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Sasakian Geometry, Homotopy Spheres and Positive Ricci Curvature

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Abstract: We discuss the Sasakian geometry of odd dimensional homotopy spheres. In particular, we give a completely new proof of the existence of metrics of positive Ricci curvature on exotic spheres that can be realized as the boundary of a parallelizable manifold. Furthermore, it is shown that on such homotopy spheres Σ^{2n+1} the moduli space of Sasakian structures has infinitely many positive components determined by inequivalent underlying contact structures. We also prove the existence of Sasakian metrics with positive Ricci curvature on each of the known 2^{2m} distinct diffeomorphism types of homotopy real projective spaces $\mathbb{RP}^{4m+1}$.

0. Introduction

Milnor's discovery of exotic spheres [Mil1] presented Riemannian geometry with a very natural question. What kind of special metrics or, more generally, geometric structures can exist on exotic spheres? Perhaps the most intriguing example of such a question concerns the existence of metrics with positive sectional curvature. In 1974 Gromoll and Meyer [GM] observed that one of the Milnor 7-spheres admits a metric of non-negative scalar curvature.Only recently Grove and Ziller [GZ] have observed that indeed all Milnor spheres, i.e., 7-spheres that are 3-sphere bundles over the 4-sphere admit metrics of non-negative sectional curvature. Yet, it is not known whether any of these metrics can be deformed to give a metric of positive sectional curvature. However, it is known that some exotic spheres cannot admit such metrics. This follows from a beautiful result of Hitchin, who observed that some of the exotic spheres, starting in dimension 9, do not even admit metric of positive scalar curvature [Hi].

A somewhat more tractible problem concerns the existence of positive Ricci curvature metrics on exotic spheres. Here, many examples have been known [Ch, Na, Her, Po]. In 1997, using surgery theory, D. Wraith [Wr] proved the existence of Riemannian metrics with positive Ricci curvature on any exotic sphere that can be realized as the boundary of a parallelizable manifold. In particular, in dimension 7 all exotic spheres admit such metric.

The question of the existence of other geometric structures on exotic spheres has been equally important and intriguing. For example, it was realized by many [Abe1-2, AE, LM, SH, Tak, Tho, Vai, YK] that Brieskorn manifolds naturally admit almost contact,

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contact, and Sasakian structures. The appoach closest to ours is that of Takahashi [Tak] who constructed weighted Sasakian structures on Brieskorn manifolds.

The purpose of this paper is to investigate the Sasakian geometry of homotopy spheres with the goal of producing Sasakian metrics with positive Ricci curvature on exotic spheres. The main ingredients of the proof are Brieskorn's description of homotopy spheres as links of isolated hypersurface singularities in $\mathbb{C}^{n+1}$, a positivity theorem in [BGN2] which is based on the orbifold or equivalently foliation version of the famous Calabi problem, and the algebraic geometry of Fano hypersurfaces in non well-formed weighted projective spaces. The link is the total space of an S^1 orbifold V-bundle over this Fano variety with a specific orbifold structure. From our point of view the differential invariants of the link are encoded in the orbifold structure, although the precise nature of this encoding still remains a mystery.

The classical Calabi Conjecture, later proved by Yau, states that if (Z, ω) is a smooth Kähler manifold then for any (1,1) form ρ' such that $[\rho'] = c_1(Z)$ there exist a Kähler metric ω' on Z whose Ricci form equals to ρ'. Moreover, $[\omega'] = [\omega]$, i.e., ω' is in the same cohomology class as ω. In particular, in the Fano case when $c_1(M) > 0$ the ω' has positive Ricci curvature. It has been often noted that the Calabi-Yau Conjecture holds in the situation when M is a Kähler orbifold (cf. [DK] or [Joy]). In the context of foliations El Kacimi-Alaoui [ElK] actually gave a proof of what one can call the "transverse Yau theorem". In [BGN2] (see Theorem 2.10 below) we showed that this theorem can be adapted to the situation, where the foliation at hand is the characteristic foliation of a Sasakian structure. In the present paper we combine all of these results to prove:

Theorem A: *For $n \geq 3$ let Σ^{2n-1} be a homotopy sphere which can be realized as the boundary of a parallelizable manifold. Then Σ^{2n-1} admits Sasakian metrics with positive Ricci curvature.*

We also show that there exists positive Sasakian structures belonging to infinitely many inequivalent contact structures. In particular, this implies that the moduli space of positive Sasakian structures on odd dimensional homotopy spheres has infinitely many components. Specifically, we prove the following:

Theorem B: *On each odd homotopy sphere $\Sigma^{2n-1} \in bP_{2n}$ there exists countably infinitely many deformation classes of positive Sasakian structures belonging to non-isomorphic underlying contact structures. Hence, the moduli space of Sasakian structures on Σ^{2n-1} has infinitely many positive components.*

By a well-known theorem of Smale there are no exotic spheres in dimension 5; however, there are 4 diffeomorphism types and 2 homeomorphism types of homotopy $\mathbb{RP}^5$'s. More generally on the homotopy projective space $\mathbb{RP}^{4m+1}$ there are at least 2^{2m} diffeomorphism types. Homotopy projective spaces have been studied in [AB,Bro,Gi1-2,LdM]. Our methods yield:

Theorem C: *On each of the known 2^{2m} diffeomorphism types of homotopy projective spaces $\mathbb{RP}^{4m+1}$ there exists deformation types of positive Sasakian structures, and each deformation class contains Sasakian metrics of positive Ricci curvature.*

Finally, we employ a method of Savel'ev [Sav] to construct homotopy spheres from rational homology spheres. Combining this with previous work [BGN4] we give positive Sasakian structures on homotopy 9-spheres belonging to deformation classes of Sasakian structures that are inequivalent to the ones constructed previously using Brieskorn spheres.

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1. Exotic Spheres

Let us briefly recall a construction of exotic differential structures on odd dimensional spheres. In 1956 Milnor stunned the mathematical world with the construction of exotic differential structures on S^7. Later the work of Milnor and Kervaire [KeMi] (see also [Hz]) showed that associated with each sphere S^n with $n \geq 5$ there is an Abelian group Θ_n consisting of equivalence classes of homotopy spheres S^n that are equivalent under oriented h-cobordism. By Smale's h-cobordism theorem this implies equivalence under oriented diffeomorphism. The group operation on Θ is connected sum. Θ_n has a subgroup bP_{n+1} consisting of equivalence classes of those homotopy n-spheres which bound parallelizable manifolds V_{n+1}. Kervaire and Milnor [KeMi] proved that $bP_{2k+1} = 0$ for $k \geq 1$. Moreover, for $m \geq 2$, bP_{4m} is cyclic of order

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1}-1) \text{ numerator } \big(\frac{4B_m}{m}\big),$$

where B_m is the m-th Bernoulli number. For bP_{4m+2} the situation is still not entirely understood. It entails computing the Kervaire invariant which is hard. It is known (see the recent review paper [La] and references therein) that $bP_{4m+2} = 0$, or $\mathbb{Z}_2$, and is $\mathbb{Z}_2$ if $4m+2 \neq 2^i - 2$ for any $i \geq 3$. Furthermore, bP_{4m+2} vanishes for $m = 1, 3, 7$, and 15.

Ten years after Milnor's landmark paper, Brieskorn [Br] showed how exotic spheres can be obtained as links of isolated hypersurface singularities. In particular if g_m denotes the Milnor generator in bP_{4m} (see [Hz]) then Brieskorn showed that for each $k \in \mathbb{Z}^+$ the link defined as the locus of points in $\mathbb{C}^{2m+1}$ that satisfies

1.1 $$\sum_{i=0}^{2m} |z_i^2| = 1, \qquad z_0^{6k-1} + z_1^3 + z_2^2 + \cdots + z_{2m}^2 = 0, \qquad k \geq 1$$

represents an element in bP_{4m}. We shall denote these homotopy $(4m-1)$-spheres by Σ_k^{4m-1}. The diffeomorphism type of $\Sigma \in bP_{4m}$ is determined [KeMi] by the signature τ of V_{4m} which is necessarily divisible by 8. Two $\Sigma, \Sigma' \in bP_{4m}$ are diffeomorphic if and only if $\tau(V') \equiv \tau(V) (\bmod\ 8|bP_{4m}|)$, and Brieskorn shows that $\tau(V_k) = (-1)^m 8k$, so that Σ_k^{4m-1} and $\Sigma_{k'}^{4m-1}$ are diffeomorphic if and only if

$$k' \equiv k (\bmod\ |bP_{4m}|).$$

The group operation in bP_{4m} is connected sum and since the signature is additive under connected sum, the element $(-1)^m \Sigma_1^{4m-1}$ is a generator $g_m \in bP_{4m}$, and $(-1)^m |bP_{4m}| g_m$ is diffeomorphic to the standard sphere S^{4m-1}.

More generally Brieskorn [Br,Di] considered the links $L(\mathbf{a})$ defined by

1.2 $$\sum_{i=0}^{n} |z_i^2| = 1, \qquad z_0^{a_0} + \cdots + z_n^{a_n} = 0.$$

To the vector $\mathbf{a} = (a_0, \cdots, a_n) \in \mathbb{Z}_+^{n+1}$ one associates a graph $G(\mathbf{a})$ whose $n+1$ vertices are labeled by $a_0, \cdots, a_n$. Two vertices a_i and a_j are connected if and only if $\gcd(a_i, a_j) > 1$.

Let C_{ev} denote the connected component of $G(\mathbf{a})$ determined by the even integers. Note that all even vertices belong to C_{ev}, but C_{ev} may contain odd vertices as well. Then Brieskorn proved that $L(\mathbf{a})$ is a $\mathbb{Z}$-homology sphere if and only if either

1. $G(\mathbf{a})$ contains at least two isolated points, or
2. $G(\mathbf{a})$ contains one isolated point and C_{ev} has an odd number of vertices and for any distinct $a_i, a_j \in C_{ev}$, $\gcd(a_i, a_j) = 2$.

Hence, if $n \geq 3$ such an $L(\mathbf{a})$ is homeomorphic to the sphere S^{2n-1}, and by Milnor's fibration theorem [Mil2] describes an element of bP_{2n}.

Clearly, the links in bP_{4m} described by 1.1 are of type 1 above, whereas, examples of type 2 links in bP_{4m+2} are given by

1.3 $$\sum_{i=0}^{2m} |z_i^2| = 1, \qquad z_0^p + z_1^2 + z_2^2 + \cdots + z_{2m+1}^2 = 0, \qquad p \geq 1.$$

We shall denote these spheres by Σ_p^{4m+1}. If $p \equiv \pm 1 \bmod 8$ then Σ_p^{4m+1} is diffeomorphic to the standard sphere S^{4m+1}, whereas assuming that $4m+2 \neq 2^i - 2$ if $p \equiv \pm 3 \bmod 8$ then Σ_p^{4m+1} is the exotic Kervaire sphere in bP_{4m+2}. This follows from Levine's [Lev] computation of the Arf invariant [Br].

The polynomials given in 1.1 and 1.3 are by no means unique. Any other Brieskorn-Pham polynomial 1.2 satisfying either of the two conditions on $G(\mathbf{a})$ above will do as well. For example, in leau of 1.1 we can consider

1.4 $$\sum_{i=0}^{2m} |z_i^2| = 1, \qquad z_0^{10k-1} + z_1^5 + z_2^2 + \cdots + z_{2m}^2 = 0, \qquad k \geq 1$$

In this case we find that Σ_k^{4m-1} and $\Sigma_{k'}^{4m-1}$ are diffeomorphic if and only if $3k' \equiv 3k (\mathrm{mod}\ |bP_{4m}|)$.

2. Sasakian Geometry on Homotopy Spheres

Recall [Bl,YK] that a Sasakian structure on a manifold M of dimension $2n+1$ is a metric contact structure (ξ, η, Φ, g) such that the Reeb vector field ξ is a Killing field and whose underlying almost CR structure is integrable. Briefly, let $(M, \mathcal{D})$ be a contact manifold, and choose a 1-form η so that $\eta \wedge (d\eta)^n \neq 0$ and $\mathcal{D} = \ker\ \eta$. The pair $(\mathcal{D}, \omega)$, where ω is the restriction of $d\eta$ to $\mathcal{D}$ gives $\mathcal{D}$ the structure of a symplectic vector bundle. Choose an almost complex structure J on $\mathcal{D}$ that is compatible with ω, that is J is a smooth section of the endomorphism bundle End $\mathcal{D}$ that satisfies

2.1 $$J^2 = -\mathbb{I}, \qquad d\eta(JX, JY) = d\eta(X, Y), \qquad d\eta(X, JX) > 0$$

for any smooth sections X, Y of $\mathcal{D}$. Notice that J defines a Riemannian metric $g_{\mathcal{D}}$ on $\mathcal{D}$ by setting $g_{\mathcal{D}}(X, Y) = d\eta(X, JY)$. One easily checks that $g_{\mathcal{D}}$ satisfies the compatibility condition $g_{\mathcal{D}}(JX, JY) = g_{\mathcal{D}}(X, Y)$. Now we can extend J to an endomorphism Φ on all of TM by putting $\Phi = J$ on $\mathcal{D}$ and $\Phi\xi = 0$. Likewise we can extend the metric $g_{\mathcal{D}}$ on $\mathcal{D}$ to a Riemannian metric g on M by setting

2.2 $$g = g_{\mathcal{D}} + \eta \otimes \eta.$$

The quadruple (ξ,η,Φ,g) is called a *metric contact structure* on M. If in addition ξ is a Killing vector field and the almost complex structure J on $\mathcal{D}$ is integrable the underlying almost contact structure is said to be *normal* and (ξ,η,Φ,g) is called a *Sasakian structure.* The fiduciary examples of compact Sasakian manifolds are the odd dimensional spheres S^{2n+1} with the standard contact structure and standard round metric g.

Actually as with Kähler structures there are many Sasakian structures on a given Sasakian manifold. In fact there are many Sasakian structures which have ξ as its characteristic vector field. If (ξ,η,Φ,g) is a Sasakian structure on a smooth manifold (orbifold) M, we consider a deformation of this structure by adding to η a continuous one parameter family of 1-forms ζ_t that are basic with respect to the characteristic foliation. We require that the 1-form $\eta_t=\eta+\zeta_t$ satisfy the conditions

2.3 $$\eta_0=\eta,\qquad \zeta_0=0,\qquad \eta_t\wedge(d\eta_t)^n\neq 0\quad\forall\quad t\in[0,1].$$

This last nondegeneracy condition implies that η_t is a contact form on M for all $t\in[0,1]$ which by Gray's Stability Theorem belongs to the same underlying contact structure as η. Moreover, since ζ_t is basic ξ is the Reeb (characteristic) vector field associated to η_t for all t. Now let us define

2.4 $$\begin{aligned}\Phi_t&=\Phi-\xi\otimes\zeta_t\circ\Phi\\ g_t&=d\eta_t\circ(\Phi_t\otimes\mathrm{id})+\eta_t\otimes\eta_t.\end{aligned}$$

In [BG3] it was proved that for all $t\in[0,1]$ and every basic 1-form ζ_t such that $d\zeta_t$ is of type $(1,1)$ and such that 2.3 holds (ξ,η_t,Φ_t,g_t) defines a continuous 1-parameter family of Sasakian structures on M belonging to the same underlying contact structure as η. Given a Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ on a manifold M, we define $\mathfrak{F}(\xi)$ to be the family of all Sasakian structures obtained by the deformations above.

Any Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ has a 1-dimensional foliation $\mathcal{F}_\xi$ associated to it, defined by the flow of the Reeb vector field ξ and called the *characteristic foliation.* Every Sasakian structure $\mathcal{S}\in\mathfrak{F}(\xi)$ defines the same basic cohomology class $[d\eta]_B\in H^2_B(\mathcal{F})$ in the basic cohomology of the foliation $\mathcal{F}(\xi)$, and conversely, any two homologous Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ and $\mathcal{S}'=(\xi,\eta',\Phi',g')$ lie in $\mathfrak{F}(\xi)$. Moreover, if $(M,\mathcal{S})$ is a compact Sasakian manifold then the cohomology class $[d\eta]_B$ is non-trivial, and cups to a non-trivial class in $H^{2n}_B(\mathcal{F}_\xi)$. We are interested in the set of Sasakian structures which correspond to the same foliation $\mathcal{F}_\xi$. This set contains $\mathfrak{F}(\xi)$, but is slightly larger. We define $\mathfrak{F}(\mathcal{F}_\xi)$ to be the set of all Sasakian structures whose characteristic foliation is $\mathcal{F}_\xi$. Clearly, we have $\mathfrak{F}(\xi)\subset\mathfrak{F}(\mathcal{F}_\xi)$. For any Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ there is the "conjugate Sasakian structure" defined by $\mathcal{S}^c=(\xi^c,\eta^c,\Phi^c,g)=(-\xi,-\eta,-\Phi,g)\in\mathfrak{F}(\mathcal{F}_\xi)$. Moreover, there is a type of homothety [YK] defined by

2.5 $$\xi'=a^{-1}\xi,\quad \eta'=a\eta,\quad \Phi'=\Phi,\quad g'=ag+(a^2-a)\eta\otimes\eta$$

for $a\in\mathbb{R}^+$. So fixing $\mathcal{S}$ we define

2.6 $$\mathfrak{F}^+(\mathcal{F}_\xi)=\bigcup_{a\in\mathbb{R}^+}\mathfrak{F}(a^{-1}\xi),$$

and $\mathfrak{F}^-(\mathcal{F}_\xi)$ to be the image of $\mathfrak{F}^+(\mathcal{F}_\xi)$ under conjugation. It is then easy to see that we have a decomposition $\mathfrak{F}(\mathcal{F}_\xi)=\mathfrak{F}^+(\mathcal{F}_\xi)\sqcup\mathfrak{F}^-(\mathcal{F}_\xi)$. Notice that for n even conjugation reverses orientation; whereas, for n odd it preserves the orientation. This discussion shows that for each $\mathcal{F}_\xi$ the subset of homology classes represented by Sasakian structures forms a line minus the origin in $H^2_B(\mathcal{F}_\xi)$. Since conjugation interchanges the positive and negative rays, we often restrict our considerations to the positive ray, and hence to $\mathfrak{F}^+(\mathcal{F}_\xi)$. We shall refer to elements of $\mathfrak{F}^\pm(\mathcal{F}_\xi)$ as *a-deformation classes* of Sasakian structures. We need

DEFINITION 2.7: *Two Sasakian structures $\mathcal{S}=(\xi,\eta,\Phi,g)$ and $\mathcal{S}'=(\xi',\eta',\Phi',g')$ in $\mathfrak{F}(\mathcal{F}_\xi)$ on a smooth manifold M are said to be a-homologous if there is an $a\in\mathbb{R}^+$ such that $\xi'=a^{-1}\xi$ and $[d\eta']_B=a[d\eta]_B$.*

The a-homology classes form a set of two elements that can be identified with positive and negative rays in $H^2_B(\mathcal{F}_\xi)$. So every Sasakian structure in $\mathfrak{F}(\mathcal{F}_\xi)$ is a-homologous to $\mathcal{S}$ or its conjugate $\mathcal{S}^c$.

Other important invariants of $\mathfrak{F}(\xi)$ are the basic Chern classes $c_i(\mathcal{F}_\xi)$ of the symplectic vector bundle $(\mathcal{D},d\eta)$ as elements of the basic cohomology ring $H^*_B(\mathcal{F}_\xi)$. In particular we are interested in the basic first Chern class $c_1(\mathcal{F}_\xi)\in H^2_B(\mathcal{F}_\xi)$. Recall [BGN2] that a Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ is *positive (negative)* if its basic first Chern class $c_1(\mathcal{F}_\xi)\in H^2(\mathcal{F}_\xi)$ can be represented by a positive (negative) definite $(1,1)$-form. On rational homology spheres there are are only two possibilities, viz.

PROPOSITION 2.8: *Let $\mathcal{S}=(\xi,\eta,\Phi,g)$ be a Sasakian structure on a rational homology sphere M^{2n-1}. Then either $c_1(\mathcal{F}_\xi)>0$ or $c_1(\mathcal{F}_\xi)<0$.*

PROOF: For rational homology spheres M the basic long exact sequence

$$\cdots \longrightarrow H^p_B(\mathcal{F}_\xi) \longrightarrow H^p(M,\mathbb{R}) \xrightarrow{j_p} H^{p-1}_B(\mathcal{F}_\xi) \xrightarrow{\delta} H^{p+1}_B(\mathcal{F}_\xi) \longrightarrow \cdots$$

implies the isomorphisms

$$H^{p-1}_B(\mathcal{F}_\xi)\approx H^{p+1}_B(\mathcal{F}_\xi)$$

for $p=1,\cdots,2n-1$, and since $H^1_B(\mathcal{F}_\xi)=0$ and $H^2_B(\mathcal{F}_\xi)$ is generated by $[d\eta]_B$, this implies the ring isomorphism

2.9 $$H^*_B(\mathcal{F}_\xi)\approx\mathbb{R}\big[[d\eta]_B\big]/([d\eta]_B)^n.$$

Thus, $\mathcal{C}(\mathcal{F}_\xi)=\mathbb{R}^+\cup\mathbb{R}^-$, and $c_1(\mathcal{F}_\xi)=a[d\eta]_B$ for some $a\in\mathbb{R}$. Since $d\eta$ is the transverse Kähler form it is positive definite, and a cannot vanish since this would imply that the basic geometric genus [BGN2] $p_g(\mathcal{F}_\xi)\neq 0$ implying $b_{n-1}(M)\neq 0$. Thus, either $a>0$ or $a<0$ implying either $c_1(\mathcal{F}_\xi)>0$ or $c_1(\mathcal{F}_\xi)<0$. ∎

In the next two sections we shall discuss how to construct positive Sasakian structures on homotopy spheres, and prove Theorem A of the Introduction. Recall that a positive Sasakian structure [BGN2] is a Sasakian structure whose basic first Chern class can be represented by a positive definite basic $(1,1)$-form. Once positive Sasakian structures are obtained Sasakian structures with positive Ricci curvature follow from the following theorem of the authors.

THEOREM 2.10 [BGN2]: *Let $\mathcal{S}=(\xi,\eta,\Phi,g)$ be a positive Sasakian structure on a compact manifold M of dimension $2n+1$. Then M admits a Sasakian structure $\mathcal{S}'=(\xi',\eta',\Phi',g')$ with positive Ricci curvature a-homologous to $\mathcal{S}$ for some $a>0$.*

3. The Characteristic Foliation and Kähler Orbifolds

This foliation encodes important invariants of the Sasakian structure. For example, (M,ξ,η,Φ,g) is said to be *quasi-regular* if all the leaves of $\mathcal{F}_\xi$ are compact. In this case the leaves are all circles, but in general they can have nontrivial holonomy. If the leaf holonomy group is trivial for every leaf, the Sasakian manifold (M,ξ,η,Φ,g) is called *regular*. It is well-known that if (M,ξ,η,Φ,g) is compact and quasi-regular, then the space of leaves $\mathcal{Z}$ is a compact Kähler orbifold, and a smooth manifold in the regular case, whose Kähler class $[\omega]$ is represented by an integral class in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ and a rational class in $H^2(\mathcal{Z},\mathbb{Q})$. The leaf holonomy groups of $\mathcal{F}_\xi$ are the local uniformizing groups of the orbifold $\mathcal{Z}$ and these are invariants of the Sasakian structure on M. A somewhat courser invariant is the *order* $\upsilon(M)$ of a compact Sasakian manifold defined to be the least common multiple of the orders of the leaf holonomy groups. So (M,ξ,η,Φ,g) is regular if and only if $\upsilon_M=1$.

For a quasi-regular Sasakian structure $\mathcal{S}$ on a compact manifold M the space of leaves $\mathcal{Z}$ is also a projective algebraic variety, and it is important to distinguish between $\mathcal{Z}$ as an orbifold and $\mathcal{Z}$ as an algebraic variety. In general the singular sets are different. In the case of the Brieskorn-Kervaire exotic spheres considered in Proposition 4.3 below, we shall see that $\mathcal{Z}$ as an algebraic variety is smooth, in fact a projective space, whereas, its orbifold singular set is actually a divisor. For hypersurfaces (or more generally complete intersections) in weighted projective spaces, the two singular sets coincide precisely when the hypersurfaces are *well-formed* (cf. [Fl]); however, the orbifolds $\mathcal{Z}$ associated to the Brieskorn spheres are never well-formed. In the non well-formed case certain pathologies can occur. For, example if $\mathcal{Z}\subset\mathbb{P}(w_0,\cdots,w_n)$ is not well-formed, there can be isomorphic coherent sheaves $\mathcal{O}_\mathcal{Z}(m)$ and $\mathcal{O}_\mathcal{Z}(n)$ with $n\neq m$.

In our previous work [BG1,BG2,BGN1,BGN2,BGN3] we have always worked with the well-formed case, and there was little need to distinguish between the orbifold context and the algebraic geometric context. In the present paper this is no longer the case. In algebraic geometry one uses a sort of poetic license not to distinguish between vector bundles and locally free sheaves. There is a similar situation in the category of orbifolds though much less well-known. First, the notion of a vector V-bundle was introduced by Satake [Sat] and essentially consists of vector bundles defined on the local uniformizing covers $(\tilde{U}_i,\Gamma_i,\phi_i)$ that patch together in a nice way, together with group homomorphisms $\Gamma_i\longrightarrow GL(n,\mathbb{C})$ that satisfy certain compatibility conditions (cf. [BG1] for details). On the other hand the notion of a locally V-free sheaf on a normal projective variety with only quotient singularities has been defined ([Bla], see also [Kaw] where it is referred to as a $\mathbb{Q}$-vector bundle), and proves to be convenient in our context. A coherent sheaf $\mathfrak{F}$ of $\mathcal{O}_\mathcal{Z}$-modules is *locally V-free* if on each uniformizing neighborhood $\tilde{U}_i$ there is a free sheaf $\tilde{\mathfrak{F}}$ together with an action of Γ_i on $\tilde{\mathfrak{F}}$ such that $\mathfrak{F}=((\phi_i)_*\tilde{\mathfrak{F}})^{\Gamma_i}$ where $(\cdot)^\Gamma$ denotes the Γ-invariant subsheaf. Notice that a locally V-free sheaf is not necessarily locally free; however, a locally free sheaf is locally V-free. Now it is straightforward to see that the usual correspondence between vector V-bundles and locally V-free sheaves still holds. We shall also refer to a locally V-free sheaf of rank one as a *V-invertible sheaf.* V-invertible sheaves correspond to holomorphic line V-bundles. In our previous work [BG1,BG2] we introduced the group $\mathrm{Pic}^{orb}(\mathcal{Z})$ of holomorphic line V-bundles on $\mathcal{Z}$. We also refer to $\mathrm{Pic}^{orb}(\mathcal{Z})$ as the group of V-invertible sheaves on $\mathcal{Z}$. The *dualizing sheaf* $\omega_\mathcal{Z}$ on $\mathcal{Z}$, defined by

3.1
$$\omega_\mathcal{Z}=\iota_*(\Omega^n\mathcal{Z}_{reg}),$$

where $\Omega^p\mathcal{Z}_{reg}$ denotes the sheaf of differential p-forms on the algebro-geometric regular set $\mathcal{Z}_{reg}$, is of particular importance. For a given orbifold structure on $\mathcal{Z}$, $\omega_\mathcal{Z}$ is V-invertible with respect to that structure, and corresponds to the canonical V-bundle $K_{\mathcal{Z}}$ as defined by Baily [Ba]. However, $\mathcal{Z}$ may admit several orbifold structures, and $\omega_{\mathcal{Z}}$ can take different forms as V-invertible sheaves.

Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action given by

3.2
$$(z_0,\ldots,z_n)\mapsto(\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n),$$

where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w}\in(\mathbb{Z}^+)^{n+1}$, and we shall assume that they are ordered $w_0\leq w_1\leq\cdots\leq w_n$ and that $\gcd(w_0,\ldots,w_n)=1$. Let f be a weighted homogeneous polynomial, that is $f\in\mathbb{C}[z_0,\ldots,z_n]$ and satisfies

3.3
$$f(\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n)=\lambda^d f(z_0,\ldots,z_n),$$

where $d\in\mathbb{Z}^+$ is the degree of f. We are interested in the *weighted affine cone* C_f defined by the equation $f(z_0,\ldots,z_n)=0$. We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity, in fact the only singularity, of f. Then the link L_f defined by

$$L_f=C_f\cap S^{2n+1},$$

where

$$S^{2n+1}=\{(z_0,\ldots,z_n)\in\mathbb{C}^{n+1}|\sum_{j=0}^n|z_j|^2=1\}$$

is the unit sphere in $\mathbb{C}^{n+1}$, is a smooth manifold of dimension $2n-1$. Furthermore, it is well-known [Mil2] that the link L_f is $(n-2)$-connected.

On S^{2n+1} there is a well-known [YK] "weighted" Sasakian structure $(\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$ which in the standard coordinates $\{z_j=x_j+iy_j\}_{j=0}^n$ on $\mathbb{C}^{n+1}=\mathbb{R}^{2n+2}$ is determined by

$$\eta_{\mathbf{w}}=\frac{\sum_{i=0}^n(x_idy_i-y_idx_i)}{\sum_{i=0}^n w_i(x_i^2+y_i^2)},\qquad \xi_{\mathbf{w}}=\sum_{i=0}^n w_i(x_i\partial_{y_i}-y_i\partial_{x_i}),$$

and the standard Sasakian structure (ξ,η,Φ,g) on S^{2n+1}. The embedding $L_f\hookrightarrow S^{2n+1}$ induces a Sasakian structure on L_f [BG3].

Given a sequence $\mathbf{w}=(w_0,\ldots,w_n)$ of ordered positive integers one can form the graded polynomial ring $S(\mathbf{w})=\mathbb{C}[z_0,\ldots,z_n]$, where z_i has or *weight* w_i. The weighted projective space [Dol, Fle] $\mathbb{P}(\mathbf{w})=\mathbb{P}(w_0,\ldots,w_n)$ is defined to be the scheme $\mathrm{Proj}(S(\mathbf{w}))$. It is the quotient space $(\mathbb{C}^{n+1}-\{0\})/\mathbb{C}^*(\mathbf{w})$, where $\mathbb{C}^*(\mathbf{w})$ is the weighted action defined in 3.2, or equivalently, $\mathbb{P}(\mathbf{w})$ is the quotient of the weighted Sasakian sphere $S^{2n+1}_{\mathbf{w}}=(S^{2n+1},\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$ by the weighted circle action $S^1(\mathbf{w})$ generated by $\xi_{\mathbf{w}}$. As such $\mathbb{P}(\mathbf{w})$ is also a compact complex orbifold with an induced Kähler structure. Furthermore, the hypersurface $\mathcal{Z}_f\subset\mathbb{P}(\mathbf{w})$ defined as the zero set of the section $f\in\mathcal{O}_{\mathbb{P}(\mathbf{w})}(d)$ is the $S^1(\mathbf{w})$ quotient of the link L_f. The assumption that the origin of $\mathbb{C}^{n+1}$ is the only singularity of f guarantees that the quotient space $\mathcal{Z}_f$ is an orbifold. In this case we also say the $\mathcal{Z}_f$ is *quasi-smooth.* We have from [BG3]

THEOREM 3.4: *The quadruple $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ gives L_f a quasi-regular Sasakian structure $\mathcal{S}_{\mathbf{w}}$ such that there is a commutative diagram*

$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 V-bundles and orbifold Riemannian submersions. Moreover, the orbifold first Chern class in $c_1(\mathcal{Z}_f) \in H^2_{orb}(\mathcal{Z}_f, \mathbb{Z})$ is related to the basic first Chern class $c_1(\mathcal{F}_{\xi_{\mathbf{w}}})$ of $\mathcal{S}_{\mathbf{w}}$ by $c_1(\mathcal{F}_{\xi_{\mathbf{w}}}) = \pi^* c_1(\mathcal{Z}_f)$.

Hence, to prove Theorem A we need to compute the orbifold first Chern class $c_1(\mathcal{Z}_f)$. This requires determining the canonical V-bundle $K_{\mathcal{Z}_f}$ by an adjunction formula. The following is well-known [Dol,BR]:

LEMMA 3.5: [Dol] *On $\mathbb{P}(\mathbf{w})$ the dualizing sheaf $\omega_{\mathbb{P}(\mathbf{w})}$ is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(-|\mathbf{w}|)$.*

WARNING: This does not imply that the anti-dualizing sheaf $\omega^{-1}_{\mathbb{P}(\mathbf{w})}$ is isomorphic to $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(|\mathbf{w}|)$. We note that this isomorphism does hold if the weighted projective space $\mathbb{P}(\mathbf{w})$ is well-formed, i.e. the gcd of any n components of $\mathbf{w} = (w_0, \cdots, w_n)$ is one (cf. [BG3]), but here we are interested in the non well-formed case.

Next we are interested in an adjunction formula. This is known to hold when the hypersurface is well-formed as defined by Fletcher [Fl]; that is a hypersurface $\mathcal{Z}_f \subset \mathbb{P}(\mathbf{w})$ is *well-formed* if $\mathbb{P}(\mathbf{w})$ is well-formed and $\mathcal{Z}_f$ contains no codimension 2 singular stratum of $\mathbb{P}(\mathbf{w})$. However, the condition that guarantees the adjunction formula is that the hypersurface $\mathcal{Z}_f$ contain no codimension 2 singular stratum of $\mathbb{P}(\mathbf{w})$, and no further condition on $\mathbb{P}(\mathbf{w})$ is needed. Indeed, in the sequel we shall make use of the adjunction formula for hypersurfaces in weighted projective spaces that are not well-formed.

LEMMA 3.6 [Adjunction formula]: *Let $\mathcal{Z}_f$ be a quasi-smooth hypersurface in $\mathbb{P}(\mathbf{w})$ defined by a weighted homogeneous polynomial f of degree d. Suppose that codim$(\mathcal{Z}_f \cap \mathbb{P}^{sing}_{\mathbf{w}}, \mathcal{Z}_f) \geq 2$. Then the adjunction formula*

$$\omega_{\mathcal{Z}_f} \approx \mathcal{O}_{\mathcal{Z}_f}(d - |\mathbf{w}|)$$

holds.

PROOF: By Proposition 5.75 of [KM], $\omega_X \simeq \mathcal{O}_X(K_X)$ for any normal variety X. In particular, for the hypersurface $\mathcal{Z}_f$ in $\mathbb{P}(\mathbf{w})$ we have

$$\omega_{\mathcal{Z}_f} \approx \mathcal{O}_{\mathcal{Z}_f}(K_{\mathcal{Z}_f}).$$

In general, $\mathcal{Z}_f$ will not be a Cartier divisor and so we cannot apply the adjunction formula directly. On the other hand, if $U \subset \mathbb{P}(\mathbf{w})$ denotes the smooth locus and $U_f = \mathcal{Z}_f \cap U$ then the adjunction formula says that

$$K_{U_f} = K_U + U_f | U_f = \mathcal{O}_{U_f}(d - |\mathbf{w}|).$$

Consequently

$$\omega_{U_f} \approx \mathcal{O}_{U_f}(d - |\mathbf{w}|)$$

The lemma then follows from Proposition 0-1-10 of [KMM] since ω_{Z_f} and $\mathcal{O}_{\mathcal{Z}_f}(d-|\mathbf{w}|)$ are reflexive rank one sheaves which agree off a set of codimension at least 2. ∎

REMARK 3.7: In several of our applications the hypersurface $\mathcal{Z}_f$ is a Cartier divisor on $\mathbb{P}(\mathbf{w})$, and in this case one can directly apply the adjunction formula in [KM].

The dualizing sheaf $\omega_{\mathcal{Z}_f}$ is a V-invertible sheaf with respect to the orbifold structure on $\mathcal{Z}_f$, and as such corresponds to the canonical V-bundle $K_{\mathcal{Z}_f}$, or equivalently to the canonical V-divisor of Baily [Ba], i.e.

3.8
$$\omega_{\mathcal{Z}_f} = \mathcal{O}(K_{\mathcal{Z}_f}).$$

Now $K_{\mathcal{Z}_f}$ is an element of $\mathrm{Pic}^{orb}(\mathcal{Z}_f)$ and as such has an inverse, namely, the anti-canonical V-bundle $K^{-1}_{\mathcal{Z}_f}$. We recall [BG1] that a compact complex orbifold $\mathcal{Z}$ is *Fano* if its anti-canonical V-bundle $K^{-1}_{\mathcal{Z}_f}$ is ample, and in this case the *orbifold Fano index* of $\mathcal{Z}$ is defined to be the largest positive integer I such that $\frac{1}{I}K^{-1}_{\mathcal{Z}}$ is an element of $\mathrm{Pic}^{orb}(\mathcal{Z})$. Now the group $\mathrm{Pic}^{orb}(\mathcal{Z})$ is isomorphic [BG4] to the divisor class group $\mathrm{Cl}(\mathcal{Z})$. Since $\mathrm{Cl}(\mathcal{Z})$ is an algebraic invariant, the orbifold Fano index is also a purely algebraic invariant. Furthermore, on a weighted projective space $\mathbb{P}(\mathbf{w})$ we have that $\mathrm{Cl}(\mathbb{P}(\mathbf{w})) \approx \mathbb{Z}$ and $\mathrm{Cl}(\mathbb{P}(\mathbf{w}))$ is generated by $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(a_{\mathbf{w}})$ [BR] where $a_{\mathbf{w}} = \mathrm{lcm}(d_0, \cdots, d_n)$ and $d_i = \gcd(w_0, \cdots, \hat{w}_i, \cdots, w_n)$. We have

LEMMA 3.9: *The following hold:*

a) *There is an isomorphism $\mathrm{Pic}^{orb}(\mathbb{P}(\mathbf{w})) \approx \mathbb{Z}$ and as a V-bundle (or equivalently V-invertible sheaf) $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(a_{\mathbf{w}})$ is the generator of $\mathrm{Pic}^{orb}(\mathbb{P}(\mathbf{w}))$ corresponding to $1 \in \mathbb{Z}$.*
b) *The (orbifold) Fano index I is independent of the orbifold structure and on $\mathbb{P}(\mathbf{w})$ equals $|\bar{\mathbf{w}}|$ where $\bar{\mathbf{w}}$ is the normalization of $\mathbf{w}$.*
c) *For any integers m, n there is a V-isomorphism of V-invertible sheaves*

$$\mathcal{O}_{\mathbb{P}(\mathbf{w})}(na_{\mathbf{w}}) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{w})}(ma_{\mathbf{w}}) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathbf{w})}((n+m)a_{\mathbf{w}})$$

describing the group structure of $\mathrm{Pic}^{orb}(\mathbb{P}(\mathbf{w}))$.
d) *$\mathrm{Pic}(\mathbb{P}(\mathbf{w}))$ is the subgroup of $\mathrm{Pic}^{orb}(\mathbb{P}(\mathbf{w}))$ generated by $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(\upsilon_{\mathbf{w}})$ where $\upsilon_{\mathbf{w}} = \mathrm{lcm}(w_0, \cdots, w_n)$ is the order of $\mathbb{P}(\mathbf{w})$ as an orbifold.*

REMARK. It is clear from the definitions that $a_{\mathbf{w}}$ divides $\upsilon_{\mathbf{w}}$.

Lemma 3.9 follows from the discussion above and the next lemma which is an orbifold version of a result of Delorme [Del].

LEMMA 3.10: *For any $j \in \mathbb{Z}$ there is a unique $J \in \mathbb{Z}$ and a V-isomorphism of V-invertible sheaves*

$$\mathcal{O}_{\mathbb{P}(\mathbf{w})}(j) \approx \mathcal{O}_{\mathbb{P}(\mathbf{w})}(Ja_{\mathbf{w}}).$$

PROOF: As an isomorphism of reflexive sheaves this is a result of Delorme [Del]. This isomorphism can be described as follows (cf. [BR]): first notice that $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(j) = (S(\mathbf{w})(j))^{\sim}$ where the graded $S(\mathbf{w})$-module $S(\mathbf{w})(j)$ is defined by $S(\mathbf{w})(j)_l = S(\mathbf{w})_{j+l}$. Now there is an equality of schemes $\mathrm{Proj}(S(\mathbf{w})) = \mathrm{Proj}(S'(\mathbf{w}))$ where $S'(\mathbf{w})$ is the subring of $S(\mathbf{w})$ defined by

$$S'(\mathbf{w}) = \bigoplus_{k\in\mathbb{Z}} S(\mathbf{w})_{ka_{\mathbf{w}}}.$$

The isomorphism in 3.10 is then induced by the equality of graded $S'(\mathbf{w})$-modules

$$\bigoplus_{k\in\mathbb{Z}} S(\mathbf{w})(j)_{ka_{\mathbf{w}}} = \bigoplus_{k\in\mathbb{Z}} z_0^{b_0(j)}\cdots z_n^{b_n(j)}\big(S(\mathbf{w})(j-\sum_i b_i(j)w_i)\big)_{ka_{\mathbf{w}}}$$

where $b_i(j)$ are the unique integers $0\leq b_i(j) < d_i$ satisfying $j = b_i(j)w_i + c_id_i$ for some $c_i\in\mathbb{Z}$. So the integer J is $j-\sum_i b_i(j)w_i$.

Let $\{U,\phi,\Gamma\}$ be a local uniformizing system. But since the isomorphic sheaves $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(j)$ and $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(Ja_{\mathbf{w}})$ are locally V-free, they pull-back to isomorphic sheaves $\phi^*\mathcal{O}_{\mathbb{P}(\mathbf{w})}(j)^{\vee\vee}$ and $\phi^*\mathcal{O}_{\mathbb{P}(\mathbf{w})}(Ja_{\mathbf{w}})^{\vee\vee}$ on the local cover U. ∎

Our next task is to compute the orbifold Fano index. It appears to be difficult to get a general formula, so we content ourselves with less. Actually all that is needed for this paper is the positivity of the index, but with a little extra effort we can do better. All of the weighted hypersurfaces considered in this paper are branched covers of the form

3.11 $$g = z_0^p + f(z_1,\cdots,z_n) = 0,$$

where f is a weighted homogeneous polynomial of degree d and weights $\mathbf{w}=(w_1,\cdots,w_n)$ that are reduced, but not necessarily well-formed. We also assume that f is quasi-smooth and that $\gcd(d,p)=1$. Under these conditions we have

LEMMA 3.12: *Let $\mathcal{Z}_f$ be a quasi-smooth hypersurface in a weighted projective space $\mathbb{P}(\mathbf{w})$. Suppose also that $\mathcal{Z}_f$ is cut out by the zero locus of the weighted homogeneous polynomial $f(z_1,\cdots,z_n)$ of degree d. Let p be a positive integer such that $\gcd(p,d)=1$. Then the hypersurface $\mathcal{Z}_g$ of degree $d_g = dp$ cut out by equation 3.11 is quasi-smooth and Fano with orbifold Fano index $|\bar{\mathbf{w}}|$ where $\bar{\mathbf{w}}$ is the normalization of $\mathbf{w}$, and as an algebraic variety $\mathcal{Z}_g$ is isomorphic to the weighted projective space $\mathbb{P}(\bar{\mathbf{w}})$.*

PROOF: It is easy to see that $\mathcal{Z}_g$ is quasi-smooth whenever $\mathcal{Z}_f$ is. To show that $\mathcal{Z}_g$ is Fano we first show that the adjunction formula holds and then compute its index. Now the hypersurface $\mathcal{Z}_g$ has degree dp in the weighted projective space $\mathbb{P}(d,p\mathbf{w})$ which is not well-formed. The map $z_0^p\mapsto z_0$ gives an isomorphism of embeddings of algebraic varieties

3.13 $$\begin{array}{ccc} \mathcal{Z}_g & \hookrightarrow & \mathbb{P}(d,p\mathbf{w}) \\ \Big\downarrow\approx & & \Big\downarrow\approx \\ \mathbb{P}(\mathbf{w}) & \hookrightarrow & \mathbb{P}(d,\mathbf{w}). \end{array}$$

Here the bottom inclusion is the hyperplane $z_0=0$, and the isomorphism $\mathcal{Z}_g\approx\mathbb{P}(\mathbf{w})$ is given by the map $z_0^p\mapsto z_0\mapsto z_0-f$. But $\mathbb{P}(\mathbf{w})$ is isomorphic as an algebraic variety to $\mathbb{P}(\bar{\mathbf{w}})$. This gives an isomorphism of algebraic varieties $\mathcal{Z}_g\approx\mathbb{P}(\bar{\mathbf{w}})$. Since the Fano index is an algebric invariant, we have $I(\mathcal{Z}_g)=|\bar{\mathbf{w}}|$. ∎

REMARK 3.14: In Section 7 we shall see that the link associated to the hypersurface $\mathcal{Z}_g$ of Lemma 3.12 is always a rational homology sphere.

4. The Kähler Orbifolds Associated to Homotopy Spheres

In this section we discuss the Kähler orbifolds associated to the characteristic foliation of the two types of homotopy spheres Σ_k^{4m-1} and Σ_p^{4m+1} discussed in section 1. We shall denote by Σ^{2n+1} these when there is no need to distinguish between them. We treat the case of Σ_k^{4m-1} first.

Proposition 4.1: *For all $k \geq 1$ and $m \geq 2$, the Sasakian structure on Σ_k^{4m-1} given by Theorem 3.5 is positive, quasi-regular of order $6(6k-1)$ and the space of leaves $\mathcal{Z}_k$ is a Fano orbifold given as a hypersurface in the weighted projective space $\mathbb{P}(6, 2(6k-1), 3(6k-1), \cdots, 3(6k-1))$ cut out by the equation*

$$z_0^{6k-1} + z_1^3 + z_2^2 + \cdots + z_{2m}^2 = 0.$$

Moreover, the natural projection $\pi_k : \Sigma_k^{4m-1} \longrightarrow \mathcal{Z}_k$ is a principal S^1 V-bundle over the Fano orbifold $\mathcal{Z}_k$ with orbifold Fano index $I = 2m+1$. The orbifold first Chern class of this V-bundle is $(c_1(\mathcal{Z}_k)/I) \in H^2_{orb}(\mathcal{Z}_k, \mathbb{Z})$ which is associated to the generator $\mathcal{O}_{\mathcal{Z}_k}(3(6k-1))$ in Pic$^{orb}(\mathcal{Z}_k)$.

Proof: With the exception of the statements about the order and the Fano condition this follows immediately from the commutative diagram

4.2
$$\begin{tikzcd} \Sigma_k^{4m-1} \arrow[r] \arrow[d, "\pi"] & S_{\mathbf{w}}^{4m+1} \arrow[d] \\ \mathcal{Z}_k \arrow[r] & \mathbb{P}(\mathbf{w}) \end{tikzcd}$$

where the weight vector $\mathbf{w}$ is given by 2.5, the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are the natural projections. Furthermore, assuming that $\mathcal{Z}_k$ is Fano, it follows by the inversion theorem of [BG1] Σ_k^{4m-1} is the total space of the principal S^1 V-bundle over the orbifold $\mathcal{Z}_k$ whose first Chern class is $c_1(\mathcal{Z}_k)/I$. So we need to compute the index I. First we note that adjunction holds since by the computation of the order in Lemma 4.3 below, $\mathcal{Z}_k$ is a Cartier divisor. Now by Lemma 3.12 $\mathcal{Z}_k$ is isomorphic as an algebraic variety to a well-formed weighted projective space. This isomorphism is determined by the sequence of transformations

$$(z_0^{6k-1}, z_1^3, z_2^2, \cdots, z_{2m}^2) \mapsto (z_0, z_1^3, z_2^2, \cdots, z_{2m}^2) \mapsto (z_0, z_1, z_2^2, \cdots, z_{2m}^2).$$

giving the sequence of isomorphisms

$$\mathbb{P}(6, 2(6k-1), 3(6k-1), \cdots, 3(6k-1)) \approx \mathbb{P}(6, 2, 3, \cdots, 3) \approx \mathbb{P}(2, 2, 1, \cdots, 1).$$

Thus, as algebraic varieties the $\mathcal{Z}_k$ are isomorphic for each $k = 1, \cdots$ to the weighted quadric

$$z_0 + z_1 + z_2^2 + \cdots + z_{2m}^2 = 0$$

in $\mathbb{P}(2, 2, 1, \cdots, 1)$ which can be transformed to $z_0 = 0$ giving the weighted projective space $\mathbb{P}(2, 1, \cdots, 1)$. There are $2m-1$ ones, so the Fano index is $2m+1$. Furthermore, $a_{\mathbf{w}} = \mathrm{lcm}(d_0, \cdots, d_{2m}) = 3(6k-1)$, so $\mathcal{O}_{\mathcal{Z}_k}(3(6k-1))$ is the positive generator in $\mathrm{Pic}^{orb}(\mathcal{Z}_k)$.

The local uniformizing groups can be computed by analyzing the singular stratum of $\mathcal{Z}_k$ and this is done in the lemma below where it is shown that the order of $\mathcal{Z}_k$ is $\upsilon(\mathcal{Z}_k) = 6(6k-1)$. The positivity of the Sasakian structure is implied by the Fano condition, namely $I > 0$. ∎

The computation of the order is given in:

LEMMA 4.3: *The (orbifold) singular stratum of $\mathcal{Z}_k$ is the union $Y_0 \cup Y_1 \cup Y_2$ where*

$$Y_0 = \{[\mathbf{z}] \in \mathcal{Z}_k |\, z_0 = 0\}, \quad Y_1 = \{[\mathbf{z}] \in \mathcal{Z}_k |\, z_1 = 0\}, \quad Y_2 = \{[\mathbf{z}] \in \mathcal{Z}_k |\, z_2 = \cdots = z_{2m} = 0\}.$$

The local uniformizing groups Γ_i on Y_i are

$$\Gamma_0 \approx \mathbb{Z}_{6k-1}, \qquad \Gamma_1 \approx \mathbb{Z}_3, \qquad \Gamma_2 \approx \mathbb{Z}_2,$$

so the order $\upsilon(\mathcal{Z}_k)$ is $6(6k-1)$.

PROOF: The condition for determining the fixed point set of Σ_k^{4m-1} under the action of finite subgroups of the weighted circle $S^1_{\mathbf{w}}$ generated by the Reeb vector field $\xi_{\mathbf{w}}$ is determined by the equations

$$4.4 \qquad \zeta^6 z_0 = z_0, \qquad \zeta^{2(6k-1)} z_1 = z_1, \qquad \zeta^{3(6k-1)} z_i = z_i \;\text{ for }\; i = 2, \cdots, 2m.$$

If $z_0 z_1 z_i \neq 0$ for some $i = 2, \cdots, 2m$ there are no fixed points, so we look at the three cases:

1. $z_0 = 0$: This gives the hypersurface $z_1^3 + z_2^2 + \cdots + z_{2m}^2 = 0$ with isotropy group $\Gamma_0 \approx \mathbb{Z}_{6k-1}$.
2. $z_1 = 0$: This is another hypersurface $z_0^{6k-1} + z_2^2 + \cdots + z_{2m}^2 = 0$ with isotropy group $\Gamma_1 \approx \mathbb{Z}_3$.
3. $z_i = 0$ for all $i = 2, \cdots, 2m$: This gives the set of points determined by $z_0^{6k-1} + z_1^3 = 0$ with isotropy group $\Gamma_2 \approx \mathbb{Z}_2$. ∎

For the case of Σ_p^{4m+1} we have

PROPOSITION 4.5: *For all $p \geq 0$ and $m \geq 1$, the Sasakian structure on Σ_p^{4m+1} given by Theorem 3.5 is positive, quasi-regular of order $2p+1$ and the space of leaves $\mathcal{Z}_p$ is a Fano orbifold given as a hypersurface in the weighted projective space $\mathbb{P}(2, 2p+1 \cdots, 2p+1)$ cut out by the equation*

$$z_0^{2p+1} + z_1^2 + z_2^2 + \cdots + z_{2m+1}^2 = 0.$$

Moreover, the natural projection $\pi_k : \Sigma_k^{4m+1} \longrightarrow \mathcal{Z}_p$ is a principal S^1 V-bundle over the Fano orbifold $\mathcal{Z}_p$ of orbifold Fano index $I = 2m+1$. The orbifold first Chern class of this V-bundle is $(c_1(\mathcal{Z}_p)/I) \in H^2_{orb}(\mathcal{Z}_p, \mathbb{Z})$ which is associated to the locally V-free sheaf $\mathcal{O}_{\mathcal{Z}_p}(2p+1)$.

PROOF: The proof of this proposition together with the following lemma are quite similar to that of Proposition 4.1 and Lemma 4.3. So we only give the computation of the index I. In this case as given in Lemma 4.4 below, the order υ is $2p+1$ and the degree $d = 2(2p+1)$, so the order divides the degree, and adjunction holds. Here we have $d_0 = 2p+1$, and $d_j = 1$ for $j > 0$, so $a_{\mathbf{w}} = 2p+1$. Thus, $\mathcal{O}_{\mathcal{Z}_p}(2p+1)$ is the positive generator of $\mathrm{Pic}^{orb}(\mathcal{Z}_p)$.

The isomorphism $\mathcal{Z}_p \approx \mathbb{P}(\bar{\mathbf{w}})$ is given by Lemma 4.12 where $\bar{\mathbf{w}} = (2, 1, \cdots, 1)$ with $2m+1$ ones, implying that the Fano index of $\mathcal{Z}_p$ is $I = 2m+1$. ∎

The order $\upsilon(\mathcal{Z}_p)$ is determined by:

LEMMA 4.6: *The (orbifold) singular stratum of $\mathcal{Z}_p$ is the quadric Q_{2m-1} given by*

$$z_1^2 + \cdots + z_{2m+1}^2 = 0$$

with the isotropy group $\mathbb{Z}_{2p+1}$. So the order $\upsilon(\mathcal{Z}_p)$ is $2p+1$.

REMARKS 4.7: The orbifolds $\mathcal{Z}_k$ of Proposition 4.1 are, as mentioned in the proof, isomorphic as algebraic varieties to the weighted projective space $\mathbb{P}(2, 1, \cdots, 1)$. Likewise, the orbifolds $\mathcal{Z}_p$ of Proposition 4.5 are isomorphic as algebraic varieties to the complex projective space $\mathbb{P}^{2m}$. This last fact was noticed previously by Thomas [Tho].

5. Some Exotic Projective Spaces

López de Medrano [LdM] has shown that the quotient space of any fixed point free involution on a homotopy sphere is homotopy equivalent to a real projective space $\mathbb{RP}^n$. Here we are interested in fixed point free involutions on Σ^{2n+1} that also preserve the Sasakian structure. In particular, we consider the well known involution T on Σ_p^{4m+1} defined by $(z_0, z_1, \cdots, z_{2m+1}) \mapsto (z_0, -z_1, \cdots, -z_{2m+1})$. One easily sees that this defines a free action of $\mathbb{Z}_2$ on Σ^{4m+1}, so by López de Medrano's theorem the quotient manifold Σ_p^{4m+1}/T is homotopy equivalent to $\mathbb{RP}^{4m+1}$. Furthermore, it is clear that this $T \approx \mathbb{Z}_2$ is a subgroup of the weighted circle group $S^1_{\mathbf{w}}$ with weights $\mathbf{w} = (2, 2p+1, \cdots, 2p+1)$. Thus, the deformation class of Sasakian structures passes to the quotient Σ_p^{4m+1}/T to give a deformation class of Sasakian structures together with their characteristic foliation $\mathcal{F}_\xi$. Furthermore, by Proposition 4.5 the Sasakian structures are positive, and so by [BGN2] the homotopy real projective spaces Σ_p^{4m+1}/T admit Sasakian metrics of positive Ricci curvature. We have arrived at:

PROPOSITION 5.1: *For each $(p, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ the homotopy projective space $\Sigma_p^{4m+1}/\mathbb{Z}_2$ admits a deformation class of positive Sasakian structures; hence, Σ_p^{4m+1}/T admits Sasakian structures with positive Ricci curvature.*

It is known [AB,Bro,Gi1-2] that there are at least 2^{2m} diffeomorphism types on a homotopy real projective space of dimension $4m+1$. Atiyah and Bott [AB] and Browder [Bro] obtained the bound 2^{2m-1} which was then extended to 2^{2m} by Giffen [Gi1-2]. Furthermore, for $0 \leq p < p' < 2^{2m}$. the homotopy real projective spaces $\Sigma_{p'}^{4m+1}/T$ and Σ_p^{4m+1}/T are not diffeomorphic. Thus, each of the diffeomorphism types is represented by deformation classes of positive Sasakian structures. This proves Theorem C. ∎

6. Exotic Contact Structures

With the exception of the last two sentences of Theorem C of the Introduction, the inequivalences of the contact structures refer to inequivalences of the underlying almost contact structures. The material discussed here is well-known and taken from Sato [Sa] and Morita [Mo]. An *almost contact structure* on an orientable manifold M^{2n-1} can be defined as a reduction of the oriented orthonormal frame bundle with group $SO(2n-1)$

to the group $U(n-1)\times 1$. Furthermore, the set $A(M)$ of homotopy classes of almost contact structures on M is in one-to-one correspondence with the set of homotopy classes of almost complex structures on $M\times\mathbb{R}$ [Sa], and when $M=\Sigma^{2n-1}$ is a homotopy sphere, the latter is known [Mo]

6.1
$$A(\Sigma^{2n-1})=\pi_{2n-1}(SO(2n)/U(n))\cong\begin{cases}\mathbb{Z}\oplus\mathbb{Z}_2 & \text{if } n\equiv 0(\text{mod } 4);\\ \mathbb{Z}_{(n-1)!} & \text{if } n\equiv 1(\text{mod } 4);\\ \mathbb{Z} & \text{if } n\equiv 2(\text{mod } 4);\\ \mathbb{Z}_{\frac{(n-1)!}{2}} & \text{if } n\equiv 3(\text{mod } 4);\end{cases}$$

Proof of Theorem B: The proof for the homotopy spheres Σ^{4m-1} and Σ^{4m+1} are somewhat different. The case Σ^{4m-1} uses the proof of Theorem 4.1(i) of Morita [Mo]. For each homotopy sphere $\Sigma^{4m-1}\in bP_{4m}$ the underlying almost contact structures are distinguished by Morita's invariant ∂, and he finds that

6.2
$$\partial(\Sigma_k^{4m-1})=\big(\frac{2}{S_m}-6\big)k,$$

where S_m is a certain rational number. Moreover, there are an infinite number of k that correspond to the same homotopy sphere $\Sigma^{4m-1}\in bP_{4m}$. Thus, we will have a countably infinite number of distinct underlying almost contact structures on the same homotopy sphere as long as $S_m\neq\frac{1}{3}$. If $S_m=\frac{1}{3}$ we simply use another description of elements in bP_{4m} by Brieskorn manifolds, for example that of 1.4. This completes the proof for this case.

Representing the homotopy spheres Σ^{4m+1} by polynomials of the form 1.3, we know that Σ^{4m+1} is the standard sphere if $p\equiv\pm1(\text{mod } 8)$, and the Kervaire sphere if $p\equiv\pm3(\text{mod } 8)$ which is exotic when $4m+2\neq 2^i-2$. In this case the Morita invariant is simply given in terms of the Milnor number μ of the link, viz.

6.3
$$\partial(\Sigma_p^{4m+1})=\frac{1}{2}\mu(\Sigma_p^{4m+1})=\frac{1}{2}(p-1).$$

Thus, as p varies through $\pm1(\text{mod } 8)$ or $\pm3(\text{mod } 8)$, the invariant $\partial(\Sigma_p^{4m+1})$ varies through $0(\text{mod } 4)$ and $3(\text{mod } 4)$, or $1(\text{mod } 4)$ and $2(\text{mod } 4)$, respectively. Since for $m>1$ $a(m)\equiv 0(\text{mod } 4)$ we can choose p such that $\partial(\Sigma_p^{4m+1})$ is any number mod $a(m)$. Thus, there are $a(m)$ distinct underlying almost contact structures which is finite in this case. To obtain an infinite number of inequivalent contact structures we use the recent results of Ustilovsky [Us] who showed by explicitly computing the contact homology of Eliashberg, Giventhal, and Hofer (cf. [Eli]) that for the standard sphere S^{4m+1} distinct p's satisfying $p\equiv\pm1(\text{mod } 8)$ give inequivalent contact structures. However, Ustilovsky's proof works equally well for any homotopy sphere Σ^{4m+1}. This completes the proof. ∎

Remarks 6.4: (1) It is interesting to note that for the 5-sphere S^5 ($m=1$ above), Ustilovsky's results distinguish infinitely many inequivalent contact structures, whereas, there is a unique underlying almost contact structure. (2) It is tempting to adapt the computations in [Us] to the case of the homotopy projective spaces $\mathbb{RP}^{4m+1}$. However, in this case complications arise, not the least of which is the fact that the simplification of the contact homology arising from the fact that all the Conley-Zehnder indices are even no longer holds.

7. Homotopy Spheres as Branched Covers

It is clear that the Brieskorn spheres described above can be thought of as cyclic branched covers of standard spheres. For example, the Kervaire spheres Σ_p^{4m+1} given by 1.3 is a p-fold cyclic cover of S^{4m+1} branched over the Stiefel manifold $V_2(\mathbb{R}^{2m+1})$ realized as an S^1 bundle over an odd complex quadric. In particular, the exotic Kervaire sphere Σ_3^{4m+1} can be realized as a 3-fold cover of S^{4m+1} branched over the Stiefel manifold $V_2(\mathbb{R}^{2m+1}) \subset S^{4m+1}$. Similarly, the homotopy spheres $\Sigma_k^{4m-1} \in bP_{4m}$ are $6k-1$-fold covers of S^{4m-1} branched over the Kervaire sphere Σ_3^{4m-3}. In both cases the branching locus is a rational homology sphere. More generally Savel'ev [Sav] noticed that under certain conditions the cyclic branched cover of a rational homology sphere is a homotopy sphere. Here we formulate Savel'ev's result in a way that is more convenient for our purpose. We make use of the work of Milnor and Orlik [MO].

THEOREM 7.1: *Let $f(z_1,\cdots,z_n)$ be a weighted homogeneous polynomial of degree d and weights $\mathbf{w}=(w_1,\cdots,w_n)$ in $\mathbb{C}^n$ with an isolated singularity at the origin. Let $p\in\mathbb{Z}^+$ and consider the link L_g of the equation*

$$g = z_0^p + f(z_1,\cdots,z_n) = 0.$$

Write the numbers $\frac{d}{w_i}$ in irreducible form $\frac{u_i}{v_i}$, and suppose that $\gcd(p,u_i)=1$ for each $i=1,\cdots,n$. Then the link L_g is a rational homology sphere with weights $\frac{(d,p\mathbf{w})}{\gcd(p,d)}$ and degree $lcm(p,d)$. Moreover, L_g is a homotopy sphere if and only if the link L_f is a rational homology sphere.

PROOF: Let us briefly recall the construction of the Alexander (characteristic) polynomial $\Delta_f(t)$ in [MO] associated to a link L_f of dimension $2n-3$. It is the characteristic polynomial of the monodromy map $\mathbb{I}-h_*: H_{n-1}(F,\mathbb{Z})\longrightarrow H_{n-1}(F,\mathbb{Z})$ induced by the $S^1_{\mathbf{w}}$ action on the Milnor fibre F. Thus, $\Delta(t)=\det(t\mathbb{I}-h_*)$. Now both F and its closure $\bar{F}$ are homotopy equivalent to a bouquet of n-spheres, and the boundary of $\bar{F}$ is the link L_f which is $n-3$-connected. The Betti numbers $b_{n-2}(L_f)=b_{n-1}(L_f)$ equal the number of factors of $(t-1)$ in $\Delta_f(t)$. Thus, L_f is a rational homology sphere if and only if $\Delta(1)\neq 0$, and a homotopy sphere if and only if $\Delta_f(1)=\pm 1$. Following Milnor and Orlik we let Λ_j denote the divisor of t^j-1. Then the divisor of $\Delta_f(t)$ is given by

7.2
$$\operatorname{div}\,\Delta_f = \prod_{i=1}^n\Big(\frac{\Lambda_{u_i}}{v_i}-1\Big)$$

Using the relations $\Lambda_a\Lambda_b=\gcd(a,b)\Lambda_{lcm(a,b)}$, equation 7.2 takes the form

$$(-1)^n+\sum a_j\Lambda_j,$$

where $a_j\in\mathbb{Z}$ and the sum is taken over the set of all least common multiples of all combinations of the $u_1,\cdots,u_n$. The Alexander polynomial is then given by

7.3
$$\Delta_f(t) = (t-1)^{(-1)^n}\prod_j (t^j-1)^{a_j},$$

and

7.4 $$b_{n-2}(L_f) \;=\; (-1)^n+\sum_j a_j.$$

Now we compute the divisor div Δ_g. We have

$$\begin{aligned}\text{div }\Delta_g \;&=\; (\Lambda_p-1)\text{div }\Delta_f \;=\; (\Lambda_p-1)\big((-1)^n+\sum a_j\Lambda_j\big)\\ &=\; \sum_j \gcd(p,j)a_j\Lambda_{\text{lcm}(p,j)}-\sum_j a_j\Lambda_j+(-1)^n\Lambda_p+(-1)^{n+1}.\end{aligned}$$

Since the j's run through all the least common multiples of the set $\{u_1,\cdots,u_n\}$ and $\gcd(p,u_i)=1$ for all i, we see that for all j, $\gcd(p,j)=1$. This implies

$$b_{n-1}(L_g) \;=\; \sum_j a_j-\sum_j a_j+(-1)^n+(-1)^{n+1} \;=\; 0.$$

Thus, L_g is a rational homology sphere. Next we compute the Alexander polynomial for L_g.

$$\Delta_g(t) \;=\; (t-1)^{(-1)^{n+1}}(t^p-1)^{(-1)^n}\prod_j (t^{pj}-1)^{a_j}(t^j-1)^{-a_j}$$

7.5 $$\;=\; (t^{p-1}+\cdots+t+1)^{(-1)^n}\prod_j\Big(\frac{t^{pj-1}+\cdots+t+1}{t^{j-1}+\cdots+t+1}\Big)^{a_j}.$$

This gives

$$\Delta_g(1) \;=\; p^{\Sigma_j a_j+(-1)^n} \;=\; 1$$

where by 7.4 the last equality holds if and only if L_f is a rational homology sphere. ∎

An easy way to insure the condition $\gcd(p,u_i)=1$ for all $i=1,\cdots,n$ is to assume that $\gcd(p,d)=1$ which we shall do henceforth.

THEOREM 7.6: *Let L_g be the link of a branched cover of S^{2n+1} branched over a smooth rational homology sphere L_f that is the link of an isolated hypersurface singularity given by a weighted homogeneous polynomial f of weights $\mathbf{w}_f$ and degree d_f as in Theorem 7.1. Suppose also that $\gcd(p,d_f)=1$, and that the corresponding hypersurface $\mathcal{Z}_f\subset\mathbb{P}(\mathbf{w}_f)$ is well-formed. Then L_g is a homotopy sphere admitting a Sasakian structure $\mathcal{S}$ with positive Ricci curvature.*

PROOF: Since $\gcd(p,d_f)=1$ the conditions of Theorem 7.1 are satisfied, so L_g is a homotopy sphere. Furthermore, since $\mathcal{Z}_f$ is well-formed it follows from Lemma 3.12 that the index $I_g=\mathbf{w}_f>0$, so the induced Sasakian structure is positive. The result then follows by Theorem 2.14. ∎

This theorem can be immediately applied to the rational homology spheres constructed in [BGN4]. We have

THEOREM 7.7: *Let L_g and L_f be as above, in particular* $\gcd(p, d_f) = 1$, *and take L_f to be one of the 184 rational homology 7-spheres listed in [BGN4]. Then L_g admits a deformation class $\mathfrak{F}(p, \mathbf{w})$ of Sasakian structures $\mathcal{S} \in \mathfrak{F}(p, \mathbf{w})$ containing Sasakian metrics with positive Ricci curvature. Furthermore, L_g is one of the following:*

1. *If p and d_f are both odd, then L_g is diffeomorphic to the standard 9-sphere S^9.*
2. *If p is odd and d_f is even, then for the 10 cases listed in the table in [BGN4] with d_f even we have:*
   a. *For*

$\mathbf{w} = (w_1, w_2, w_3, w_4, w_5)$	d_f
$(97, 1531, 2201, 2775, 3253)$	9856
$(101, 439, 559, 579, 619)$	2296
$(103, 1321, 2337, 2845, 3251)$	9856
$(115, 341, 523, 591, 727)$	2296

   L_g is diffeomorphic to the standard 9-sphere S^9 if $p \equiv \pm 1(mod\ 8)$, and to the exotic Kervaire 9-sphere Σ^9 if $p \equiv \pm 3(mod\ 8)$.

   b. *For*

$\mathbf{w} = (w_1, w_2, w_3, w_4, w_5)$	d_f
$(155, 1075, 3532, 5835, 7064)$	17660
$(187, 2416, 8177, 10965, 19328)$	41072
$(221, 2416, 5491, 13617, 19328)$	41072
$(316, 1727, 9577, 13345, 24648)$	49612
$(316, 2041, 6751, 15857, 24648)$	49612

   L_g is diffeomorphic to the standard 9-sphere S^9 if $p^2 \equiv \pm 1(mod\ 8)$, and to the exotic Kervaire 9-sphere Σ^9 if $p^2 \equiv \pm 3(mod\ 8)$.

   c. *For*

$\mathbf{w} = (w_1, w_2, w_3, w_4, w_5)$	d_f
$(49, 334, 525, 668, 763)$	2338

   L_g is diffeomorphic to the standard 9-sphere S^9 if $p^3 \equiv \pm 1(mod\ 8)$, and to the exotic Kervaire 9-sphere Σ^9 if $p^3 \equiv \pm 3(mod\ 8)$.

3. *If p is even and d_f is odd, then L_g is diffeomorphic to the*

$$\begin{cases} S^9 & \text{if } |H_3(L_f, \mathbb{Z})| \equiv \pm 1(mod\ 8); \\ \Sigma^9 & \text{if } |H_3(L_f, \mathbb{Z})| \equiv \pm 3(mod\ 8). \end{cases}$$

   However, the 174 examples given in [BGN4] with odd degree all have $|H_3(L_f, \mathbb{Z})| \equiv 1(mod\ 8)$; hence, in this case L_g is always a standard sphere.

In items 1 and 2 above there is a countable infinity of deformation classes of Sasakian structures with positive Ricci curvature on each of the homotopy 9-spheres.

PROOF: The first part follows immediately from Theorem 7.6 by choosing p such that $\gcd(p, d_f) = 1$. To compute the differential invariant we use a Theorem of Levine [Lev, Mil2] which asserts that L_g is the standard 9-sphere if $\Delta_g(-1) \equiv \pm 1(\text{mod } 8)$ and the exotic Kervaire sphere if $\Delta_g(-1)\equiv \pm 3 (\mathrm{mod}\ 8)$. So we need to compute $\Delta_g(-1)$. First, by factoring we can reduce 7.5 further:

7.7 $$\Delta_g(t) \ = \ (t^{p-1}+\cdots+t+1)^{-1}\prod_j \bigl(t^{(p-1)j}+\cdots+t^j+1\bigr)^{a_j},$$

where j ranges through the set of all least common multiples of the u_i's.

Case 1: Here we have both p and d_f odd. Since d_f is odd all the u_i's are odd. Thus, j in the product in 7.7 is always odd. This implies that $\Delta_g(-1)=1$ so L_g is the standard 9-sphere.

Case 2: Again p is odd, but now d is even, so both even and odd j's occur in 7.7. The odd j's contribute a 1 to the product. Thus, we have

$$\Delta_g(-1)=\prod_{j \text{ even}} p^{a_j}=p^{\Sigma_{\text{even j}}a_j}.$$

To proceed further we must look at the list of rational homology 7-spheres in [BGN4]. There are precisely 10 on the list with even degree. Moreover, there are 2 types that occur as indicated by Lemmas 3.4 and 3.12 of [BGN4], and they are distinguished by the divisor of their Alexander polynomials. For the first type we have div $\Delta=\Lambda_d-1$, and 4 of the 10 are of this type, precisely the ones occurring in 2a of the Theorem. In this case there is one even j, namely $j=d$ and $a_d=1$. Thus, we have $\Delta_g(-1)=p$, so 2a follows by Levine's result. The remaining 6 rational homology 7-spheres with even degree are of type two and for these we have

7.8 $$\text{div }\Delta = n(\mathbf{w})\Lambda_d+\Lambda_{m_3}-n(\mathbf{w})\Lambda_{m_2}-1$$

where $n(\mathbf{w})\in\mathbb{Z}^+$, and $d=m_2m_3$ with m_2 and m_3 relatively prime. One can see from the definition of m_2 and m_3 in [BG4] that m_3 must be even and m_2 odd. So if follow from 7.8 that $\sum_{j\text{ even}}a_j=n(\mathbf{w})+1$. Now $n(\mathbf{w})$ can be read off from the table in [BGN4], since the order of the third homology group $H_3(L_f,\mathbb{Z})$ is $m_3^{n(\mathbf{w})+1}$; hence, 2b and 2c follow.

Case 3: Here p is even and d_f is odd. In this case the form of 7.7 is indeterminant at $t=-1$, so we need some further manipulation. We also need to treat the two types of rational homology 7-spheres separately. For type 1 we have precisely one j which is odd and equal to d, and $a_d=1$. Thus, from 7.5 we see that the denominator in the product evaluates to 1 at $t=-1$ since j is odd. Then we have

$$\Delta_g(-1) \ = \ \lim_{t\to -1}(t^{p-1}+\cdots+t+1)^{-1}\bigl(t^{pd_f-1}+\cdots+t+1\bigr) \ = \ \lim_{t\to -1}t^{p(d_f-1)}+\cdots+t^p+1=d_f.$$

For rational homology 7-spheres L_f of type 1 the order of $H_3(L_f,\mathbb{Z})$ is precisely the degree d_f. For rational homology 7-spheres L_f of type 2, equation 7.8 above holds. Moreover, since d_f is odd, so are m_2 and m_3. So we have 3 odd j's, namely, d,m_2,m_3 and $a_d=n(\mathbf{w})=-a_{m_2},\ a_{m_3}=1$. Again the terms in the denominator inside the product in 7.5 evaluate to 1, and we find

$$\begin{aligned}\Delta_g(-1)&=\lim_{t\to -1}\Bigl(\frac{t^{pm_3-1}+\cdots+t+1}{t^{p-1}+\cdots+t+1)}\Bigr)\Bigl(\frac{t^{pd_f-1}+\cdots+t+1}{t^{pm_2-1}+\cdots+t+1}\Bigr)^{n(\mathbf{w})}\\ &=\lim_{t\to -1}(t^{p(m_3-1)}+\cdots+t^p+1)(t^{p(d_f-1)}+\cdots+t^{pm_2}+1)^{n(\mathbf{w})}=m_3^{n(\mathbf{w})+1}.\end{aligned}$$

But $m_3^{n(\mathbf{w})+1}$ is the order of $H_3(L_f,\mathbb{Z})$ for L_f of type 2. Combining this with Levine's result mentioned above proves 3 and the theorem. ∎

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