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The Twistor Space of a 3-Sasakian Manifold

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Abstract. Any compact 3-Sasakian manifold $\mathcal{S}$ is a principal circle V-bundle over a compact complex orbifold $\mathcal{Z}$. This orbifold has a contact Fano structure with a Kähler-Einstein metric of positive scalar curvature and it is the twistor space of a positive compact quaternionic Kähler orbifold $\mathcal{O}$. We show that many results known to hold when $\mathcal{Z}$ is a smooth manifold extend to this more general singular case. However, we construct infinite families of examples with $b_2(\mathcal{Z})$=2 which sharply differs from the smooth case, where there is only one such $\mathcal{Z}$.

Introduction

Ever since Salamon [4,36] generalized Penrose's twistor construction to an arbitrary quaternionic dimension, twistor geometry has played a fundamental role in the study of quaternionic Kähler geometry. In particular, Salamon [36] showed that the twistor space of a quaternionic Kähler manifold with positive scalar curvature is a Fano contact manifold with a Kähler-Einstein metric. However, it was only recently that LeBrun [28] proved the following inversion theorem:

Theorem [28]: *Let $\mathcal{Z}$ be a Fano contact manifold. Then $\mathcal{Z}$ is the twistor space of a quaternionic Kähler manifold M of positive scalar curvature precisely when it admits a Kähler-Einstein metric.*

Moreover, there is a 1-1 correspondence between compact Kähler-Einstein Fano contact manifolds up to biholomorphism and compact quaternionic Kähler manifolds of positive scalar curvature up to homothety.

It is also well known by now that 3-Sasakian geometry is intimately related to this setup [6,7]. In fact, the twistor space is just the total space of a certain 2-sphere bundle over a quaternionic Kähler manifold, and the 3-Sasakian manifold is the associated principal $SU(2)$ or $SO(3)$ bundle [6,27]. Thus, LeBrun's theorem should have a 3-Sasakian version. However, there is an essential difference. It is now necessary to work in the category of orbifolds or V-manifolds on the level of the twistor and quaternionic Kähler geometries. Fortunately, the foundational work of Satake [38] and Baily [2,3] has set the ground work for doing differential geometry on such objects. Many of the known results about quaternionic Kähler manifolds and their twistor manifolds carry over to the orbifold setting. For example, as we shall show below, both Salamon's construction and LeBrun's

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inversion theorem have orbifold counterparts. However, since it is 3-Sasakian geometry that interests us most, one of our main results gives the precise correspondence between 3-Sasakian geometry and twistor geometry. This is carried out in section 4.

Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian manifold of dimension $4n+3$. (See [6,7] and references therein for full details.) Let S^1_a denote the locally free circle action generated by the vector field ξ^a. Then $\mathcal{S}$ has a one dimensional foliation with compact leaves, and the space of leaves $\mathcal{Z}$, which we call *the twistor space* of $\mathcal{S}$, is fairly well behaved. In fact, much more is true:

THEOREM [6]: *Let $\mathcal{S}$ be a compact 3-Sasakian manifold of dimension $4n+3$. Then the space of leaves $\mathcal{Z}_a = \mathcal{S}/S^1_a$ is a compact complex orbifold of complex dimension $2n+1$ with a Kähler-Einstein metric of positive scalar curvature $8(2n+1)(n+1)$. Furthermore, the local uniformizing groups of the orbifold are all cyclic.*

This Theorem is a generalization of the Boothby-Wang fibration to the orbifold setting with a bit more structure. The proof given in [6] is indirect in that it uses the relationship between 3-Sasakian geometry and quaternionic Kähler geometry developed by Ishihara and Konishi [19,20,27]. It is certainly possible to give a direct proof. Since the circle actions S^1_a and S^1_b corresponding to ξ^a and ξ^b, respectively are conjugate under an $Sp(1)$ transformation, one sees easily that the corresponding twistor spaces $\mathcal{Z}_a$ and $\mathcal{Z}_b$ are biholomorphic. Henceforth, we shall denote the twistor space by $\mathcal{Z}$ without the subscript and refer to it as the twistor space associated to a 3-Sasakian manifold.

Our twistor spaces "live" in three distinct categories: They are compact topological spaces; they are compact complex orbifolds; and as we prove in section 3 they are normal projective algebraic varieties which, as it turns out, are of a type known in Mori theory as $\mathbb{Q}$-factorial varieties. Each picture has its own particular advantage. We show in section 3, as in the regular case, that as a topological space a twistor space $\mathcal{Z}$ is always simply connected. However, in the category of orbifolds, Thurston [41] has introduced the notion of an "orbifold fundamental group" π_1^{orb} and this does not necessarily vanish for our twistor spaces. Another important invariant is Pic, the Picard group of line bundles on an algebraic variety. For our twistor spaces Pic is a free Abelian group of rank equal to the second Betti number of $\mathcal{Z}$. However, on a complex orbifold there is another group Pic^{orb} consisting of equivalence classes of complex line V-bundles. This group always contains Pic as a subgroup, but carries additional information about the orbifold structure.

Although the natural projection $\mathcal{S}\longrightarrow\mathcal{Z}$ is not a fibration, it may be expected that the rational cohomology of $\mathcal{S}$ and $\mathcal{Z}$ behave as though it were. In this regard we are able to generalize a result of Galicki and Salamon [15] relating the Betti numbers of $\mathcal{S}$ and $\mathcal{Z}$ in the case of regular foliations to our more general setting. The proofs given here use harmonic theory which hold equally well on orbifolds [2]. Our main result in this regard is Theorem 2.4 below.

Thus, many results that hold in the regular case still hold in the general case. However, there are places where the analogy breaks down. The most notable are the recent finiteness theorems of LeBrun and Salamon [29,30] which stem from the known finiteness theorems of Fano manifolds. For example, if a twistor space $\mathcal{Z}$ is a manifold with second Betti number $b_2(\mathcal{Z}) = 2$, then $\mathcal{Z}$ must be the flag manifold $F_{1,2,n}$. This is certainly not true in the non-regular case. In section 5 we give infinite families of what we call "weighted flag varieties" which have $b_2(\mathcal{Z}) = 2$ and are inequivalent as twistor spaces. Nevertheless, we show that our weighted flag varieties have the same rational cohomology groups as the homogeneous flag $F_{1,2,n}$. Of course, all of the weighted flag varieties are singular, and their singularity structure is described in section 6. Further, in the regular case we must have $b_2(\mathcal{Z}) \leq 2$. At the end of section 4 we describe a reduction procedure which, using Kirwan's geometric invariant theory [23,32], permits a study of the rational cohomology of such twistor spaces. We hope to be able to use this procedure to see if the bound on the second Betti number persists in the non-regular case.

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§1. Preliminaries on Complex Orbifolds

In this section we give the necessary background on Riemannian and complex orbifolds (or V-manifolds). For more details we refer to [2,3,38]. Since we shall deal mainly with complex orbifolds, we give the definition in the complex analytic category.

Definition 1.1: *Let $\tilde{U}$ be an open subset of a Hausdorff space. A local uniformizing system of $\tilde{U}$ is a triple $\{U, \Gamma, \varphi\}$, where U is connected open subset of $\mathbb{C}^n$ containing the origin, Γ is a finite group of analytic transformations of U, and $\varphi : U \longrightarrow \tilde{U}$ is a continuous map onto $\tilde{U}$ such that $\varphi \circ \gamma = \varphi$ for all $\gamma \in \Gamma$ and the induced natural map of U/Γ onto $\tilde{U}$ is a homeomorphism. The finite group Γ is called a local uniformizing group. Given local uniformizing systems $\{U, \Gamma, \varphi\}$ and $\{U', \Gamma', \varphi'\}$ for $\tilde{U}$ and $\tilde{U}'$, respectively such that $\tilde{U} \subset \tilde{U}'$, a biholomorphic map λ from U onto its image in U' such that for any $\gamma \in \Gamma$ there is $\gamma' \in \Gamma'$ such that $\lambda \circ \gamma = \gamma' \circ \lambda$, and $\varphi = \varphi' \circ \lambda$ is called an injection. Then a complex orbifold (or complex V-manifold) is a second countable Hausdorff space X together with a family $\mathcal{F}$ of local uniformizing systems for a collection of open subsets $\{\tilde{U}\}$ of X that satisfy:*

(i) The open sets $\{\tilde{U}\}$ for which there exist local uniformizing systems $\{U, \Gamma, \varphi\}$ form a basis for the topology of X.

(ii) If $\{U, \Gamma, \varphi\}$ and $\{U', \Gamma', \varphi'\}$ are two local uniformizing systems of $\mathcal{F}$ such that $\tilde{U} \subset \tilde{U}'$, then there exists an injection $\lambda : \{U, \Gamma, \varphi\} \longrightarrow \{U', \Gamma', \varphi'\}$.

If an injection as defined above is also surjective then the two uniformizing systems are said to be *equivalent*. It is also straightforward to define the notion of equivalence of orbifolds (V-manifolds) [38]. Often we shall say orbifold to mean an equivalence class of orbifolds. Let X be an orbifold and choose a local uniformizing system $\{U, \Gamma, \varphi\}$. Let $x \in X$ be any point, and let $p \in \varphi^{-1}(x)$, then the isotropy subgroup $\Gamma_p \in \Gamma$ depends only on x, and accordingly we shall denote this isotropy subgroup by Γ_x. A point of X whose isotropy subgroups $\Gamma_x \neq \mathrm{id}$ is called a *singular* point. Those points with $\Gamma_x = \mathrm{id}$ are called *regular* points. An orbifold X is a smooth manifold or in the complex analytic category a complex manifold if and only if $\Gamma_x = \mathrm{id}$ for all $x \in X$. In this case we can take $\Gamma = \mathrm{id}$ and $\varphi = \mathrm{id}$, and the definition of an orbifold reduces to the usual definition of a smooth manifold.

Many of the usual differential geometric concepts that hold for smooth or complex analytic manifolds also hold in the orbifold category. In particular, the important notion of a fiber bundle.

Definition 1.2: *A V-bundle over an orbifold X consists of a bundle B_U over U for each local uniformizing system $\{U, \Gamma, \varphi\}$ with Lie group G and fiber F (independent of U) together with an anti-monomorphism $h_U : \Gamma \longrightarrow G$ satisfying:*

(i) If b lies in the fiber over $x \in U$ then for each $\gamma \in \Gamma$, $h_U(\gamma)b$ lies in the fiber over $\gamma^{-1}x$.

(ii) If $\lambda : \{U, \Gamma, \varphi\} \longrightarrow \{U', \Gamma', \varphi'\}$ is an injection, then there is a bundle map $\lambda^* : B_U|\lambda(U) \longrightarrow B_U$ satisfying the condition that if $\gamma \in \Gamma$, and $\gamma' \in \Gamma'$ is the unique element such that $\lambda \circ \gamma = \gamma' \circ \lambda$, then $h_U(\gamma) \circ \lambda^* = \lambda^* \circ h_{U'}(\gamma')$, and if $\lambda' : \{U', \Gamma', \varphi'\} \longrightarrow \{U'', \Gamma'', \varphi''\}$ is another injection then $(\lambda \circ \lambda')^* = \lambda'^* \circ \lambda^*$.

If the fiber F is a vector space and G acts on F as linear transformations, then the V-bundle is called a vector V-bundle. Similarly, if F is the Lie group G with its right action, then the V-bundle is called a principal *V-bundle.*

Now by choosing the local uniformizing neighborhoods small enough, we can always take B_U to be the product $U \times F$ in which case we say that the local uniformizing system $\{U, \Gamma, \varphi\}$ is *fine*. Then a V-bundle is determined [3] by the following:

Lemma 1.3: *Assuming the notation used in 1.2, a V-bundle B over X with a fine local uniformizing system is characterized precisely by the following data:*

$$h_U(\gamma)(x, u) = (\gamma^{-1}x, \eta_U(\gamma)(x) \cdot u),$$

$$\lambda^*(\lambda(x), u) = (x, \xi_\lambda(\lambda(x)) \cdot u),$$

where $(x, u) \in U \times F$ and $\eta_U(\gamma) : U \longrightarrow G$ and $\xi_\lambda : \lambda(U) \longrightarrow G$ are holomorphic maps (or smooth in the real case) that satisfy the following conditions:

(i) $\eta_U(\gamma_2)(\gamma_1^{-1}x)\eta_U(\gamma_1)(x) = \eta_U(\gamma_1\gamma_2)(x)$.

(ii) $\eta_U(\gamma)(z)\xi_\lambda(\lambda(x)) = \xi_\lambda(\lambda(\gamma^{-1}x))\eta_{U'}(\gamma')(\lambda(x))$.

(iii) $\xi_\lambda(\lambda(x))\xi_{\lambda'}(\lambda' \circ \lambda(x)) = \xi_{\lambda' \circ \lambda}(\lambda' \circ \lambda(x))$.

We shall be particularly interested in the case of complex line V-bundles in which case $\eta_U(\gamma)$ and ξ_U are non-zero holomorphic functions. Notice, however, that generally a V-bundle is not necessarily a fibration, so many of the standard topological techniques cannot be applied directly.

The notion of equivalence of V-bundles is particularly easy to give [3] in terms of the data of Lemma 1.3:

DEFINITION 1.4: *Two V-bundles B_1 and B_2 over X with the same Lie group G are said to be equivalent if for each local uniformizing system $\{U, \Gamma, \varphi\}$ there is a smooth (holomorphic) map $\vartheta_U : U \longrightarrow G$ such that if $\gamma \in \Gamma$ and $x \in U$ then*

$$\eta_U^1(\gamma)\vartheta_U(x) = \vartheta_U(\gamma^{-1}x)\eta_U^2(\gamma)(x)$$

and

$$\xi_\lambda^1(\lambda(x))\vartheta_{U'}(\lambda(x)) = \vartheta_U(x)\xi_\lambda^2(\lambda(x)),$$

where the superscripts $1, 2$ correspond to the V-bundles B_1, B_2, respectively.

The trivial V-bundle over X is isomorphic (in the orbifold category) to the product $X \times F$. In this case we can take $\eta_U(\gamma) = \mathrm{id}$ for all U. This is a special case of what Baily calls an *absolute* V-bundle which corresponds to the ordinary notion of a bundle over X. Baily [3] proves

LEMMA 1.5: *A V-bundle B' is absolute if and only if B' is equivalent to a V-bundle B with corresponding functions $\eta_U(\gamma) = id$ and ξ_λ invariant under γ for all $\gamma \in \Gamma$.*

We are particularly interested in equivalence classes of complex line V-bundles over X, which, in accordance with standard terminology, we denote by $\mathrm{Pic}^{orb}(X)$. We have [3]

LEMMA 1.6: *The set of equivalence classes $Pic^{orb}(X)$ of complex line V-bundles over X forms an Abelian group.*

PROOF: Let L_1 and L_2 be two line V-bundles over X with corresponding functions $\eta_U^1(\gamma), \xi_\lambda^1$ and $\eta_U^2(\gamma), \xi_\lambda^2$. Then the product bundle which we denote by $L_1 \otimes L_2$ is determined by the functions $\eta_U^1(\gamma)\eta_U^2(\gamma), \xi_\lambda^1\xi_\lambda^2$. The inverse is also clearly defined and one can check that the product and inverse pass to equivalence classes. ∎

The *total space* of a V-bundle over X is an orbifold E with local uniformizing systems

$\{U \times F, \Gamma^*, \varphi^*\}$ such that the following diagram commutes:

1.7
$$\begin{array}{ccc} U \times F & \xrightarrow{\pi} & U \\ \Big\downarrow \varphi^* & & \Big\downarrow \varphi \\ \widetilde{U \times F} & \xrightarrow{\tilde{\pi}} & \tilde{U}, \end{array}$$

where π and $\tilde{\pi}$ are the obvious projections. The injections λ^* for the local uniformizing systems of a V-bundle satisfy the condition $\lambda^*(p,q) = (\lambda(p), g_\lambda(p)(q))$ for some smooth map $g_\lambda : U \longrightarrow G$. If the total space E is a smooth manifold, we call E a *desingularizing* V-bundle.

DEFINITION 1.8: *Let E be a V-bundle over an orbifold X, then a section σ of E over the open set $V \subset X$ is a section σ_U of the bundle B_U for each local uniformizing system $\{U, \Gamma, \varphi\} \in \mathcal{F}_V$ such that any $x \in U$,*

(i) For each $\gamma \in \Gamma$ $\sigma_U(\gamma^{-1}x) = h_U(\gamma)\sigma_U(x)$.

(ii) If $\lambda : \{U, \Gamma, \varphi\} \longrightarrow \{U', \Gamma', \varphi'\}$ is an injection, then $\lambda^* \sigma_{U'}(\lambda(x)) = \sigma_U(x)$.

If each of the local sections σ_U is continuous, smooth, holomorphic, etc., we say that σ is continuous, smooth, holomorphic, etc., respectively. Given local sections σ_U of a vector V-bundle we can always construct Γ-invariant local sections by "averaging over the group", i.e., we define

1.9a
$$\sigma_U^I = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sigma_U \circ \gamma.$$

A similar procedure holds for product structures. For example, if L is a holomorphic line V-bundle on X, and if σ is a holomorphic section, we can construct local invariant sections σ_U^I of $L^{|\Gamma|}$ by taking products, viz.,

1.9b
$$\sigma_U^I = \frac{1}{|\Gamma|} \prod_{\gamma \in \Gamma} \sigma_U \circ \gamma.$$

The standard notions of tangent bundle, cotangent bundle, and all the associated tensor bundles all have V-bundle analogues [2,38]. In particular, if V is an open subset of $\varphi(U)$ then the integral of an n-form (measurable) σ is defined by

1.10
$$\int_V \sigma = \frac{1}{|\Gamma|} \int_{\varphi^{-1}(V)} \sigma_U.$$

All of the standard integration techniques, such as Stokes' theorem, hold on V-manifolds.

Riemannian metrics also exist by the standard partition of unity argument, and we shall always work with Γ-invariant metrics. Moreover, all the standard differential geometric objects involving curvature and metric concepts, such as the Ricci tensor, Hodge star operator, etc., hold equally well. On a complex orbifold there is a Γ-invariant tensor field J of type $(1,1)$ which describes the complex structure on the tangent V-bundle TX. The almost complex structure J gives rise in the usual way to the V-bundles $A^{r,s}$ of differential forms of type (r,s). The standard concepts of Hermitian and Kähler metrics hold equally well on V-manifolds, and all the special identities involving Kähler, Einstein, or Kähler-Einstein geometry hold. In particular, the standard Weizenböck formulas hold.

Finally, there is associated to every compact orbifold X an integer m_0 called the *order* of X and defined to be the least common multiple of the orders of the local uniformizing groups.

§2. Harmonic Theory and Betti Numbers

Let $\{U, \Gamma, \phi\}$ be a local uniformizing system for $\mathcal{Z}$, then since $\Gamma \subset S^1$, the induced metric g_U on U is Γ-invariant. So Baily's generalization [2] of harmonic theory applies to the compact complex orbifold $\mathcal{Z}$. More generally we let X denote a compact complex orbifold with a Hermitian metric invariant under the local uniformizing groups. Let $\Lambda^r(X)$ and $\Lambda^{p,q}(X)$ denote the $C^\infty(X)$-module of smooth sections of the V-bundles of exterior differential r-forms and exterior differential forms of type (p,q), respectively, and let Ω^p denote the sheaf of germs of holomorphic p-forms on X. By [2] the following Laplacians are well-defined self adjoint elliptic operators on the appropriate Sobolev completions of $\Lambda^r(X)$ and $\Lambda^{p,q}(X)$, respectively,

2.1a $$\triangle = d\delta + \delta d,$$

2.1b. $$\bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

In the usual way we define the complex harmonic spaces $\mathcal{H}^r(X) = \text{Ker } \triangle$ and $\mathcal{H}^{p,q}(X) = \text{Ker } \bar{\Box}$. Then Baily's generalization of Hodge theory gives:

Baily-Hodge-Dolbeault Theorem: *Let X be as described above. Then the following isomorphisms hold:*

$$H^r(X, \mathbb{C}) \simeq \mathcal{H}^r(X),$$

$$H^q(X, \Omega^p) \simeq H^{p,q}_{\bar{\partial}}(X) \simeq \mathcal{H}^{p,q}(X).$$

In particular, these groups are finite dimensional, and give well-defined Betti and Hodge numbers, viz.,

$$b_r(X) = dim_{\mathbb{C}} H^r(X, \mathbb{C}) = dim_{\mathbb{C}} \mathcal{H}^r(X),$$

$$h^{p,q}(X) = dim_{\mathbb{C}} H^q(X, \Omega^p) = dim_{\mathbb{C}} \mathcal{H}^{p,q}(X).$$

Here $H^r(X,\mathbb{C})$ denotes de Rham cohomology, $H^{p,q}_{\bar\partial}(X)$ Dolbeault cohomology, and $H^q(X,\Omega^p)$ denotes the corresponding sheaf cohomology. There are many important consequences of the above theorem. We list several below:

Poincaré Duality: *There are conjugate linear isomorphisms*

$$H^r(X,\mathbb{C}) \simeq H^{m-r}(X,\mathbb{C}),$$

where m is the real dimension of X.

Serre Duality: *Let E be a holomophic vector V-bundle over X. Then there are conjugate linear isomorphisms*

$$H^r(X,E\otimes\Omega^p) \simeq H^{n-r}(X,E^*\otimes\Omega^{n-p}),$$

where n denotes the complex dimension of X. In particular, $h^{p,q}(X) = h^{n-p,n-q}(X)$.

In the case that the orbifold X is Kähler much more can be said. The Kähler condition is a local condition on curvature, and all the standard results hold equally well for orbifolds. In particular, the operators in 2.1 satisfy $\triangle = 2\bar\Box$. Recall also that the space $\Lambda(X) = \oplus\Lambda^r(X)$ carries a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in terms of the operators L,Λ and $B=\sum_{p=0}^{2n}(n-p)\Pi^p$, where L denotes wedging with the Kähler form, Λ denotes the L^2-dual of L, and Π^p denotes projector onto $\Lambda^p(X)$. When X is Kähler these operators commute with the Laplacian $\triangle$ (and hence with $\bar\Box$), and there is an induced representation on the subspace of harmonic forms. Letting $\mathcal{H}^r_0(X)$ and $\mathcal{H}^{p,q}_0(X)$ denote the space of *primitive* r and (p,q)-forms, that is the kernel of Λ on $\mathcal{H}^r(X)$ and $\mathcal{H}^{p,q}(X)$, respectively, we have

Lefschetz Decomposition Theorem: *Let X be a compact complex orbifold of complex dimension n with a compatible Kähler metric. Then the following direct sum decompositions hold:*

$$\mathcal{H}^r(X) = \sum_{s\ge(r-n)^+} L^s\mathcal{H}^{r-2s}_0(X),$$

$$\mathcal{H}^{p,q}(X) = \sum_{s\ge(p+q-n)^+} L^s\mathcal{H}^{p-s,q-s}_0.$$

In particular, $b_{2r}(X)\ge b_2(X)\ge 1$. Moreover, for $p<n$ the map

$$L^{n-p}: H^p(X,\mathbb{C})\longrightarrow H^{2n-p}(X,\mathbb{C})$$

is an isomorphism.

We also have the well-known:

Hodge Decomposition Theorem: *Let X be a compact complex orbifold with a compatible Kähler metric. Then $\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}$, and the following direct sum decompositions hold:*

$$H^r(X,\mathbb{C}) \simeq \sum_{p+q=r} H^{p,q}(X,\mathbb{C}),$$

$$\mathcal{H}^r(X,\mathbb{C}) \simeq \sum_{p+q=r} \mathcal{H}^{p,q}(X,\mathbb{C}).$$

In particular, $b_r(X) = \sum_{p+q=r} h^{p,q}$ *and* $h^{p,q}(X) = h^{q,p}(X)$.

The last statement in the Lefschetz decomposition theorem implies Poincaré duality. The proofs of all of the above theorems follow word for word from the usual proofs (cf. [43]) by using Baily's generalization of harmonic theory to the compact orbifold category. Another important result of Baily [3] is:

Kodaira-Baily Vanishing Theorem: *Let* X *be a compact complex orbifold,* K *the canonical V-line bundle and* L *any holomorphic V-line bundle on* X*. If* $L\otimes K^{-1}$ *is a positive V-line bundle then*

$$H^q(X,\mathcal{O}(L)) \;=\; 0 \quad \text{for } q\geq 1.$$

We are now ready to apply the above results to the situation at hand, namely the twistor space $\mathcal{Z}$ of a 3-Sasakian manifold. Our first result depends only on the fact that $\mathcal{Z}$ is a Kähler orbifold with positive definite Ricci curvature. It is a special case of the Kodaira-Baily vanishing theorem.

Corollary 2.2: *On the twistor space of a compact* 3*-Sasakian manifold we have* $h^{0,0}=1$*, and* $h^{p,0}(\mathcal{Z}) = h^{0,p}(\mathcal{Z}) = 0$*, so there are no holomorphic* p*-forms on* $\mathcal{Z}$ *for* $0<p\leq 2n+1$*. In particular, the holomorphic Euler number* $\chi(\mathcal{Z},\mathcal{O})=1$*.*

Next we relate the Betti numbers of $\mathcal{Z}$ to those of $\mathcal{S}$. First we remark that if we define the primitive Betti numbers by $b^r_0 = \dim_{\mathbb{C}}\mathcal{H}^r_0(X)$, then the Lefschetz decomposition theorem implies the equality

2.3
$$b^r_0(\mathcal{Z}) = b_r(\mathcal{Z}) - b_{r-2}(\mathcal{Z})$$

for $0\leq r\leq 2n+1$ with $b_i=0$ for $i<0$. The following theorem was proven in [15] for the case that $\mathcal{S}$ is regular by a Gysin sequence argument. Here we give a different proof using harmonic theory which is valid in the general case.

Theorem 2.4: *Let* $\mathcal{S}$ *be a compact* 3*-Sasakian manifold of dimension* $4n+3$ *and let* $\mathcal{Z}$ *be its twistor space, then for* $0\leq r\leq 2n+1$ *we have*

$$b_r(\mathcal{S}) = b_r(\mathcal{Z}) - b_{r-2}(\mathcal{Z}).$$

This is an immediate consequence of 2.3 and

Lemma 2.5: *For* $0\leq r\leq 2n+1$ *an* r*-form on* $\mathcal{S}$ *is harmonic if and only if it is the lift of a primitive harmonic form on* $\mathcal{Z}$*.*

PROOF: A theorem of Tachibana [40] says that any harmonic form α on $\mathcal{S}$ is horizontal, i.e., $\xi^1 \lfloor \alpha = 0$. But then since ξ^1 is a Killing vector field, we also have $\xi^1 \lfloor d\alpha = 0$. Thus, α is also projectible. So if $0 \neq \alpha \in \mathcal{H}^r(\mathcal{S})$ we let β be the unique closed form on $\mathcal{Z}$ such that $\alpha = \pi^*\beta$. Now the metrics of $\mathcal{S}$ and $\mathcal{Z}$ are related by

2.6
$$g_{\mathcal{S}} = \pi^* g_{\mathcal{Z}} + (\eta^1)^2,$$

and this implies the following relations between the Hodge star operators:

2.7
$$\star_{\mathcal{S}}\alpha = \eta^1 \wedge \pi^* \star_{\mathcal{Z}} \beta.$$

This relation holds for any basic horizontal form α. We claim that the form β is a primitive harmonic form on $\mathcal{Z}$. We have

2.8
$$d \star_{\mathcal{S}} \alpha = d\eta^1 \wedge \pi^* \star_{\mathcal{Z}} \beta - \eta^1 \wedge d(\pi^* \star_{\mathcal{Z}} \beta).$$

Now the second term on the right is in the ideal of $\Lambda\mathcal{S}$ generated by η^1 and the first term is not. So if $\alpha \in \mathcal{H}^r(\mathcal{S})$ the two terms on the right must vanish separately. The vanishing of the second term implies that β is co-closed. Recall that $d\eta^1 = \pi^*\omega$, the pullback of the Kähler form on $\mathcal{Z}$. So the vanishing of the first term implies that β is in the kernel of $L\star_{\mathcal{Z}}$, but this is the same as the kernel of Λ, implying that β is primitive. Conversely, if β is harmonic and primitive on $\mathcal{Z}$, then 2.8 shows that α is harmonic on $\mathcal{S}$. Moreover, α vanishes identically only if β does. ∎

REMARK 2.9: It is seen easily from the proof that Theorem 2.4 holds under the more general situation where $\mathcal{S}$ is a compact Sasakian manifold and the base orbifold $\mathcal{Z}$ is Kähler.

We now have an immediate corollary of Theorem 2.4 and Theorem A of [15]:

COROLLARY 2.10: *The twistor space $\mathcal{Z}$ of a compact 3-Sasakian manifold $\mathcal{S}$ has vanishing odd Betti numbers, and for $1 \leq r \leq n$ the even Betti numbers satisfy*

$$b_{2r}(\mathcal{Z}) = \sum_{k=1}^{r} b_{2k}(\mathcal{S}) + 1.$$

In the next section we shall see that the more refined result of Kobayashi generalizes to show that not only $b_1(\mathcal{Z}) = 0$, but that $\mathcal{Z}$ is, in fact, simply connected. Another corollary of our results is:

COROLLARY 2.11: *The Euler number $\chi(\mathcal{Z})$ satisfies*

$$\chi(\mathcal{Z}) = 2 + 2\sum_{k=1}^{n} b_{2k}(\mathcal{Z}) \geq 2 + 2nb_2(\mathcal{Z}) \geq 2(n+1).$$

We remark that $\chi(\mathcal{Z})$ denotes the usual Euler number of $\mathcal{Z}$ and not the "orbifold Euler number" found in the literature [38,41]. Finally, let us note that it seems quite likely that the more general Akizuki-Nagano vanishing theorem will hold on $\mathcal{Z}$ and allow a generalization of Salamon's proof [36] that all the cohomology of $\mathcal{Z}$ is type (p,p), but we have not checked the details of this argument. However, it is clear from our results above that all the cohomology in dimension two is of type $(1,1)$.

§3. The Twistor space as a Projective Algebraic Variety

In this section we describe the twistor space $\mathcal{Z}$ in terms of projective algebraic varieties. At the heart of the proof of this fact is Baily's generalization [3] of the Kodaira embedding theorem to the orbifold (or V-manifold) category:

KODAIRA-BAILY EMBEDDING THEOREM: *Let X be a compact complex orbifold and suppose that X has a positive V-line bundle. Then X is an algebraic variety.*

Once it is known that the twistor space is a projective algebraic variety, algebraic geometric techniques can be brought to bare on the study of 3-Sasakian geometry. In this regard we make free use of Serre's GAGA to pass between the complex analytic and algebraic pictures. We are now ready for:

THEOREM 3.1: *Let $\mathcal{Z}$ denote the twistor space associated to a compact 3-Sasakian manifold. Then $\mathcal{Z}$ is a simply connected normal projective algebraic variety. Hence, the singular locus of $\mathcal{Z}$ has complex codimension at least two.*

PROOF: Since $\mathcal{Z}$ has a Kähler-Einstein metric of positive scalar curvature, K^{-1} is positive. Since $\mathcal{Z}$ is compact, the Kodaira-Baily embedding theorem (with L trivial) implies that $\mathcal{Z}$ is a projective algebraic variety.

To prove normality we consider the local ring structure for compact complex orbifolds. Let $\{U,\Gamma,\phi\}$ be a local uniformizing system. Let $\mathcal{O}_{\mathcal{Z}}$ or just $\mathcal{O}$ denote the structure sheaf of $\mathcal{Z}$ considered as a complex space. For $z\in\mathcal{Z}$ the stalk $\mathcal{O}_z$ of $\mathcal{O}$ is isomorphic to the ring $\mathcal{O}^{\Gamma_z}_{\mathbb{C}^n}$, of Γ_z-invariant germs of locally convergent power series in n complex variables, where n is the complex dimension of $\mathcal{Z}$. Since the local uniformizing groups are the identity on the dense set of regular points, the ring $\mathcal{O}_z$ is isomorphic to the ring $\mathcal{O}_{\mathbb{C}^n}$ of convergent power series in $\mathbb{C}^n$ when $z\in\mathcal{Z}$ is regular. In particular, $\mathcal{O}_{\mathbb{C}^n}$ is a UFD. However, at a singular point $z\in\mathcal{Z}$, the ring $\mathcal{O}_z$ is isomorphic to the ring $\mathcal{O}^{\Gamma_z}_{\mathbb{C}^n}$, which for $\Gamma_z\neq \mathrm{id}$ is not a UFD. Thus, the complex space $(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$ is not locally factorial. Nevertheless, since Γ_z is finite, the local rings $\mathcal{O}^{\Gamma_z}_{\mathbb{C}^n}$ are integrally closed [11, pg 323] so $(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$ is normal; hence, by a standard result the singular locus of $\mathcal{Z}$ has complex codimension at least two.

The proof of simple connectivity is essentially a translation of Kobayashi's proof [25] in the case of manifolds to the case of compact complex orbifolds. By positivity of the Ricci tensor $\mathcal{Z}$ has at most a finite cover. By Corollary 2.2 the holomorphic Euler number $\chi(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})=1$, and this must also hold on any finite cover. But from the above $\mathcal{Z}$ is projective algebraic, hence complete. So the version of the Hirzebruch-Riemann-Roch Theorem for singular algebraic varieties due to Baum, Fulton, and Macpherson [13,pg 354] applies and we have

$$1 = \chi(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) =< \mathrm{Td}(\mathcal{Z}), [\mathcal{Z}] >, \tag{3.2}$$

where $\mathrm{Td}(\mathcal{Z})$ is the Todd class. Moreover, 3.2 holds on any k-fold cover $\tilde{\mathcal{Z}}$ as well, and $< \mathrm{Td}(\tilde{\mathcal{Z}}), [\tilde{\mathcal{Z}}] >= k < \mathrm{Td}(\mathcal{Z}), [\mathcal{Z}] >$. This forces $k = 1$. ∎

We caution the reader that the simple connectivity of $\mathcal{Z}$, that is $\pi_1(\mathcal{Z}) = 0$ should not be confused with "orbifold simple connectivity" introduced by Thurston [41] who proves the existence of a "universal covering orbifold" $\tilde{X}$ for any orbifold X, which is generally not a covering space in the usual sense. He then defines the *orbifold fundamental group* $\pi_1^{orb}(X)$ as the group of deck transformations of $\tilde{X}$. So we may have $\pi_1^{orb}(\mathcal{Z})$ non-vanishing. A simple example of this is the twistor space $K\backslash\mathbb{P}^{2n+1}$, where $K \subset SO(3)$ is a finite subgroup. This is simply connected by the Theorem above, but $\pi_1^{orb}(\mathcal{Z}) = K$. Further examples are given in section 5.

PROPOSITION 3.3: *Let $\mathcal{S}$ be a complete 3-Sasakian manifold and $\mathcal{Z}$ its twistor space. Then, $\pi_1^{orb}(\mathcal{Z})$ is finite. Moreover, if $\mathcal{S}$ is simply connected $\pi_1^{orb}(\mathcal{Z}) = 0$.*

PROOF: This follows from the homotopy exact sequence

$$\pi_1(S^1) \longrightarrow \pi_1(\mathcal{S}) \longrightarrow \pi_1^{orb}(\mathcal{Z}) \longrightarrow \{e\}$$

established by Haefliger and Salem [17]. ∎

Suppose we are given a holomorphic line V-bundle on a compact complex orbifold X. This is not necessarily a line bundle (or invertible sheaf) in the sense of complex manifolds or algebraic geometry. It is, however, a reflexive sheaf of rank 1. A holomorphic line bundle $\mathcal{L}$ on X is what Baily calls an *absolute* line bundle. The local transition functions for $\mathcal{L}$ are everywhere Γ-invariant. More generally the same distinction holds for holomorphic vector bundles.

Similarly, we can define [3] a Baily divisor on X to be a Γ-equivariant divisor. On each local uniformizing system $\{U, \Gamma, \varphi\}$ we consider a divisor D_U on U. Since U is an open subset of $\mathbb{C}^n$, D_U corresponds to an invertible sheaf $\mathcal{D}$ on U. We let $\mathcal{D}_x$ denote the stalk of $\mathcal{D}$ at $x \in X$. Then

DEFINITION 3.4: *A Baily divisor on $\mathcal{Z}$ is a divisor D_U on each local uniformizing system $\{U, \Gamma, \varphi\}$ that satisfies the two conditions*

(i) If for each $x \in X$ and $\gamma \in \Gamma$, $f \in \mathcal{D}_{\gamma x}$ then $f \circ \gamma \in \mathcal{D}_x$.

(ii) If $\lambda : \{U, \Gamma, \varphi\} \longrightarrow \{U', \Gamma', \varphi'\}$ is an injection and $f \in \mathcal{D}'_{\lambda(x)}$ then $f \circ \lambda \in \mathcal{D}_x$.

A Baily divisor is called absolute if on each local uniformizing system $\{U, \Gamma, \varphi\}$ the divisor D can be written as $D = (f)$, where f is the quotient of Γ-invariant holomorphic functions on U.

Absolute Baily divisors on $\mathcal{Z}$ are nothing but Cartier divisors. Analogous to the usual case, for every Baily divisor D there corresponds [3] a complex line V-bundle [D]. If D is absolute so is $[D]$. Since $\mathcal{Z}$ is a normal projective variety, Weil divisors also exist on $\mathcal{Z}$; however, since $\mathcal{O}_{\mathcal{Z}}$ is not necessarily locally factorial, Weil divisors are not always Cartier divisors.

PROPOSITION 3.5: *On $\mathcal{Z}$ Weil divisors and Baily divisors coincide.*

PROOF: Let D be a Weil divisor on $\mathcal{Z}$. Then its restriction $D_{\tilde{U}}$ to $\tilde{U}$ is a Weil divisor on $\tilde{U}$. Its inverse image $\varphi^{-1}(D_{\tilde{U}})$ is a divisor on U. Moreover, the Γ invariance of φ implies that condition (i) of Definition 3.4 is satisfied. Similarly, one easily checks that condition (ii) of 3.4 is satisfied. Conversely, let D be a Baily divisor which we assume to be irreducible. Then, the invariance condition (i) guarantees that the image of D under φ cuts out a closed subscheme of $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of codimension one. Moreover, this subscheme is integral since D is irreducible. ∎

Varieties whose local rings are essentially Γ-invariant subrings of the ring of convergent power series in $\mathbb{C}^n$, where Γ is a finite group have recently played an increasingly important role in algebraic geometry. Mumford [34] used Matsusaka's [31] theory of Q-varieties to study the geometry of the moduli space of curves. More recently, the related notion of a $\mathbb{Q}$-factorial variety has become important in Mori's Program on the Minimal Model Problem (cf. [26]).

DEFINITION 3.6: *A normal projective algebraic variety V is called a $\mathbb{Q}$-factorial variety if for every Weil divisor D on V, there is a positive integer q such that qD is a Cartier divisor. The smallest such integer is called the index of the divisor D, denoted by Ind (D). If in addition the anticanonical divisor $-K$ is ample, we call V a $\mathbb{Q}$-factorial Fano variety.*

We remark that although the term $\mathbb{Q}$-Fano is ubiquitous in the Mori theory literature, it appears to have many apparently inequivalent definitions. The one closest to our needs is that given by Miyaoka and Mori [33]: a normal projective variety Y is a *$\mathbb{Q}$-Fano variety* if $i_*K_{Y^0}^{\otimes r}$ is an ample invertible sheaf for some negative integer r, where K denotes the canonical sheaf, and $i : Y^0 \longrightarrow Y$ is the natural inclusion of the smooth locus of Y. Clearly, a $\mathbb{Q}$-factorial Fano variety is $\mathbb{Q}$-Fano, but the converse is not necessarily true. Recall [33] an n dimensional variety X is *uniruled* if there are an $n-1$ dimensional variety W and a dominant rational map $f : \mathbb{P}^1 \times W \longrightarrow X$. We have

THEOREM 3.7: *The twistor space $\mathcal{Z}$ of a 3-Sasakian manifold $\mathcal{S}$ is a simply connected $\mathbb{Q}$-factorial Fano variety. In particular, $\mathcal{Z}$ is uniruled and the Kodaira dimension $\kappa(\mathcal{Z}) = -\infty$.*

PROOF: Since $\mathcal{Z}$ has a Kähler-Einstein metric of positive scalar curvature, K^{-1} is ample. By Proposition 3.5 Weil divisors are Baily divisors. So if D is a Weil divisor, on each local uniformizing system $\{U, \Gamma, \varphi\}$ we have $D = (f_U)$. Now f_U is not necessarily Γ-invariant; however, the product f_U^I defined by 1.7b is invariant. Then condition (ii) of 3.4 implies that $(f_U^I)^{\frac{m_0}{m}} = m_0 D$, where $m = |\Gamma|$. Hence, $m_0 D$ is a Cartier divisor. It is known [33] that $\mathbb{Q}$-Fano varieties are uniruled. That $\kappa(\mathcal{Z}) = -\infty$ follows directly from uniruledness. ∎

An explicit ruling of $\mathcal{Z}$ by a real family of rational curves can be given by realizing $\mathcal{Z}$ as a V-bundle over a quaternionic Kähler orbifold M. Along the singular locus of $\mathcal{Z}$ the fibers are possibly singular rational curves of the form $K\backslash\mathbb{P}^1$, where K is a finite subgroup of $SO(3)$ or $SU(2)$ that preserves both the complex and metric structures on $\mathbb{P}^1$. We have

PROPOSITION 3.8: *$\mathcal{Z}$ is ruled by a real family of rational curves C with possible singularities on the singular locus of $\mathcal{Z}$. All the curves C are simply connected, but $\pi_1^{orb}(C) = K$ with K as above, and all the isotropy groups are cyclic.*

PROOF: This is well known off the singular locus in which case all the curves are $\mathbb{P}^1$'s. On the singular locus the fibers are compact complex orbifolds of the form $K\backslash\mathbb{P}^1$. Moreover, the generalization of Kobayashi's argument used in the proof of 3.1 shows that $K\backslash\mathbb{P}^1$ is simply connected. The remainder follows from Thurston [41]. ∎

An important invariant of any algebraic variety is the Picard group Pic $(\mathcal{Z})$. This is the usual Picard group, not $\mathrm{Pic}^{orb}(\mathcal{Z})$ defined in section 1. In fact since ordinary line bundles are identified with absolute line V-bundles, Pic $(\mathcal{Z})$ is a subgroup of $\mathrm{Pic}^{orb}(\mathcal{Z})$. By applying the Kodaira-Baily vanishing theorem to the long exact cohomology sequence of the exponential sequence we easily see that Pic $(\mathcal{Z}) \simeq H^2(\mathcal{Z}, \mathbb{Z})$. Furthermore, since $\mathcal{Z}$ is simply connected, Pic $(\mathcal{Z}) \simeq H^2(\mathcal{Z}, \mathbb{Z})$ is torsionfree by universal coefficients. Collecting our results we have

PROPOSITION 3.9: *Let $\mathcal{Z}$ be the twistor space of a compact 3-Sasakian manifold, and let $k = b_2(\mathcal{Z})$ the second Betti number of $\mathcal{Z}$. Then*

(i) Pic $(\mathcal{Z})$ is a subgroup of $\mathrm{Pic}^{orb}(\mathcal{Z})$.

(ii) Pic $(\mathcal{Z}) \simeq H^2(\mathcal{Z}, \mathbb{Z}) \simeq \mathbb{Z}^k$ with $k > 0$.

(iii) $\mathrm{Pic}^{orb}(\mathcal{Z}) \otimes \mathbb{Q} \simeq$ Pic $(\mathcal{Z}) \otimes \mathbb{Q} \simeq \mathbb{Q}^k$.

In the smooth case there are very strong constraints [29,30] on $b_2(\mathcal{Z})$, namely, $b_2(\mathcal{Z}) \leq 2$, and equals 2 in only one case, that of the complex flag manifold $F_{1,2,n+1}$. Furthermore, there are only a finite number of inequivalent twistor spaces. This translates into similar results for both quaternionic Kähler manifolds [29] and regular 3-Sasakian manifolds [15]. An important open problem is to understand what happens in the singular case, or correspondingly in the general 3-Sasakian case. In section 5 we describe an infinite number of inequivalent twistor spaces with $b_2(\mathcal{Z}) = 2$ and $\pi_1^{orb}(\mathcal{Z}) = 0$. We mention that the larger group $\mathrm{Pic}^{orb}(\mathcal{Z})$ should fit into the general orbifold cohomology of Haefliger [16].

§4. The Complex Contact Structure

Next we discuss the complex contact structure on the twistor space $\mathcal{Z}$ of a 3-Sasakian manifold $\mathcal{S}$. This was first observed in the case of a regular foliation by Ishihara and Konishi [21], and for the related twistor space of a quaternionic Kähler manifold by Ward [42] in dimension 4 and by Salamon [36] in the general case. Since then it has been used extensively [28,29,30,35] in the study of quaternionic Kähler manifolds.

Fixing a Sasakian structure, say (Φ^1, ξ^1) in the 3-Sasakian structure, we notice that subbundle $\mathcal{H} = \ker\, \eta^1$ of $T\mathcal{S}$ together with $I = -\Phi^1|\mathcal{H}$ define a CR structure on $\mathcal{S}$. Furthermore, one easily sees that this CR structure is strictly pseudoconvex with vanishing Webster torsion. Since ξ^1 is an infinitesimal automorphism of the CR structure, the orbifold $\mathcal{Z}$ obtained as the quotient of $\mathcal{S}$ by the circle action generated by ξ^1 inherits a complex structure $\mathcal{I}$ from the CR-structure I. This complex structure coincides with the complex structure constructed from the quaternionic Kähler orbifold $\mathcal{O} \simeq \mathcal{S}/Sp(1)$. Actually a 3-Sasakian structure gives a special kind of CR structure, namely, a CR structure with a compatible holomorphic contact structure. Notice that the complex valued one form on $\mathcal{S}$ defined by $\eta^+ = \eta^2 + i\eta^3$ is type $(1,0)$ on $\mathcal{S}$. Moreover, one checks that η^+ is holomorphic with respect to the CR structure I. Although the 1-form η^+ is not invariant under the circle action generated by ξ^1, the trivial complex line bundle L^+ generated by η^+ is invariant. Thus, the complex line bundle L^+ pushes down to a nontrivial complex V-line bundle $\mathcal{L}$ on $\mathcal{Z}$. Let V denote the one dimensional complex vector space generated by L^+. Writing the circle action as $\exp\,(i\phi\xi^1)$ shows that V is the representation with character $e^{-2i\phi}$, and since $\mathcal{S}$ is a principal S^1 V-bundle over $\mathcal{Z}$, the twisted product $\mathcal{L} \simeq \mathcal{S} \times_{S^1} V$ is a complex line V-bundle on $\mathcal{Z}$. Now we can define a map of V-bundles $\theta : T^{(1,0)}\mathcal{Z} \longrightarrow \mathcal{L}$ by

4.1
$$\theta(X) = \eta^+(\hat{X}),$$

where $\hat{X}$ is the horizontal lift of the vector field X on $\mathcal{Z}$. Notice that $\theta(X)$ is not a function on $\mathcal{Z}$ but a section of $\mathcal{L}$. Now a straightforward computation shows that $\eta^+ \wedge (d\eta^+)^n$ is a nowhere vanishing section of $\Lambda^{(2n+1,0)}\mathcal{H}$ on $\mathcal{S}$, and thus $\theta \wedge (d\theta)^n$ is a nowhere vanishing section of $K \otimes \mathcal{L}^{n+1}$, where K is the canonical V-line bundle [3] on $\mathcal{Z}$. Hence, in $\mathrm{Pic}^{orb}(\mathcal{Z})$ we have $\mathcal{L} = K^{-\frac{1}{n+1}}$. (Here we are using the multiplicative tensor notation when thinking of K as a V-line bundle). Alternatively, the subbundle $\ker\, \theta$ is a holomorphic subbundle of $T^{(1,0)}\mathcal{Z}$ which is maximally non-integrable. This defines the complex contact structure on $\mathcal{Z}$.

For a 3-Sasakian manifold $\mathcal{S}$ let $\mathrm{Aut}_0(\mathcal{S})$ denote the connected component of the subgroup of the group of isometries of $\mathcal{S}$ that preserve the 3-Sasakian structure $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$. Then since $\mathrm{Aut}_0(\mathcal{S})$ commutes with ξ^1 one easily proves

PROPOSITION 4.2: *Let $\mathcal{S}$ be a compact simply connected 3-Sasakian manifold and let $\mathcal{Z}$ be its twistor space. Suppose that Γ is a discrete subgroup of $\mathrm{Aut}_0(\mathcal{S})$ that acts freely on $\mathcal{S}$. Then $\mathcal{S}/\Gamma$ is a compact 3-Sasakian manifold with fundamental group Γ, and $\mathcal{Z}/\Gamma$ is its twistor space with $\pi_1^{orb}(\mathcal{Z}/\Gamma) = \Gamma/\Gamma_0$, where Γ_0 is the normal subgroup of Γ that acts as the identity on $\mathcal{Z}$.*

It is clear from this Proposition that Γ does not necessarily act effectively on $\mathcal{Z}$. So it is possible that two different 3-Sasakian manifold have the same twistor space. The only known case when this happens is for $\mathcal{Z}=\mathbb{P}^{2n+1}$ and $\mathcal{S}$ is S^{4n+3} or $\mathbb{RP}^{2n+1}$. We shall show below that this is the only case.

Proposition 4.3: *Let $\mathcal{Z}$ be the twistor space of a 3-Sasakian manifold of dimension $4n+3$. If the contact line V-bundle $\mathcal{L}$ or equivalently its dual $\mathcal{L}^{-1}$ has a root in $Pic^{orb}(\mathcal{Z})$, then it must be a square root. Moreover, in this case we must have $\mathcal{Z}=\mathbb{P}^{2n+1}$.*

Proof: By Proposition 3.8 $\mathcal{Z}$ is ruled by rational curves C which on the singular locus take the form $K\backslash\mathbb{P}^1$. Now the restriction $\mathcal{L}^{-1}|C$ is $\mathcal{O}(-2)$ which is a V-bundle if C is singular. In either case it has only a square root namely the tautological V-bundle $\mathcal{O}(-1)$. Since these curves C cover $\mathcal{Z}$ this proves the first statement.

The second statement follows from a modification of an argument due to Kobayashi and Ochiai [24] and used by Salamon [36]. The main point is that we can apply Kawasaki's Riemann-Roch [22] theorem to arbitrary powers of the line V-bundle $\mathcal{L}^{\frac{1}{2}}$:

4.4 $$\chi(\mathcal{Z},\mathcal{O}(\mathcal{L}^{\frac{r}{2}}))=\sum_{\tilde{U}}\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\int_{U^\gamma}f_{\tilde{U}}\mathrm{Td}^\gamma(U,\mathcal{L}_U^{\frac{r}{2}}),$$

where as usual $\{U,\Gamma,\varphi:U\longrightarrow\tilde{U}\}$ is a local uniformizing system, $f_{\tilde{U}}$ is a partition of unity, U^γ is the fixed point set of $\gamma\in\Gamma$ in U, and $\mathrm{Td}^\gamma(X,E)$ denotes the equivariant Todd class of the holomorphic V-bundle E. Now as in the usual case the equivariant Todd class [1], and hence, the right hand side of 4.4 is a polynomial in r of degree $\leq 2n+1$, and the left hand side is the arithmetic genus $=\sum_i(-1)^ih^i(\mathcal{L}^{\frac{r}{2}})$. Using the Kodaira-Baily vanishing theorem the argument in [24] goes through as before to show that $h^0(\mathcal{L})=(n+1)(2n+3)$. Now given the complex contact form θ there is a well known isomorphism between the Lie algebra $\mathfrak{c}$ of infinitesimal complex contact transformations and the holomorphic sections of $\mathcal{L}$ given by $X\mapsto\theta(X)$. A slight modification of Salamon's argument [36] will give the result. First, by compactness the Lie algebra $\mathfrak{c}$ integrates to a complex Lie group action on $\mathcal{Z}$; second, Matsushima reductiveness theorem holds for positive Kähler-Einstein orbifolds [12], so there is a compact isometry group G on $\mathcal{Z}$ of dimension $(n+1)(2n+3)$. Thus, the linear isotropy group G_0 has dimension greater than or equal to $(n+1)(2n+3)-2(2n+1)=2n^2+n+1$. Passing to the quaternionic Kähler orbifold shows as in [36] that $G_0=Sp(n)\times U(1)$. It follows by dimension counting that the isometry group G acts locally transitively on $\mathcal{Z}$. But since $\mathcal{Z}$ is simply connected G acts transitively on $\mathcal{Z}$ and from Boothby's list [10] we see that $G=Sp(n+1)$ and $\mathcal{Z}=\mathbb{P}^{2n+1}$. ∎

This Proposition allows us to give an extension of the Marchiafava-Romani class to orbifolds. We assign to each twistor space $\mathcal{Z}$ an element $\varepsilon(\mathcal{Z})\in\mathbb{Z}/2$ by defining $\varepsilon(\mathcal{Z})=0$ if $\mathcal{L}$ has a square root in $\mathrm{Pic}^{orb}(\mathcal{Z})$ and $\varepsilon(\mathcal{Z})=1$ otherwise. Of course, Proposition 4.3 says that $\varepsilon=1$ unless $\mathcal{Z}=\mathbb{P}^{2n+1}$. We now wish to formulate a converse to the construction in the beginning of the section.

Definition 4.5: *A complete $\mathbb{Q}$-factorial Fano contact variety $\mathcal{Z}$ is said to be good if the* *total space of the principal circle bundle $\mathcal{S}$ associated with the contact V-line bundle $\mathcal{L}$ is a smooth compact manifold.*

Thus, for good $\mathbb{Q}$-factorial Fano contact varieties, $\mathcal{S}$ desingularizes $\mathcal{Z}$. Notice also that in this case $\mathcal{S}$ is necessarily compact. We now are ready for:

THEOREM 4.6: *A good $\mathbb{Q}$-factorial Fano contact variety $\mathcal{Z}$ is the twistor space associated to a compact 3-Sasakian manifold if and only if it admits a compatible Kähler-Einstein metric h.*

PROOF: Let $\mathcal{Z}$ be a good $\mathbb{Q}$-factorial Fano contact variety with a compatible Kähler-Einstein metric h. Choose the scale of h so that the scalar curvature is $8(2n+1)(n+1)$. Let $\pi : \mathcal{S} \to \mathcal{Z}$ denote the principal orbifold circle bundle associated to $\mathcal{L}$. It is a smooth compact submanifold embedded in the dual of the contact V-line bundle $\mathcal{L}^{-1}$. The Kähler-Einstein metric h has Ricci form $\rho = 4(n+1)\omega$, where ω is the Kähler form on $\mathcal{Z}$, and ρ represents the first Chern class of K^{-1}. Now K^{-1} has $\mathcal{L}$ as an $(n+1)$-st root in $\mathrm{Pic}^{orb}(\mathcal{Z})$. Let η^1 be the connection in $\pi : \mathcal{S} \to \mathcal{Z}$ with curvature form $2\pi^*\omega$. Then 2.6 can be used to define the Riemannian metric $g_{\mathcal{S}}$ on $\mathcal{S}$. It is standard (see the proof in Example 1 of section 4.2 in [5]) that $g_{\mathcal{S}}$ is Sasakian-Einstein. By Proposition 2.2.4 of [39] the bundle $\mathcal{L}\otimes\Lambda^{(1,0)}\mathcal{Z}$ has a section θ such that the Kähler-Einstein metric $g_{\mathcal{Z}}$ decomposes as $g_{\mathcal{Z}} = |\theta|^2 + h_D$, where h_D is a metric in the V-bundle D. Let us write $\pi^*\theta = \eta^+$. Since $\mathcal{S}$ is a circle bundle in $\mathcal{L}^{-1}$, the contact bundle $\mathcal{L}$ trivializes when pulled back to $\mathcal{S}$. This together with the condition that $\theta \wedge (d\theta)^n$ is nowhere vanishing on $\mathcal{Z}$ implies that η^+ is a nowhere vanishing complex valued 1-form on $\mathcal{S}$. So the metric $g_{\mathcal{S}}$ on $\mathcal{S}$ can be written as

4.7 $$g_{\mathcal{S}} = (\eta^1)^2 + |\eta^+|^2 + \pi^* h_D.$$

We claim that this metric is 3-Sasakian. To see this consider the total space M of the dual of the contact V-line bundle minus its 0 section which is $\mathcal{S} \times \mathbb{R}^+$. Put the cone metric $dr^2 + r^2 g$ on M. The natural $\mathbb{C}^*$ action on M induces homotheties of this metric. Now using a standard Weitzenböck argument, LeBrun [28] shows that M has a parallel holomorphic symplectic structure and his argument works just as well in our case. Let ϑ denote the pullback of the contact form θ to M which is a holomorphic 1-form on M that is homogeneous of degree 1 with respect to the $\mathbb{C}^*$ action. Thus $\Upsilon = d\vartheta$ is a holomorphic symplectic form on M which is parallel with respect to the Levi-Civita connection of the cone metric. Hence, $(M, dr^2 + r^2 g)$ is hyperkähler. Furthermore, if $\{I^a\}_{a=1}^3$ denote hyperkähler endomorphisms on M, ϑ^2, ϑ^3 are the real and imaginary parts of ϑ, and ϑ^1 is the pullback of η^1 to M, then LeBrun shows that

$$\vartheta^1 I^1 = \vartheta^2 I^2 = \vartheta^3 I^3.$$

It then follows from our previous work [6] that g is 3-Sasakian. But by construction $\mathcal{Z}$ is the space of leaves of the foliation generated by ξ^1, so $\mathcal{Z}$ must be the twistor space of the compact 3-Sasakian manifold $\mathcal{S}$. ∎

Henceforth, by a *twistor space* we shall mean a $\mathbb{Q}$-factorial Fano contact variety with a Kähler-Einstein metric of scalar curvature $8(2n+1)(n+1)$. Our previous results immediately give

Corollary 4.8: *If $\varepsilon(\mathcal{Z}) = 1$ there is a one-to-one correspondence between compact 3-Sasakian manifolds $\mathcal{S}$ and their twistor spaces $\mathcal{Z}$, whereas if $\varepsilon(\mathcal{Z}) = 0$ then $\mathcal{Z} = \mathbb{P}^{2n+1}$ and there are precisely two 3-Sasakian manifolds with this twistor space, S^{4n+3} and $\mathbb{RP}^{4n+3}$.*

We now have the following generalization of LeBrun's Theorem:

Corollary 4.9: *Let $\mathcal{Z}$ be a good $\mathbb{Q}$-factorial Fano contact variety. Then $\mathcal{Z}$ is the twistor space of a good quaternionic Kähler orbifold of positive scalar curvature if and only if it admits a Kähler-Einstein metric.*

Another consequence of our result is:

Proposition 4.10: *Let $\mathcal{Z}$ be the twistor space of a 3-Sasakian manifold $\mathcal{S}$ and suppose that $\varepsilon(\mathcal{Z}) = 1$. Then if $\pi_1^{orb}(\mathcal{Z}) = 0$ the 3-Sasakian manifold $\mathcal{S}$ is simply connected.*

Proof: Suppose that $\mathcal{S}$ is not simply connected, then there exist a simply connected 3-Sasakian manifold $\tilde{\mathcal{S}}$ covering $\mathcal{S}$ nontrivially. Let $\tilde{\mathcal{Z}}$ be the twistor space of $\tilde{\mathcal{S}}$. Then there is a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{S}} & \longrightarrow & \tilde{\mathcal{Z}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{S} & \longrightarrow & \mathcal{Z}. \end{array}$$

By Proposition 3.3 $\pi_1^{orb}(\tilde{\mathcal{Z}}) = 0$ and by hypothesis $\pi_1^{orb}(\mathcal{Z}) = 0$. But then it follows from Thurston's uniqueness of universal orbifold covers [41] that $\tilde{\mathcal{Z}} \simeq \mathcal{Z}$. But this contradicts Corollary 4.8. ∎

We end this section with a brief description of a contact reduction procedure due to Hitchin [18] which when combined with Kirwan's geometric invariant theory [23,32] allows one to compute the rational cohomology of $\mathcal{Z}$. When applied to the twistor space of a 3-Sasakian manifold, this complex contact reduction technique fits together in the expected way with both 3-Sasakian [7] and quaternionic Kähler reduction [14]. Given a compact 3-Sasakian manifold $\mathcal{S}$, let $K \subset \mathrm{Aut}_0(\mathcal{S})$ be a fixed subgroup of the group of 3-Sasakian isometries. We assume that K acts freely on the zero level set of the 3-Sasakian moment map, in which case the 3-Sasakian reduction $\mathcal{S}/\!\!/\!\!/K$ is a compact manifold. The complexification $K^{\mathbb{C}}$ of K acts on the twistor space $\mathcal{Z}$ of $\mathcal{S}$ as biholomorphic transformations preserving the complex contact structure θ. One can then define a holomorphic K-equivariant moment map $\mu : \mathcal{Z} \longrightarrow \mathfrak{k}^* \otimes \mathcal{L}$ by $< \mu, X >= \theta(X)$, where $X \in \mathfrak{k}$ the Lie algebra of K. This moment map is essentially the push forward of the component $\mu_{\mathcal{S}}^- = \mu_{\mathcal{S}}^2 - i\mu_{\mathcal{S}}^3$ of the 3-Sasakian moment map $\mu_{\mathcal{S}}$ defined in [7]. Let $Z \subset \mathcal{Z}$ denote the zero set of μ. Now K acts on Z and there is an ordinary symplectic (Kähler) moment map $\mu_1 : Z \longrightarrow \mathfrak{k}^*$, which is the push forward of $\mu_{\mathcal{S}}^1$ of [7]. The reduced space $\mu_1^{-1}(0)/K$ is a compact Kähler-Einstein orbifold. In the case that Z is smooth there is a well known theorem of Kirwan which states that there is a homeomorphism $Z^{ss}/K^{\mathbb{C}} \simeq \mu_1^{-1}(0)/K$, where Z^{ss} denotes the subset of semistable points of Z. This then is identified with the reduced twistor space $\tilde{\mathcal{Z}} = \mathcal{Z}/\!\!/\!\!/K$ of the reduced 3-Sasakian manifold $\mathcal{S}/\!\!/\!\!/K$. Then using rational equivariant cohomology theory Kirwan [23] proves that there is an isomorphism of Abelian groups

4.11 $$H^*_K(Z,\mathbb{Q})\cong H^*(Z,\mathbb{Q})\otimes H^*(BK,\mathbb{Q}),$$

where BK is the classifying space of K and H^*_K denotes equivariant cohomology. Furthermore, there is a surjective map $H^*_K(Z,\mathbb{Q})\longrightarrow H^*_K(Z^{ss},\mathbb{Q})\simeq H^*(\tilde{\mathcal{Z}},\mathbb{Q})$, so one can use this procedure to determine the rational cohomology of $\tilde{\mathcal{Z}}$ [23,32].

§5. The Weighted Flag Varieties $\mathcal{Z}(\mathbf{p})$

We recall how the manifolds $\mathcal{S}(\mathbf{p})$ can be obtained by 3-Sasakian reduction [7]. All known examples of 3-Sasakian manifolds are either homogeneous or obtained by 3-Sasakian reduction from the sphere S^{4n-1}, or are finite quotients of such [6,7]. Let $\mathbf{u}=(u_1,...,u_n)\in\mathbb{H}^n$, be the quaternionic coordinate on the n-dimensional quaternionic vector space, equipped with the flat metric. We consider the unit sphere in $\mathbb{H}^n\supset S^{4n-1}(1)=\{\mathbf{u}\in\mathbb{H}^n\mid \sum_{\alpha=1}^n\overline{u}_\alpha u_\alpha=1\}$, with its standard 3-Sasakian structure from the right. The subgroup of the connected isometry group $\mathrm{Isom}(S^{4n-1})=SO(4n)$ which preserves this 3-Sasakian structure is $Sp(n)\cdot Sp(1)$. If we represent the quaternionic coordinates $\mathbf{u}$ as a column vector, the action of this group on the S^{4n-1} is the $\mathbb{Z}/2$ central quotient of the action $\big((\mathbb{A},\sigma);\mathbf{u}\big)\longrightarrow\mathbb{A}\mathbf{u}\sigma$, where $\mathbb{A}\in Sp(n)$ is a quaternionic $n\times n$ matrix of the defining representation of $Sp(n)$, and $\sigma\in Sp(1)$ is a unit quaternion. Fixing a direction $\sigma\in Sp(1)$ fixes a CR structure on S^{4n-1} and hence, a complex structure on the corresponding twistor space $\mathbb{P}^{2n-1}$. The subgroup of $Sp(n)\cdot Sp(1)$ which stabilizes this direction is $\mathrm{Aut}_0(S^{4n-1})=Sp(n)$. Choosing any closed subgroup $K\subset Sp(n)$ then gives rise to a 3-Sasakian moment map $\mu: S^{4n-1}\longrightarrow\mathfrak{k}^*\otimes\mathbb{R}^3$, where $\mathfrak{k}^*$ denotes the dual of the Lie algebra $\mathfrak{k}$ of K. If K is a general circle subgroup of the maximal torus T^n in $Sp(n)$, then the action is determined by a *weight vector* $\mathbf{p}=(p_1,...,p_n)\in\mathbb{Z}^n$. Explicitly, we have $\theta_{\mathbf{p}}: S^1\times S^{4n-1}\longrightarrow S^{4n-1}$, where

5.1 $$\theta_{\mathbf{p}}(\tau,\mathbf{u})=(\tau^{p_1}u_1,\ldots,\tau^{p_n}u_n).$$

The circle action (5.1) gives the 3-Sasakian moment map $\mu(\mathbf{p}): S^{4n-1}\longrightarrow\mathbb{R}^3$ as

5.2 $$\begin{aligned}2\mu^1(\mathbf{p})(\mathbf{z},\mathbf{w}) &= \sum_{j=1}^n p_j(|z_j|^2-|w_j|^2),\\ \mu^-(\mathbf{p})(\mathbf{z},\mathbf{w}) &= \sum_{j=1}^n p_j\bar{w}_j z_j,\quad \text{where } \mu^-=\mu^2-i\mu^3,\end{aligned}$$

where we identify $\mathbb{H}^n\simeq\mathbb{C}^n\times\overline{\mathbb{C}^n}\ni(\mathbf{z},\overline{\mathbf{w}})$ by $\mathbf{u}=\mathbf{z}+j\overline{\mathbf{w}}$. The zero level set of the moment map $\mu(\mathbf{p})$ defined by $N(\mathbf{p})=\mu(\mathbf{p})^{-1}(0)$ is a real codimension 3 compact submanifold in S^{4n-1} diffeomorphic to the Stiefel manifold $U(n)/U(n-2)=N(\mathbf{1})$ of complex 2-frames in $\mathbb{C}^n$. If all the weights are non-zero and pairwise relatively prime then the quotient of

$N(\mathbf{p})$ by the circle action (5.2) is a manifold $\mathcal{S}(\mathbf{p})$. The geometry and topology of these spaces were discussed in [7]. We mention that the Weyl group $\mathcal{W} \simeq \Sigma_n \rtimes (\mathbb{Z}/2)^n$ of $Sp(n)$ acts on the weight space, so if $w \in \mathcal{W}$ then $\mathcal{S}(w\mathbf{p}) \simeq \mathcal{S}(\mathbf{p})$ as 3-Sasakian manifolds. Here we want to describe the twistor space $\mathcal{Z}(\mathbf{p})$ associated to $\mathcal{S}(\mathbf{p})$.

The twistor space can be viewed as the leaf space of a T^2-torus action $A(\mathbf{p})$ on the Stiefel manifold $N(\mathbf{p})$ given by

5.3
$$\begin{pmatrix} z_1 & \cdots & z_n \\ w_1 & \cdots & w_n \end{pmatrix} \xrightarrow{(\tau,\rho)} \begin{pmatrix} \tau^{p_1} z_1 \rho & \cdots & \tau^{p_n} z_n \rho \\ \tau^{p_1} w_1 \overline{\rho} & \cdots & \tau^{p_n} w_n \overline{\rho} \end{pmatrix},$$

where $(\tau, \rho) = (e^{2\pi i t}, e^{2\pi i s})$, $t, s \in [0, 1)$ are the complex coordinates on a 2-torus. There is a real structure on the twistor space $\mathcal{Z}(\mathbf{p})$ obtained from left multiplication by the second quaternionic unit j on the quaternionic vector $\mathbf{u} \in \mathbb{H}^n$ or in terms of the complex coordinates

5.4
$$\mathcal{J}(\mathbf{z}, \mathbf{w}) = (-\overline{\mathbf{w}}, \overline{\mathbf{z}}).$$

Now $\mathcal{J}$ intertwines the equivalent actions $A(\mathbf{p})$ and $A(-\mathbf{p})$, so the order 4 map $\mathcal{J}$ on $N(\mathbf{p})$ passes to an involution $\mathcal{J}$ on the twistor space $\mathcal{Z}(\mathbf{p})$.

In [7] we wrote $\mathcal{S}(\mathbf{p})$, $\mathbf{p} \in (\mathbb{Z}^+)^n$, as a certain biquotient of the unitary group $U(n)$. Similarly we can write $\mathcal{Z}(\mathbf{p}) = U(1)_{\mathbf{p}} \backslash U(n)/U(n-2) \times U(1)$, where $U(1)_{\mathbf{p}} \subset U(n)_L$ and $U(n-2) \times U(1) \subset U(n)_R$. The action is given by the formula

$$\mathbb{W} \xrightarrow{(\tau,\mathbb{B},\mathbb{A})} \begin{pmatrix} \tau^{p_1} & & \\ & \ddots & \\ & & \tau^{p_n} \end{pmatrix} \mathbb{W} \begin{pmatrix} \mathbb{A} & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix}.$$

Here $\mathbb{W} \in U(n)$ and $(\tau, \mathbb{B}, \mathbb{A}) \in U(1) \times U(n-2) \times U(1)$, with $\mathbb{A} = \begin{pmatrix} \rho & 0 \\ 0 & \bar{\rho} \end{pmatrix}$. In particular $\mathcal{Z}(\mathbf{1})$ is homogeneous and it is the complex flag manifold $F_{1,2,n}$. It is therefore natural to refer to the general $\mathcal{Z}(\mathbf{p})$ as *weighted flag varieties*. In fact, they have the same rational cohomology as $F_{1,2,n}$. Using Theorem 2.4 and Poincaré duality together with the fact that the full integral cohomology of $\mathcal{S}(\mathbf{p})$ was calculated in [7] we have

THEOREM 5.5 : *For each $\mathbf{p} \in (\mathbb{Z}^+)^n$ such that $\gcd(p_i, p_j) = 1$ for $i \neq j$, we have isomorphisms of Abelian groups*

$$H^*(\mathcal{Z}(\mathbf{p}); \mathbb{Q}) \simeq H^*(\mathcal{Z}(\mathbf{1}); \mathbb{Q}) = H^*(F_{1,2,n}; \mathbb{Q}).$$

In particular, $b_2(\mathcal{Z}(\mathbf{p})) = 2$ and Pic $(\mathcal{Z}(\mathbf{p})) = \mathbb{Z} \oplus \mathbb{Z}$. Moreover, $\pi_1^{orb}(\mathcal{Z}(\mathbf{p})) = 0$.

There is an alternative proof of Theorem 5.5 that avoids knowing the cohomology of the 3-Sasakian manifold $\mathcal{S}(\mathbf{p})$. In fact this method together with 2.4 then gives the rational cohomology of $\mathcal{S}(\mathbf{p})$ independently from the calculations in [7]. The method uses the complex contact reduction procedure together with Kirwan's geometric invariant theory [23,32] discussed at the end of section 4. The moment map 5.2 is now interpreted as an object on the complex projective space $\mathbb{P}^{2n-1}$ with homogeneous coordinates $[z_1 : \cdots : z_n : \bar{w}_1 : \cdots : \bar{w}_n]$. Now μ^1 is a function on $\mathbb{P}^{2n-1}$, whereas μ^- is a holomorphic section of the line bundle $\mathcal{L} \simeq \mathcal{O}(2)$. The zero set of μ^- is just the complex quadric $Q \subset \mathbb{P}^{2n-1}$, and the subset Q^{ss} of semi-stable points is $Q - \mathbb{P}^{n-1} \sqcup \mathbb{P}^{n-1}$. So by Kirwan's Theorem $\mathcal{Z}(\mathbf{p}) \simeq Q^{ss}/\mathbb{C}^*$. Moreover, all quadrics are equivalent and it is easy to see from [23, pg 105] that the semi-stable points are independent of $\mathbf{p}$. So the isomorphism of the rational cohomology groups follows from the discussion at the end of section 4. The orbifold simple connectivity of the weighted flag varieties $\mathcal{Z}(\mathbf{p})$ follows immediately from Proposition 3.3 and the fact [7] that the 3-Sasakian manifolds $\mathcal{S}(\mathbf{p})$ are simply connected. ∎

REMARK 5.6: The condition that the components of $\mathbf{p}$ be pairwise relatively prime in Theorem 5.5 is in no way essential as can be seen from the second proof. If $\mathbf{p} \in (\mathbb{Z}^+)^n$ is an arbitrary n-tuple of non-zero integers, the weighted flag varieties $\mathcal{Z}(\mathbf{p})$ are still well-defined. Moreover, it is easy to see that $\mathcal{Z}(\mathbf{p})$ is independent of the common g.c.d. of the n-tuple $(p_1, \cdots, p_n)$, so without loss of generality we set $\gcd(p_1, \ldots, p_n) = 1$. In this more general case $\mathcal{S}(\mathbf{p})$ is no longer a manifold, but an orbifold. Nevertheless, by applying the homotopy exact sequence of [17] to the T^2 and S^1 actions on $N(\mathbf{p})$ we get both $\pi_1^{orb}(\mathcal{Z}(\mathbf{p})) = \pi_1^{orb}(\mathcal{S}(\mathbf{p})) = 0$.

Using either the equivalence results of [8] or the singularity analysis in the next section, it is not difficult to prove

THEOREM 5.7: *Let $\mathbf{p}, \mathbf{p}' \in (\mathbb{Z}^+)^n$ and let $\mathcal{Z}(\mathbf{p})$ and $\mathcal{Z}(\mathbf{p}')$ be twistor spaces. Then they are equivalent as complex orbifolds if and only if $\mathbf{p}' = \sigma\mathbf{p}$ for some $\sigma \in \Sigma_n$, where Σ_n is the symmetric group on n-letters.*

In [8] it was shown that $N(\mathbf{p})$ has a natural hypercomplex structure, and the connected component of the group of automorphisms of this complex structure was computed. Moreover, this was used [9] to compute the connected component $\mathrm{Aut}_0(\mathcal{S}(\mathbf{p}))$ of the group of 3-Sasakian automorphisms of the manifolds $\mathcal{S}(\mathbf{p})$. It was shown that $\mathrm{Aut}_0(\mathcal{S}(\mathbf{p})) = S(U(k) \times U(1)^{n-k})$, where k is the number of 1's in $\mathbf{p}$, and S denotes removing the central $U(1)$ subgroup determined by the action 5.1. Using this it is not difficult to prove:

THEOREM 5.8: *Let $\mathcal{Z}(\mathbf{p})$ be the twistor space of the compact 3-Sasakian manifold $\mathcal{S}(\mathbf{p})$ of dimension $4n-5$. Then the connected component $K_0(\mathcal{Z}(\mathbf{p}))$ of the group of automorphisms of $\mathcal{Z}(\mathbf{p})$ preserving both the complex contact and the Kähler structures is isomorphic to $\mathrm{Aut}_0(\mathcal{S}(\mathbf{p})) = S(U(k) \times U(1)^{n-k})$, where k is the number of 1's in $\mathbf{p}$.*

We note that it is easy to generalize this result to those orbifolds discussed in Remark 5.6.

Simply connected 3-Sasakian manifolds admit a lot of discrete quotients. For the 3-Sasakian homogeneous spaces $\mathcal{S} = G/K$ classified in [7] all such quotients are of the form $\Gamma\backslash\mathcal{S} = \Gamma\backslash G/K$, where $\Gamma \subset G$ is a discrete subgroup of the group of 3-Sasakian isometries $G = \mathrm{Aut}_0(G/K)$. The twistor space of $\Gamma\backslash G/K$ is $\Gamma\backslash G/K \cdot U(1)$, *i.e.,* $\Gamma\backslash\mathcal{Z}$, where $\mathcal{Z} = G/K \cdot U(1)$ is the twistor space of $\mathcal{S}$. Clearly, the fundamental group of $\Gamma\backslash\mathcal{S}$ is just Γ and the orbifold fundamental group of $\Gamma\backslash\mathcal{Z}$ is given by Proposition 4.2.

The spherical space forms $\Gamma\backslash S^{4n-1}$, where $\Gamma \subset Sp(n)$, were considered already by Sasaki [37]. However, we point out that its is not just the 3-Sasakian spheres that admit such discrete quotients. Below we give an example of some discrete quotients of the 3-Sasakian space $\mathcal{S}(1,1,k)$. Let Γ be the cyclic subgroup of order m of $\mathrm{Aut}_0(\mathcal{S}(\mathbf{p}))$ defined by the action on the quaternionic coordinates by $(u_1,u_2,u_3) \mapsto (\zeta u_1, u_2, \bar{\zeta} u_1)$, where ζ is a primitive of $\mathbb{Z}/m$. Now we restrict this action to $N(1,1,k)$ and notice that the condition to have fixed points on $\mathcal{S}(1,1,k)$ is

$$\text{5.9} \qquad \begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} \zeta\tau z_1 & \tau z_2 & \bar{\zeta}\tau^k z_3 \\ \zeta\tau w_1 & \tau w_2 & \bar{\zeta}\tau^k w_3 \end{pmatrix}.$$

From this it is not difficult to see that $\Gamma \simeq \mathbb{Z}/m$ acts on $\mathcal{S}(1,1,k)$ without fixed points if m is chosen so that $\gcd(m, k+1) = 1$. For such m we have 3-Sasakian manifolds $\mathcal{S}(1,1,k)/\Gamma$ with $\pi_1 = \mathbb{Z}/m$. It is straightforward to check that this Γ action induces an effective action of Γ on $\mathcal{Z}(1,1,k)$. Thus, $\pi_1^{orb}(\mathcal{Z}(1,1,k)/\Gamma) = \mathbb{Z}/m$ when m is subject to the above conditions. Note that when $k=1$ the condition on m reduces to it being odd. In this case we get examples of $\mathbb{Z}/m$ quotients of the homogeneous 3-Sasakian manifold $\mathcal{S}(1,1,1) \simeq U(3)/U(1)\times U(1)$ by a cyclic subgroup Γ of its isometry group $SU(3)$. Many further examples of quotients of $\mathcal{S}(\mathbf{p})$ can be easily worked out in a similar way.

§6. The Singular Locus of $\mathcal{Z}(\mathbf{p})$

We now want to describe the singular locus $\Sigma(\mathbf{p})$ of the twistor space $\mathcal{Z}(\mathbf{p})$. As with any group action there is a natural stratification of the twistor space $\mathcal{Z}(\mathbf{p})$ defined by the toral action.

Definition 6.1: *We say that two points $x, y \in \mathcal{Z}(\mathbf{p})$ are in the same stratum if their isotropy groups (leaf holonomy groups) Γ_x and Γ_y are conjugate by the action $A(\mathbf{p})$ of T^2.*

Of course, since T^2 is Abelian this means that the isotropy groups actually coincide. Furthermore, the smooth locus $\mathcal{Z}_0(\mathbf{p})$ comprises the dense open subset of points of $\mathcal{Z}$ whose isotropy subgroup is the identity. The equation for fixed points reads

$$\text{6.2} \qquad \begin{pmatrix} z_1 & \cdots & z_n \\ w_1 & \cdots & w_n \end{pmatrix} = \begin{pmatrix} \tau^{p_1} z_1 \rho & \cdots & \tau^{p_n} z_n \rho \\ \tau^{p_1} w_1 \overline{\rho} & \cdots & \tau^{p_n} w_n \overline{\rho} \end{pmatrix}.$$

First we prove a general result about the set of points with non-trivial isotropy.

Lemma 6.3 : *Let $\mathbf{p} \in (\mathbb{Z}^+)^n$ such that $\gcd(p_i,p_j) = 1$ for all $i \neq j$ and suppose that $(\mathbf{z},\mathbf{w}) \in N(\mathbf{p})$ is on a singular T^2-orbit. There are two cases:*

(i) If all the weights p_i are odd, then for each $i = 1, ..., n$ either $z_i = 0$ or $w_i = 0$.

(ii) If one of the weights, say p_1, is even, then either $z_1 = 0$ or $w_1 = 0$, and if $(z_1, w_1) \neq (0,0)$ then for each $i = 2, ..., n$ either $z_i = 0$ or $w_i = 0$.

PROOF : We prove case (i) only, case (ii) is proven similarly. Suppose not. Then for some $i \neq j$ we have

$$\tau^{p_i}\rho = \tau^{p_i}\overline{\rho} = 1, \qquad \tau^{p_j}\rho = \tau^{p_j}\overline{\rho} = 1$$

which yields $\rho^2 = 1$, and $\tau^{p_j} = \tau^{p_i}$. As $\gcd(p_i, p_j) = 1$ and both are odd odd we get $(\tau, \rho) = \{(1,1), (-1,-1)\}$ as the only solution. But this is exactly the $\mathbb{Z}/2$ factor that acts trivially on $\mathcal{Z}(\mathbf{p})$. ∎

For simplicity, let us consider the $n = 3$ case, that is $\mathcal{Z}(p_1, p_2, p_3)$ is a complex 3-fold with orbifold singularities. The general case can be treated similarly and we will discuss it later. In view of Lemma 6.3 we consider two cases. In the first case all of p_1, p_2, and p_3 are odd and $(-1,-1) \in T^2$ fixes every point of $N(\mathbf{p})$. We will factor out this $\mathbb{Z}/2$ to pass to an effective action of the quotient group. In the second case we take p_1 to be even.

Let us consider the case of all odd weights first. We introduce the following notation: Let $i, j = 1, 2, 3$. We define

6.4 $$p_i^j = \frac{1}{2}(p_i + p_j) \qquad i \neq j,$$

and

6.5 $$d_{ik}^j = \gcd(p_i^j, p_k^j), \qquad d_j^{ik} = \gcd(p_j^i, p_j^k).$$

As we will show later, the six integers $p_1^2 = p_2^1$, $p_1^3 = p_3^1$, $p_3^2 = p_2^3$, $d_{23}^1 = d_1^{23}$, $d_{13}^2 = d_2^{13}$, $d_{12}^3 = d_3^{12}$ are the orders of the cyclic groups associated to the orbifold stratification of $\mathcal{Z}(p_1, p_2, p_3)$. We can write $\mathcal{Z}(p_1, p_2, p_3)$ as

$$\mathcal{Z}(p_1, p_2, p_3) = \mathcal{Z}_0(p_1, p_2, p_3) \cup \Sigma(p_1, p_2, p_3),$$

where $\mathcal{Z}_0(p_1, p_2, p_3)$ denotes the regular part and $\Sigma(p_1, p_2, p_3)$ is the singular locus. The singular strata of $\Sigma(p_1, p_2, p_3)$ can be described as follows: Let

6.6 $$\tilde{\Sigma}_2^1 = \left\{ \begin{pmatrix} z_1 & 0 & 0 \\ 0 & w_2 & 0 \end{pmatrix} \middle| \quad |z_1|^2 = \frac{p_2}{p_1 + p_2}, \quad |w_2|^2 = \frac{p_1}{p_1 + p_2} \right\} \simeq T^2.$$

Then all the points on Σ_1^2 have the same isotropy subgroup described by

6.7 $$\Gamma_2^1 = \{(\tau, \rho) \in T^2 | \quad \tau^{p_1 + p_2} = 1, \quad \rho = \tau^{p_2}\}$$

which, after dividing by the non-effective $\mathbb{Z}/2$, is isomorphic to the cyclic group $\mathbb{Z}/p_1^2$. Similarly, let

6.8 $$\tilde{\Sigma}_1^2 = \left\{ \begin{pmatrix} 0 & z_2 & 0 \\ w_1 & 0 & 0 \end{pmatrix} \middle| \quad |z_2|^2 = \frac{p_1}{p_1 + p_2}, \quad |w_1|^2 = \frac{p_2}{p_1 + p_2} \right\} \simeq T^2,$$

with

$$\text{6.9} \qquad \Gamma_1^2 = \{(\tau, \rho) \in T^2 | \quad \tau^{p_1+p_2} = 1, \quad \rho = \tau^{p_1}\}.$$

Note that although both groups are abstractly isomorphic to the cyclic group of order $p_1^2 = p_2^1$, they are not the same. As T^2 is Abelian, they cannot be conjugate in T^2. We define $\tilde{\Sigma}_1^3$, $\tilde{\Sigma}_3^1$, $\tilde{\Sigma}_2^3$, and $\tilde{\Sigma}_3^2$ and the corresponding isotropy subgroups in a similar way. Again, note that abstractly both Γ_j^i and Γ_i^j are abstractly isomorphic to a cyclic group of order $p_i^j = p_i^j$. But they are never conjugate unless they are trivial. All of the sets described so far are single T^2 orbits so they correspond to a point in the twistor space. Notice that under the $\mathcal{J}$-map defined by 5.4 we have $\mathcal{J}(\tilde{\Sigma}_i^j) = \tilde{\Sigma}_j^i$. Next, consider

$$\text{6.10} \qquad \tilde{\Sigma}_3^{12} = \left\{ \begin{pmatrix} z_1 & z_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix} \middle| \quad \begin{matrix} (p_1+p_3)|z_1|^2 + (p_2+p_3)|z_2|^2 = p_3 \\ |w_3|^2 = 1 - |z_1|^2 - |z_2|^2, z_1 \neq 0, z_2 \neq 0 \end{matrix} \right\}.$$

Then all the points on $\tilde{\Sigma}_3^{12}$ are fixed by the isotropy group

$$\text{6.11} \qquad \Gamma_3^{12} = \{(\tau, \rho) \in T^2 | \quad \tau^{p_1+p_3} = \tau^{p_2+p_3} = 1, \quad \rho = \tau^{p_3}\}.$$

Similarly, we have

$$\text{6.12} \qquad \tilde{\Sigma}_{12}^3 = \left\{ \begin{pmatrix} 0 & 0 & z_3 \\ w_1 & w_2 & 0 \end{pmatrix} \middle| \quad \begin{matrix} (p_1+p_3)|w_1|^2 + (p_2+p_3)|w_2|^2 = p_3 \\ |z_3|^2 = 1 - |w_1|^2 - |w_2|^2, \ w_1 \neq 0, \ w_2 \neq 0 \end{matrix} \right\},$$

on which every point has isotropy group

$$\text{6.13} \qquad \Gamma_{12}^3 = \{(\tau, \rho) \in T^2 | \quad \tau^{p_1+p_3} = \tau^{p_2+p_3} = 1, \quad \overline{\rho} = \tau^{p_3}\}.$$

Again, note that both Γ_3^{12} and Γ_{12}^3 are abstractly isomorphic to a cyclic group of order $d_3^{12} = d_{12}^3$, but they are not conjugate in T^2. We define Γ_{ij}^k, $\tilde{\Sigma}_{ij}^k$, Γ_k^{ij}, and $\tilde{\Sigma}_k^{ij}$ for $k = 1, 2$ in a similar fashion. Note that $\mathcal{J}(\tilde{\Sigma}_{ij}^k) = \tilde{\Sigma}_k^{ij}$. Also the vertices and the edges are put together so that, for example,

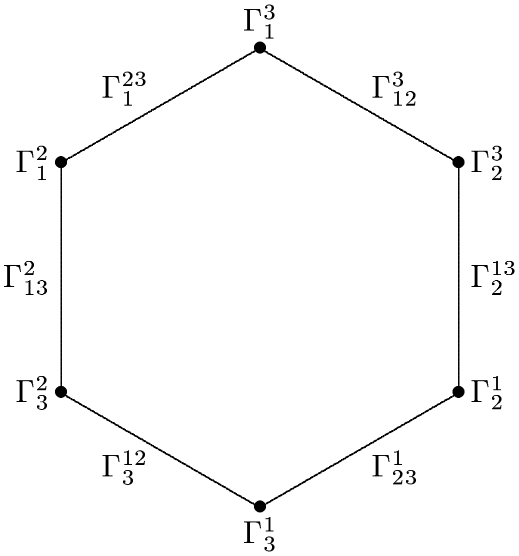
$$\text{6.14} \qquad N(\Gamma_k^{ij}) \equiv \tilde{\Sigma}_k^i \cup \tilde{\Sigma}_k^{ij} \cup \tilde{\Sigma}_k^j \simeq S^3 \times S^1$$

is the Hopf surface with another copy of it obtained as the image of the $\mathcal{J}$-map of 5.4. Thus on $N(p_1, p_2, p_3)$ we have six copies of the Hopf surface $S^3 \times S^1$ intersecting at 2-tori and paired by the $\mathcal{J}$-map.

Lemma 6.3 shows that in this case there are no more singular orbits. If we define $\Sigma_i^j = \tilde{\Sigma}_i^j / T^2$, $\Sigma_i^{jk} = \tilde{\Sigma}_i^{jk} / T^2$, and $\Sigma_{jk}^i = \tilde{\Sigma}_{jk}^i / T^2$ then, on $\mathcal{Z}(\mathbf{p})$ we can write

$$\text{6.15} \quad \Sigma(p_1, p_2, p_3) = \Sigma_1^2 \cup \Sigma_1^{23} \cup \Sigma_1^3 \cup \Sigma_{12}^3 \cup \Sigma_2^3 \cup \Sigma_2^{13} \cup \Sigma_2^1 \cup \Sigma_{23}^1 \cup \Sigma_3^1 \cup \Sigma_3^{12} \cup \Sigma_3^2 \cup \Sigma_2^{13}.$$

Naturally, if any of the isotropy groups associated with Σ_i^j or Σ_i^{jk} is trivial that set will be a part of $\mathcal{Z}_0(p_1, p_2, p_3)$ rather than $\Sigma(p_1, p_2, p_3)$. Even with this in mind (6.15) not always represents the orbifold stratification of $\mathcal{Z}(p_1, p_2, p_3)$, where different strata are defined to be unions of points with conjugate isotropy groups. However, it coincides with such stratification unless $\mathbf{p} = (1, 1, k)$, see Remark 6.17. Schematically, we can describe the whole stratification by a hexagon:



6.16. The singular set for p_1, p_2, p_3 odd

One can think of this diagram in two ways. On one hand it is a representation of the set of points with nontrivial isotropy subgroups in $N(p_1, p_2, p_3)$. On the other hand it is also a representation of the singular locus $\Sigma(p_1, p_2, p_3)$ of the orbifold $\mathcal{Z}(p_1, p_2, p_3)$. It is understood here that, if any of the positive integers attached to the lines and vertices of the hexagon are equal to one, than they should be removed as they would correspond to the regular part.

Remark 6.17 : Depending on the weights, we have six different types of stratifications:

(1) If $\mathbf{p} = (1, 1, 1)$ then the twistors space $\mathcal{Z}(\mathbf{p})$ is a manifold and it is the complex flag $F_{1,2,3} = U(3)/T^3$.

(2) If $\mathbf{p} = (1, 1, k)$, $k > 1$ and odd. In this case there are two isotropy groups, both cyclic of order $\frac{k+1}{2}$. The singular stratum $\Sigma(1, 1, k)$ consists of two disjoint copies of S^2 and its schematic picture can be obtained by removing from the above hexagon two opposite vertices and four edges.

(3) If at least two of the weights are greater than 1, and $d_1^{23} = d_2^{13} = d_3^{12} = 1$ then $\Sigma(p_1, p_2, p_3)$ consists of six isolated points as, for instance, in the case of $\mathcal{Z}(1, 9, 13)$ or $\mathcal{Z}(1, 5, 9)$.

(4) If at least two of the weights are greater than 1 and one of the three integers

$\{d_1^{23}, d_2^{13}, d_3^{12}\}$ is greater than 1, then $\Sigma(p_1, p_2, p_3)$ consists of of six points, two pairs of which are joined by an S^2 as in the case of $\mathcal{Z}(1, 3, 5)$.

(5) If two of the three integers $\{d_1^{23}, d_2^{13}, d_3^{12}\}$ are greater than 1, then $\Sigma(p_1, p_2, p_3)$ consists of of six points joined by 2-spheres in a way that is given by the diagram 6.16 when we remove two opposite edges. This is the case, for example, for $\mathcal{Z}(\mathbf{p}) = \mathcal{Z}(1, 9, 11)$.

(6) If all three of the integers $\{d_1^{23}, d_2^{13}, d_3^{12}\}$ is greater than 1, then $\Sigma(p_1, p_2, p_3)$ consists of of six points joined by the 2-spheres in a way shown by diagram (6.16), as, for instance, in the case of $\mathcal{Z}(1, 71, 209)$.

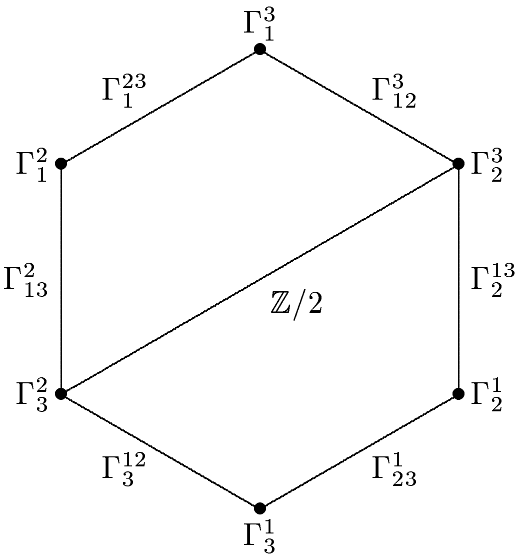
When one of the weights is even, there is one more singular orbit. Let p_1 be even and define

$$p_i^j = (p_i + p_j) \qquad i \neq j,$$

with d_{ik}^j, d_j^{ik} as before. Now, there are always exactly two vertices with for which the orbifold group has even order equal to $p_2 + p_3$. These two vertices are always connected by a 2-sphere on which the isotropy group is just $\mathbb{Z}/2$. The singular orbit on $N(p_1, p_2, p_3)$ can be described as

$$6.18 \quad \tilde{\Sigma}(\mathbb{Z}/2) = \left\{ \begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \in N(p_1, p_2, p_3) \middle| \quad w_1 = z_1 = 0 \quad \begin{array}{c} (z_2, w_3) \neq (0, 0) \\ (z_3, w_2) \neq (0, 0) \end{array} \right\}.$$

This set is mapped to itself by the $\mathcal{J}$-map. Clearly, $\tilde{\Sigma}(\mathbb{Z}/2)$ together with the tori $\tilde{\Sigma}_2^3 \cup \tilde{\Sigma}_3^2$ is another Hopf surface and we obtain the following diagram



6.19. The singular set for p_1 even

REMARK 6.20: In this case we always have at least 6 singular points, two of them necessarily connected by a 2-sphere. The five types of the stratification described in Remark

5.23 all occur in the presence of this additional $\mathbb{Z}/2$ singularity. Again, the $\mathbf{p}=(1,1,k)$, k even is special. The orbifold $\mathcal{Z}(1,1,k)$ has three disjoint strata in $\Sigma(1,1,k)$, one of them has isotropy group $\mathbb{Z}/2$ and the other two $\mathbb{Z}/(k+1)$ (but the two groups are not conjugate).

In view of the Lemma 6.3 we can describe the general case in the same way by introducing top dimensional "strata". Take the n-element set $\{1,2,...,n\}$ and divide it into two non-empty subsets

6.21 $$\{1,2,...,n\}=\{\alpha_1,...,\alpha_k\}\cup\{\beta_1,...,\beta_j\}=\alpha\cup\beta.$$

We introduce the subset of $N(\mathbf{p})$

6.22 $$N\big(\Gamma_\alpha^\beta\big)=\left\{\begin{pmatrix} z_1 & \cdots & z_n\\ w_1 & \cdots & w_n\end{pmatrix}\in N(\mathbf{p})\Big|\quad \begin{matrix} w_{\beta_1}=\cdots=w_{\beta_j}=0\\ z_{\alpha_1}=\cdots=z_{\alpha_k}=0\end{matrix}\right\}$$

which consist of points with the following isotropy subgroup

6.23 $$\Gamma_\alpha^\beta=\{(\tau,\rho)\in T^2|\quad \tau^{p_{\beta_1}}\rho=\cdots=\tau^{p_{\beta_j}}\rho=\tau^{p_{\alpha_1}}\overline{\rho}=\cdots=\tau^{p_{\alpha_k}}\overline{\rho}=1\}.$$

In the case when all the weights are odd we take $\Gamma_\alpha^\beta/\mathbb{Z}_2$. In this way we get (2^n-2) "singular sets" $N\big(\Gamma_\alpha^\beta\big)$ which correspond to different partitions of $\{1,...,n\}$ into two non-empty subsets. One can easily see that

6.24 $$N\big(\Gamma_\alpha^\beta\big)\simeq S^{2|\beta|-1}\times S^{2|\alpha|-1}.$$

The isotropy group Γ_α^β may be trivial in which case almost all of the $N\big(\Gamma_\alpha^\beta\big)$ will be in the non-singular stratum. They are paired by the $\mathcal{J}$-map, which corresponds to interchanging upper and lower multi-indices. The top-dimensional singular sets may intersect in a lower dimensional one with the isotropy group of the new set containing the other two groups as subgroups. In particular, let

6.25 $$\{1,2,...,n\}=\alpha\cup\beta=\gamma\cup\delta$$

be two different partitions and let $N\big(\Gamma_\alpha^\beta\big)$, $N\big(\Gamma_\gamma^\delta\big)$ be the corresponding singular sets. Then on the intersection $N\big(\Gamma_\alpha^\beta\star\Gamma_\gamma^\delta\big)=N\big(\Gamma_\alpha^\beta\big)\cap N\big(\Gamma_\gamma^\delta\big)$ we get

6.26 $$(z_i,w_i)=(0,0),\qquad i\in K,$$

where $K=(\alpha\cap\delta)\cup(\beta\cap\gamma)$. Then this intersection is just the product $S^{2(|\beta-\gamma|)-1}\times S^{2(|\alpha-\delta|)-1}$, and the new isotropy group $\Gamma_\alpha^\beta\star\Gamma_\gamma^\delta$ can be obtained from Γ_α^β or Γ_γ^δ by dropping the relations on (τ,ρ) which involve all p_i with $i\in K$:

6.27 $$\Gamma_\alpha^\beta\star\Gamma_\gamma^\delta=\{(\tau,\rho)\in T^2|\quad \tau^{p_{\beta_i}}\rho=\tau^{p_{\alpha_j}}\overline{\rho}=1,\quad\forall\beta_i\in\beta\setminus\gamma,\quad\forall\alpha_j\in\alpha\setminus\delta\}.$$

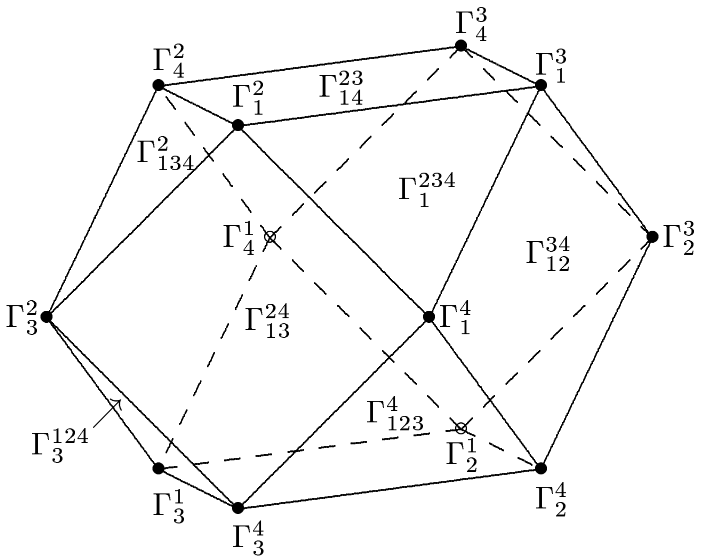
Hence, we can write $\Gamma_\alpha^\beta\star\Gamma_\gamma^\delta=\Gamma_{\alpha\setminus\delta}^{\beta\setminus\gamma}$ and "$\star$" is simply a "contraction" of indices. Note that if any of the sets $(\alpha\setminus\delta)$ or $(\beta\setminus\gamma)$ is empty then $N\big(\Gamma_\alpha^\beta\big)\cap N\big(\Gamma_\gamma^\delta\big)=\emptyset$ and $\Gamma_\alpha^\beta\star\Gamma_\gamma^\delta$ is not defined. For instance, $N(\Gamma_\alpha^\beta)\cap N(\Gamma_\beta^\alpha)=\emptyset$, which says that $N(\Gamma_\alpha^\beta)$ does not intersect its image under $\mathcal{J}$.

In the case when p_1 is even there is an extra $\mathbb{Z}/2$-set which is just a the Stiefel manifold $N(p_2,...,p_n)$. On the twistor space each $N(\Gamma_\alpha^\beta)/T^2\subset\Sigma(\mathbf{p})$ is simply a singular Kähler reduction of the complex projective space $\mathbb{C}P^{n-1}$ by a circle action, which is automatically a compact Kähler orbifold of complex dimension $n-2$ (but not Kähler-Einstein when $n>3$). In the case of all weights being odd this is the largest possible dimension of the singular stratum in $\mathcal{Z}(\mathbf{p})$ so that the orbifold singularities are at least of complex codimension $n-1$. When p_1 is even, we get $N(p_2,...,p_n)/T^2\simeq\mathcal{Z}(p_2,...,p_n)$, which is of complex codimension 2 in $\mathcal{Z}(\mathbf{p})$ and is a compact Kähler-Einstein suborbifold. It is a coincidence that when $n=3$ the $\mathbb{Z}/2$ set looks the same as all the other singular sets of $\Sigma(\mathbf{p})$.

Just as in the $n=3$ case, the rules under which the top strata are put together can be expressed in terms of a polytope in $\mathbb{R}^{n-1}$, similar to the hexagons of Fig. 6.16 and 6.20. This polytope has 2^n-2 faces. There are $2n$ faces which are $(n-1)$-topes and the rest are n-topes. These faces intersect in $\binom{n}{k}(2^{n-k}-2), k=1,2,...,n-2$, strata of dimension $(n-k-2)$, all of them polytopes in $\mathbb{R}^{n-k-1}$. Note that all lower dimensional sets must come from the natural inclusions

$$\Sigma(p_1,...,\hat{p}_i,...,p_n)\subset\Sigma(\mathbf{p}),\qquad i=1,2,...,n. \tag{6.28}$$

Thus one gets an inductive way of building the singular set $\Sigma(\mathbf{p})$ for general n starting with $n=3$ and at each step considering only the top dimensional singular sets. The next example illustrates this with $n=4$.



6.29. The singular set of $\mathcal{Z}(p_1,...,p_4)$ for all p_i odd

Example 6.30 Let us describe the singular locus $\Sigma(p_1,p_2,p_3,p_4)$ of $\mathcal{Z}(p_1,p_2,p_3,p_4)$. Assume here that all the weights are odd. On $N(p_1,p_2,p_3,p_4)$ we have $2^4-2=14$ faces. We have six faces which are squares. They are $N(\Gamma^{34}_{12})$, $N(\Gamma^{24}_{13})$, and $N(\Gamma^{14}_{23})$ (plus their images by the $\mathcal{J}$-map). Then there are 8 faces which are triangles, that is $N(\Gamma^{123}_4)$, $N(\Gamma^{234}_1)$, $N(\Gamma^{134}_2)$, and $N(\Gamma^{124}_3)$ (plus their images by the $\mathcal{J}$-map). The faces intersect in 24 Hopf surfaces $S^3\times S^1$ (edges), and 12 tori $S^1\times S^1$ (vertices) according to the rules described by 6.26-27. The whole stratum can be put together into a semi-regular polyhedron called a cuboctahedron. In Figure 6.29 we have indicated the isotropy group associated to each front face and the isotropy groups of all 12 vertices. The isotropy groups of the "opposite" faces are obtained by switching upper and lower indices (the $\mathcal{J}$-map) and all the edges can be easily inscribed by either Γ^{jk}_i or Γ^i_{jk} using the "contraction" rules given in 5.33-34. For instance the vertices of the $\triangle\Gamma^4_1\Gamma^3_1\Gamma^2_1$ would read $\Gamma^{234}_1\star\Gamma^{34}_{12}=\Gamma^{34}_1$, $\Gamma^{234}_1\star\Gamma^{23}_{14}=\Gamma^{23}_1$, $\Gamma^{234}_1\star\Gamma^{24}_{13}=\Gamma^{24}_1$. Note that the hexagon $\Sigma(p_1,p_2,p_3)$ includes in $\Sigma(p_1,p_2,p_3,p_4)$ as a "section" of the cuboctahedron with vertices $\Gamma^1_2\Gamma^3_2\Gamma^3_1\Gamma^2_1\Gamma^2_3\Gamma^1_3$. The same holds for the 3 remaining hexagons and they build all the vertices and edges of the cuboctahedron.

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