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# On Strongly Inhomogeneous Einstein Manifolds

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## 1. Introduction

There are few known constructions of compact inhomogeneous Einstein manifolds of positive scalar curvature. The first explicit example of such a metric was introduced by Page [P]. His construction was later generalized by Bérard Bergery [BB, Bes] to obtain cohomogeneity one Einstein metrics on $S^2$-bundles over certain Kähler manifolds. When the dimension of $M$ is even, this is the only known method that produces such Einstein metrics explicitly. All other known even-dimensional examples come from Kähler-Einstein geometry. Here the only explicit examples are due to Sakane [Sa, Bes] who used a variant of Bérard Bergery's construction to obtain Kähler-Einstein metrics of cohomogeneity one. It was pointed out to us by Wang that the Einstein equations in Sakane's construction can actually be solved explicitly. Koiso and Sakane [KoiSa1, KoiSa2] later generalized this construction and showed the existence of inhomogeneous Kähler-Einstein metrics of positive scalar curvature on many other compact complex manifolds. Their method yields Einstein metrics of arbitrary cohomogeneity; however, we are not aware of any explicit solutions to the Einstein equations with cohomogeneity higher than one. Tian and Yau [TY] proved another existence result, that there are inhomogeneous Kähler-Einstein metrics on the del Pezzo surfaces $\mathbb{CP}^2\#k(-\mathbb{CP}^2)$ for $3 \leq k \leq 8$.

When $M$ is odd dimensional the only explicit examples of inhomogeneous Einstein manifolds of positive scalar curvature with arbitrary cohomogeneity were obtained by the authors in [BGM1, BGM2, BGM3]. All of these examples are 3-Sasakian manifolds. A different construction of odd-dimensional inhomogeneous Einstein spaces is due to Wang and Ziller [WZ1, WZ2]. They obtained Einstein metrics of positive scalar curvature on certain torus bundles over products of Kähler-Einstein spaces. This construction is actually quite explicit, but in order to get explicit inhomogeneous metrics one needs an explicit inhomogeneous Kähler-Einstein metric on the base. Here the only such examples are the Sakane's metrics mentioned above which are of cohomogeneity one and thus yield a cohomogeneity one metric in the Wang-Ziller torus bundle. A similar construction of Einstein metrics on certain principal $\mathbb{RP}^3$-bundles over products of quaternionic Kähler manifolds is due to Wang [W2].

Following Eschenberg [E1] one says that $M$ is strongly inhomogeneous if $M$ is *not* homotopy equivalent to any homogeneous space $G/H$. In [BGM2, BGM3] we constructed inhomogeneous Einstein metrics of positive scalar curvature on compact simply connected

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3-Sasakian manifolds $(\mathcal{S}(\mathbf{p}), g(\mathbf{p}))$ in dimension $4n-5$ for all $n \geq 3$. The metrics obtained there are all inhomogeneous as can be easily seen from the geometry of the construction. The cohomogeneity of $g(\mathbf{p})$ depends on $\mathbf{p}$ and is a number between $3n-7$ and 0. The highest cohomogeneity is realized by infinitely many $g(\mathbf{p})$'s in every dimension while the lowest is realized by the homogeneous metric on $\frac{U(n)}{U(n-2)\times U(1)}$. In dimension 7 our $\mathcal{S}(\mathbf{p})$ manifolds are all diffeomorphic to Eschenberg's bi-quotients of $U(3)$ first introduced in [E1] and later generalized in [E2]. Furthermore, using a cohomology calculation of [BGM3] which determined $H^*(\mathcal{S}(\mathbf{p});\mathbb{Z})$ and a result of Eschenberg [E1] on 7-dimensional, homogeneous spaces, we proved the following Theorem [BGM2, BGM3]:

Theorem: *There are countable subfamilies of pairwise homotopy distinct $\mathcal{S}(\mathbf{p})$ manifolds each of which is a 7-dimensional, compact, simply connected, strongly inhomogeneous, Einstein manifold of positive scalar curvature.*

The positive scalar curvature Einstein metrics on $\mathcal{S}(\mathbf{p})$ are obtained explicitly. To our knowledge these are the only explicit examples of cohomogeneity greater than one. Furthermore, these are the first such examples of strongly inhomogeneous Einstein manifolds in odd dimension. LeBrun [L] has pointed out to us that circle bundles over some del Pezzo surfaces are also strongly inhomogeneous. It then follows from the results of Tian and Yau [TY] and Wang and Ziller [WZ1] that they admit Einstein metrics of positive scalar curvature. However, there is no known explicit construction for these metrics. In addition, we do not know if the torus bundle examples over the manifolds constructed by Sakane [Sa] can be shown to be strongly inhomogeneous.

While Eschenberg's analysis of homogeneous spaces was restricted to dimension 7, the cohomology analysis of the $\mathcal{S}(\mathbf{p})$ manifolds is complete in every possible dimension. Thus, it is natural to ask if these 3-Sasakian examples give the first known explicit examples of strongly inhomogeneous Einstein manifolds in higher dimensions as well. It is the purpose of this note is to prove

Theorem A: *For all $n > 2$ there are countable subfamilies of pairwise homotopy distinct $\mathcal{S}(\mathbf{p})$ manifolds each of which is a $4n-5$-dimensional, compact, simply connected, strongly inhomogeneous, Einstein manifold of positive scalar curvature.*

In the course of proving Theorem A we encounter certain homogeneous spaces which we denote by $\mathcal{M}(a_1, a_2)$. These manifolds topologically are higher dimensional generalizations of the 7-dimensional Aloff-Wallach manifolds [AW] and are defined in section three. Theorem A is actually a Corollary of the following

Theorem B: *Let $\mathcal{S}(\mathbf{p})$ be any 3-Sasakian example constructed in [BGM3]. Then either*

1. *$\mathcal{S}(\mathbf{p})$ is strongly inhomogeneous,*

*or*

2. *$\mathcal{S}(\mathbf{p})$ is homotopy equivalent to some $\mathcal{M}(a_1, a_2)$.*

The $\mathcal{S}(\mathbf{p})$ examples and their main properties are recalled in section two while Theorem B is proved in section three and Theorem A in section four. Actually, the conclusion of Theorem B holds for any space $X$ that is the topological quotient of the Stiefel manifold $\mathbb{V}^{\mathbb{C}}_{n,2}$ of 2 frames in $\mathbb{C}^n$ by a free circle action, where $\pi_1(X) = 0$ and $\pi_2(X) = \mathbb{Z}$. Therefore, the evident generalization of Theorem B also applies to the 7-dimensional bi-quotients of $SU(3)$ constructed by Eschenberg in [E1] and, more generally, to the bi-quotients of $U(3)$

considered in [E2]. Hence, not only does Theorem B allow us to extend our earlier results on the existence of strongly inhomogeneous Einstein manifolds to higher dimensions but also sharpens the strongly inhomogeneity results of both [E1] and [BGM3] (while Eschenberg actually proves Theorem B in dimension 7, he does not state this stronger result explicitly). The implications of Theorem B in dimension 7 are discussed in section five.

We would like to thank M. Wang for bringing Sakane's paper to our attention and for pointing out that the Einstein equations arising in his construction can be solved explicitly. We would like to thank C. LeBrun for telling us about his examples mentioned above and to thank the referee both for bringing reference [O] to our attention and also for making other suggestions which have improved this note. Finally, the first and the second named authors would like to thank the Erwin Schrödinger International Institute for Mathematical Physics in Vienna for their hospitality. The final version of this article was completed during their visits there.

## 2. The Examples

To begin we recall some relevant facts from [BGM3].

DEFINITION 2.1: *Let $n \geq 3$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n$ be an $n$-tuple of non-decreasing, pairwise relatively prime, positive integers. Such a sequence is called admissible. Let $\mathcal{S}(\mathbf{p})$ be the left-right quotient of the unitary group $U(n)$ by $U(1) \times U(n-2) \subset U(n)^2 = U(n)_L \times U(n)_R$, where the action is given by the formula*

2.2 $$\mathbb{W} \xrightarrow{(\tau,\mathbb{B})} \begin{pmatrix} \tau^{p_1} & & \\ & \ddots & \\ & & \tau^{p_n} \end{pmatrix} \mathbb{W} \begin{pmatrix} \mathbb{I}_2 & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix}.$$

*Here $\mathbb{W} \in U(n)$ and $(\tau, \mathbb{B}) \in U(1) \times U(n-2)$.*

Equivalently, $\mathcal{S}(\mathbf{p})$ is the quotient of the complex Stiefel manifold $\mathbb{V}^{\mathbb{C}}_{n,2}$ of 2-frames in $\mathbb{C}^n$ by the free circle action given in equation 2.2. Thus, there is a fibration sequence

2.3 $$S^1 \longrightarrow \mathbb{V}^{\mathbb{C}}_{n,2} \longrightarrow \mathcal{S}(\mathbf{p}) \xrightarrow{f(\mathbf{p})} B_{S^1},$$

where $f(\mathbf{p})$ is the classifying map of the associated principal circle bundle over $\mathcal{S}(\mathbf{p})$. Furthermore, since $\mathbb{V}^{\mathbb{C}}_{n,2}$ is $2n-4$ connected, 2.3 implies that

2.4 $$\pi_i(\mathcal{S}(\mathbf{p})) = \begin{cases} 0 & \text{if } i = 0, 1, \\ \mathbb{Z} & \text{if } i = 2, \\ 0 & \text{if } 2 < i < 2n-4. \end{cases}$$

THEOREM 2.5: [BGM3] *Let $n \geq 3$ and $\mathbf{p}$ be an admissible sequence. Then $\mathcal{S}(\mathbf{p})$ is a compact, simply connected, $(4n-5)$-dimensional smooth manifold which admits an Einstein metric $g(\mathbf{p})$ of positive scalar curvature and a compatible Sasakian 3-structure. In addition, $\mathcal{S}(\mathbf{p})$ also admits a second Einstein metric of positive scalar curvature, $g_1(\mathbf{p})$, which is non-homothetic to $g(\mathbf{p})$. Furthermore, both $\big(\mathcal{S}(\mathbf{p}), g(\mathbf{p})\big)$ and $\big(\mathcal{S}(\mathbf{p}), g_1(\mathbf{p})\big)$ are inhomogeneous Einstein manifolds as long as $\mathbf{p} \neq (1, \ldots, 1)$.*

In [BGM4] the authors studied the automorphism groups of certain hypercomplex structures $\mathcal{I}(\mathbf{p})$ on the Stiefel manifolds $\mathbb{V}^{\mathbb{C}}_{n,2}$. Using Theorems C and 3.21 of [BGM4] one can easily determine the connected component of the 3-Sasakian isometry group $I_0$ of $\big(\mathcal{S}(\mathbf{p}), g(\mathbf{p})\big)$. The 3-Sasakian isometries are isometries of $g(\mathbf{p})$ which commute with the action of $Sp(1)$ defined by the 3-Sasakian vector fields. Using a result of Tanno [T], the connected component of the full isometry group can then be determined. We have

THEOREM 2.6: *Let $I_0$ be the group of 3-Sasakian isometries of $\big(\mathcal{S}(\mathbf{p}), g(\mathbf{p})\big)$ and let $k$ be the number of 1's in $\mathbf{p}$. Then the connected component of $I_0$ is $S(U(k)\times U(1)^{n-k})$, where we define $U(0)=\{e\}$. Thus, the connected component of the isometry group is the product $I_0\times SO(3)$ if the sums $p_i+p_j$ are even for all $1\leq i,j\leq n$, and $I_0\times Sp(1)$ otherwise.*

In the case that $\mathbf{p}$ has no repeated 1's, the cohomogeneity can easily be determined, viz.

COROLLARY 2.7: *If the number of 1's in $\mathbf{p}$ is 0 or 1 then the dimension of the principal orbit in $\mathcal{S}(\mathbf{p})$ equals $n+2$ and the cohomogeneity of $g(\mathbf{p})$ is $3n-7$.*

## 3. Homotopic Homogeneous Spaces

In this section we assume that $\mathcal{S}(\mathbf{p})$ is *not* strongly inhomogeneous; that is, we assume that there is a homotopy equivalence

3.1 $$h:\mathcal{S}(\mathbf{p})\longrightarrow A/B,$$

where $A$ is a compact Lie group and $B$ is a closed subgroup. Our main purpose here is to deduce as much as possible about $A/B$ under this assumption; more precisely, to prove Theorem B. The 7-dimensional case is discussed in section five so we assume $n>3$ for the remainder of this section. To begin, using arguments similar to Eschenberg [E: §4] and equation 2.4 one can assume that

3.2 $$\frac{A}{B}=\frac{G}{K\times S^1},$$

where both $G$ and $K$ are compact, simply-connected, semi-simple Lie groups and $K\times S^1$ is a closed subgroup of $G$.

Furthermore, since $n>3$, equation 2.4 implies there is the following short exact sequence

3.3 $$0\longrightarrow\pi_3(K)\xrightarrow{\ \cong\ }\pi_3(G)\longrightarrow 0.$$

Thus, $G$ and $K$ have the same number of simple factors and we have

3.4 $$\frac{G}{K}\cong\frac{G_1\times\cdots\times G_l}{K_1\times\cdots\times K_l},$$

where each $G_i$ and $K_j$ is a compact, simple, simply-connected Lie group for $1\leq i,j\leq l$.

Since $S^1 \subset K \times S^1 \subset G$ is a subgroup and the map in 3.1 is a homotopy equivalence we have the following commutative diagram classifying principal circle bundles:

3.5
$$\begin{tikzcd}
S^1 \arrow[d] & S^1 \arrow[d] \\
\mathbb{V}^{\mathbb{C}}_{n,2} \arrow[d] & G/K \arrow[d] \\
\mathcal{S}(\mathbf{p}) \arrow[r, "h"] \arrow[d, "f(\mathcal{S}(\mathbf{p}))"] & \frac{G}{K\times S^1} \arrow[d, "f(G,K)"] \\
B_{S^1} \arrow[r, "id"] & B_{S^1}.
\end{tikzcd}$$

Here $\pi_1(\mathbb{V}^{\mathbb{C}}_{n,2}) = \pi_2(\mathbb{V}^{\mathbb{C}}_{n,2}) = \pi_1(G/K) = \pi_2(G/K) = 0$ and $h$ is a homotopy equivalence between simply-connected spaces so $f(\mathcal{S}(\mathbf{p})) \sim f(G,K) \circ h$ and both circle bundles in 3.5 are classified by the same element

$$1 \in H^2(\mathcal{S}(\mathbf{p}); \mathbb{Z}) = [\mathcal{S}(\mathbf{p}), B_{S^1}] \cong \mathbb{Z}.$$

But this implies that there is a homotopy equivalence of the total spaces

3.6
$$\hat{h} : \mathbb{V}^{\mathbb{C}}_{n,2} \longrightarrow G/K.$$

LEMMA 3.7: *$G/K$ is diffeomorphic to $\mathbb{V}^{\mathbb{C}}_{n,2} \cong U(n)/U(n-2)$ for $n > 3$.*

PROOF: The lemma is a direct consequence of results of Onishchik, specifically Theorem 12 and Theorem 13 of chapter seven of [O]. We are grateful to the referee who pointed out to us that the techniques of [O] actually classify all homogeneous spaces homotopy equivalent to $\mathbb{V}^{\mathbb{C}}_{n,2}$ although this is not explicitly stated in [O]. ∎

DEFINITION 3.8: *For all $n \geq 3$ let $(a_1, a_2)$ be a pair of relatively prime integers with $a_1 \geq a_2$ and $\mathcal{M}(a_1, a_2)$ be the quotient of the unitary group $U(n)$ by $U(n-2) \times S^1$ where the action is given by the formula*

3.9
$$\mathbb{W} \xrightarrow{(\mathbb{B},\tau)} \mathbb{W}\begin{pmatrix} \mathbb{A}(\tau) & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix}.$$

*Here $\mathbb{W} \in U(n)$, $\mathbb{B} \in U(n-2)$, $\tau \in S^1$, and* $\mathbb{A}(\tau) = \begin{pmatrix} \tau^{a_1} & 0 \\ 0 & \tau^{a_2} \end{pmatrix}$.

Remark 3.10: Aloff and Wallach first studied the $\mathcal{M}(a_1, a_2)$ manifolds in dimension 7 and their main result was to prove they had metrics of positive sectional curvature [AW]. Later, Wang [W1] showed that the 7-dimensional Aloff-Wallach manifolds also supported an Einstein metric and Kreck and Stolz [KrSt] analyzed their homeomorphism and diffeomorphism types. However, we do not know whether in higher dimensions the manifolds $\mathcal{M}(a_1, a_2)$ have either metrics of positive sectional curvature or Einstein metrics.

Proof of Theorem B: Lemma 3.7 implies that the target of the homotopy equivalence $h$ in equation 3.1 is

3.11
$$\frac{G}{K \times S^1} \cong \frac{U(n)}{U(n-2) \times S^1},$$

where $U(n-2) \times S^1$ is a subgroup of $U(n)$. Since the $S^1$ factor acts as a subgroup which commutes with the $U(n-2)$ factor, up to conjugation, the most general action one can have is given by equation 3.9, where $a_1$ and $a_2$ are relatively prime. ∎

## 4. Homology Calculations

In order to prove Theorem A of the introduction we need to compare the cohomology rings for $\mathcal{S}(\mathbf{p})$ and $\mathcal{M}(a_1, a_2)$. The following was proved in [BGM3].

Theorem 4.1: *Let $\mathbf{p}$ be an admissible $n$-tuple. Then, as rings,*

4.2
$$H^*(\mathcal{S}(\mathbf{p}); \mathbb{Z}) \;\cong\; \left( \frac{\mathbb{Z}[b_2]}{[b_2^n = 0]} \otimes E[f_{2n-1}] \right) / \mathcal{R}(\mathcal{S}(\mathbf{p})),$$

*where the subscripts on $b_2$ and $f_{2n-1}$ denote the cohomological dimension of each generator. The relations $\mathcal{R}(\mathcal{S}(\mathbf{p}))$ are given by $\sigma_{n-1}(\mathbf{p})b_2^{n-1} = 0$ and $f_{2n-1}b_2^{n-1} = 0$. Here $\sigma_{n-1}(\mathbf{p})$ is the $(n-1)^{st}$ elementary symmetric polynomial in $\mathbf{p}$.*

Corollary 4.3: *$H^{2n-2}(\mathcal{S}(\mathbf{p}); \mathbb{Z}) \cong \mathbb{Z}_{\sigma_{n-1}(\mathbf{p})}$ for all admissible sequences $\mathbf{p}$.*

We now prove that the $\mathcal{M}(a_1, a_2)$ manifolds have similar cohomology rings.

Theorem 4.4: *For all pairs of relatively prime integers $a_1$ and $a_2$ with $a_1 \geq a_2$*

4.5
$$H^*(\mathcal{M}(a_1, a_2); \mathbb{Z}) \;\cong\; \left( \frac{\mathbb{Z}[b_2]}{[b_2^n = 0]} \otimes E[f_{2n-1}] \right) / \mathcal{R}(\mathcal{M}(a_1, a_2)),$$

*where the subscripts on $b_2$ and $f_{2n-1}$ denote the cohomological dimension of each generator. The relations $\mathcal{R}(\mathcal{M}(a_1, a_2))$ are given by $\alpha(a_1, a_2)b_2^{n-1} = 0$ and $f_{2n-1}b_2^{n-1} = 0$. Here*

4.6
$$\alpha(a_1, a_2) = \sum_{i=0}^{n} a_1^{n-i} a_2^i = \begin{cases} n+1 & \text{if } a_1 = a_2 = 1, \\ \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2} & \text{otherwise.} \end{cases}$$

Corollary 4.7: *$H^{2n-2}(\mathcal{M}(a_1, a_2); \mathbb{Z}) \cong \mathbb{Z}_{\alpha(a_1, a_2)}$.*

The notational convention in Corollary 4.7 is that $\mathbb{Z}_0 = \mathbb{Z}$ and $\mathbb{Z}_1 = 0$. This occurs only when $\alpha(1,-1) = 0$ (if $n$ is odd) or $\alpha(1,-1) = 1$ (if $n$ is even). These are the only cases when $H^{2n-2}(\mathcal{M}(a_1,a_2);\mathbb{Z})$ is not a finite cycle group. If $(a_1,a_2) \neq (1,-1)$ then, when $n = 3$, one can assume that both $a_1$ and $a_2$ are positive. This is not possible in higher dimensions; however, we can and will assume that $|a_1| > |a_2|$ and $a_1 > 0$ whenever $n > 3$.

Proof of Theorem 4.4: When $n = 3$ this computation is carried out by Kreck and Stolz [KrSt: section 4] and the generalization for $n > 3$ is straightforward. ∎

When $n > 3$ Theorem A follows from Theorem B and Corollaries 4.3 and 4.7 as $\sigma_{n-1}(\mathbf{p})$ grows linearly in the entries of $\mathbf{p}$ whereas $\alpha(a_1,a_2)$ grows to the $n^{th}$ order in $a_1$ and $a_2$ (even if $a_1$ and $a_2$ differ in sign).

## 5. The Seven Dimensional Case

Eschenberg constructed the first examples of inhomogeneous metrics of positive sectional curvature on certain compact simply connected 7-dimensional circle bi-quotients of $SU(3)$ [E1]. In order to prove that these examples are inhomogeneous he showed

Theorem 5.1: *Let $M^7$ be a free circle quotient of $SU(3)$ and $H^4(M^7;\mathbb{Z}) = \mathbb{Z}_r$, where $r \equiv 2 \pmod 3$. Then $M$ is strongly inhomogeneous.*

The proof of Theorem 5.1 given in [E1] actually implies Theorem B in dimension 7 although this is not stated explicitly (see [E1: section 4 and appendix 3], [E2: pp.163-164]). To see that Theorem B is indeed stronger note that Theorem 5.1 does not show the strong inhomogeneity of $\mathcal{S}(1,1,4)$ while Theorem B and a simple calculation using the quadratic formula do imply that $\mathcal{S}(1,1,4)$ is strongly inhomogeneous. Moreover, as $\alpha(a_1,a_2)$ increases quadratically when $n = 3$, Theorem B implies

Proposition 5.2: *Let $d$ run over the positive integers. The families*

5.3 $$\{\mathcal{S}(1,1,3d)\} \quad \textit{and} \quad \{\mathcal{S}(1,1,3d+1)\}$$

*must each contain infinitely many homotopy distinct strongly inhomogeneous elements.*

We can also apply Theorem B to obtain strong inhomogeneity results for some bi-quotient manifolds constructed by Eschenberg [E1]. For example, using the notation of [E1], the bi-quotient $M_{1,1,2,2}$ has the property that $H^4(M_{1,1,2,2};\mathbb{Z}) = \mathbb{Z}_9$ so that, while Theorem 5.1 does not apply in this case, Theorem B implies that $M_{1,1,2,2}$ is also strongly inhomogeneous. More generally, we have

Proposition 5.4: *Let $d$ run over the positive integers. The family*

5.5 $$\{M_{d+1,d+1,d,d}\}$$

*of Eschenberg bi-quotients must contain infinitely many homotopy distinct strongly inhomogeneous elements.*

Proof: It is easy to check that $(d+1, d+1, d, d)$ satisfies the hypothesis of [E1: Proposition 2.1] so each element of 5.5 is a smooth manifold. Each element of 5.5 is topologically the quotient of a free circle action on $SU(3)$ with $\pi_1(M_{d+1,d+1,d,d}) = 0$ and $\pi_2(M_{d+1,d+1,d,d}) =$

$\mathbb{Z}$. Therefore, the evident generalization of Theorem B extends to cover these elements. Furthermore, [E1: Proposition 3.6] shows that

$$5.6 \qquad H^4(M_{d+1,d+1,d,d};\mathbb{Z}) \;=\; \mathbb{Z}_{6d+3}$$

and, as $6d+3$ increases linearly in $d$, the proposition follows from Corollary 4.7. ∎

None of the strongly inhomogeneous manifolds in Propositions 5.2 and 5.4 can be shown to be strongly inhomogeneous via Theorem 5.1. However, both results follow from the discussion of the $U(3)$ bi-quotients $U_{x,y}$ introduced in [E2] and our remark that the proof of Theorem 5.1 in [E1] implies Theorem B in dimesions 7.

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