Classical and Quantum Gravity
Volume 8 (1991), 1529-1543.

Multi-centre metrics with negative cosmological constant

Krzysztof Galicki

Abstract: We describe a hyperkähler quotient construction of the multi-Taub-NUT gravitational instantons of Gibbons and Hawking and show how it relates to the quotient construction of the multi-center asymptotically locally Euclidean (ALE) gravitational instanton. Then we use the quaternionic quotient method to obtain a family of Einstein metrics with negative scalar curvature and self-dual Weyl tensor and with possible orbifold type singularities. Our metrics give the multi-center Taub-NUT metrics in the limit of the scalar curvature going to zero.

1. Introduction

A lot of progress has recently been made in our understanding of 4-manifolds with $SU(2)$-holonomy. This condition implies that such metrics are self-dual and Ricci-flat. For that reason they are sometimes called gravitational instantons. The first example was constructed by Eguchi and Hanson [EH]. They produced an $SU(2)$-holonomy metric on the cotangent bundle of the complex projective line. Later Gibbons and Hawking employed an ansatz on the form of the metric and constructed multi-center generalizations of the Eguchi-Hanson metric [GH]. Their ansatz has a very simple and elegant geometric interpretation: It is equivalent to the requirement of the existence of, at least one, triholomorphic Killing vector [H1]. Then the symmetry induced by this Killing vector can be used to define three momentum mappings associated with the three Kähler forms. The momentum mappings define a local fiberation over $\mathbb{R}^3$

$$\mu:\ M \longrightarrow \mathbb{R}^3,$$

i.e., the manifold M has a circle bundle structure. It is then a simple consequence of the hyperkähler geometry that the metric can be cast in the following form

$$ds^2 = V d\mathbf{x}\cdot d\mathbf{x} + V^{-1}(d\tau - A)^2 \tag{1.1}$$

During the preparation of this work the author was supported by NSF grant DMS-8906806.

where A is a connection 1-form, τ is a local coordinate on the fiber, $d\mathbf{x} \cdot d\mathbf{x}$ is the standard flat metric on $\mathbb{R}^3$, and V is a function on $\mathbb{R}^3$. Moreover, the pair (A, V) has to be a monopole solution, *i.e.,* $*dV = dA$. In other words, V has to satisfy the Laplace equation on $\mathbb{R}^3$. Any such solution gives (locally) a metric with $SU(2)$-holonomy. If one requires that the metrics be complete there are two different cases. When

$$V(\mathbf{x}) = 1 + \sum_{i=1}^{k} \frac{m}{|\mathbf{x} - \mathbf{x}_i|} \tag{1.2}$$

we get the k-center multi-Taub-NUT Asymptotically Locally Flat (ALF) $(3k-5)$-parameter family $(k > 1)$ of hyperkähler metrics and when

$$V(\mathbf{x}) = \sum_{i=1}^{k} \frac{m}{|\mathbf{x} - \mathbf{x}_i|} \tag{1.3}$$

we get the k-center multi-Eguchi-Hanson Asymptotically Locally Euclidean (ALE) family with $(3k-6)$ parameters. In both cases the manifold is diffeomorphic to the minimal resolution of the singularity in $\mathbb{C}^2/\mathbb{Z}_k$. Putting $k = 1$ in (1.2) and (1.3) corresponds to the Taub-NUT metric and the flat $\mathbb{R}^4$ metric respectively.

At the same time, Hitchin constructed the multi-Eguchi-Hanson family using different methods [H2]. He showed that the minimal resolution of $\mathbb{C}^2/\Gamma$, where $\Gamma = \mathbb{Z}_k$, admits a family of Kähler and Ricci flat metrics and constructed them explicitly using twistor techniques. He also conjectured that such metrics exist for other discrete subgroup of $SU(2)$. Recently, Kronheimer constructed them for all Γ's using hyperkähler quotients and showed that these are all the hyperkähler ALE spaces [K1, K2].

For $\Gamma = \mathbb{Z}_k$ every ALE gravitational instanton has its ALF analogue and locally the connection between the two is provided by the Gibbons-Hawking ansatz. It is not known, however, if all ALE spaces constructed by Kronheimer have their ALF analogues. For $\Gamma = \mathbb{Z}_k$ we shall describe a relation between the hyperkähler quotient construction in both cases. We demonstrate how an ALF multi-Taub-NUT metric can be obtained from its ALE multi-Eguchi-Hanson partner.

In the main part of this article we show that there exist families of self-dual Einstein metrics of negative scalar curvature which are quaternionic analogues of the multi-Taub-NUT metrics. In general they may have orbifold-type singularities and they give the multi-Taub-NUT families when we take the limit of the scalar curvature going to zero.

Our paper is organized as follows. In Chapter 2 we review and compare the hyperkähler and the quaternionic Kähler quotient constructions. In Chapter 3 we describe the quotient construction of the multi-Taub-NUT family. We also produce some new $4n$-dimensional hyperkähler metrics which are hyperkähler deformations of higher dimensional analogue of the Taub-NUT metric. Finally, in Chapter 3 we give the construction of our new metrics and show that they produce the Taub-NUT family as the scalar curvature going to zero limit.

2. Hyperkähler versus quaternionic quotient

We begin with a review of the hyperkähler quotient construction. Let us recall that a hyperkähler manifold is a Riemannian manifold M with three complex structures $I, J, K \in \mathrm{End}(TM)$,

$$I^2 = J^2 = K^2 = -\mathbb{1}, \quad IJ = -JI = K, \tag{2.1}$$

that are covariantly constant and a Riemannian metric g that is Hermitian with respect to all three complex structures. Consequently, M is a Kähler manifold with respect to these three complex structures. Hence, we can define three closed non-degenerate symplectic 2-forms on M

$$\omega^i(X,Y) = g(A^i X, Y); \quad X, Y \in \Gamma(TM); \quad A^1 = I, A^2 = J, A^3 = K. \tag{2.2}$$

Let G be a subgroup of the isometry group of M such that it commutes with all three complex structures. We shall call such isometries triholomorphic. Then G is also a symplectic action for all three symplectic forms which means that

$$\mathcal{L}_V \omega^i = 0 \qquad \forall\ i \tag{2.3}$$

where V is any vector field generated by G. Suppose there exists an equivariant momentum mapping for each of the symplectic structures

$$\mu^i :\ M \ \longrightarrow \mathfrak{g}^* \tag{2.4}$$

defined as

$$\left\langle d\mu^i, V \right\rangle = i_V \omega^i \tag{2.5}$$

where i_V denotes contraction with V and $\mathfrak{g}^*$ is the Lie co-algebra of G. Equivalently, we can consider a mapping

$$\mu :\ M \ \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3. \tag{2.6}$$

Let us introduce a "ξ-momentum level", $\mu^{-1}(\xi)$, where $\xi \in \mathfrak{g}^* \otimes \mathbb{R}^3$ is a G-invariant element. The following result is due to Hitchin et al. [HKLR]:

Theorem 2.7 *If $\widehat{M} = \mu^{-1}(\xi)/G$ is a manifold then its induced metric is hyperkähler.*

In fact both the metric and the hyperkähler structure are given by the pullbacks of the projection and inclusion maps.

$$M \xleftarrow{\ i\ } \ \mu^{-1}(\xi) \ \xrightarrow{\ \pi\ } \widehat{M}. \tag{2.8}$$

In particular, if $\widehat{M}$ is a 4-manifold, it is a hyperkähler gravitational instanton and this is the construction used by Kronheimer to obtain all gravitational instantons conjectured by Hitchin [K1].

In the quaternionic Kähler geometry there exist a similar construction which we shall briefly describe now [G1, GL]. Let M be a $4n$ dimensional manifold with holonomy group contained in $Sp(n) \cdot Sp(1)$. Then there exists a real rank 3 subbundle $\mathcal{V}$ of $\mathrm{End}(TM)$. Locally, at each $x \in M$, $\mathcal{V}$ has a basis $\{I, J, K\}$ satisfying

$$I^2 = J^2 = -1\!\!1, \qquad IJ = -JI = K. \tag{2.9}$$

The metric g on M is compatible with the bundle $\mathcal{V}$ in the sense that for each $A \in \mathcal{V}_x$ g is Hermitian with respect to A, *i.e.,* $g(AX, AY) = g(X, Y)$ for all $X, Y \in T_xM$. It is clear that a compatible metric always exist. One can use the metric to define an isomorphism

$$\mathrm{End}(TM) \cong \Lambda T^*M \otimes \Lambda T^*M$$

under which $\mathcal{V}$ is isometrically embedded in $\Lambda^2 T^*M$. Explicitly, any element $A \in \mathcal{V}_x$ is mapped into ω_A by

$$\omega_A(X, Y) = g(AX, Y), \qquad X, Y \in T_xM.$$

Let $\{\omega_1, \omega_2, \omega_3\}$ be a local orthogonal frame of $\mathcal{V} \subset \Lambda^2 T^*M$. Let us define

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3. \tag{2.10}$$

This Ω is a globally defined, non-degenerate 4-form on M and it is parallel. It is usually called the fundamental 4-form or the quaternionic structure on M as its parallelism determines the reduction of the structure group on M. The condition $\nabla\Omega = 0$ can be used to define quaternionic Kähler geometry in dimension bigger than 4. It is equivalent to the requirement that the holonomy group be a subgroup of $Sp(n) \cdot Sp(1)$. In dimension 4 we shall say that M is quaternionic Kähler if it is self-dual and Einstein.

Let G be a group acting on M by isometries which preserve the fundamental 4-form. Let V be any Killing vector field of this action. Define

$$\Theta_V \equiv \sum_i (i_V \omega_i) \otimes \omega_i. \tag{2.11}$$

Clearly Θ_V remains invariant under local change of frame field.

Theorem 2.12 *Assume that the scalar curvature of M is not zero. Then to each V there corresponds a unique section $f_V \in \Gamma(\mathcal{V})$ such that*

$$\nabla f_V = \Theta_V \ . \tag{2.13}$$

This unique section can be used to define a map

$$\mu : \quad M \longrightarrow \mathfrak{g}^* \otimes \mathcal{V}. \tag{2.14}$$

whose value at $m \in M$ is the homomorphism $V \longrightarrow f_V(m)$. Because of the uniqueness of the section f_V the map $V \longrightarrow f_V$ transforms naturally under G and consequently our momentum mapping μ is always G-equivariant. Again, let us introduce a "zero momentum level set"

$$\mu^{-1}(0) = \{m \in M : \ \mu(m) = 0\}. \tag{2.15}$$

One can show [GL] that

Theorem 2.16 *If $\widehat{M} = \mu^{-1}(0)/G$ is a manifold then it its induced metric is quaternionic Kähler.*

In particular if $\widehat{M}$ is a 4-manifold it is necessarily self-dual and Einstein, *i.e.*, a gravitational instanton.

3. Quotient construction of the multi-Taub-NUT metrics

Suppose that some smooth hyperkähler manifold M_ξ was obtained as a quotient of the flat space $\mathbb{H}^k$ by an action of a subgroup G of $Sp(k)$ where $Sp(k)$ acts linearly on $\mathbf{u} \in \mathbb{H}^k$ from the left. Here we think of $Sp(k)$ as the group of quaternionic transformations preserving the canonical quaternionic Hermitian product on $\mathbb{H}^k$. Let $\xi \in \mathfrak{g}^* \otimes \mathbb{R}^3$ be a G-invariant element with respect to which the reduction is carried out. We shall write

$$M_\xi = \mathbb{H}^k /\!/\!/ G. \tag{3.1}$$

We can then take $\mathbb{H}^{k+1} = \mathbb{H}^k \times \mathbb{H}$ and introduce a non-compact action of $\widetilde{G} = G \times \mathbb{R}$ defined as follows

$$\widetilde{G} = G \times \mathbb{R} \ni (A, a) : \quad (A, a) \cdot (\mathbf{u}, w) = (e^{ia} A \cdot \mathbf{u}, w + \Lambda a), \qquad (\mathbf{u}, w) \in \mathbb{H}^k \times \mathbb{H} \tag{3.2}$$

provided

$$A e^{ia} = e^{ia} A \quad \text{for all} \quad A \in G. \tag{3.3}$$

It is easy to see that the above extension makes $\widetilde{G}$ act freely and properly on $\mathbb{H}^{k+1}$. It is also the action by triholomorphic isometries. Choose any $\sigma \in \mathbb{R} \otimes \mathbb{R}^3$. The hyperkähler quotient with respect to (ξ, σ) gives a smooth hyperkähler manifold $\widetilde{M}_{(\xi,\Lambda,\sigma)}$. We write

$$\widetilde{M}_{(\xi,\Lambda,\sigma)} = \mathbb{H}^{k+1} /\!/\!/ \widetilde{G}. \tag{3.4}$$

One can easily verify that $\widetilde{M}_{(\xi,\Lambda,\sigma)}$ is diffeomorphic to M_ξ. However, they need not be isometric as Riemannian manifolds. The action of $\mathbb{R}$, when restricted to $\mathbf{u} \in \mathbb{H}^k$, is just a circle action. Now, if this action can be generated by a one parameter subgroup of G, then, in fact, M_ξ is isometric to $\widetilde{M}_{(\xi,\Lambda,\sigma)}$. If not, the two spaces are different.

Now let $M_\xi = M_\xi(\mathbb{Z}_k)$ be the ALE space given explicitly by the reduction of $\mathbb{H}^k$ by $G(\mathbb{Z}_k) = U(1)^{k-1} \subset U(k)$ [K1, K2]. Then $\widetilde{M}_{(\xi,\Lambda,\sigma)}$ is the corresponding ALF space (the multi-Taub-NUT metrics).

Before describing this construction in more detail we make the following observation. Let M_1, M_2 be two hyperkähler manifolds. Let K act on M_1 by triholomorphic isometries and let H act on $M = M_1 \times M_2$ by triholomorphic isometries. Moreover, let the actions of K and H on M_1 commute, *i.e.,* $G = K \times H$ acts on M and it is a hyperkähler action. We have the following momentum mappings

$$\begin{cases} \mu_1 : \ M_1 \ \longrightarrow \ \mathfrak{k}^*, \\ \mu_2 : \ M \ \longrightarrow \ \mathfrak{h}^*, \\ \ \mu : \ M \ \longrightarrow \ \mathfrak{g}^*. \end{cases} \tag{3.5}$$

Let $\xi_1 \in \mathfrak{k}^*$, $\xi_2 \in \mathfrak{h}^*$, $\xi = \xi_1 \oplus \xi_2 \in \mathfrak{h}^* \oplus \mathfrak{k}^* \cong \mathfrak{g}^*$ be the respective co-algebras' elements defining the three momentum levels. Now it is easy to see that

Lemma 3.6 *If all three quotients are smooth hyperkähler manifolds then the following spaces are isomorphic*

$$M/\!/\!/G \cong \left(\left(M_1/\!/\!/K\right) \times M_2\right)/\!/\!/H \cong \left(\left(M_1 \times M_2\right)/\!/\!/H\right)/\!/\!/K.$$

Let us consider some examples. First, we take the $4k$-dimensional generalization of the Taub-NUT metric. It is given by the hyperkähler quotient of $\mathbb{H}^{k+1}$ by the action of the group of translations where for $a \in \mathbb{R}$

$$a \cdot (\mathbf{u}, w) = (e^{ia}\mathbf{u}, w + \Lambda a), \qquad (\mathbf{u}, w) \in \mathbb{H}^k \times \mathbb{H}, \tag{3.7}$$

where Λ is any non-zero real constant. Choose any $\sigma \in \mathbb{R} \otimes \mathbb{R}^3$ and take the quotient

$$M_{(\Lambda,\sigma)} = (\mathbb{H}^{k+1}, ds^2)/\!/\!/\mathbb{R} = (\mathbb{H}^k, ds^2_{TN}), \tag{3.8}$$

where ds^2 is the standard flat metric. $M_{(\Lambda,\sigma)}$ is homeomorphic to $\mathbb{H}^k$ and, as a consequence of the Theorem 2.7, its induced metric is hyperkählerian. When $k = 1$, ds^2_{TN} is the Taub-NUT metric and when $k > 1$ it is its higher dimensional generalization introduced by Roček [**R**]. Notice that $M_{(\Lambda,\sigma)}$ has $U(k)$ as the group of hyperkähler isometries. Furthermore, notice that one can deform $M_{(\Lambda,\sigma)}$ through hyperkähler deformations just by taking a different $\mathbb{R}$-action

$$a \cdot (\mathbf{u}, w) = (e^{aT}\mathbf{u}, w + a) \tag{3.9}$$

where T is any element of the Lie algebra of $Sp(k)$. Let $M_{(T,\sigma)}$ denote a hyperkähler quotient of $\mathbb{H}^{k+1}$ by the action of (3.9) taken with respect to the σ-momentum level.

Proposition 3.10 $M_{(T,\sigma)}$ *is a smooth hyperkähler manifold for all* (T, σ).

Proof. If $M_{(T,\sigma)}$ is a smooth manifold then its metric is hyperkähler as a consequence of Theorem 2.7. It is then enough to show that it is a smooth manifold. The σ-momentum level is given by the following constraints

$$\mu^{-1}(\sigma) = \left\{(\mathbf{u}, w) \in \mathbb{H}^{k+1} : \ \ \mathbf{u}^* \cdot T\mathbf{u} + (w - w^*) = \sigma\right\} \tag{3.11}$$

which is clearly a submanifold in $\mathbb{H}^{k+1}$. The one-parameter action defined in (3.9) is free on $\mu^{-1}(\sigma)$. The action is not compact. We show, however, the existence of a slice. Let

$$\mathcal{S} = \Big\{(\mathbf{u}, w) \in \mu^{-1}(\sigma); \quad w^* + w = 0\Big\} \simeq \mathbb{H}^k. \tag{3.12}$$

Every orbit of $\mathbb{R}$ passes through $\mathcal{S}$ and if p_1, p_2 are any two points on $\mathcal{S}$ then they belong to different orbits and $\mathcal{S}$ is transversal to the orbits. Hence, $\mathcal{S}$ is a global slice and $\mu^{-1}(\sigma)/\mathbb{R} \simeq \mathcal{S}$. ■

Notice that the previous $M_{(\Lambda,\sigma)}$ corresponds to taking $T = i(1/\Lambda)\mathbb{1}$ and $\mathbb{R}^{4k} \simeq \mathbb{H}^k$ with the standard flat metric corresponds to $T \equiv 0$. Since e^{aT} can be chosen to act diagonally with respect to some unitary quaternionic basis of $\mathbb{H}^k$ with weights q_α $(1 \le \alpha \le k)$ $M_{(T,\sigma)}$ describes a k-parameter family of hyperkähler metrics with k commuting triholomorphic Killing vectors. As pointed out in [PP], such metrics correspond to solutions of the generalized Bogomolny equations just as in the 4-dimensional case they correspond to the solutions of the usual monopole equations.

The group of hyperkähler isometries of $M_{(T,\sigma)}$ depends on the choice of T. In particular, any $A \in Sp(k)$ such that $[A, T] = 0$ is a hyperkähler isometry of our manifold. Again, notice that for $T = i(1/\Lambda)\mathbb{1}$, $G = U(k)$ and when $k = 1$ it is well known that the Taub-NUT metric has only one triholomorphic Killing vector (even though it is $U(2)$-symmetric).

Our second example is the Kronheimer's construction for $\Gamma = \mathbb{Z}_k$ [K1]. It was originally introduced by Roček as an $N = 2$ supersymmetric sigma model in four dimensions [R]. The quotient can be described as follows. Take $\mathbb{H}^k$ and consider the action of $G \subset U(k) \subset Sp(k)$, $G = T^{k-1}$ being the maximal torus subgroup of $SU(k)$. T^{k-1} acts on $\mathbf{u} \in \mathbb{H}^k$ as

$$\varphi_{\mathbf{s}}(\mathbf{u}) = \exp\Big(\sum_{A=1}^{k-1} 2\pi i s^A T_A\Big)\mathbf{u}, \quad A = 1, ..., k-1 \tag{3.13}$$

where

$$T_1 = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \; T_2 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \ldots, T_{k-1} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix},$$

and $\mathbf{s} = (s^1, ..., s^{k-1})$ is a coordinate chart on T^{k-1}.

Choose any $\xi \in \mathbb{R}^{k-1} \otimes \mathbb{R}^3$, $\xi = (\xi_1, ..., \xi_A)$ being a purely imaginary quaternionic $(k-1)$-vector. The ξ-momentum level set is now given by the following constraints:

$$\mu^{-1}(\xi) = \{\mathbf{u} \in \mathbb{H}^k \; : \; \mathbf{u}^* \cdot iT_A \mathbf{u} = \xi^A \quad A = 1, ..., k-1\}. \tag{3.14}$$

where “$*$” denotes the quaternionic conjugation. $\mu^{-1}(\xi)$ is a T^{k-1}-invariant submanifold in $\mathbb{H}^k$. If one chooses ξ appropriately then the action of T^{k-1} is free on $\mu^{-1}(\xi)$ and $M_\xi(\mathbb{Z}_k) = \mu^{-1}(\xi)/T^{k-1}$ is a hyperkähler manifold. The value of ξ has to be in, what Kronheimer calls, a “good set” [K1]. Namely, choose any $\mathbf{s} = (s^1, ..., s^A)$ and suppose that $\mathbf{s} \neq 0$ and

$$\varphi_{\mathbf{s}}(\mathbf{u}) = \mathbf{u}$$

or, using the coordinates,

$$\begin{aligned} u_1 &= e^{2\pi i s^1} u_1, \\ u_2 &= e^{2\pi i (s^2 - s^1)} u_2, \\ u_3 &= e^{2\pi i (s^3 - s^2)} u_3, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{k-1} &= e^{2\pi i (s^{k-1} - s^{k-2})} u_{k-1}, \\ u_k &= e^{-2\pi i s^{k-1}} u_k. \end{aligned} \tag{3.15}$$

Suppose no two components of $\mathbf{u}$ can vanish on $\mu^{-1}(\xi)$ at the same time. Then (3.15) implies that $\mathbf{s} \equiv 0$ and thus the action is free on the ξ-level set. One can easily write down the conditions that the components of ξ must satisfy in order to exclude vectors with more than one vanishing component from the level set. If one considers the Lie algebra of T^{k-1} as the Cartan subalgebra of $\mathfrak{su}(k)$ then we must have

$$\xi \notin \bigcup_\theta (\mathbb{R}^3 \otimes P_\theta) \tag{3.16}$$

where P_θ is a hypersurface describing a wall of the Weyl chamber given by any root θ [K1].

$M_\xi(\mathbb{Z}_k)$ gives a $(3k-6)$-parameter family of hyperkähler metrics on a space that is diffeomorphic to the minimal resolution of singularity of $\mathbb{C}^2/\mathbb{Z}_k$. Lemma 3.6 can be now used to observe that the above construction gives a sequence of $U(1)$ quotients

$$\mathbb{H}^k \overset{U(1)}{\longrightarrow} M_{\xi_1}(\mathbb{Z}_2) \times \mathbb{H}^{k-2} \overset{U(1)}{\longrightarrow} M_{(\xi_1,\xi_2)}(\mathbb{Z}_3) \times \mathbb{H}^{k-3} \overset{U(1)}{\longrightarrow} \ldots\ldots \overset{U(1)}{\longrightarrow} M_\xi(\mathbb{Z}_k). \tag{3.17}$$

Finally, we employ the idea described at the beginning of this chapter. Let us introduce $T^{k-1} \times \mathbb{R}$ acting on $\mathbb{H}^k \times \mathbb{H}$ such that T^{k-1} acts only on $\mathbb{H}^k$ as in (3.13) and $\mathbb{R}$ acts on $\mathbb{H}^k \times \mathbb{H}$ as in (3.7). Choose ξ as before and any $\sigma \in \mathbb{R} \otimes \mathbb{R}^3$. Then (3.6) says that

$$\begin{aligned} \widetilde{M}_{(\xi,\Lambda,\sigma)}(\mathbb{Z}_k) &= (\mathbb{H}^{k+1}, ds^2) /\!/\!/ T^{k-1} \times \mathbb{R} = \{(\mathbb{H}^{k+1}, ds^2) /\!/\!/ \mathbb{R}\} /\!/\!/ T^{k-1} \\ &= \{M_\xi(\mathbb{Z}_k) \times \mathbb{H}\} /\!/\!/ \mathbb{R}. \end{aligned} \tag{3.18}$$

This shows that one can view our construction of the family of multi-Taub-NUT spaces as the quotient construction of Kronheimer for $\Gamma = \mathbb{Z}_k$ with a different, non-flat metric,

$$\widetilde{M}_{(\xi,\Lambda,\sigma)}(\mathbb{Z}_k) = M_{(\Lambda,\sigma)} /\!/\!/ T^{k-1} = (\mathbb{H}^k, ds^2_{TN}) /\!/\!/ T^{k-1}. \tag{3.19}$$

But we can also think of $\widetilde{M}_{(\xi,\Lambda,\sigma)}(\mathbb{Z}_k)$ as the quotient of $M_\xi(\mathbb{Z}_k)\times\mathbb{H}$ by $\mathbb{R}$ and it is clear that, in order for $\widetilde{M}_{(\xi,\Lambda,\sigma)}(\mathbb{Z}_k)$ not to be isometric to $M_\xi(\mathbb{Z}_k)$, $\mathbb{R}$ has to act on the latter as a triholomorphic isometry in a nontrivial fashion. This is the reason why a similar extension will not work for other, non-Abelian, Γ's when $M_\xi(\Gamma)$ has only discrete hyperkähler symmetries (no triholomorphic Killing vectors.)

4. Negative scalar curvature case

Before we discuss the multicenter case let us describe a negative scalar curvature quaternionic Kähler analogue of the $4k$-dimensional generalization of the Taub-NUT metric $M_{(T,\sigma)}$. Consider the $4(k+1)$-dimensional quaternionic projective ball $\mathbb{HH}^{k+1} \stackrel{\text{def}}{=} \mathbb{P}(\mathbb{H}^{1,1}\times\mathbb{H}^k)$. Let $(w_0,w_1,\mathbf{u})\in\mathbb{P}(\mathbb{H}^{1,1}\times\mathbb{H}^k)$ be a homogeneous coordinate, *i.e.*,

$$\mathbf{u}^*\cdot\mathbf{u}+w_1^*w_1-w_0^*w_0=-\frac{1}{\kappa^2},$$

$$(w_0,w_1,\mathbf{u}) \sim (w_0,w_1,\mathbf{u})\nu \tag{4.1}$$

where $\nu\in Sp(1)\subset\mathbb{H}^*$ is a unit quaternion $\nu^*\nu=1$. $\mathbb{HH}^{k+1}$ is homeomorphic to an open quaternionic $(k+1)$-ball $\mathcal{B}^{k+1}(1/\kappa)$ of radius $1/\kappa$. We can put the standard $Sp(1,k+1)$-invariant hyperbolic metric on it

$$\begin{aligned} ds^2 =&\mathrm{Re}\left(\sum_{i=1}^k du_i^*\otimes du_i+dw_1^*\otimes dw_1-dw_0^*\otimes dw_0\right)\\ &+\kappa^2\mathrm{Re}\left(\left(\sum_{i=1}^k u_i^*du_i\right)\otimes\left(\sum_{j=1}^k u_j^*du_j\right)+w_1^*dw_1\otimes dw_1^*w_1-w_0^*dw_0\otimes dw_0^*w_0\right).\end{aligned}\tag{4.2}$$

We consider a non-compact action on $\left(\mathcal{B}^{k+1}(1/\kappa),ds^2\right)$ by $\mathbb{R}$ where $\mathbb{R}\simeq SO(1,1)\subset U(1,k+1)\subset Sp(1,k+1)$ given explicitly by

$$\mathbb{R}\times\left(\mathcal{B}^{k+1}(1/\kappa),ds^2\right)\longrightarrow\left(\mathcal{B}^{k+1}(1/\kappa),ds^2\right)$$

$$\varphi_s(w_0,w_1,\mathbf{u})=\left(\begin{array}{cc|c} e^{i\beta s}\cosh\lambda s & e^{i\beta s}\sinh\lambda s & \\ & & \mathbb{O}\\ e^{i\beta s}\sinh\lambda s & e^{i\beta s}\cosh\lambda s & \\ \hline & & \\ \multicolumn{2}{c|}{\mathbb{O}} & e^{sD} \end{array}\right)\begin{pmatrix} w_0\\ w_1\\ \\ \mathbf{u}\end{pmatrix}, \tag{4.3}$$

where λ,β are any two real constants and D any $\mathfrak{sp}(k)$-matrix. The action (4.3) is a quaternionic isometry of the metric ds^2 and therefore, according to the Theorem (2.12), it defines a momentum section μ. The zero-section level set $\mu^{-1}(0) \subset \mathcal{B}^{k+1}(1/\kappa)$ is given by following constraints

$$\mu^{-1}(0) = \left\{ (w_0, w_1, \mathbf{u}) \in \mathcal{B}^{k+1}(1/\kappa) : \quad \mu(w_0, w_1, \mathbf{u}) = 0 \right\}$$

where

$$\mu(w_0, w_1, \mathbf{u}) = \beta(-w_0^* i w_0 + w_1^* i w_1) + \lambda(w_1^* w_0 - w_0^* w_1) + \mathbf{u}^* \cdot D\mathbf{u}, \tag{4.4}$$

and it is clearly an $\mathbb{R}$-invariant submanifold in $\mathcal{B}^{k+1}(1/\kappa)$.

Let us introduce global "inhomogeneous" coordinates $(y, \mathbf{x}) \in \mathcal{B}^{k+1}(1/\kappa)$ defined as

$$\begin{aligned} \mathbf{x} &= \frac{1}{\kappa} \mathbf{u} w_0^{-1}, \\ y &= \frac{1}{\kappa} w_1 w_0^{-1}. \end{aligned} \tag{4.5}$$

Then (4.1) yields

$$w_0^* w_0 = \frac{1}{\kappa^2} \Big[1 - \kappa^2 (y^* y + \mathbf{x}^* \cdot \mathbf{x}) \Big]^{-1} \tag{4.6}$$

and therefore

$$\mathbf{x}^* \cdot \mathbf{x} + y^* y < \frac{1}{\kappa^2}$$

as a simple consequence of (4.1). $\mu^{-1}(0)$ is now described in terms of $(y, \mathbf{x})$ as

$$\mu^{-1}(0) = \left\{ (y, \mathbf{x}) \in \mathcal{B}^{k+1}(1/\kappa) : \ \ \mathbf{x}^* \cdot D\mathbf{x} + \beta y^* i y + \frac{\lambda}{\kappa}(y^* - y) = \frac{i\beta}{\kappa^2} \right\}. \tag{4.7}$$

The transformation of $(y, \mathbf{x})$ under (4.3) is

$$\begin{aligned} \varphi_s(y) &= \Big((e^{i\beta s} \cosh \lambda s) y + \kappa^{-1} e^{i\beta s} \sinh \lambda s) \Big) \Big((\kappa e^{i\beta s} \sinh \lambda s) y + e^{i\beta s} \cosh \lambda s \Big)^{-1}, \\ \varphi_s(\mathbf{x}) &= e^{sD} \mathbf{x} \Big((\kappa e^{i\beta s} \sinh \lambda s) y + e^{i\beta s} \cosh \lambda s \Big)^{-1}. \end{aligned} \tag{4.8}$$

The transformation of y in (4.8) is a special case of the quaternionic analogue of the fractional linear transformations of the complex upper-half plane. If y is the affine coordinate on the quaternionic unit ball $Sp(1,1)/Sp(1) \cdot Sp(1)$ then we can consider the following transformation

$$y \longrightarrow \quad (ay + b)(cy + d)^{-1} \tag{4.9}$$

where $a, b, c, d \in \mathbb{H}$ and the quaternionic 2×2-matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1,1).$$

In our case
$$\begin{pmatrix} e^{i\beta s}\cosh\lambda s & e^{i\beta s}\sinh\lambda s \\ e^{i\beta s}\sinh\lambda s & e^{i\beta s}\cosh\lambda s \end{pmatrix} \in U(1,1) \subset Sp(1,1).$$

Now we shall demonstrate that the above action is free on $\mathcal{B}^{k+1}(1/\kappa)$. Consider the following equation
$$\varphi_s(y) = y. \tag{4.10}$$

It can be written as
$$\Big((e^{i\beta s}\cosh\lambda s)y + \kappa^{-1}e^{i\beta s}\sinh\lambda s)\Big) = y\Big((\kappa e^{i\beta s}\sinh\lambda s)y + e^{i\beta s}\cosh\lambda s\Big)$$

or
$$\sinh\lambda s\Big(\kappa^{-1}e^{i\beta s} - \kappa y e^{i\beta s} y\Big) + \cosh\lambda s\Big(e^{i\beta s}y - ye^{i\beta s}\Big) = 0.$$

If $y = z+uj$, where j is the second quaternionic unit, then the above quaternionic equation is equivalent to two complex equations
$$\sinh\lambda s\left(\frac{1}{\kappa}e^{i\beta s} - \kappa z^2 e^{i\beta s} + \kappa|u|^2 e^{-i\beta s}\right) = 0, \tag{4.11}$$

$$-\kappa\sinh\lambda s\Big(ue^{-i\beta s}\overline{z} + \overline{z}e^{i\beta s}u\Big) + \cosh\lambda s\Big(e^{i\beta s}u - ue^{-i\beta s}\Big) = 0. \tag{4.12}$$

If $\sinh\lambda s \neq 0$ then the (4.11) gives
$$\frac{1}{\kappa^2} - |z|^2 + |u|^2 e^{-2i\beta s} = 0. \tag{4.13}$$

This implies that
$$|u|^2 = \left|\frac{1}{\kappa^2} - |z|^2\right| \tag{4.14}$$

which means that $|z|^2 + |u|^2 \geq \frac{1}{\kappa^2}$ or $y^*y \geq 1/\kappa^2$. But then y is not in the ball. Hence, the only solution of (4.10) is $s = 0$. As we have just shown, the action of $\mathbb{R}$ on $\mathcal{B}^{k+1}(1/\kappa)$ is free everywhere and not only on $\mu^{-1}(0)$. As long as we choose $D \in \mathfrak{sp}(k)$ in such a way that e^{sD} generates a circle action on $(0,\mathbf{x}) \in \mathcal{B}^{k+1}(1/\kappa)$ $\mathbb{R}$ acts on $\mu^{-1}(0)$ properly. However, this need not be the case and we still can demonstrate the existence of a slice on $\mu^{-1}(0)$. Namely, let
$$\mathcal{S} = \Big\{(y,\mathbf{x}) \in \mu^{-1}(0): \quad y^* + y = 0\Big\}. \tag{4.15}$$

$\mathcal{S}$ is a smooth manifold, diffeomorphic to a 4k-dimensional open ball when $\beta = 0$. If one chooses $\mathbf{x}$ then $(y - y^*)$ is fixed by the condition that $(y,\mathbf{x})$ be in $\mu^{-1}(0)$. Every orbit of $\mathbb{R} \simeq SO(1,1)$ passes through $\mathcal{S}$ because
$$\varphi_s(y) + \varphi_s(y)^* = 0 \tag{4.16}$$

has solutions for all $|y| < 1/\kappa$. Every orbit meets $\mathcal{S}$ only once because $\varphi_s(y)^* + \varphi_s(y) = 0$ and $y^* + y = 0$ implies $s = 0$. Hence, $\mathcal{S}$ is a slice and $X_{(D,\lambda,\beta)} = \mu^{-1}(0)/\mathbb{R}$ is a quaternionic Kähler manifold diffeomorphic to $\mathcal{S}$. $X_{(D,\lambda,\beta)}$ describes a $(k+1)$-parameter family of quaternionic Kähler metrics with negative scalar curvature. Let us point out that the existence of nontrivial deformations in this case contrasts the compact case in which there are no nontrivial deformations through quaternionic Kähler manifolds [L].

$X_{(D,\lambda,0)} \simeq \mathbb{H}^k \simeq M_{(T,\sigma)}$ are all diffeomorphic but they differ as Riemannian manifolds, the first being an Einstein manifold with negative cosmological constant, the second the flat Euclidean space, and the third hyperkähler (Ricci flat and Kähler in particular).

In the $k = 1$ case $X_{(D,\lambda,0)}$ is the metric constructed by Pedersen [P] and discussed in [G2] from the $N = 2$ supergravity point of view. It is easy to observe that if we set $i\beta = \kappa^2\sigma$, $\lambda = \kappa$ and $T = D$ then, as Riemannian manifolds,

$$\lim_{\kappa\to 0} X_{(T,\lambda,\beta)} = M_{(T,\sigma)}. \tag{4.17}$$

The isometry group of $X_{(D,\lambda,\beta)}$ equals to $G \times Sp(1)$ where G is a subgroup of $Sp(k)$ commuting with D.

Now let us consider our final example: The negative scalar curvature multicenter metrics. Again, we start with a quaternionic hyperbolic ball of $\mathbb{HH}^{k+1} = Sp(1,k+1)/Sp(1) \times Sp(k+1)$ as our ambient space. Let $(w_0, w_1, \mathbf{u})$ be the homogeneous coordinate on it as in (4.1). We also think of it as a Riemannian manifold with the quaternionic Kähler $Sp(1,k+1)$-invariant metric (4.2). We introduce $G \times \mathbb{R}$-action on $\mathcal{B}^{k+1}(1/\kappa)$ as follows

$$\varphi_s(w_0, w_1, \mathbf{u}) = \exp\Big(\sum_{A=1}^{k-1} 2\pi i s^A \widehat{T}_A\Big)(w_0, w_1, \mathbf{u})^t, \quad A = 1, ..., k-1 \tag{4.18}$$

$$\varphi_\tau(w_0, w_1, \mathbf{u}) = \left(\begin{array}{cc|c} \cosh\lambda\tau & \sinh\lambda\tau & \\ & & \mathbb{O} \\ \sinh\lambda\tau & \cosh\lambda\tau & \\ \hline & & \\ & \mathbb{O} & e^{i\tau}\,\mathbb{1} \\ & & \end{array}\right) \begin{pmatrix} w_0 \\ \\ w_1 \\ \\ \\ \mathbf{u} \end{pmatrix} \tag{4.19}$$

where

$$\widehat{T}_1 = \begin{pmatrix} p_1 & & & & & & \\ & p_1 & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}, \quad \widehat{T}_2 = \begin{pmatrix} p_2 & & & & & & \\ & p_2 & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}, \ldots.$$

$$
\ldots, \widehat{T}_{k-1} = \begin{pmatrix} p_{k-1} & & & & & & \\ & p_{k-1} & & & & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & & \\ & & & & & 1 & \\ & & & & & & -1 \end{pmatrix}, \tag{4.20}
$$

$\tau \in \mathbb{R}$, $\mathbf{s} = (s^1, ..., s^{k-1})$ is a coordinate chart on G in the neighborhood of the identity element, *i.e.*, $\mathbf{s} \in \mathfrak{g}$, and $\mathbf{p} = (p_1, ..., p_{k-1}) \in \mathbb{Q}^{k-1}$. Let us point out that choosing p corresponds to the choice of a homomorphism from T^{k-1} into $U(1)$. If $\mathbf{p} = 0$, we have the trivial homomorphism.

The quaternionic momentum section μ for this action is given in terms of $3k$ quadratic constraints

$$
\mu^{-1}(0) = \left\{ (w_0, w_1, \mathbf{u}) \in \mathcal{B}^{k+1}(1/\kappa) : \quad \mu_A(w_0, w_1, \mathbf{u}) = 0; \quad A = 0, 1, ..., k-1 \right\} \tag{4.21}
$$

where

$$
\mu_0(w_0, w_1, \mathbf{u}) = \lambda(w_1^* w_0 - w_0^* w_1) + \mathbf{u}^* \cdot i\mathbf{u} \tag{4.22}
$$

and

$$
\mu_A(w_0, w_1, \mathbf{u}) = \mathbf{u}^* \cdot iT_A \mathbf{u} + w_1^* ip_A w - w_0^* ip_A w_0, \quad A = 1, ..., k-1. \tag{4.23}
$$

Now T_A's are as in (3.13).

Just as before we introduce the inhomogeneous global quaternionic coordinate chart on the ball $\mathcal{B}^{k+1}(1/\kappa) \ni (y, \mathbf{x})$. In these coordinates our zero-momentum section is described by the following constraints

$$
\widetilde{\mu}_0(y, \mathbf{x}) = \mathbf{x}^* \cdot i\mathbf{x} + \frac{\lambda}{\kappa}(y^* - y) = 0 \tag{4.24}
$$

$$
\widetilde{\mu}_A(y, \mathbf{x}) = \mathbf{x}^* \cdot iT_A \mathbf{x} + y^* ip_A y = \frac{ip_A}{\kappa^2}, \quad A = 1, ..., k-1. \tag{4.25}
$$

In terms of $(y, \mathbf{x})$ (4.18) and (4.19) read as

$$
\begin{cases} \varphi_\tau(y) = \Big((\cosh\lambda\tau)y + \kappa^{-1}\sinh\lambda\tau)\Big)\Big((\kappa\sinh\lambda\tau)y + \cosh\lambda\tau\Big)^{-1} \\ \varphi_\tau(\mathbf{x}) = e^{i\tau}\mathbf{x}\Big((\kappa\sinh\lambda\tau)y + \cosh\lambda\tau\Big)^{-1} \\ \varphi_{s,t}(y, \mathbf{x}) = \Big(e^{2\pi i(\mathbf{s}\cdot\mathbf{p})t} y e^{-2\pi i(\mathbf{s}\cdot\mathbf{p})t}, \exp\Big(2\pi i t s^A T_A\Big)\mathbf{x} e^{-2\pi(\mathbf{s}\cdot\mathbf{p})t}\Big) \end{cases} \tag{4.26}
$$

where $\mathbf{s}\cdot\mathbf{p} = \sum_A p_A s^A$ and $\varphi_{\mathbf{s},t}(y, \mathbf{x})$ is one parameter flow in the direction of $\mathbf{s} \in \mathfrak{g}$. Just as before, $\varphi_\tau(y, \mathbf{x})$ describes a free action on the whole ball $\mathcal{B}^{k+1}(1/\kappa)$. Also, any one-parameter subgroup of $G \times \mathbb{R}$ acting nontrivially by $\mathbb{R}$-part is free. The proof of this fact is the same as the proof of (4.10). We can also show that $\varphi_{s,t}(y,\mathbf{x})$ generates a locally free action for any $\mathbf{s}$ for some choice of parameter $\mathbf{p}$. Let us first calculate the vector field generated by this action

$$\begin{aligned}\frac{1}{2\pi}V(y,\mathbf{x}) =& \frac{1}{2\pi}\frac{d}{dt}\varphi_{\mathbf{s},t}(y,\mathbf{x})\bigg|_{t=0} = \\ =& \Big((\mathbf{s}\cdot\mathbf{p})iy - yi(\mathbf{s}\cdot\mathbf{p}),\quad is^{A}T_{A}\mathbf{x} - \mathbf{x}(\mathbf{s}\cdot\mathbf{p})i\Big).\end{aligned} \tag{4.27}$$

Vanishing of the vector filed V at $(y,\mathbf{x})$ is then equivalent to the following set of equations

$$\begin{aligned}&y^{*}iy(\mathbf{s}\cdot\mathbf{p}) = |y|^{2}i(\mathbf{s}\cdot\mathbf{p})\\ &is^{A}T_{A}\mathbf{x} = \mathbf{x}(\mathbf{s}\cdot\mathbf{p})i.\end{aligned} \tag{4.28}$$

Suppose $\mathbf{s}\cdot\mathbf{p} \neq 0$. Then equations (4.25) give

$$\mathbf{x}^{*}\cdot is^{A}T_{A}\mathbf{x} = -y^{*}(\mathbf{s}\cdot\mathbf{p})iy + \frac{(\mathbf{s}\cdot\mathbf{p})i}{\kappa^{2}} \tag{4.29}$$

and, comparing with (4.28), we get

$$\mathbf{x}^{*}\cdot is^{A}T_{A}\mathbf{x} = -|y|^{2}(\mathbf{s}\cdot\mathbf{p})i + \frac{(\mathbf{s}\cdot\mathbf{p})i}{\kappa^{2}} = |\mathbf{x}|^{2}(\mathbf{s}\cdot\mathbf{p})i$$

or

$$\Big(|\mathbf{x}|^{2} + |y|^{2}\Big) = \frac{1}{\kappa^{2}}. \tag{4.30}$$

But this contradicts $(y,\mathbf{x}) \in \mathcal{B}^{k+1}(1/\kappa)$. Hence, in order for V to vanish on $\mu^{-1}(0)$, $\mathbf{s}\cdot\mathbf{p}$ must vanish. Suppose that it does. Then the vector field

$$\frac{1}{2\pi}V(y,\mathbf{x}) = (0, is^{A}T_{A}\mathbf{x}) \tag{4.31}$$

vanishes on $\mu^{-1}(0)$ whenever

$$s^{A}T_{A}\mathbf{x} = 0. \tag{4.32}$$

Let us write (4.32) explicitly in terms of coordinates

$$\begin{aligned}&0 = s^{1}x_{1},\\ &0 = (s^{2} - s^{1})x_{2},\\ &\dots\dots\dots\dots\dots\dots\\ &0 = (s^{k-1} - s^{k-2})x_{k-1},\\ &0 = s^{k-1}x_{k}.\end{aligned} \tag{4.33}$$

If we can choose p in such a way that no two components of $\mathbf{x}$ can vanish on $\mu^{-1}(0)$ then (4.33) would imply that $\mathbf{s} \equiv 0$ and our action would be locally free. That would mean that

$$\frac{p_A}{\kappa^2}\Big(i - \kappa^2 y^* i y\Big), \quad A = 1, ..., k-1,$$

must satisfy the same conditions as ξ_A in (3.14) for any fixed y such that $(y, \mathbf{x}) \in \mathcal{B}^{k+1}(1/\kappa)$. But that is equivalent to the assumption that ip_A/κ satisfy these conditions since $i - \kappa^2 y^* i y \neq 0$ for any $y \in \mathcal{B}^{k+1}(1/\kappa)$. This concludes the proof that the action of G is locally free on $\mu^{-1}(0)$ for a generic choice of $\mathbf{p}$. In general, there may be some orbifold singularities in the quotient. Away from singular points, however,

$$\mathcal{M}_{(\mathbf{p},\lambda,\kappa)}(\mathbb{Z}_k) = \mu^{-1}(0)/G \times \mathbb{R} \tag{4.34}$$

has a self-dual and Einstein metric with constant negative scalar curvature. Moreover, if we set $p_A = -i\kappa^2\xi_A$ and $\lambda = \Lambda\kappa$ then we can take $\kappa \to 0$ limit to obtain

$$\lim_{\kappa \to 0} \mathcal{M}_{(p,\lambda,\kappa)}(\mathbb{Z}_k) = M_{(\xi,\Lambda,0)}(\mathbb{Z}_k) = M_{(\xi,\Lambda,\sigma)}(\mathbb{Z}_k). \tag{4.35}$$

Acknowledgments

I would like to thank the Max-Planck-Institut für Mathematik in Bonn and Prof. F. Hirzebruch for their hospitality. Some of this work was done there. I would also like to thank Dr. T. Nitta for some interesting discussions.

References

[EH] T. Eguchi and A. J. Hanson, *Self-Dual Solutions to Euclidean Gravity*, Ann. of Phys. 120 (1979), 82-106.

[G1] K. Galicki, *A generalization of the momentum mapping construction for quaternionic Kähler manifolds*, Commun.Math.Phys. 108 (1987), 117-138.

[G2] K. Galicki, *New Matter Couplings in N=2 Supergravity*, Nucl.Phys. B289 (1987), 573-588.

[GL] K. Galicki and B. H. Lawson, *Quaternionic reduction and quaternionic orbifolds*, Mathematische Annalen 282 (1988), 1-21.

[GH] G. W. Gibbons and S. W. Hawking, *Gravitational multi-instantons*, Phys.Lett. B78 (1978), 430.

[H1] N. J. Hitchin, *Monopoles, minimal surfaces, and algebraic curves*, SMS, vol. 105, Montreal 1987.

[H2] N. J. Hitchin, *Polygons and gravitons*, Math. Proc. Cambridge Philos. Soc. 83 (1979), 468-476.

[HKLR] N. J. Hitchin, A. Karlhede, U. Lindström, U., and M. Roček, *HyperKähler metrics and supersymmetry*, Commun. Math. Phys. 108, (1987), 535.

[K1] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J.Differential Geometry 29 (1986), 665-683.

[K2] P. B. Kronheimer, *A Torelli-type theorem for gravitational instantons*, J.Differential Geometry 29 (1986), 685-697.

[L] C. LeBrun, C, *A rigidity theorem for quaternionic Kähler manifolds*, preprint.

[P] H. Pedersen, *Einstein Metrics, Spinning Top Motions and Monopoles*, Math. Ann. 274 (1986), 35.

[PP] H. Pedersen and Y. S. Poon, *Hyper-Kähler Metrics and a Generalization of the Bogomolny Equations*, Commun.Math.Phys. 117 (1989), 569-580.

[R] M. Roček, *Supersymmetry and nonlinear σ-models*, Physica D15 (1985), 75.

Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131
email: galicki@math.unm.edu

September 1990