result determines precisely the admissible weight matrices in the subset $\mathcal{A}_{k\times k+2}(1;\mathbb{Z})$. Any such matrix takes the form

2.20 $$\Omega = \begin{pmatrix} 1 & 0 & \ldots & 0 & a^1 & b^1 \\ 0 & 1 & \ldots & 0 & a^2 & b^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a^k & b^k \end{pmatrix}.$$

In this case the admissibility conditions simplify. The remaining subsets $\mathcal{A}_{k\times k+2}(m;\mathbb{Z})$ for $m\neq 1$ are more difficult to treat. For low values of k it is not difficult to construct admissible weight matrices by hand, but there appears to be no general algorithm.

PROPOSITION 2.21: *Let $\Omega\in\mathcal{A}_{k\times k+2}(1;\mathbb{Z})$ be in the normal form 2.20, then Ω is admissible if and only if $\gcd(a^j,b^j)=1$ for all $j=1,\cdots,k$, and if for any pairs $a^i=\pm a^j$ or $b^i=\pm b^j$ then we must have $b^i\neq\pm b^j$ or $a^i\neq\pm a^j$, respectively.*

PROOF: To check admissibility we need to select $k+1$ columns and then look at the gcd of the resulting $k+1$ minor determinants. If we select the first $k+1$ columns or the $k+1$ columns obtained by deleting column $k+1$, the gcd of the corresponding $k+1$ minor determinants is identically 1. The remaining conditions are all obtained by deleting one of the first k columns and then considering the corresponding $k+1$ k by k minor determinants. So the conditions for admissibility imply that $\det\begin{pmatrix} a^i & b^i\\ a^j & b^j\end{pmatrix}\neq 0$ for all $1\leq i<j\leq k$, and that

$$\gcd(a^j,b^j,\det\begin{pmatrix} a^1 & b^1\\ a^j & b^j\end{pmatrix},\cdots,\det\begin{pmatrix} a^j & b^j\\ a^k & b^k\end{pmatrix})=1$$

for all $j=1,\cdots k$, where of course the term $\det\begin{pmatrix} a^j & b^j\\ a^j & b^j\end{pmatrix}$ is deleted. The gcd condition is satisfied if and only if $\gcd(a^j,b^j)=1$ for all $j=1,\cdots,k$, and then the remaining conditions follow easily from the non vanishing of the corresponding 2 by 2 minor determinants. ∎

EXAMPLE 2.22 The following example was first introduced in [BGM1] and it is related to Kronheimer's construction of the ALE-spaces for $\Gamma=\mathbb{Z}_{k+1}$. Consider the weight matrix

2.23
$$\Omega(\mathbf{a})=\begin{pmatrix} 1 & -1 & 0 & \dots & 0 & a_1\\ 0 & 1 & -1 & \dots & 0 & a_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & -1 & a_k\end{pmatrix},$$

where $\mathbf{a}=(a_1,...,a_k)$. Clearly $\Omega(\mathbf{a})\in\mathcal{A}_{k\times k+2}(1;\mathbb{Z})$ and the corresponding 3-Sasakian quotient $\mathcal{S}(\Omega(\mathbf{a}))$ is 7-dimensional. Note that $\Omega(\mathbf{a})$ is equivalent to a matrix of the form 2.20 with $a^i=-1$ and $b^i=a_i+\cdots+a_k$. Condition (2) of the Definition 2.6 is now automatically satisfied. Condition (1) means that $\mathbf{a}$ must be chosen away from a union of $\frac{1}{2}k(k+1)$ hyperplanes in $\mathbb{R}^k$ defined by

2.24
$$\forall\ j=0,1,...,k-1\quad \forall i\leq k-j\qquad a_i+a_{i+1}+\cdots+a_{i+j}\neq 0.$$

The construction of $\mathcal{S}(\Omega(\mathbf{a}))$ is related to the quotient construction of the gravitational instantons of Gibbons and Hawking [GH] and further analyzed by Hitchin [Hi1]. The non-Abelian case is due to Kronheimer [Kr]. We briefly describe the quotient construction

[HKLR] as described by Kronheimer [Kr] for general Γ and then in more details for $\Gamma = \mathbb{Z}_{k+1}$. For any discrete subgroup Γ of $SU(2)$ Kronheimer introduces a subgroup $G(\Gamma) \subset U(|\Gamma|)$ of dimension $|\Gamma|-1$ which then gives a linear action on the model flat hyperkähler manifold, the quaternionic vector space $\mathbb{H}^{|\Gamma|}$. Now $G(\Gamma) \subset U(|\Gamma|)$ and acts on $\mathbb{H}^{|\Gamma|}$ by hyperkähler isometries. Hence we obtain the hyperkähler moment map

$$\mu(\Gamma) : \mathbb{H}^{|\Gamma|} \longrightarrow \mathcal{G}^*(\Gamma) \otimes \mathbb{R}^3,$$

where $\mathcal{G}^*(\Gamma)$ is the dual of the Lie algebra of $G(\Gamma)$. Let $T(\Gamma) \subset G(\Gamma)$ be the center of $G(\Gamma)$. Now, if $\xi \in \mathcal{T}^*(\Gamma) \otimes \mathbb{R}^3$ is in the so-called "good set" then the hyperkähler quotient $M(\Gamma, \xi) = \mu(\Gamma)^{-1}(\xi)/G(\Gamma)$ is a 4-dimensional hyperkähler manifold. Let us review the $\Gamma = \mathbb{Z}_{k+1}$ case. Then $G(\Gamma) = T^k \subset SU(k+1) \subset U(k+1)$ is just the maximal torus in $SU(k+1)$ acting on $\mathbb{H}^{k+1}$ as $\mathbb{A}(\tau)$ matrix multiplication from the left, where

$$\mathbb{A}(\tau) = \begin{pmatrix} \tau_1 & 0 & 0 & \dots & 0 \\ 0 & \overline{\tau}_1\tau_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \overline{\tau}_k \end{pmatrix},$$

and $\tau = (\tau_1, ..., \tau_k) \in T^k$. The moment map for this action can be easily computed and one gets

$$\mu^j(\mathbf{u}) = \overline{u}_j i u_j - \overline{u}_{j+1} i u_{j+1}, \qquad j = 1, ..., k,$$

with $\mu = \sum_j \mu^j e_j$, where $\{e_j\}_{j=1}^k$ denote the standard basis for $\mathfrak{t}_k^* \simeq \mathbb{R}^k$.

Let us denote by $\mu_k^{-1}(\xi)$ the inverse image of some element $\xi = (\xi_1, ..., \xi_k) \in \mathfrak{t}_k^* \otimes \mathbb{R}^3$. We say that ξ is in a *good set* if conditions 2.24 with a_i replaced by ξ_i are satisfied. It can be easily checked (see also [BD] for more general cases of toral quotients) that T^k acts freely on the level set $\mu_k^{-1}(\xi)$ if and only if ξ is in the good set. The quotient space $M(\mathbb{Z}_{k+1}, \xi)$ is a simply-connected hyperkähler manifold diffeomorphic to the minimal resolution of Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_{k+1}$. The first and second integral cohomology groups are $H^1(M(\mathbb{Z}_{k+1}, \xi); \mathbb{Z}) = 0$ and $H^2(M(\mathbb{Z}_{k+1}, \xi); \mathbb{Z}) = \oplus^k \mathbb{Z}$.

Clearly, the 3-Sasakian quotient $\mathcal{S}(\Omega(\mathbf{a}))$ is a modification of the above construction and it was first discussed in [GN]. Namely, choose $\mathbf{a} = (a_1, ..., a_k) \in \mathbb{Z}^k$ and consider a homomorphism $h_{\mathbf{a}} : T^k \longrightarrow T^1$, given by

$$h_{\mathbf{a}}(\tau) = \prod_{j=1}^k \tau_j^{a_j}.$$

This allows one to extend Kronheimer's action of the T^k on $\mathbb{H}^{k+1}$ to another action of the T^k on $\mathbb{H}^{k+2}$ as follows:

$$\big(\tau;\ (\mathbf{u}, u_{k+2})\big) \longrightarrow \big(\mathbb{A}(\tau)\mathbf{u}, h_{\mathbf{a}}(\tau) u_{k+2}\big), \qquad \mathbf{u} \in \mathbb{H}^{k+1}, u_{k+2} \in \mathbb{H},$$

and the hyperkähler moment map of this action restricted to the S^{4k+7} gives the corresponding 3-Sasakian moment map and we get $\mathcal{S}(\Omega(\mathbf{a}))$.

§3. A Stratification of $\mathcal{S}(\Omega)$

In this section we describe a stratification of the orbit space $\mathcal{S}(\Omega) = N(\Omega)/T^k$ in the case $n = k+2$, that is $\dim \mathcal{S}(\Omega) = 7$. To do this we consider the space $Q(\Omega)$ defined as the space of orbits of $N(\Omega)$ by the isometry group $T^n \cdot Sp(1)$. For the moment we consider general n. Let $\Omega \in \mathcal{A}_{k\times n}(\mathbb{Z})$ and consider the manifold $N(\Omega)$. It follows from Proposition 1.4 that the groups $T^n \times Sp(1)$ and $T^{n-k} \times Sp(1)$ act as isometry groups on $N(\Omega)$ and $\mathcal{S}(\Omega)$, respectively. Let $Q(\Omega)$ denote orbit space $N(\Omega)/T^n \times Sp(1)$. We have the following commutative diagram

3.1
$$\begin{tikzcd} & N(\Omega) \arrow[dl] \arrow[dr] & \\ Q(\Omega) & & \mathcal{S}(\Omega). \arrow[ll] \end{tikzcd}$$

The southeast arrow is a principal bundle with fibres T^k, but the other arrows are not fibrations. The southwest arrow has generic fibres $T^n \cdot Sp(1) \simeq T^n \times S^3$, while the west arrow has generic fibres $T^{n-k} \cdot Sp(1)$ homeomorphic either to $T^{n-k} \times \mathbb{RP}^3$ or $T^{n-k} \times S^3$ depending on Ω. The dimension of the orbit space $Q(\Omega)$ is $3(n-k)-4$, and by analyzing this diagram we shall obtain very simple stratifications of both $N(\Omega)$ and $\mathcal{S}(\Omega)$ in the case $n = k+2$, when $\dim Q(\Omega) = 2$. This stratification is related to the stratification by orbit types [Bre], but is cruder and will provide the quotient $Q(\Omega)$ with a very simple CW decomposition. Let us define the following subsets of $N(\Omega)$: (Recall from the proof of Lemma 2.15 that in the case $n = k+2$ at most one quaternionic coordinate can vanish.)

3.2
$$\begin{aligned} N_0(\Omega) &= \{\mathbf{u} \in N(\Omega) \mid u_\alpha = 0 \text{ for some } \alpha = 1, \cdots, k+2\}, \\ N_1(\Omega) &= \{\mathbf{u} \in N(\Omega) \mid \text{for all } \alpha = 1, \cdots, k+2,\ u_\alpha \neq 0 \text{ and} \\ &\qquad \text{there is a pair } (u_\alpha, u_\beta) \text{ that lies on the same} \\ &\qquad \text{complex line in } \mathbb{H}\}, \\ N_2(\Omega) &= \{\mathbf{u} \in N(\Omega) \mid \text{for all } \alpha = 1, \cdots, k+2,\ u_\alpha \neq 0 \text{ and} \\ &\qquad \text{no pair } (u_\alpha, u_\beta) \text{ lies on the same complex line} \\ &\qquad \text{in } \mathbb{H}\}. \end{aligned}$$

Here the strata are labeled by the dimension of the cells in the resulting CW decomposition of $Q(\Omega)$. Clearly, $N(\Omega) = N_0(\Omega) \sqcup N_1(\Omega) \sqcup N_2(\Omega)$ and $N_2(\Omega)$ is a dense open submanifold of $N(\Omega)$.

Lemma 3.3: *Let $n = k+2$ and $\Omega \in \mathcal{A}_{k\times k+2}(\mathbb{Z})$, and let $\rho_N : N(\Omega) \longrightarrow Q(\Omega)$ denote the natural projection.*

(i) The orbits in $N_2(\Omega)$ are $T^{k+2} \cdot Sp(1) \simeq T^{k+2} \times S^3$, and ρ_N restricted to $N_2(\Omega)$ is a principal $T^{k+2} \cdot Sp(1)$ bundle over a two dimensional base space $Q_2(\Omega)$.

(ii) The orbits in $N_1(\Omega)$ are $T^{k+1} \cdot Sp(1) \simeq T^{k+1} \times S^3$.

(iii) The orbits in $N_0(\Omega)$ are $T^k{\cdot}Sp(1) \simeq T^k \times S^3$.

The $\simeq$ above indicates diffeomorphism.

PROOF: The action of $T^{k+2} \times Sp(1)$ on $\mathbb{H}^{k+2}$ is given in quaternionic coordinates u_α by $u_\alpha \mapsto e^{i\theta_\alpha} u_\alpha q$ for $\alpha = 1, \cdots, k+2$, where $0 \leq \theta_\alpha \leq 2\pi$ and $q \in Sp(1)$. This action clearly restricts to the zero set of the moment map $N(\Omega)$. The action is not effective since the element $e^{i\theta_\alpha} = -1, q = -1$ acts as the identity on $\mathbb{H}^{k+2}$. We do get an effective action of the factor group $T^{k+2}{\cdot}Sp(1)$.

In the description of $N_2(\Omega)$ and $N_1(\Omega)$ we can use the $Sp(1)$ action to choose a complex line in $\mathbb{C}^2$ for a chosen quaternionic coordinate u_α. For example writing $u_\alpha = z_\alpha + w_\alpha j$, we choose $u_{k+2} = z_{k+2}$ or equivalently $w_{k+2} = 0$. Then 3.2 becomes

$$
\begin{aligned}
&N_2(\Omega) = \{\mathbf{u} \in N(\Omega) |\ u_\alpha \neq 0 \text{ for all } \alpha = 1, \cdots, k+2, \text{ and } z_\alpha w_\alpha \neq 0 \text{ for some } \alpha\}, \\
3.4 \quad &N_1(\Omega) = \{\mathbf{u} \in N(\Omega) |\ \text{for all } \alpha = 1, \cdots, k+2,\ u_\alpha \neq 0 \text{ and } z_\alpha w_\alpha = 0\}, \\
&N_0(\Omega) = \{\mathbf{u} \in N(\Omega) |\ u_\alpha = 0 \text{ for some } \alpha = 1, \cdots, k+2\}.
\end{aligned}
$$

We show that $T^{k+2}{\cdot}Sp(1)$ acts freely on $N_2(\Omega)$. The subgroup of $Sp(1)$ that stabilizes the complex line determined by $w_{k+2} = 0$ is an S^1 given by multiplication from the right, and the stabilizer in the full isometry group is $G = T^{k+2}{\cdot}S^1$. In this case the action of G on $N_2(\Omega)$ is given by

$$
3.5 \qquad z_\alpha \mapsto e^{i(\theta_\alpha + \phi)} z_\alpha, \qquad w_\alpha \mapsto e^{i(\theta_\alpha - \phi)} w_\alpha,
$$

where $e^{i\phi}$ parameterizes the S^1 from the right. At any point of $N_2(\Omega)$ there is an α such that $z_\alpha w_\alpha \neq 0$. So if this point is fixed 3.5 forces either the identity or $e^{i\phi} = -1$ and $e^{i\theta_\alpha} = -1$. But then since for all $\beta = 1, \cdots, k+2$ either z_β or w_β is non vanishing we get that either $e^{i\theta_\beta} = 1$ or $e^{i\theta_\beta} = e^{\pm i\phi} = -1$. Thus, $T^{k+2}{\cdot}Sp(1)$ acts freely on $N_2(\Omega)$, and restricting ρ_N to this stratum gives a principal $T^{k+2}{\cdot}Sp(1)$ bundle.

Now consider the orbits in $N_1(\Omega)$. Again setting $w_{k+2} = 0$ we have the stabilizer $G = T^{k+2}{\cdot}S^1$ and Eq.3.5 gives that $e^{i\theta_{k+2}} = e^{-i\phi}$. Now since for each α precisely one of either z_α or w_α is non vanishing, Eq.3.5 implies that for each $\alpha = 1, \cdots, k+1$ we have $e^{i\theta_\alpha} = e^{\pm i\phi}$. Hence, the isotropy group at every point of $N_1(\Omega)$ is an S^1, and the inclusion map $\iota : S^1 \longrightarrow T^{k+2}{\cdot}S^3$ has the following form: On the first factor ι has the form $\tau \mapsto (\tau^{\pm 1}, \cdots, \tau^{-1})$ where the signs are determined by certain choices. On the second factor ι is the standard inclusion in the Hopf fibration. It follows that the fibres in $N_1(\Omega)$ are $(T^{k+2}{\cdot}S^3)/S^1 \simeq T^{k+1}{\cdot}S^3$.

Next we analyze the orbits in $N_0(\Omega)$. Any $\Omega \in \mathcal{A}_{k\times k+2}(\mathbb{Z})$ has the form 2.20 with $n = k+2$. As far as the equations $2.4'$ describing $N(\Omega)$ is concerned we can pass from $\mathbb{Z}$ to its field of fractions $\mathbb{Q}$. So for the equations $2.4'$ we can choose Ω to be in the form 2.20 by inverting the first k by k block by an element of $GL(k, \mathbb{Q})$. Now as before if $u_{k+2} \neq 0$, we can use the $Sp(1)$ action to set $w_{k+2} = 0$. If u_{k+2} is the coordinate that vanishes do this for u_{k+1}. Then our transformed equations for the moment map take the form

3.6
$$
\begin{aligned}
|z_j|^2 - |w_j|^2 + f_j(|z_{k+1}|^2 - |w_{k+1}|^2) + g_j|z_{k+2}|^2 &= 0 \\
\bar{w}_j z_j + f_j \bar{w}_{k+1} z_{k+1} &= 0
\end{aligned}
$$

for $j = 1, \cdots, k$ and $f_j, g_j \in \mathbb{Q} - \{0\}$. The second equation immediately implies that with the choices made on $N(\Omega)$:

(1) If $z_\alpha w_\alpha = 0$ for some α it vanishes for all α.

From the proof of Lemma 2.15 at most one quaternionic coordinate u_α can vanish. So suppose that $u_\alpha = 0$ then the circle $e^{i\theta_\alpha}$ contributes an S^1 to the isotropy subgroup of any point of $N_0(\Omega)$. Moreover, since (1) above holds the analysis for orbits in $N_1(\Omega)$ above shows that the isotropy subgroup of any point in $N_0(\Omega)$ is the 2-torus T^2, so analyzing as before shows that these orbits are $T^k \cdot S^3$. ∎

Our next lemma relates our stratification with the finer stratification by orbit types [Bre].

Lemma 3.7: *Assuming the hypothesis of Lemma 3.3, the stratification $N(\Omega)$ by orbit types refines the stratification defined by 3.2 as follows:*

(i) $N_2(\Omega)$ is precisely the stratum of principal orbits.

(ii) $N_1(\Omega)$ is disconnected and decomposes into components $\sqcup_{\alpha=1}^{k+2} N_1(\alpha, \Omega)$, and each component consists of a single orbit type.

(iii) $N_0(\Omega)$ consists of $k+2$ disjoint copies of the group $T^k \cdot S^3$, and each component is an orbit type.

(iv) There are no exceptional orbits.

Proof: (i) and (iv) are obvious from Lemma 3.3. Consider the orbits in $N_0(\Omega)$. Recall that at most one quaternionic coordinate u_α can vanish. Suppose $u_{k+2} \neq 0$ (if this is the coordinate that vanishes interchange the role of u_{k+1} and u_{k+2}). As before we analyze the moment map equations by considering a slice of the $Sp(1)$ action yielding Eqs. 3.4. We need to consider various cases. We assume that both $f_j, g_j > 0$. (The other cases are handled similarly.) First, suppose that $u_{k+1} = 0$ and choose $w_{k+2} = 0$ as before, then Eqs. 3.6 imply $w_j \neq 0, z_j = 0$ for $j = 1, \cdots, k$. Moreover, the first of Eqs. 3.6 together with the sphere constraint determine the moduli

3.8
$$
|z_{k+2}|^2 = \frac{1}{1 + \sum_j g_j}, \qquad |w_j|^2 = \frac{g_j}{1 + \sum_j g_j}.
$$

The only free parameters are the phases of w_j and these determine the T^k of the fibre. So there is a single orbit with fibre $T^k \cdot S^3$. A similar analysis can be done at the vertices

$u_j = 0$. In this case which of the coordinates z_i, w_i vanish depends on the sign of $g_i f_j - f_i g_j$, and one sees again that k phases are undetermined. This proves (iii).

Now consider orbits in $N_1(\Omega)$. Again we use the form 3.4 and set $u_{k+2} = z_{k+2} \neq 0$. The isotropy subgroup $G_{\mathbf{u}} \simeq S^1$ only depends on whether w_α or z_α vanish (recall from (1) above that the product vanishes for all α). Choosing this for each α chooses a component of $N_1(\Omega)$, and this component is a trivial principal $T^{k+1}{\cdot}S^3$ bundle. Hence, on any component all fibres are equivalent. We could do a similar, albeit more tedious analysis than above, to count the number of components. However, it will follow easily from Lemma 3.11 below that there are precisely $k+2$ components. ∎

Since T^{k+2} is the center of $T^{k+2}{\cdot}Sp(1)$, diagram 3.1 and Lemmas 3.3 and 3.7 imply

Lemma 3.9: *Let $n = k+2$ and $\Omega \in \mathcal{A}_{k\times k+2}(\mathbb{Z})$, and consider the natural projection $\rho_{\mathcal{S}} : \mathcal{S}(\Omega) \longrightarrow Q(\Omega)$. Then there is a stratification $\mathcal{S}(\Omega) = \mathcal{S}_0(\Omega) \sqcup \mathcal{S}_1(\Omega) \sqcup \mathcal{S}_2(\Omega)$ such that*

(i) The orbits in $\mathcal{S}_2(\Omega)$ are precisely the principal orbits of type

$$\frac{T^{k+2}{\cdot}Sp(1)}{T^k} = G(\Omega).$$

Moreover, $H_*(G(\Omega);\mathbb{Q}) \cong H_*(T^2 \times Sp(1);\mathbb{Q})$.

(ii) The orbits in $\mathcal{S}_1(\Omega)$ are of the form $(G(\Omega)/S^1)$ and $\mathcal{S}_1(\Omega)$ decomposes into components $\sqcup_{\alpha=1}^{k+2}\mathcal{S}_1(\alpha,\Omega)$, where each component is an orbit type. Furthermore, the Weyl group $\mathcal{W}(Sp(k+2))$ acts transitively on the set of components.

(iv) $\mathcal{S}_0(\Omega)$ consists of $k+2$ disjoint copies of the form $(G(\Omega)/T^2)$ and each component is an orbit type. Again the Weyl group $\mathcal{W}(Sp(k+2))$ acts transitively on the set of components.

(v) There are no exceptional orbits.

Remark 3.10: The group $G(\Omega)$ defined in Lemma 3.9 is isomorphic to either $T^2 \times Sp(1)$ or $T^2 \times SO(3)$. For example, if Ω has the form of 2.20, we see that

$$G(\Omega) \simeq \begin{cases} T^2 \times SO(3) & \text{if } \sum a^i \text{ and } \sum b^i \text{ are both odd} \\ T^2 \times Sp(1) & \text{otherwise.} \end{cases}$$

Next we analyze the quotient space $Q(\Omega)$. The stratification given by 3.2 induces a stratification of the quotient, namely $Q(\Omega) = Q_0(\Omega) \sqcup Q_1(\Omega) \sqcup Q_2(\Omega)$. We now have

Lemma 3.11: *Under the hypothesis of Lemma 3.3, the following hold:*

(i) The orbit space $Q(\Omega)$ is homeomorphic to the closed disc $\bar{D}^2$, and the subset of singular orbits $Q_1(\Omega) \sqcup Q_0(\Omega)$ is homeomorphic to the boundary $\partial\bar{D}^2 \simeq S^1$.

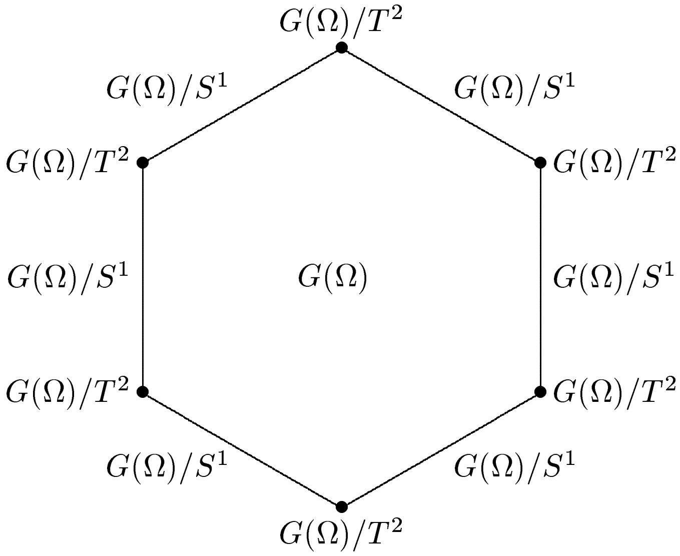
(ii) $Q_2(\Omega)$ *is homeomorphic to the open disc* D^2.

(iii) $Q_1(\Omega)$ *is homeomorphic to the disjoint union of* $k+2$ *copies of the open unit interval.*

(iv) $Q_0(\Omega)$ *is a set of* $k+2$ *points.*

PROOF: Consider the natural projection $\rho_{\mathcal{S}} : \mathcal{S}(\Omega) \longrightarrow Q(\Omega)$. Since, $\mathcal{S}(\Omega)$ has finite fundamental group [BGM2], it follows from Corollary II.6.5 of Bredon [Bre] that $H_1(Q(\Omega), \mathbb{Q}) = 0$. Moreover, from (v) of Lemma 3.9 there are no exceptional orbits. So as in the proof of Theorem IV.8.6 of Bredon [Bre] $Q(\Omega)$ is homeomorphic to the closed disc $\bar{D}^2$, and the subset of singular orbits $Q_1(\Omega) \sqcup Q_0(\Omega)$ is homeomorphic to the boundary $\partial\bar{D}^2 \simeq S^1$. It follows from (iv) of Lemma 3.9 that $Q_0(\Omega)$ is a set of $k+2$ points. But then $Q_1(\Omega) \simeq \partial\bar{D}^2 - Q_0(\Omega)$ is homeomorphic to $k+2$ disjoint arcs of a circle. The remainder of the lemma as well as filling the gap in the proof of Lemma 3.7 now follow easily. ∎

Lemma 3.11 implies that the stratification of $Q(\Omega)$ provides it with a fairly natural CW decomposition. This CW decomposition is depicted in the following diagram.



3.12. The quotient $Q(\Omega)$ for $k = 4$ with the fibres of $\rho_{\mathcal{S}}$ indicated

§4. The Rational Homology of $\mathcal{S}(\Omega)$

In this section we restrict our attention to the case where $n = k+2$. Thus, each $\mathcal{S}(\Omega)$ is a 7-dimensional 3-Sasakian manifold. As mentioned in the introduction the results of [BGM2] and [GS1] imply the rational homology of any such manifold is given by

4.1 $$H_*(\mathcal{S}(\Omega); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } * = 0, 7; \\ 0, & \text{if } * = 1, 3, 4, 6; \\ \mathbb{Q}^r & \text{if } * = 2, 5; \end{cases}$$

for some integer r. Hence, the only rational homology invariant for these manifolds is the Betti number $b_2(\mathcal{S}(\Omega)) = b_5(\mathcal{S}(\Omega))$. Our main result of this section is that every possible Betti number is realized by the examples constructed by the admissible weight matrices of Proposition 2.21.

THEOREM 4.2: *Let $n = k+2$ and $\Omega \in \mathcal{A}_{k\times k+2}(\mathbb{Z})$. Then*

$$H_2(\mathcal{S}(\Omega);\mathbb{Q}) \;\cong\; \mathbb{Q}^k \;\cong\; H_5(\mathcal{S}(\Omega);\mathbb{Q}).$$

REMARK 4.3: Theorem 4.2 actually holds in the more general case of 3-Sasakian orbifolds where the the second admissibility condition fails as explained in remark 2.19.

PROOF: We can use the natural projection $\rho_{\mathcal{S}} : \mathcal{S}(\Omega) \longrightarrow Q(\Omega)$ to construct a Leray spectral sequence converging to $H_*(\mathcal{S}(\Omega);\mathbb{Q})$. Let B_i be the natural CW decomposition of $Q(\Omega)$ given in section 3 so that, in the notation of Lemma 3.11,

$$B_0 = Q_0(\Omega), \quad B_1 = Q_0(\Omega) \cup Q_1(\Omega), \quad \text{and} \quad B_2 = Q(\Omega).$$

Then, we can filter $\mathcal{S}(\Omega)$ by $X_i = \rho_{\mathcal{S}}^{-1}(B_i)$ to obtain the increasing filtration

$$X_0 = \mathcal{S}_0(\Omega), \quad X_1 = \mathcal{S}_0(\Omega) \cup \mathcal{S}_1(\Omega), \quad \text{and} \quad X_2 = \mathcal{S}(\Omega).$$

The Leray spectral sequence associated to this filtration has E^1 term given by

$$E^1_{s,t} \;\cong\; H_{s+t}(X_t, X_{t-1};\mathbb{Q})$$

with differential $d_1 : H_{s+t}(X_t, X_{t-1};\mathbb{Q}) \longrightarrow H_{s+t-1}(X_{t-1}, X_{t-2};\mathbb{Q})$ where we use the convention that $X_{-1} = \emptyset$.

To compute these E^1 terms notice that all the pairs (X_t, X_{t-1}) are relative manifolds so that one can apply the Alexander-Poincaré duality theorem. Hence, by Lemma 3.9,

$$\begin{aligned}
H_s(X_0;\mathbb{Q}) &\;\cong\; H_s((Sp(1)^{k+2};\mathbb{Q}); \\
H_s(X_1, X_0;\mathbb{Q}) &\;\cong\; H^{5-s}((S^1 \times Sp(1))^{k+2};\mathbb{Q}); \\
H_s(X_2, X_1;\mathbb{Q}) &\;\cong\; H^{7-s}(S^1 \times S^1 \times Sp(1);\mathbb{Q}).
\end{aligned}$$

Hence,

$$E^1_{s,t} \;\cong\; \begin{cases} \mathbb{Q}^{k+2} & \text{if } t=0 \text{ and } s=0,3; \\ \mathbb{Q}^{k+2} & \text{if } t=1 \text{ and } s=0,1,3,4; \\ \mathbb{Q} & \text{if } t=2 \text{ and } s=0,2,3,5; \\ \mathbb{Q}^2 & \text{if } t=2 \text{ and } s=1,4; \\ 0 & \text{otherwise.} \end{cases}$$

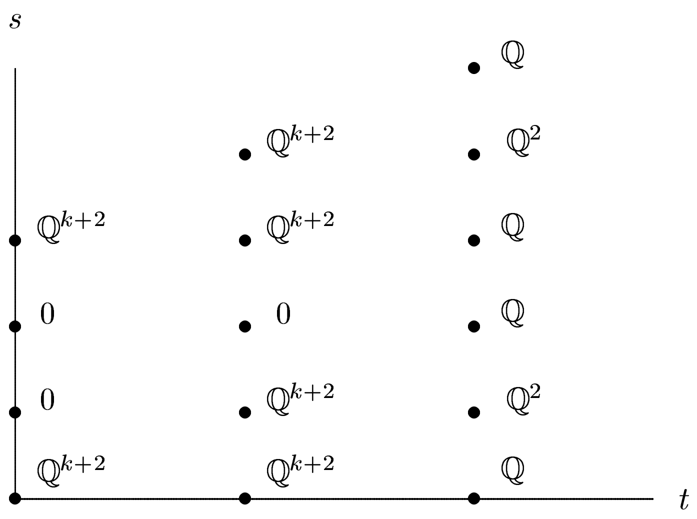


Diagram 4.4

The d_1 differentials, which move horizontally one column to the left in diagram 4.4, are completely determined by equation 4.1. Thus,

$$E^2_{s,t} \cong \begin{cases} \mathbb{Q} & \text{if } t = 0 \text{ and } s = 0, 3; \\ \mathbb{Q}^k & \text{if } t = 1 \text{ and } s = 1, 4; \\ \mathbb{Q} & \text{if } t = 2 \text{ and } s = 3, 5; \\ 0 & \text{otherwise.} \end{cases}$$

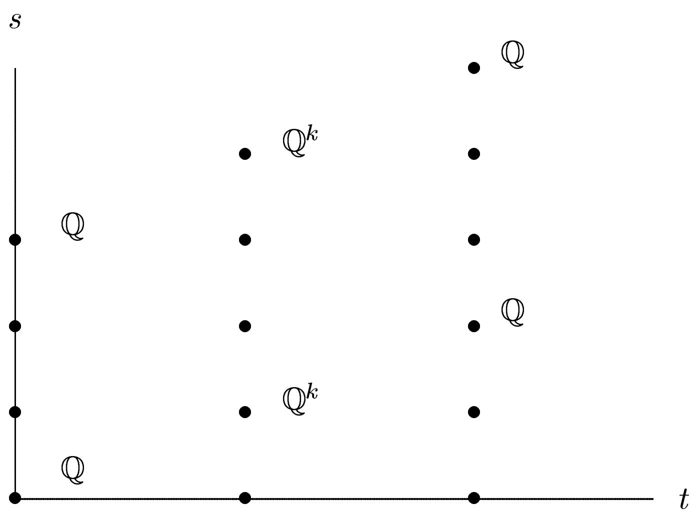


Diagram 4.5

Finally, there is only one possibly non-trivial d^2 differential

$$\begin{array}{ccc} H_4(X_2, X_1; \mathbb{Q}) & \xrightarrow{d^2_{4,2}} & H_3(X_0; \mathbb{Q}) \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ \mathbb{Q} & \longrightarrow & \mathbb{Q}. \end{array}$$

Equation 4.1 implies $d^2_{4,2}$ must be an isomorphism, there are no other possible differentials, so $E^3 \cong E^\infty$, and the result follows. ∎

It should be possible to refine the computations of this section to compute the integral cohomology ring $H^*(\mathcal{S}(\Omega); \mathbb{Z})$, at least in the $n = k + 2$ case. It appears that, just like in the $k = 1$ case, that there is torsion in $H^4(\mathcal{S}(\Omega); \mathbb{Z})$ which depends non-trivially on the weight matrix Ω. This would imply that these $\mathcal{S}(\Omega)$ examples run through infinitely many distinct homotopy types for each fixed k.

After completing this work we received a preprint from R. Bielawski [Bi] where he computes the Betti numbers for 3-Sasakian orbifolds of any possible dimension arising as toral quotients of spheres. As expected the second Betti number of such an orbifold is equal to the dimension of the torus. However, the existence of smooth manifolds with $n > k+2$ is much more delicate. For example, the subset $\mathcal{A}_{k\times k+3}(1, \mathbb{Z})$, defined in section two, is empty for $k > 4$. Bielawski's result implies for example that, in dimension 11, the Betti number relation $b_2 = b_4$ valid for regular 3-Sasakian manifolds [GS] is not true in general.

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