Geometriae Dedicata
Volume 63 (1996), 45-68

Betti Numbers of 3-Sasakian Manifolds

Krzysztof Galicki and Simon Salamon

Introduction

In 1960, Sasaki introduced a particular type of Riemannian manifold endowed with an almost contact structure [48]. The importance of Sasakian manifolds was soon recognized, and the special case of 3-Sasakian manifolds was first distinguished by Kuo [35] and Udrişte [55]. A 3-Sasakian manifold $(\mathcal{S}, g)$ is a $(4n+3)$-dimensional Riemannian manifold with three orthonormal Killing fields ξ^1, ξ^2, ξ^3 which define a local $SU(2)$-action and satisfy a curvature condition. One says that $\mathcal{S}$ is regular if the vector fields ξ^i are complete and the corresponding 3-dimensional foliation is regular, so that the space of leaves is a smooth $4n$-dimensional manifold M. Konishi and Ishihara [30] noticed that the induced metric on the latter is quaternion-Kähler with positive scalar curvature.

A complete regular 3-Sasakian manifold $\mathcal{S}$ fits into a diagram of Riemannian submersions

$$\begin{tikzcd} \mathcal{S} \arrow[r, "S^1"] \arrow[d, "\mathbb{RP}^3"] & \mathcal{Z} \arrow[dl, "\mathbb{CP}^1"'] \\ M & \end{tikzcd}$$

except that the fibre $\mathbb{RP}^3$ has to be replaced by S^3 when $\mathcal{S}$ is the sphere. The manifold $\mathcal{Z}$ is Kähler, and arises as the quotient of $\mathcal{S}$ by the circle group associated to any one of the vector fields ξ^a. All three Riemannian manifolds $\mathcal{S}, M, \mathcal{Z}$ are Einstein with positive scalar curvature. Conversely, starting with a quaternion-Kähler manifold M of positive scalar curvature, the manifolds $\mathcal{S}$ and $\mathcal{Z}$ can each be recovered as the total spaces of bundles naturally associated to M. In particular, $\mathcal{Z}$ is the twistor space of M, and its geometry was investigated in [45].

The above situation has been generalized to the orbifold category by Boyer, Galicki and Mann [7]–[13]. Indeed, 3-Sasakian geometry provides a natural language in which to describe results about quaternion-Kähler orbifolds, since there are situations in which $\mathcal{S}$ is a manifold but M and $\mathcal{Z}$ are not. In particular, for any $n \geq 1$ and any $(n+2)$-tuple of pairwise relatively prime positive integers $\boldsymbol{p} = (p_1, \dots, p_{n+2})$, there exists a non-regular $(4n+3)$-dimensional 3-Sasakian manifold $\mathcal{S}(\boldsymbol{p})$ whose leaf space is a quaternion-Kähler orbifold studied in [21].

The above examples show that there are infinitely many homotopy types of 3-Sasakian manifolds in every allowable dimension. Their existence motivated the present paper whose purpose is to establish some new results about the topology of 3-Sasakian manifolds.

THEOREM A. *Let $(\mathcal{S}, g, \xi^a)$ be a compact 3-Sasakian manifold of dimension $4n+3$. Then the odd Betti numbers b_{2k+1} of $\mathcal{S}$ are all zero for $0 \leq k \leq n$.*

This theorem is proved in Section 2 and follows from the fact that harmonic forms on $\mathcal{S}$ are restricted to lie in very specific submodules of the exterior algebra. This type of result is familiar for manifolds with reduced holonomy with which $\mathcal{S}$ is associated, but its own holonomy group is significantly never a proper subgroup of $SO(4n+3)$. Moreover, Theorem A holds without assuming that $\mathcal{S}$ is itself regular in the sense defined above.

In the regular case it is easy to relate the Betti numbers of the three manifolds $\mathcal{S}$, $\mathcal{Z}$, M, and we show in Section 3 that those of $\mathcal{S}$ coincide with the so-called primitive or effective Betti numbers of both M and $\mathcal{Z}$. Theorem A then provides an alternative, and in some ways more elementary, proof of the fact that the odd Betti numbers of M and $\mathcal{Z}$ all vanish [45]. On the other hand, the existence of a complex structure on $\mathcal{Z}$ does lead to quick proofs of a number of non-trivial results, such as the fact that $\mathbb{RP}^{4n+3}$ is the only compact regular 3-Sasakian manifold which is not simply-connected. In a similar vein, the following theorems are re-interpretations of further results of LeBrun [36, 37].

THEOREM B. (i) *Up to isometries, there are only finitely many compact regular 3-Sasakian manifolds in each dimension $4n+3$, $n \geq 0$.*
(ii) *The only compact regular 3-Sasakian manifolds for which $b_2 > 0$ are the spaces $U(m)/(U(m-2) \times U(1))$, $m \geq 3$.*

Friedrich and Kath [18] showed that a 3-Sasakian metric on a 7-manifold (the case $n=1$) is characterized by the existence of at least three independent Killing spinors. As explained by Bär [4], a Riemannian 7-manifold admits at least one Killing spinor if and only if it has weak holonomy G_2 in the terminology of Gray [24]. To illustrate the calculus of differential forms in Section 2, we identify a second Einstein metric with weak holonomy G_2 on a 3-Sasakian 7-manifold. The latter arises from a metric with holonomy $Spin(7)$ constructed in [15] on the spin bundle of the orbifold M.

The classification of 3-Sasakian homogeneous spaces was given in [7] and is explained briefly in Section 4. It is known that *any* compact regular 3-Sasakian $(4n+3)$-manifold $\mathcal{S}$ is homogeneous if $n \leq 2$. The case $n=1$ was proved in [18] and follows from the classification of compact self-dual Einstein manifolds of positive scalar curvature [27, 19, 5]. The case $n=2$ was obtained in [7] and follows from the corresponding result for positive quaternion-Kähler 8-manifolds [42, 37]. Related to the latter is the fact that the Betti numbers of a compact regular 3-Sasakian 11-manifold must satisfy $b_2 = b_4$. Furthermore

THEOREM C. *Let $\mathcal{S}$ be a compact regular 3-Sasakian manifold of dimension $4n+3$.*
(i) *The Betti numbers of $\mathcal{S}$ satisfy* $\sum\limits_{k=1}^{n} k(n+1-k)(n+1-2k)b_{2k} = 0$.
(ii) *If $b_4 = 0$ and $n = 3$ or 4 then $\mathcal{S}$ is the sphere or real projective space.*

The intriguing constraint in (i) is deduced from an analogous one for M, namely [37, Theorem 0(iii)] that was proved by considering coupled Dirac operators. It may be that equivariant index theory can be used to provide a more direct proof of Theorem C(i), and to extend its validity to the non-regular case. It should, however, be noted that Theorem B(ii) is known not to hold in general [7, 8]. A proof of Theorem C(ii) proceeds in Section 5 by showing that the corresponding quaternion-Kähler manifold M must be the projective space $\mathbb{HP}^n$.

Acknowledgements. The first author was supported by NSF grant DMS–9224811, and would like to thank Charles Boyer and Ben Mann for many discussions and conversations concerning this work. The second author thanks Claude LeBrun for ideas that led to parts of this article that had their origin in a section of the preprint version of [37]. Both authors are also grateful to the Erwin Schrödinger Institute in Vienna for the hospitality that they received there in the autumn of 1994.

1. Preliminaries

This section includes definitions of Sasakian structures and examines some of their basic properties.

Let $(\mathcal{S}, g)$ be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. Given a vector field ξ, let η denote the 1-form dual to ξ, and Φ the endomorphism of the tangent bundle defined by $\Phi(X) = \nabla_X \xi$. The condition that Φ be skew-symmetric is precisely equivalent to saying that ξ is a Killing vector field.

1.1 Definition. *$(\mathcal{S}, g, \xi)$ is a Sasakian manifold if ξ is a Killing vector field of unit length and*

$$(\nabla_X \Phi)(Y) = \eta(Y)X - g(X,Y)\xi$$

for all vector fields X, Y.

Suppose that X is a vector field orthogonal to ξ. Since ξ has length 1,

$$0 = g(\nabla_X \xi, \xi) = -g(\nabla_\xi X, \xi) = g(X, \nabla_\xi \xi),$$

and it follows that $\Phi(\xi) = 0$. Using this, one obtains

$$g(\Phi^2 Y, X) = -g(\nabla_X \xi, \Phi Y) = g(\xi, (\nabla_X \Phi)Y),$$

and Definition 1.1 implies that

$$\Phi^2 = -\mathbf{1} + \eta \otimes \xi. \tag{1}$$

This shows that Φ acts as an almost complex structure on the codimension 1 distribution orthogonal to ξ, and it follows that $\mathcal{S}$ has dimension $2m+1$. If ϕ denotes the 2-form corresponding to Φ, we have

$$\phi(X,Y) = g(\Phi X, Y) = \tfrac{1}{2}((\nabla_X \eta)Y - (\nabla_Y \eta)X) = d\eta(X,Y). \tag{2}$$

It makes sense to consider the 'Nijenhuis tensor'

$$N_\Phi(X,Y) = [\Phi X, \Phi Y] + \Phi^2[X,Y] - \Phi[X, \Phi Y] - \Phi[\Phi X, Y]$$

associated to Φ. The close relationship between Sasakian and complex structures derives from the easily-proven formula

$$N_\Phi(X,Y) = -2\phi(X,Y)\xi$$

which implies that if the infinitesimal automorphism ξ arises from a free circle action then the quotient $\mathcal{S}/U(1)$ is a complex manifold. The latter actually possesses a Kähler metric. We next describe a more specialized situation that is the subject of this article.

1.2 Definition. *A 3-Sasakian manifold is a Riemannian manifold* $(\mathcal{S},g)$ *that admits three distinct Sasakian structures whose vector fields* ξ^1,ξ^2,ξ^3 *are mutually orthogonal and satisfy* $[\xi^a,\xi^b]=2\sum\limits_{c=1}^{3}\epsilon^{abc}\xi^c$ *for* $a,b=1,2,3$.

The equation in Definition 1.1 is not preserved by a homothety $g\mapsto tg$ with $t>0$ constant, and so the metric g in Definition 1.2 cannot be re-scaled so as to remain 3-Sasakian. The Lie bracket equation may be rewritten

$$\Phi^a\xi^b=-\nabla_{\xi^a}\xi^b=-\sum_c\epsilon^{abc}\xi^c, \tag{3}$$

and the third Sasakian structure ξ^3 is determined up to sign by ξ^1 and ξ^2.

There is an orthogonal decomposition

$$T\mathcal{S}=F\oplus F^\perp \tag{4}$$

of the tangent bundle of a 3-Sasakian manifold $\mathcal{S}$, where F is the subbundle spanned by ξ^1,ξ^2,ξ^3. It follows from (1) and (3) that the restrictions $(\Phi^a)^\perp$ to $F^\perp$ of the endomorphisms $\Phi^a=\nabla\xi^a$ are almost complex structures satisfying the quaternion identities

$$(\Phi^a)^\perp\circ(\Phi^b)^\perp=-\delta^{ab}\mathbf{1}+\sum_c\epsilon^{abc}(\Phi^c)^\perp. \tag{5}$$

As a consequence, the rank of $F^\perp$ is necessarily a multiple of 4, and the dimension of $\mathcal{S}$ equals $4n+3$.

It follows from Definition 1.2 that the Killing fields ξ^a span the Lie algebra $\mathfrak{sp}(1)$ of a local action of $Sp(1)\cong SU(2)$ on $\mathcal{S}$. Suppose now that the ξ^a are complete, so that $Sp(1)$ acts globally on $\mathcal{S}$. We let $\mathcal{F}$ denote the corresponding 3-dimensional foliation tangent to the 'vertical' distribution F. If $\mathcal{S}$ is complete and $\mathcal{F}$ is regular, the leaves of $\mathcal{F}$ are diffeomorphic either to $SO(3)\cong\mathbb{RP}^3$ or to $Sp(1)\cong S^3$ [52] and, as explained in the Introduction, the quotient $M=\mathcal{S}/Sp(1)$ is a quaternion-Kähler manifold of positive scalar curvature. The only case in which the foliation $\mathcal{F}_3$ has S^3 leaves is that of the Hopf fibration $S^{4n+3}\to\mathbb{HP}^n$, a fact which may be proved by applying [45, Theorem 6.1]. With this exception, given a compact quaternion-Kähler manifold M of positive scalar curvature, the total space of the natural $\mathbb{RP}^3$ bundle is the unique 3-Sasakian manifold $\mathcal{S}$ fibering over M, and was studied by Konishi [33]. Its 3-Sasakian structure is completely determined by a result of [39].

Each $\tau\in\mathfrak{sp}(1)$ singles out a Killing field, a circle subgroup $U(1)_\tau\subset Sp(1)$, and a 1-dimensional foliation $\mathcal{F}_\tau$ subordinate to $\mathcal{F}$ giving $\mathcal{S}$ the structure of a Seifert fibred space. If $\mathcal{F}$ is regular then $\mathcal{F}_\tau$ is regular for all τ. When $\mathcal{S}$ is complete the converse is also true; if any of the foliations $\mathcal{F}_\tau$ is regular than so is $\mathcal{F}$, and in this situation we say that $\mathcal{S}$ is a regular 3-Sasakian manifold [52]. It is well known that the quotient $\mathcal{S}/U(1)_\tau$ is then a Kähler-Einstein manifold which, for any τ, is isomorphic to the twistor space $\mathcal{Z}$ of $\mathcal{S}/Sp(1)$. Moreover, $\mathcal{S}$ can be identified with the total space of the circle bundle of unit vectors in the holomorphic line bundle $L=\kappa^{-1/(n+1)}$ over $\mathcal{Z}$, described in [37].

In the more general orbifold setting, it is necessary to interpret the above objects as V-bundles [49, 2, 3]. Proofs of some of the corresponding properties summarized below can be found in [5, 28, 29, 30, 31, 33, 54] and [7]–[9].

1.3 Theorem. *Let $(\mathcal{S},g,\xi^a)$ be a 3-Sasakian manifold of dimension $4n+3$ such that the vector fields ξ^a are complete for $a=1,2,3$. Then*

(i) *g is Einstein with scalar curvature $2(2n+1)(4n+3)$;*

(ii) *g is bundle-like (in the sense of [43]) with respect to each $\mathcal{F}_\tau$ and $\mathcal{F}$;*

(iii) *each leaf of $\mathcal{F}$ is totally geodesic and of constant curvature 1, and the space of leaves is a quaternion-Kähler orbifold M of scalar curvature $16n(n+2)$;*

(iv) *the space of leaves of $\mathcal{F}_\tau$ is a complex orbifold $\mathcal{Z}$ of dimension $2n+1$ that is independent of τ, and has a Kähler-Einstein metric of scalar curvature $8(2n+1)(n+1)$. Moreover, $\mathcal{Z}$ may be identified with the orbifold twistor space of M, and is a projective algebraic variety.*

From (i) and the theorem of Myers, it follows that if $\mathcal{S}$ is complete as a Riemannian manifold, then it is compact and $\pi_1(\mathcal{S})$ is finite. Furthermore, if $\mathcal{S}$ is complete and regular there is a fibration $\mathcal{S}\to M$ with $\pi_1(M)$ trivial, and it follows that $\pi_1(\mathcal{S})$ has order 1 or 2. In the latter case, there is a lift to an S^3-bundle, and from above M is isometric to $\mathbb{HP}^n$ and $\mathcal{S}$ is a sphere. In fact, both $\mathcal{S}$ and $\mathcal{Z}$ admit a second Einstein metric formed by re-scaling g relative to (4) [8]; we shall justify this statement in the case $n=2$ at the end of the next section.

While renewed interest in 3-Sasakian manifolds is rather recent, there has been more extensive effort made to understand the geometry of compact quaternion-Kähler manifolds of positive scalar curvature. It was proved by Alekseevsky [1] that the only homogeneous spaces of this type are the symmetric ones classified by Wolf [58]. There are no other known examples, and in real dimension 8 all positive quaternion-Kähler manifolds are indeed symmetric [42]. More recent results in [36, 37] support a conjecture that there are no complete non-symmetric quaternion-Kähler manifolds of positive scalar curvature. Given the above correspondence between 3-Sasakian and quaternion-

Kähler manifolds, Theorem B is a direct consequence of (i) [37, Theorem 0.1] and (ii) [37, Theorem 0.3(ii)], and the latter in turn relies on a classification due to Wiśniewski [57] of compact complex Fano manifolds. In the present paper we comment no further on the proof of Theorem B. As noted in the Introduction, there are infinitely many homotopy types of complete irregular 3-Sasakian manifolds with $b_2(\mathcal{S})=1$ (see Section 4), and we know of no restriction on $\pi_2(\mathcal{S})$ when $\mathcal{S}$ is not regular.

2. Differential forms

For the remainder of this article, we shall suppose that $(\mathcal{S},g,\xi^a)$ is a complete (equivalently, compact) $(4n+3)$-dimensional 3-Sasakian manifold, which implies that the vector fields ξ^1,ξ^2,ξ^3 are complete.

Let $\Omega^p(\mathcal{S})$ denote the space of p-forms on $\mathcal{S}$; throughout this section we shall suppose that $p\le 2n+1$. Referring to (4), we shall say that a p-form $u\in\Omega^p(\mathcal{S})$ has *bidegree* $(i,p-i)$ if it is a section of the subbundle of $\bigwedge^p T^*\mathcal{S}$ isomorphic to the dual of $\bigwedge^i F\otimes\bigwedge^{p-i}(F^\perp)$. In particular, u is called *3-horizontal* if has bidegree $(0,p)$, or equivalently if $\xi^a\lfloor u=0$ for $a=1,2,3$. An element $u\in\Omega^p(\mathcal{S})$ is called *invariant* if $h^*\omega=\omega$ for all $h\in Sp(1)$. In the regular case, there is an $Sp(1)$-bundle $\pi\colon\mathcal{S}\to M$, and ω is both 3-horizontal and invariant if and only if it is the pullback $\pi^*\hat\omega$ of a form $\hat\omega$ on the quaternion-Kähler base M.

Let $\phi^a=d\eta^a$ denote the 2-form associated to the skew-symmetric endomorphism $\Phi^a=\nabla\xi^a$, defined by (2) for $a=1,2,3$. Definition 1.2 then implies that

$$\overline{\phi}^a=\phi^a+\sum_{b,c}\epsilon^{abc}\eta^b\wedge\eta^c$$

is 3-horizontal. It follows from (3) that the closed 2-forms

$$\omega^a=r^2\phi^a+2rdr\wedge\eta^a,\quad a=1,2,3, \tag{6}$$

are associated to a triple of almost complex structures I^1,I^2,I^3 on $\mathbb{R}^+\times\mathcal{S}$ orthogonal with respect to the metric

$$\tilde g=dr^2+r^2g. \tag{7}$$

Using [46, Lemma 8.4], we immediately deduce

2.1 Proposition. *If $(\mathcal{S},g)$ is 3-Sasakian, the cone $(\mathbb{R}^+\times\mathcal{S},\tilde g)$ is hyper-Kähler.*

When $\mathcal{S}$ is regular, the metric $\tilde{g}$ is locally equivalent to the natural hyper-Kähler metric associated to the quaternion-Kähler manifold M described by Swann [50]. Proposition 2.1, first proved in [8], provides the most direct link between 3-Sasakian and quaternionic geometry.

The Killing fields ξ^a transform according to the adjoint representation of $Sp(1)$, and the same is true of the associated triples η^a, ϕ^a, and Φ^a. For example, if $h \in Sp(1)$, we may write

$$h_*\Phi^a = \sum_b h^{ab}\Phi^b, \quad a = 1, 2, 3, \tag{8}$$

where h^{ab} are components of the image of h in $Sp(1)/\mathbb{Z}_2 \cong SO(3)$. The 3-forms

$$\begin{aligned} \Upsilon &= \eta^1 \wedge \eta^2 \wedge \eta^3, \\ \Theta &= \sum_a \eta^a \wedge \overline{\phi}^a = \sum_a \eta^a \wedge \phi^a + 6\Upsilon \end{aligned} \tag{9}$$

have respective bidegrees $(3,0), (1,2)$, and are clearly invariant. Their exterior derivatives are

$$d\Upsilon = \eta^1 \wedge \eta^2 \wedge \overline{\phi}^3 + \eta^2 \wedge \eta^3 \wedge \overline{\phi}^1 + \eta^3 \wedge \eta^1 \wedge \overline{\phi}^2, \tag{10}$$

and

$$d\Theta = \Omega + 2d\Upsilon, \tag{11}$$

where the 4-form

$$\Omega = \sum_a \overline{\phi}^a \wedge \overline{\phi}^a$$

is 3-horizontal and invariant. In fact, Ω is the canonical 4-form determined by the quaternionic structure (5) of the subbundle $F^\perp$, and is the pullback of the fundamental 4-form $\hat{\Omega}$ on the quaternion-Kähler orbifold M.

We next aim to prove Theorem A. Let

$$\mathcal{H}^p(\mathcal{S}) = \{u \in \Omega^p(\mathcal{S}) : du = 0 = d * u\}$$

denote the finite-dimensional space of harmonic p-forms on $\mathcal{S}$. Any harmonic form u is necessarily invariant because of the homotopy invariance of cohomology [59]. Fix $a \in \{1, 2, 3\}$ and set $\xi = \xi^a$ so as to consider $\mathcal{S}$ temporarily as a Sasakian manifold. The tensor $\Phi = \nabla\xi$ extends to an endomorphism of $\Omega^p(\mathcal{S})$ by setting

$$(\Phi u)(X_1, X_2, \ldots, X_p) = \sum_{i=1}^{p} u(X_1, \ldots, \Phi X_i, \ldots, X_p). \tag{12}$$

With this notation, we state the fundamental results of Tachibana [53].

2.2 THEOREM. *Let* $u \in \mathcal{H}^p(\mathcal{S})$, $p \leq 2n+1$. *Then*
(i) $\xi \lfloor u = 0$, *and*
(ii) $\Phi u \in \mathcal{H}^p(\mathcal{S})$.

It is easy to prove (i) in the simple case $p=1$. Indeed, let

$$u = \alpha + f\eta$$

be a closed invariant 1-form, where $\alpha(\xi) = 0$, and f is a function. The vanishing of the Lie derivative of u along ξ implies that $0 = d(\xi \lfloor u) = df$, so that f is a constant and $0 = d\alpha + f\phi$ where $\phi = d\eta$. Then

$$0 = \int_{\mathcal{S}} d(\alpha \wedge \phi^{2n} \wedge \eta) = -\int_{\mathcal{S}} f\phi^{2n+1} \wedge \eta.$$

Since $\phi^{2n+1} \wedge \eta$ is a non-zero multiple of the volume form of $\mathcal{S}$, we obtain $f=0$ and $\xi \lfloor u = 0$. A first step in the proof in [53] of the general case is the assertion that, with the hypotheses of Theorem 2.2, $\xi \lfloor u$ is coclosed.

Theorem 2.2(i) implies that any harmonic p-form with $p \leq 2n+1$ on the compact 3-Sasakian manifold $\mathcal{S}$ is 3-horizontal. Apply (12) so as to obtain

$$\Phi^a\colon \mathcal{H}^p(\mathcal{S}) \to \mathcal{H}^p(\mathcal{S}), \quad a = 1,2,3, \quad p \leq 2n+1,$$

and define further

$$(\mathcal{I}^a u)(X_1, X_2, \ldots, X_p) = u(\Phi^a X_1, \Phi^a X_2, \ldots, \Phi^a X_p).$$

The basic identity (1) combines with Theorem 2.2 to show that $\mathcal{I}^a u$ is a linear combination of $(\Phi^a)^k u$ for $0 \leq k \leq p$. Thus $\mathcal{I}^a$ also maps $\mathcal{H}^p(\mathcal{S})$ into itself. Moreover, (5) translates into the identity

$$\mathcal{I}^b \circ \mathcal{I}^a = (-\delta^{ab})^p \mathbf{1} + \sum_c (\epsilon^{abc})^p \mathcal{I}^c. \tag{13}$$

As pointed out by Kuo, when p is odd these relations endow $\mathcal{H}^p(\mathcal{S})$ with an almost quaternionic structure.

2.3 THEOREM. *Let* $u \in \mathcal{H}^p(\mathcal{S})$, $p \leq 2n+1$.
(i) *If* p *is odd then* $u \equiv 0$.
(ii) *If* p *is even then* $\mathcal{I}^a u = u$ *for* $a = 1,2,3$.

Proof. Let $u \in \mathcal{H}^p(\mathcal{S})$. We shall in fact show that $\mathcal{I}^1 u = \mathcal{I}^2 u$ irrespective of whether p is even or odd; the result then follows from the identities (13) and symmetry between the indices 1,2,3. By (8), we may choose an isometry $h \in Sp(1)$ so that $h_*\Phi^1 = \Phi^2$. Both u and $\mathcal{I}^1 u$ are harmonic, so $h^* u = u$ and

$$\begin{aligned}(\mathcal{I}^1 u)(X_1, \ldots, X_p) &= (h_*(\mathcal{I}^1 u))(X_1, \ldots, X_p)\\ &= u((h_*\Phi^1)(X_1), \ldots, (h_*\Phi^1)(X_p))\\ &= u(\Phi^2 X_1, \ldots, \Phi^2 X_p)\\ &= (\mathcal{I}^2 u)(X_1, \ldots, X_p).\end{aligned}$$

This completes the proof. QED

Theorem 2.3(i) immediately implies Theorem A of the Introduction. We may interpret Theorem 2.3(ii) as follows. When $p = 2k$, the $Sp(n)Sp(1)$-structure of the 'horizontal' bundle $F^\perp$ (see (5) and (8)) allows us to write

$$\bigwedge^p (F^\perp)^* \cong \bigoplus_{i=0}^{k} V_i \otimes \Sigma_i, \tag{14}$$

where each V_i is a real subbundle arising associated to a representation of $Sp(n)$, and Σ_i is the real irreducible representation of $Sp(1)/\mathbb{Z}_2 \cong SO(3)$ of dimension $2i+1$. Then any harmonic form takes values in the subbundle V_0 of forms which are invariant by the action of $Sp(1)$, and which are of type (k,k) relative to $\mathcal{I}^a$ for $a = 1, 2, 3$.

To complete this section, we show how differential forms can be used to relate 3-Sasakian 7-manifolds and G_2-structures. A G_2-structure on a 7-manifold is determined by a 3-form φ which is 'non-degenerate' and 'positive' in an appropriate sense [46]. Such a form determines automatically an orientation and a Riemannian metric g, and the latter is said to have *weak holonomy* G_2 if

$$d\varphi = c * \varphi, \tag{15}$$

where $*$ is the star operator relative to g (and therefore φ) and c is a constant whose sign is fixed by the orientation convention. The terminology is due to Gray [24], who showed that any such metric is Einstein with positive scalar curvature. The equation (15) implies that φ is 'nearly parallel' in the sense that only a 1-dimensional component of $\nabla\varphi$ is different from zero [17].

The sphere S^7 with its constant curvature metric is isometric to the isotropy irreducible space $Spin(7)/G_2$. The fact that G_2 leaves invariant (up to constants) a unique 3-form and a unique 4-form on $\mathbb{R}^7$ implies immediately that this space has weak holonomy G_2, although extra 3-forms materialize when the symmetry group $Spin(7)$ is reduced to one of its proper subgroups $SU(4)$ or $Sp(2)$. These observations can be applied in a more general setting, since a metric g has weak holonomy G_2 if and only if the metric $\tilde{g}$ of (7) has holonomy contained in $Spin(7)$ [44, 4]. Special types of weak G_2 structures then arise if the holonomy of $\tilde{g}$ reduces to $SU(4)$ or $Sp(2)$, in which case the metric g on M is subordinate to respectively 2 or 3 independent G_2 3-forms. The situation has been characterized in terms of Killing spinors by Bär [4], who refers to these two non-generic cases as type $(2,0)$ or $(3,0)$ respectively.

Since $Sp(2)$ is the holonomy group of a hyper-Kähler manifold, Proposition 2.1 confirms that any 3-Sasakian metric on a 7-manifold has weak holonomy G_2 of type $(3,0)$. We may determine one of the corresponding 3-forms by setting

$$\omega^1\wedge\omega^1+\omega^2\wedge\omega^2-\omega^3\wedge\omega^3=c\,r^4*\varphi+4r^3dr\wedge\varphi,$$

since the left-hand side is a 4-form invariant by $Spin(7)$ [14]. This gives $c=-4$ and

$$\varphi=\eta^1\wedge\overline{\phi}^1+\eta^2\wedge\overline{\phi}^2-\eta^3\wedge\overline{\phi}^3-2\Upsilon,$$

and two other forms are obtained by changing the position of the first minus sign. On the other hand, it is well known that S^7, regarded as the space $Sp(2)/Sp(1)$ and fibering over S^4, admits a 'squashed' Einstein metric which does not have constant curvature. This metric also has weak holonomy G_2 since the associated cone metric has holonomy equal to $Spin(7)$. An analogous metric with holonomy (equal to or contained in) $Spin(7)$ exists on an open set of the spin bundle of any 4-dimensional self-dual Einstein orbifold [15], and the next result is predicted by this fact.

2.4 Proposition. *A 7-dimensional 3-Sasakian manifold $(\mathcal{S},g)$ admits a metric g' with weak holonomy G_2, but not homothetic to g.*

Proof. We may find locally an orthonormal basis $\{\alpha^1,\alpha^2,\alpha^3,\alpha^4\}$ of 1-forms spanning the annihilator in $T^*\mathcal{S}$ of F in (4) such that

$$\begin{aligned}\overline{\phi}^1&=2(\alpha^1\wedge\alpha^2-\alpha^3\wedge\alpha^4),\\ \overline{\phi}^2&=2(\alpha^1\wedge\alpha^3-\alpha^4\wedge\alpha^2),\\ \overline{\phi}^3&=2(\alpha^1\wedge\alpha^4-\alpha^2\wedge\alpha^3).\end{aligned}$$

In addition, set

$$\alpha^5=\lambda\eta^1,\quad \alpha^6=\lambda\eta^2,\quad \alpha^7=\lambda\eta^3,$$

where λ is a positive constant to be determined. The Riemannian metric g' for which the α^i's form an orthonormal basis is related to g by a non-trivial change of scale on the subbundle F of $T\mathcal{S}$. With respect to g', the 3-form

$$\varphi = \tfrac{1}{2}\lambda\Theta + \lambda^3\Upsilon$$

can be expressed as $125 - 345 + 136 - 426 + 147 - 237 + 567$, and therefore defines a compatible G_2-structure. Moreover,

$$\begin{aligned} d\varphi &= \tfrac{1}{2}\lambda\Omega + \lambda(\lambda^2+1)d\Upsilon, \\ *\varphi &= -\tfrac{1}{2}\lambda^2 d\Upsilon - \tfrac{1}{24}\Omega, \end{aligned}$$

and (15) is solved by taking $\lambda = 1/\sqrt{5}$. QED

By Theorem 2.3 and Poincaré duality, any compact 3-Sasakian 7-manifold has b_1, b_3, b_4, b_6 all zero. Moreover, any harmonic 2-form lies in the 3-dimensional subspace V_0, which may be identified with the subbundle Λ^+ of $\bigwedge^2 F^\perp$, on which the $*$ operator acts as $+1$. These remarks highlight the interest in searching for compact manifolds with $b_3 \neq 0$ that admit an Einstein metric with weak holonomy G_2.

3. Regular 3-Sasakian cohomology

Throughout this section we assume that $(\mathcal{S}, g, \xi^a)$ is a compact 3-Sasakian manifold for which the foliations $\mathcal{F}_\tau$, $\mathcal{F}$ are regular, so that $\mathcal{Z} = \mathcal{S}/U(1)$ and $M = \mathcal{S}/Sp(1)$ are manifolds endowed with their canonical Kähler and quaternion-Kähler structure respectively. Because $\mathcal{S}$ is a specific $SO(3)$-bundle over M and $U(1)$-bundle over $\mathcal{Z}$, it is straightforward to relate the Betti numbers $b_p(\mathcal{S})$, $b_p(M)$, $b_p(\mathcal{Z})$. Some of the results we discuss next can be found in [30].

Let $u \in \mathcal{H}^p(\mathcal{S})$, supposing always that $p \leq 2n+1$. Since u is both 3-horizontal and invariant, it is the pullback $\pi^*\hat{u}$ of a form $\hat{u}$ on the quaternion-Kähler manifold M. We claim that $\hat{u}$ is itself harmonic. Certainly $\hat{u}$ is closed, as $\pi^* d\hat{u} = du = 0$ and π^* is injective at the level of forms. It therefore suffices to prove that $*\hat{u}$ is closed, where $*$ here denotes the star operator on M. In fact, since π is a Riemannian submersion,

$$*u = \pi^*(*\hat{u}) \wedge \Upsilon,$$

and

$$0 = d(*u) = \pi^*(d*\hat{u}) \wedge \Upsilon + (-1)^p \pi^*(*\hat{u}) \wedge d\Upsilon.$$

The terms on the right have respective bidegrees $(3, 4n-p+1)$, $(2, 4n-p+2)$ relative to (4), and the result follows. In this way, we see that the induced mapping $\pi^*: H^*(M) \to H^*(\mathcal{S})$ is surjective.

The formula (11) shows that π^* is not injective on cohomology since it maps the fundamental class $[\hat{\Omega}]$ to zero. In fact, Theorem 3.2 below implies that the kernel of π^* is exactly the ideal generated by $[\hat{\Omega}]$; we prove this with more topological methods.

3.1 Lemma. *Let $p \le 2n+1$. Then $b_p(\mathcal{S}) = b_p(\mathcal{Z}) - b_{p-2}(\mathcal{Z})$.*

Proof. Applying the Gysin sequence to the fibration $\mathcal{S} \to \mathcal{Z}$ gives the following exact sequence:

$$\cdots \to H^{p-1}(\mathcal{S}) \to H^{p-2}(\mathcal{Z}) \xrightarrow{\delta} H^p(\mathcal{Z}) \to H^p(\mathcal{S}) \to H^{p-1}(\mathcal{Z}) \to \cdots$$

Now, $\mathcal{S}$ is covered by the total space of the circle bundle in the canonical bundle κ of $\mathcal{Z}$, and the Kähler-Einstein metric of $\mathcal{Z}$ arises from κ in accordance with Kobayashi's theorem [45, 32]. It follows that the connecting homomorphism δ is given by wedging with a non-zero multiple of the Kähler form of $\mathcal{Z}$, and this is well known to be injective so long as $p \le 2n+2$. The Gysin sequence therefore reduces to a series of short exact sequences up to and including $H^{2n+1}(\mathcal{S})$, and the lemma follows. QED

3.2 Theorem. *Let $(\mathcal{S}, g, \xi^a)$ be a compact regular 3-Sasakian manifold of dimension $4n+3$, with quotients M, $\mathcal{Z}$. Then the odd Betti numbers of both M and $\mathcal{Z}$ all vanish, and*

$$b_{2k}(M) - b_{2k-4}(M) = b_{2k}(\mathcal{S}) = b_{2k}(\mathcal{Z}) - b_{2k-2}(\mathcal{Z}), \quad k \le n.$$

Proof. The vanishing of $b_{2k+1}(\mathcal{Z})$ for $k \le n$ follows by applying Theorem 2.3(i) to the equations $b_{2k+1}(\mathcal{Z}) = b_{2k+1}(\mathcal{S}) + b_{2k-1}(\mathcal{Z})$ for $k \le n$ (with the obvious convention that $b_p = 0$ if $p < 0$). The result for $k > n$ follows from Poincaré duality as $\mathcal{Z}$ has even real dimension. Applying the Gysin sequence of the fibration $\mathcal{Z} \to M$ gives

$$b_p(\mathcal{Z}) = b_p(M) + b_{p-2}(M),$$

and the remainder of the theorem follows. QED

Let M be a compact quaternion-Kähler $4n$-manifold with positive scalar curvature. The proof that the odd Betti numbers of M all vanish was first given in [45] by showing that the Hodge numbers $h^{p,q}$ of the twistor space $\mathcal{Z}$ are zero except when $p = q$. The last result can in fact be deduced from Theorem 2.3(ii), which also implies that a harmonic $2k$-form on M is of bidegree (k, k) relative to *every* almost complex structure subordinate to the $Sp(n)Sp(1)$-structure at a point (cf. (14)).

The proof of Lemma 3.1 shows clearly that for $k \leq n$, each Betti number $b_{2k}(\mathcal{S})$ coincides with the dimension of the *primitive cohomology group* $H_0^{2k}(\mathcal{Z}, \mathbb{R})$, isomorphic to the cokernel of the injective mapping

$$H^{2k-2}(\mathcal{Z}, \mathbb{R}) \hookrightarrow H^{2k}(\mathcal{Z}, \mathbb{R}), \quad k \leq n,$$

defined by wedging with the Kähler 2-form. On a quaternion-Kähler manifold, wedging with the canonical 4-form Ω determines in an analogous manner an injection

$$H^{2k-4}(M, \mathbb{R}) \hookrightarrow H^{2k}(M, \mathbb{R}), \qquad k \leq n+1,$$

[6, 34, 20]. It follows that the Betti numbers of $\mathcal{S}$ may also be regarded as the 'primitive Betti numbers' of M; this fact appears in a different guise in [30].

Theorem 3.2 implies that the Euler characteristics of M and $\mathcal{Z}$ are related by the formula

$$\chi(\mathcal{Z}) = 2\chi(M).$$

For future reference, we note that the signature $\tau(M) = b_{2n}^+(M) - b_{2n}^-(M)$ may be expressed in terms of the Betti numbers of $\mathcal{S}$. Indeed, the intersection form on $H^{2n}(M, \mathbb{R})$ is known to be definite [38, 20], and with the appropriate orientation convention, $\tau(M) = (-1)^n b_{2n}(M)$ [37, (5.6)]. It follows that

$$\tau(M) = (-1)^n \sum_{k=0}^{[n/2]} b_{2n-4k}(\mathcal{S}), \tag{16}$$

a formula which is illustrated in Section 4.

3.3 Theorem. *Let $\mathcal{S}$ be a compact regular 3-Sasakian manifold of dimension $4n+3$, with $n \geq 2$. The Betti numbers of M, $\mathcal{Z}$ and $\mathcal{S}$ satisfy the following constraints:*

(i) $\displaystyle\sum_{k=0}^{n-1}(6k^2 + 6k - 6nk + n^2 - 4n + 3)b_{2k}(M) \ + \ \tfrac{1}{2}n(n-1)b_{2n}(M) = 0;$

(ii) $\displaystyle\sum_{k=0}^{n}(6k^2 - 6nk + n^2 - n)b_{2k}(\mathcal{Z}) = 0;$

(iii) $\displaystyle\sum_{k=1}^{n} k(n-k+1)(n-2k+1)b_{2k}(\mathcal{S}) = 0.$

Proof. Part (i) was proved in [37, Section 5]. Parts (ii) and (iii) follow directly from this after substituting by means of the equation in Theorem 3.2. As an example, we verify (iii) by deducing (i) from it. Let $c_k=k(n-k+1)(n-2k+1)$. Then

$$\sum_{k=0}^{n} c_k b_{2k}(\mathcal{S}) = \sum_{k=0}^{n-2}(c_k-c_{k+2})b_{2k}(M) \ + \ c_{n-1}b_{2n-2}+c_nb_{2n},$$

and for $0\le k\le n-2$,

$$\begin{aligned} c_k-c_{k+2} &= k(n+1-k)(n+1-2k)-(k+2)(n-1-k)(n-3-2k)\\ &= k(6n-2-8k)-2(n-1-k)(n-3-2k)\\ &= -2(6k^2+6k-6nk+n^2-4n+3). \end{aligned}$$

The values of c_{n-1} and c_n are also consistent with (i). QED

The relations of Theorem 3.3(iii) for low values of n are given in Table 1. The last three lines correspond to the dimensions of the exceptional homogeneous spaces mentioned in the next section, and to save space list only the coefficients of $(b_2,\dots,b_n)$ with any common factors removed. Because it is invariant by the symmetry $k\leftrightarrow n+1-k$, the constraint is satisfied by any regular 3-Sasakian manifold $\mathcal{S}$ whose Betti numbers satisfy the 'duality'

$$b_{2k}(\mathcal{S}) = b_{2n-2k+2}(\mathcal{S}),\quad 1\le k\le n. \tag{17}$$

Given (17), we might say that the cohomology of $\mathcal{S}$ is 'balanced'. This is true of S^{4n+3} by default, and Proposition 4.1 below implies that all the homogeneous 3-Sasakian manifolds have balanced cohomology except for those with isometry group of type D and E. Another consequence is that

$$b_{2k}(\mathcal{S})=0,\quad 1\le k\le [\tfrac12(n+1)],$$

implies that $b_p(\mathcal{S})=0$ whenever $0<p<4n+3$.

n	relation or coefficients thereof
2	$b_2 = b_4$
3	$b_2 = b_6$
4	$2b_2 + b_4 = b_6 + 2b_8$
5	$5b_2 + 4b_4 = 4b_8 + 5b_{10}$
6	$5b_2 + 5b_4 + 2b_6 = 2b_8 + 5b_{10} + 5b_{12}$
7	$7b_2 + 8b_4 + 5b_6 = 5b_{10} + 8b_{12} + 7b_{14}$
8	$28b_2 + 35b_4 + 27b_6 + 10b_8 = 10b_{10} + 27b_{12} + 35b_{14} + 28b_{16}$
9	$12b_2 + 16b_4 + 14b_6 + 8b_8 = 8b_{12} + 14b_{14} + 16b_{16} + 12b_{18}$
10	$15, 21, 20, 14, 5$
16	$40, 65, 77, 78, 70, 55, 35, 12$
28	$126, 225, 299, 350, 380, 391, 385, 364, 330, 285, 231, 170, 104, 35$

Table 1

4. Examples

The classification of compact 3-Sasakian homogeneous spaces, i.e. compact 3-Sasakian manifolds for which the group of automorphisms of the 3-Sasakian structure acts transitively, was obtained in [8]. Any such manifold is necessarily regular, and the quaternion-Kähler quotient is one of the symmetric spaces described by Wolf [58]. The space forms S^{4n+3} and $\mathbb{RP}^{4n+3}$ are both homogeneous 3-Sasakian manifolds fibering over the quaternionic projective space $\mathbb{HP}^n$. However, a compact homogeneous 3-Sasakian manifold of non-constant curvature is necessarily the total space of a unique $\mathbb{RP}^3$-bundle over a Wolf space.

Any compact homogeneous 3-Sasakian manifold $\mathcal{S}$ has the form G/K, where G is (possibly a finite cover of) its isometry group, and K a subgroup of G. Table 2 lists one pair (G, K) for each homogeneous space $\mathcal{S}$, but ignoring the non-simply-connected example $\mathbb{RP}^{4n+3}$. In each case, the group of isometries generated by the vector fields ξ^a may be regarded as a subgroup $Sp(1)$ of G such that $K \cap Sp(1) \cong \mathbb{Z}_2$, and the fibration $\mathcal{S} \to M$ takes the form

$$G/K \longrightarrow G/(K\,Sp(1)),$$

where $K\,Sp(1)$ denotes $K \times_{\mathbb{Z}_2} Sp(1)$. The 3-Sasakian metrics on these coset spaces are as always Einstein, and were considered in this context by Besse [5]. However, with the exception of the constant curvature case, the metrics are not the normal homogeneous ones, as they are not naturally reductive and thus are not obtained from the bi-invariant metric on G by Riemannian submersion. All other compact homogeneous manifolds admitting a (non-homogeneous) 3-Sasakian structure are covered by spheres [51].

$\dim\mathcal{S}$	G	K
$4n+3$	$A_{n+1}=SU(n+2)$	$A_{n-1}\times_{\mathbb{Z}_n}T^1$
$8k-1$	$B_{k+1}=SO(2k+3)$	$B_{k-1}\times A_1$
$4n+3$	$C_{n+1}=Sp(n+1)$	C_n
$8\ell+3$	$D_{\ell+2}=SO(2\ell+4)$	$D_\ell\times A_1$
43	E_6	A_5
67	E_7	$\widetilde{D}_6$
115	E_8	E_7
31	F_4	C_3
11	G_2	A_1

Table 2

4.1 PROPOSITION. *The Poincaré polynomials of the homogeneous 3-Sasakian manifolds are as given in Table 3.*

Proof. An expression for the Poincaré polynomial of a coset space G/H with G and H compact Lie groups of equal ranks can be found in [25]. The latter also explains how to compute the minimal model for the de Rham algebra, and we shall illustrate the essence of this for the 115-manifold $\mathcal{S}=E_8/E_7$. Other entries of Table 3 may be verified by the same methods.

The exponents

$$\begin{aligned}\mathfrak{e}_7 &: \ 3,11,15,19,23,27,35;\\ \mathfrak{e}_8 &: \ 3,15,23,27,35,39,47,59\end{aligned}$$

represent the degrees of the generators of the invariant forms on the respective Lie algebras (see for example [16]). Consider the freely-generated graded algebra

$$A=\bigwedge(x_3,x_{15},x_{23},x_{27},x_{35},x_{39},x_{47},x_{59})\otimes \mathsf{S}(z_4,z_{12},z_{16},z_{20},z_{24},z_{28},z_{36}),$$

where the degrees of elements are represented by subscripts, and the even generators correspond to the exponents of $\mathfrak{e}_7$ with a shift in degree of $+1$. A differential is defined on A by setting $dz_j=0$, and dx_i equal to a combination of the even generators that reflects restriction of the associated invariant polynomials on $\mathfrak{e}_8$ to those on $\mathfrak{e}_7$. Taking it for granted that z_{i+1} appears with a non-zero coefficient in dx_i, one merely strikes out pairs (x_i, z_{i+1}) leaving

$$\tilde{A} \cong \bigwedge(x_{39}, x_{47}, x_{59}) \otimes \mathsf{S}(z_{12}, z_{20}),$$

and by construction the non-zero differentials $dx_{39}, dx_{47}, dx_{59}$ become combinations of decomposable elements. The Betti numbers of $\mathcal{S}$ coincide with those of the differential graded algebra $\tilde{A}$; they are all zero except that $b_p = 1$ when p or $115 - p$ equals one of $0, 12, 20, 24, 32, 36, 44, 56$, as indicated below. QED

G	$P(\mathcal{S}, t)$
A_{n+1}	$\sum_{i=0}^{n} (t^{2i} + t^{4n+3-2i})$
B_{k+1}	$\sum_{i=0}^{k-1} (t^{4i} + t^{8k-1-4i})$
C_{n+1}	$1 + t^{4n+3}$
$D_{\ell+2}$	$t^{2\ell} + t^{6\ell+3} + \sum_{i=0}^{\ell} (t^{4i} + t^{8\ell+3-4i})$
E_6	$1 + t^6 + t^8 + t^{12} + t^{14} + t^{20} + \cdots$
E_7	$1 + t^8 + t^{12} + t^{16} + t^{20} + t^{24} + t^{32} + \cdots$
E_8	$1 + t^{12} + t^{20} + t^{24} + t^{32} + t^{36} + t^{44} + t^{56} + \cdots$
F_4	$1 + t^8 + t^{23} + t^{31}$
G_2	$1 + t^{11}$

Table 3

It is clear from Table 3 that all the homogeneous 3-Sasakian manifolds except for the one with isometry group $D_{\ell+2}$ with $2|\ell$ have $b_p(\mathcal{S}) \le 1$. Conversely, this inequality would severely limit the possible Betti numbers of a regular Sasakian 3-manifold, by the constraint of Theorem 3.3(iii). The latter is especially striking for the spaces of type D and E, since Table 3 reveals that their Betti numbers satisfy

$$b_{2n-2k} = b_{2k}, \quad 0 \le k \le n,$$

and are unbalanced (see (17)). For example for E_8/E_7 the arithmetic reduces to

$$391 + 285 + 170 - 104 - 231 - 385 - 126 = 0;$$

the reader may enjoy checking that the remaining entries of Tables 1 and 3 exhibit consistent information.

Using (16), the signatures of the Wolf spaces can be read off from Table 3 and confirm results of the paper [26] and its appendix. For example, the Wolf space $E_8/(E_7\,Sp(1))$ has dimension 112 and signature equal to 8.

We conclude this section by mentioning the construction of non-homogeneous 3-Sasakian manifolds that is described in more detail in [8]. Let $n \geq 3$ and suppose that $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n$ is an n-tuple of non-decreasing, pairwise relatively-prime, positive integers. Let $\mathcal{S}(\boldsymbol{p})$ be the left-right quotient of the unitary group $U(n)$ by $U(1) \times U(n-2) \subset U(n)_l \times U(n)_r$, where the action is given by the formula

$$(\tau, \mathbb{B})\mathbb{W} = \begin{pmatrix} \tau^{p_1} & & \\ & \ddots & \\ & & \tau^{p_n} \end{pmatrix} \mathbb{W} \begin{pmatrix} \mathbb{I}_2 & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix}.$$

Here, $\tau \in U(1)$, $\mathbb{B} \in U(n-2)$, and $\mathbb{W} \in U(n)$. Then $\mathcal{S}(\boldsymbol{p})$ is a compact, simply connected, $(4n-5)$-dimensional smooth manifold which admits an Einstein metric $\breve{g}(\boldsymbol{p})$ of positive scalar curvature and a compatible 3-Sasakian structure. Furthermore, $\pi_2(\mathcal{S}(\boldsymbol{p})) = \mathbb{Z}$ and

$$H^{2n-2}(\mathcal{S}(\boldsymbol{p}), \mathbb{Z}) \cong \mathbb{Z}/r,$$

where $r = \sigma_{n-1}(\boldsymbol{p}) = \sum_{j=1}^{n} p_1 \cdots \hat{p}_j \cdots p_n$ is the $(n-1)$st elementary symmetric polynomial in the entries of $\boldsymbol{p}$.

All the $\mathcal{S}(\boldsymbol{p})$ spaces are inhomogeneous unless $\boldsymbol{p} = (1, \ldots, 1)$. The 3-Sasakian structure on $\mathcal{S}(\boldsymbol{p})$ is obtained explicitly thus providing the first such examples of Einstein metrics with positive scalar curvature and arbitrary cohomogeneity. In fact there are subfamilies in $\mathcal{S}(\boldsymbol{p})$ which are strongly inhomogeneous, that is not homotopy equivalent to any compact homogeneous space [9].

5. Further results

Of the theorems in the Introduction, it remains to establish part (ii) of Theorem C. This will follow immediately from Theorem 5.1 below, but we first review some necessary index theory from [47, 37].

Let M be a quaternion-Kähler manifold. The spin bundle of M is given by $\bigoplus_{p+q=n} R^{p,q}$, where

$$R^{p,q} = \textstyle\bigwedge_0^p E \otimes \mathsf{S}^q H,$$

and E, H are standard $Sp(n)$, $Sp(1)$ vector bundles. Provided $n+p+q$ is even, we may consider the index $i^{p,q}$ of the Dirac operator on M coupled to $R^{p,q}$. The Atiyah-Singer index equates $i^{p,q}$ to the evaluation of $\mathrm{ch}(R^{p,q})\hat{A}$ on the fundamental class $[M]$. On the other hand, according to [45],

$$i^{p,q} = \begin{cases} 0, & p+q<n \\ (-1)^p(b_{2p}(M)+b_{2p-2}(M)), & p+q=n \\ d, & p=n+2,\ q=0, \end{cases}$$

where d is the dimension of the isometry group of M. These indices are expressed below in terms of the Chern classes $c_{2i}=c_{2i}(E)$, $1\le i\le n$, of E, and the class $u=-c_2(H)$ which is a real multiple of $[\hat{\Omega}]$ (see (11)).

5.1 Theorem. *Let M be a compact quaternion-Kähler manifold of positive scalar curvature of dimension 12 or 16 with $b_4(M)=1$. Then M is isometric to $\mathbb{HP}^3$ or $\mathbb{HP}^4$.*

Proof. First suppose that $\dim M = 12$. The complex Grassmannian has $b_4=2$, so by [37, Theorem 0.3(ii)] we may assume that $b_2(M)=0$. Table 1 now shows that M has the same Betti numbers as $\mathbb{HP}^3$.

If V is an integral combination of the $R^{p,q}$ of virtual rank r, then

$$\mathrm{ch}(V)\hat{A} = r\hat{A}_3 + \mathrm{ch}_2\hat{A}_2 + \mathrm{ch}_4\hat{A}_1 + \mathrm{ch}_6, \tag{18}$$

where $\mathrm{ch}_k \in H^{2k}(M,\mathbb{R})$ is the appropriate component of the Chern character $\mathrm{ch}(V)$. We put this into practice by considering the virtual bundles

$$\begin{aligned} V_1 &= (E-3H)(\mathsf{S}^2H-3) \cong R^{1,2} - 3R^{1,0} - 3R^{0,3} + 6R^{0,1}, \\ V_2 &= (\mathsf{S}^2H-3)^2H \cong R^{0,5} - 4R^{0,3} + 5R^{0,1}, \\ V_3 &= (\mathsf{S}^2H-3)H \cong R^{0,3} - 2R^{0,1}. \end{aligned}$$

All these have $r=0$, and $\mathrm{ch}_2(V_i)=0$ for $i=1,2$; thus (18) does not require $\hat{A}_3$. Identifying $H^{12}(M,\mathbb{Z})$ with $\mathbb{Z}$, we obtain

$$\begin{gathered} -\tfrac{2}{3}(3u^3+2c_2u^2+c_4u) = -(b_2(M)+1)-3 = -4, \\ \tfrac{8}{3}(11u^3+c_2u^2) = d-4, \\ \tfrac{1}{90}(56u^3+32c_2u^2+3c_2^2u-c_4u) = 1. \end{gathered}$$

Now suppose that M is not isometric to $\mathbb{HP}^3$, and therefore [1] not homogeneous. Applying the inequality $c_2u^2 \geq u^3$ (which is equivalent to [45, Lemma 7.6]), and the fact that d is necessarily less than $36 = \dim Sp(4)$ to the last equation gives $u^3 < 1$. Recalling that $4u \in H^4(M,\mathbb{Z})$, we deduce that $4u = m\xi$ where ξ is an indivisible integral class and $1 \leq m \leq 3$. Since $b_4 = 1$, the characteristic classes c_2 and u must be proportional, so we may suppose that $c_2 = \lambda u$ for some $\lambda \in \mathbb{Q}$. Now $4c_2$ is an integral class, so we may write $\lambda = p/q$ with $p \in \mathbb{Z}$, $q \in \{1,2,3\}$. Eliminating c_4 and then setting $c_2 = \lambda u$ gives

$$
\begin{aligned}
(59 + 34\lambda + 3\lambda^2)u^3 &= 96,\\
8(11+\lambda)u^3 &= 3d - 12,
\end{aligned}
$$

and

$$
d = 4 + \frac{256(\lambda + 11)}{3\lambda^2 + 34\lambda + 59}.
$$

A computer check reveals that there are no integers d with $4 < d < 36$ and $6\lambda \in \mathbb{Z}$.

The 16-dimensional case proceeds in exactly the same way, using the virtual bundles

$$
\begin{aligned}
V_1 &= (\textstyle\bigwedge_0^2 E - 3EH + 6\mathsf{S}^2H)(\mathsf{S}^2H - 3)\\
&\cong R^{2,2} - 3R^{1,3} + 6R^{0,4} - 3R^{2,0} + 6R^{1,1} - 12R^{0,2} + 6\\
V_2 &\cong (\mathsf{S}^2H - 3)^2 = R^{0,4} - 5R^{0,2} + 10\\
V_3 &\cong (\mathsf{S}^2H - 3)^3 = R^{0,6} - 7R^{0,4} + 21R^{0,2} - 35\\
V_4 &\cong (E - 4H)(\mathsf{S}^2H - 3)H = R^{1,3} - 4R^{0,4} - 2R^{1,1} + 4R^{0,2} + 8.
\end{aligned}
$$

This gives

$$
\begin{aligned}
4u^4 + 3c_2u^3 + 2c_4u^2 + c_6u &= 10,\\
\tfrac{1}{45}(26u^4 + 17c_2u^3 + 3c_2^2u^2 - c_4u^2) &= 1,\\
\tfrac{16}{3}(8u^4 + c_2u^3) &= d - 7,\\
-\tfrac{1}{90}(224u^4 + 152c_2u^3 + 9c_2^2u^2 + 56c_4u^2 + 6c_2u + 3c_2c_4u) &= -5.
\end{aligned}
$$

Assuming that M is not isometric to $\mathbb{HP}^4$, one proves that $d < 55$ and $u^4 < 1$. Set $c_2 = \lambda u$ as before; this time $2\lambda u \in H^4(M,\mathbb{Z})$, forcing $\lambda \in \mathbb{Z}$. Eliminating c_6u,

$$
\begin{aligned}
(200 + 134\lambda + 9\lambda^2)u^4 + (44 + 3\lambda)c_4u^2 &= 390,\\
(26 + 17\lambda + 3\lambda^2)u^4 - c_4u^2 &= 45,\\
\tfrac{16}{3}(8+\lambda)u^4 &= d - 7.
\end{aligned}
$$

Eliminating c_4u^2 and solving for d yields

$$d = 7 + \frac{80(\lambda+8)(9\lambda+158)}{3(\lambda+4)(3\lambda^2+52\lambda+112)},$$

which has no integral solutions with $7 < d < 55$. QED

To conclude this article, we extend the class of geometrical structures so far considered by bringing the Riemannian product $\mathcal{H} = S^1 \times \mathcal{S}$ of a circle and a compact 3-Sasakian manifold into the picture. This product is a hyper-Hermitian manifold which is locally conformally hyper-Kähler [8]: if w^1, ω^2, ω^3 denote the 2-forms associated to the product metric g of $\mathcal{H}$ and the complex structures, then there exists a closed 1-form α such that

$$d\omega^a = \omega^a \wedge \alpha, \quad a = 1, 2, 3.$$

The 'Lee form' α is not exact, for otherwise g could be scaled into a hyper-Kähler metric, but $b_1(\mathcal{H}) = 1$ and $\mathcal{H}$ cannot admit any Kähler metric. A result of Gauduchon [22, 23] implies that that any locally conformally hyper-Kähler manifold which is not hyper-Kähler is a generalized Hopf manifold, which means that $\nabla\alpha = 0$.

In his study of generalized Hopf manifolds, Vaisman identified the 2-dimensional foliation $\mathcal{E}$ generated by $\alpha^\sharp$ and $J\alpha^\sharp$ [56]. Assuming leaf compactness, $\mathcal{H}$ is the total space of an analytic V-submersion onto a Kähler V-manifold in the sense of Satake [49], and all the fibers of the submersion are complex 1-dimensional tori. Similarly, assuming that all leaves of the 1-dimensional foliation $\mathcal{F}_\alpha$ generated by $\alpha^\sharp$ are compact, $\mathcal{H}$ is a Seifert fibered space of an analytic V-submersion onto a Sasakian V-manifold. In particular, if $\mathcal{F}_\alpha$ and $\mathcal{E}$ are regular (in this case $\mathcal{H}$ is called 'strongly regular'), then $\mathcal{H}$ is a flat circle bundle over a Sasakian space and there is diagram of fibrations

$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d, "T^2"] & \mathcal{S} \arrow[dl, "S^1"'] \\ \mathcal{Z} & \end{tikzcd}$$

described in [56]. Moreover, $\mathcal{H}$ is a flat principal circle bundle over a principal circle bundle over $\mathcal{Z}$ whose Chern class differs only by a torsion element form the Chern class induced by the Hopf fibration.

If $(\mathcal{H}, h)$ is locally conformally hyper-Kähler but has no hyper-Kähler metric in the conformal class of h one can get an automatic extension of the above diagram [41, 40]. The hypercomplex structure allows one to define the following foliations of $\mathcal{H}$: (i) the 1-dimensional foliation $\mathcal{F}_\alpha$ defined by α, (ii) 2-dimensional foliations $\mathcal{E}^i$ defined by $\{\alpha^\sharp, J_i\alpha^\sharp\}$ for each $i = 1, 2, 3$, (all three are equivalent), and (iii) a 4-dimensional foliation $\mathcal{F}_4$ defined by $\{\alpha^\sharp, J_1\alpha^\sharp, J_2\alpha^\sharp, J_3\alpha^\sharp\}$. The leaf space $\mathcal{S} = \mathcal{H}/\mathcal{F}_\alpha$ is easily seen to have a natural Sasakian 3-structure [40] and the leaf space $M = \mathcal{H}/\mathcal{F}_4$ is a quaternion-Kähler orbifold of positive scalar curvature [41]. In the strongly regular case we get the following diagram of fibrations [40]

$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d, "T^2"] & \mathcal{S} \arrow[d, "\mathbb{RP}^3"] \arrow[dl, "S^1"'] \\ \mathcal{Z} \arrow[r] & M. \end{tikzcd}$$

Our theorems have a number of interpretations for the structure of $\mathcal{H}$, now assumed to be a compact strongly regular generalized Hopf manifold of dimension $4n+4$ which is locally conformally hyper-Kähler. Applying the Gysin sequence to the double circle fibration described above yields

$$b_p(\mathcal{H}) = b_p(\mathcal{Z}) + b_{p-1}(\mathcal{Z}) - b_{p-2}(\mathcal{Z}) - b_{p-3}(\mathcal{Z}), \quad p \le 2n+2.$$

Using the twistor fibration $\mathcal{Z} \to M$, one can derive similar relations between the Betti numbers of $\mathcal{H}$ and the quaternion-Kähler base M which will be a manifold under strong regularity assumption on $\mathcal{E}$. As a counterpart to Theorem 3.2, one has

5.2 COROLLARY [40]. *Let $\mathcal{H}$ be a compact strongly regular generalized Hopf surface of dimension $4n+4$, which is locally conformally hyper-Kähler. Then*

$$b_{2k}(\mathcal{H}) = b_{2k+1}(\mathcal{H}) = b_{2k}(M) - b_{2k-4}(M), \quad k \le n.$$

In the strongly regular case, when all spaces of the diagram are smooth manifolds and $\mathcal{H}$ is a flat circle bundle over $\mathcal{S}$, we get $b_p(\mathcal{H}) = b_p(\mathcal{S}) + b_{p-1}(\mathcal{S})$. The following extends results of [40].

5.3 COROLLARY. *Let $\mathcal{H}$ be a compact locally conformally hyper-Kähler manifold of real dimension $4n+4$ which is not hyper-Kähler, and suppose that $\mathcal{H}$ is strongly regular. Then*

(i) $b_2(\mathcal{H}) = b_3(\mathcal{H}) = 0$ *unless* $\mathcal{H}$ *is* $S^1 \times U(n+1)/(U(n-1) \times U(1))$*;*
(ii) $b_4(\mathcal{H}) = 0$ *and* $n = 3$ *or 4 implies that* $\mathcal{H}$ *is either* $S^1 \times S^{4n+3}$ *or one of the two flat circle bundles over* $\mathbb{RP}^{4n+3}$*;*

(iii) $\displaystyle\sum_{k=1}^{n} k(n-k+1)(n-2k+1)b_{2k}(\mathcal{H}) = 0.$

References

1. D.V. Alekseevsky: Compact quaternion spaces, Functional Anal. Appl. **2p** (1968), 106–114.
2. W.L. Baily: The decomposition theorem for V-manifolds, Amer. J. Math. **78** (1956), 862–888.
3. W.L. Baily: On the imbedding of V-manifolds in projective space, Amer. J. Math. **79** (1957), 403–430.
4. C. Bär: Real Killing spinors and holonomy, Commun. Math. Phys. **154** (1993), 509–521.
5. A. Besse: Einstein Manifolds, Springer (1987).
6. E. Bonan: Sur l'algèbra extérieure d'une variété presque hermitienne quaternionique, C. R. Acad. Sci. Paris **295** (1982), 115–118.
7. C.P. Boyer, K. Galicki, B.M. Mann: Quaternionic reduction and Einstein manifolds, Communications in Analysis and Geometry, **1** (1993), 1–51.
8. C.P. Boyer, K. Galicki, B.M. Mann: The geometry and topology of 3-Sasakian manifolds, J. reine angew. Math. **455** (1994), 183–220.
9. C.P. Boyer, K. Galicki, B.M. Mann: New examples of inhomogeneous Einstein manifolds of positive scalar curvature, J. London Math. Soc., to appear.
10. C.P. Boyer, K. Galicki, B.M. Mann: Hypercomplex structures on Stiefel manifolds, Ann. Global Anal. Geom., to appear.
11. C.P. Boyer, K. Galicki, B.M. Mann: New examples of inhomogeneous Einstein manifolds of positive scalar curvature, Math. Res. Lett. **1** (1994), 115–121.
12. C.P. Boyer, K. Galicki, B.M. Mann: Some new examples of inhomogeneous hypercomplex manifolds, Math. Res. Lett. **1** (1994), 531–538.
13. C.P. Boyer, K. Galicki, B.M. Mann: 3-Sasakian manifolds, Proc. Japan Acad. **69** (1993), 335–340.
14. R.L. Bryant, R. Harvey: Submanifolds in hyper-Kähler geometry, J. Amer. Math. Soc. **1** (1989), 1–31.
15. R.L. Bryant, S.M. Salamon: On the construction of some complete metrics with exceptional holonomy, Duke Math. J. **58** (1989), 829–850.
16. C. Chevalley: Determination of the Betti numbers of the exceptional Lie groups, in Proc. International Congress of Mathematicians 1950, Volume 2.
17. M. Fernández, A. Gray: Riemannian manifolds with structure group G_2, Ann. Mat. Pura Appl. **32** (1982), 19–45.
18. Th. Friedrich, I. Kath: Compact seven-dimensional manifolds with Killing spinors, Commun. Math. Phys. **133** (1990), 543–561.
19. Th. Friedrich, H. Kurke: Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature, Math. Nach. **106** (1982), 271-299.

20. A. Fujiki: On the de Rham cohomology group of a compact Kähler symplectic manifold, in Algebraic Geometry, Sendai, 1985 (Advanced Studies in Pure Mathematics 10), ed. T. Oda, North Holland (1987).
21. K. Galicki, B.H. Lawson: Quaternionic reduction and quaternionic orbifolds, Math. Ann. **282** (1988), 1–21.
22. P. Gauduchon: Structures de Weyl-Einstein espaces de twisteurs at variétés type $S^1 \times S^3$, preprint, École Polytechnique (1993).
23. P. Gauduchon: La 1-forme de torsion d'une variété hermitienne compacte, Math. Ann. **267** (1984), 495–518.
24. A. Gray: Weak holonomy groups, Math. Z. **123** (1971), 290–300.
25. W. Greub, S. Halperin, R. Vanstone: Curvature, Connections and Characteristic classes, Volume 3, Academic Press (1976).
26. F. Hirzebruch, P. Sladowy: Ellipic genera, involutions, and homogeneous spin manifolds, Geom. Dedicata **35** (1990), 309–343.
27. N.J. Hitchin: Kählerian twistor spaces, Proc. Lond. Math. Soc. **43** (1981), 133-150.
28. S. Ishihara: Quaternion Kählerian manifolds and fibered Riemannian spaces with Sasakian 3-structure, Kodai Math. Sem. Rep. **25** (1973), 321–329.
29. S. Ishihara: Quaternion Kählerian manifolds, J. Diff. Geometry **9** (1974), 483–500.
30. S. Ishihara, M. Konishi: Fibered Riemannian spaces with Sasakian 3-structure, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo (1972), 179–194.
31. T. Kashiwada: A note on a Riemannian space with Sasakian 3-structure, Nat. Sci. Reps. Ochanomizu Univ. **22** (1971), 1–2.
32. S. Kobayashi: On compact Kähler manifolds with positive definite Ricci tensor, Ann. Math. **74** (1961), 570–574.
33. M. Konishi: On manifolds with Sasakian 3-structure over quaternion Kählerian manifolds, Math. Sem. Reps. **26** (1975), 194–200.
34. V. Kraines: Topology of quaternionic manifolds, Trans. Amer. Math. Soc. **122** (1966), 357–367.
35. Y-Y. Kuo: On almost contact 3-structure, Tôhoku Math. J. **22** (1970), 325–332.
36. C.R. LeBrun: A rigidity theorem for quaternion-Kähler manifolds, Proc. Amer. Math. Soc. **103** (1988), 1205–1208.
37. C. LeBrun, S. Salamon: Strong rigidity of positive quaternion-Kähler manifolds, Invent. Math. 118 (1994), 109–132.
38. T. Nagano, M. Takeuchi: Signature of quaternionic Kaehler manifolds, Proc. Japan Acad. **59** (1983), 384–386.
39. Y. Nagatomo: Rigidity of c_1-self-dual connections on quaternionic Kähler manifolds, J. Math. Phys. **33** (1992), 4020–4025.
40. L. Ornea, P. Piccini: Locally conformal Kähler tructures in quaternionic geometry, Trans. Amer. Math. Soc., to appear.
41. H. Pedersen, Y.S. Poon, A. Swann: The Einstein-Weyl equations in complex and quaternionic geometry, Diff. Geom. Appl. **3** (1993), 309–321.
42. Y.S. Poon, S.M. Salamon: Eight-dimensional quaternionic Kähler manifolds with positive scalar curvature, J. Diff. Geometry **33** (1991), 363–378.
43. B.L. Reinhart: Foliated manifolds with bundle-like metrics, Ann. of Math. **69** (1959), 119–132.
44. R. Reyes Carrión: Some special geometries defined by Lie groups, D.Phil. thesis, University of Oxford, 1993.
45. S. Salamon: Quaternionic Kähler manifolds, Invent. Math. **67** (1982), 143–171.

46. S.M. Salamon: Riemannian Geometry and Holonomy Groups, Pitman Research Notes Math. **201**, Longman, 1989.
47. S.M. Salamon: The Dirac operator and quaternionic Kähler manifolds, in Differential Geometry and its Applications, volume 1, eds. O. Kowalski, D. Krupka, Silesian University at Opava (1993)
48. S. Sasaki: On differentiable manifolds with certain structures which are closely related to almost contact structure, Tôhoku Math. J. **12** (1960), 459–476.
49. I. Satake: The Gauss-Bonnet Theorem for V-manifolds, J. Math. Soc. Japan **9** (1957), 464–476.
50. A.F. Swann: Hyperkähler and quaternionic Kähler geometry, Math. Ann. **289** (1991), 421–450.
51. S. Tanno: On the isometry of Sasakian manifolds, J. Math. Soc. Japan **22** (1970), 579–590.
52. S. Tanno: Killing vectors on contact Riemannian manifolds and fiberings related to the Hopf fibrations, Tôhoku Math. J. **23** (1971), 313–333.
53. S. Tachibana: On harmonic tensors in compact Sasakian spaces, Tôhoku Math. J. **17** (1965), 271–284.
54. S. Tachibana, W.N. Yu: On a Riemannian space admitting more than one Sasakian structure, Tôhoku Math. J. **22** (1970), 536–540.
55. C. Udrişte: Structures presque coquaternioniennes, Bull. Math. Soc. Sci. Math. Roumanie **12** (1969), 487–507.
56. I. Vaisman: Generalized Hopf manifolds, Geom. Ded. **13** (1982), 231–255.
57. J.A. Wiśniewski: On Fano manifolds of large index, Manuscripta Math. **70** (1991), 145–152.
58. J.A. Wolf: Complex homogeneous contact structures and quaternionic symmetric spaces, J. Math. Mech. **14** (1965) 1033–1047.
59. K. Yano, S. Bochner: Curvature and Betti numbers, Annals of Math. Studies 32, Princeton University Press, 1953.

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131
`galicki@math.unm.edu`

Mathematical Institute, 24–29 St Giles', Oxford OX1 3LB
`salamon@maths.ox.ac.uk`