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# Einstein Manifolds of Positive Scalar Curvature with Arbitrary Second Betti Number

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Abstract. We announce a quotient construction of new families of compact, irreducible, inhomogeneous, Einstein 7-manifolds of positive scalar curvature with arbitrary second Betti number. For infinitely many $(\mathbf{a},\mathbf{b}) \in (\mathbb{Z}^*)^k \oplus (\mathbb{Z}^*)^k$ we obtain a compact 3-Sasakian 7-manifold $\mathcal{S}(\mathbf{a},\mathbf{b})$ with $b_2(\mathcal{S}(\mathbf{a},\mathbf{b})) = k$. The manifold $\mathcal{S}(\mathbf{a},\mathbf{b})$ has two more compact positive scalar curvature Einstein spaces (orbifolds) naturally associated to it: (1) the twistor space $\mathcal{Z}(\mathbf{a},\mathbf{b})$ which is a $\mathbb{Q}$-Fano 3-fold with a complex contact structure and (2) the self-dual Einstein orbifold $\mathcal{O}(\mathbf{a},\mathbf{b})$. We show that $b_2(\mathcal{S}(\mathbf{a},\mathbf{b}))=b_2(\mathcal{O}(\mathbf{a},\mathbf{b}))=b_2(\mathcal{Z}(\mathbf{a},\mathbf{b}))-1=k$. These appear to be the first examples of such objects with arbitrarily large total Betti number.

## 1. Introduction

Amongst all Riemannian geometries the class of Einstein metrics stands out as perhaps the most natural and interesting [Bes]. Even so there are still many open questions about the relationship between the topology of a compact manifold and the existence of Einstein metrics. One such question concerns the existence of Einstein manifolds of positive scalar curvature on manifolds with large total Betti number. In the case of Einstein manifolds of negative scalar curvature, such examples are plentiful. The celebrated theorem of Aubin and Yau says that a Kähler manifold with $c_1$ negative definite always admits a Kähler-Einstein metric. For example, there are compact complex surfaces of general type which have $c_1$ negative and arbitrarily high second Betti number. It is well known that Yau's proof of the Calabi conjecture does not apply when $c_1 > 0$, and there appear to be no known examples of compact Einstein manifolds (in any dimension) of positive scalar curvature with an arbitrarily large total Betti number. Such examples are of interest in view of a remarkable theorem of Gromov [Gro] which implies that if a positive Einstein manifold admits a metric whose sectional curvatures are bounded below by a negative constant then the Betti numbers must be bounded. This, combined with results described herein, implies that given a number $\kappa < 0$ there are an infinite number of positive Einstein manifolds that do not admit metrics with sectional curvatures bounded below by $\kappa$.

The technique that we use to construct our examples is the 3-Sasakian reduction procedure [BGM2] starting from the standard 3-Sasakian sphere $(S^{4n-1}, g_{can})$. Thus, the positive Einstein manifolds that we describe are 3-Sasakian. Our construction is described

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in the next section and several corollaries are given. A brief outline of the proof is given in section 3 and full details are in [BGMR]. Finally, in section 4 we give some consequencess concerning related geometries. In particular, we announce the existence of $\mathbb{Q}$-factorial Fano 3-folds with arbitrarily large second Betti number, as well as self-dual Einstein orbifolds with arbitrarily large second Betti number. These results are of interest in view of the following known Betti number bounds. Mori and Mukai [MM] showed that smooth Fano 3-folds must have $b_2 \leq 10$, and LeBrun [Le,LeSal] showed that the second Betti number of any quaternionic Kähler (self-dual Einstein in dimension 4) manifold is different from zero in only one case.

## 2. The Construction and Results

We begin with the quaternionic vector space $\mathbb{H}^{k+2}$ and the unit sphere $S^{4k+7}$ given in quaternionic coordinates $\mathbf{u} = (u_1, ..., u_{k+2}) \in \mathbb{H}^{k+2}$ by

$$S^{4k+7} = \{\mathbf{u} \in \mathbb{H}^{k+2} \quad | \quad \sum_{\alpha=1}^{k+2} \overline{u}_\alpha u_\alpha = 1\},$$

where $\overline{u}$ denotes the quaternionic conjugate. Let us choose a purely imaginary direction, say $i$, in the unit quaternions, and consider the complete intersection of quadrics in $S^{4k+7}$ given by

$$N(\mathbf{a},\mathbf{b}) = \Big\{(u_1, ..., u_{k+2}) \in S^{4k+7} \mid \overline{u}_j i u_j + a_j \overline{u}_{k+1} i u_{k+1} + b_j \overline{u}_{k+2} i u_{k+2} = 0, \ \forall j = 1, ..., k\Big\},$$

where $a_j, b_j$ are nonvanishing integers for all $j$. Here $\mathbf{a}, \mathbf{b}$ denote vectors in $\mathbb{Z}^k$ with components $a_j, b_j$, respectively. If for all $i, j = 1, \cdots, k$ the $2\times 2$ minor determinants $\det\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix}$ are nonvanishing, then $N(\mathbf{a},\mathbf{b})$ is a smooth compact manifold of dimension $k+7$. Henceforth, we shall assume this to be the case.

Consider the $k$-torus action on $S^{4k+7}$ defined by

$$2.1 \qquad \varphi_{(\tau_1,\ldots,\tau_k)}(u_1, ..., u_k, u_{k+1}, u_{k+2}) = \Big(\tau_1 u_1, ..., \tau_k u_k, \prod_{j=1}^{k} \tau_j^{a_j} u_{k+1}, \prod_{j=1}^{k} \tau_j^{b_j} u_{k+2}\Big)$$

for $\tau_j \in S^1$. This action restricts to a locally free action on $N(\mathbf{a},\mathbf{b})$. Furthermore, if $\gcd(a_j, b_j) = 1$ for all $j = 1, \cdots, k$ the action is free on $N(\mathbf{a},\mathbf{b})$. Henceforth, we shall also assume this to be the case. Thus, the quotient $\mathcal{S}(\mathbf{a},\mathbf{b})$ of $N(\mathbf{a},\mathbf{b})$ by the action 2.1 is a smooth compact manifold of dimension 7.

Now consider the canonical metric $g_{can}$ on $S^{4k+7}$ and restrict this metric to a metric $\hat{g}$ on $N(\mathbf{a},\mathbf{b})$. The $k$-torus action given in 2.1 is an action by isometries of $\hat{g}$. So there is a metric $g(\mathbf{a},\mathbf{b})$ on $\mathcal{S}(\mathbf{a},\mathbf{b})$ such that the projection $\pi : N(\mathbf{a},\mathbf{b}) \longrightarrow \mathcal{S}(\mathbf{a},\mathbf{b})$ is a Riemannian submersion. Our main result is:

Theorem A: *Let $k$ be a positive integer, and let $(\mathbf{a},\mathbf{b})\in(\mathbb{Z}^*)^k\oplus(\mathbb{Z}^*)^k$ whose components $(a^i,b^i)$ are pairs of relatively prime integers for $i=1,\cdots,k$ that satisfy the condition that if for some pair $i,j$ $a^i=\pm a^j$ or $b^i=\pm b^j$ then we must have $b^i\neq\pm b^j$ or $a^i\neq\pm a^j$, respectively. Then the Riemannian manifolds $\big(\mathcal{S}(\mathbf{a},\mathbf{b}),g(\mathbf{a},\mathbf{b})\big)$ admit a 3-Sasakian structure and have second Betti number $b_2(\mathcal{S}(\mathbf{a},\mathbf{b}))=k$. In particular, there exist simply connected compact Einstein 7-manifolds of positive scalar curvature with arbitrary second Betti number.*

There are several important corollaries of Theorem A. The first follows immediately from Theorem A and a Theorem 2A and 2B of Gromov [Gro].

Corollary B: *There are infinitely many compact simply-connected Einstein 7-manifolds of positive scalar curvature, namely the $\mathcal{S}(\mathbf{a},\mathbf{b})$ of Theorem A, that do not admit metrics of nonnegative sectional curvature. Furthermore, for any negative real number $\kappa$ there are infinitely many 3-Sasakian manifolds $\mathcal{S}(\mathbf{a},\mathbf{b})$ which do not admit metrics whose sectional curvatures are all greater than or equal to $\kappa$.*

The question whether or not there exists compact Riemannian manifolds of "nonnegative Ricci curvature" which do not admit metrics of nonnegative sectional curvature was problem 5 of Yau's famous problem section of the 1979-80 Princeton Seminar [Y]. This question was answered affirmatively in 1989 by Sha and Yang [SY], but to the best of the authors' knowledge our construction gives the first examples for Einstein manifolds of positive scalar curvature.

Our next corollary is a partial classification result. It follows immediately from Theorem A and results of [GS].

Corollary C: *In dimension seven there exist 3-Sasakian manifolds with every allowable rational homology type.*

It is clear that our examples do not satisfy the necessary conditions that guarentee many of the well-known finiteness results (cf. [Che]). However, one can contrast the examples given here which do not admit metrics of positive sectional curvature with our previous examples [BGM2,BGM3] as well as the Einstein manifolds of [Wa]. In those examples one has positive Einstein manifolds with $b_2=1$, and with infinitely many distinct homotopy types. However, many of those examples admit metrics with positive sectional curvature. Furthermore, the manifolds in [Wa] are diffeomorphic to the homogeneous Aloff-Wallach manifolds of positive sectional curvature. It was also shown in [BGM2,BGM4] that most of our previous examples are not homotopy equivalent to any homogeneous spaces. Regarding homogeneity it is not difficult to see that any compact homogeneous manifold must satisfy $b_2\leq\frac{1}{2}\mathrm{dim}$. Thus, we have

Corollary D: *If $k>3$ the 3-Sasakian manifolds $\mathcal{S}(\mathbf{a},\mathbf{b})$ are not homotopy equivalent to any homogeneous space.*

## 3. Idea of proof

The proof of Theorem A uses the 3-Sasakian reduction procedure [BGM2]. The manifold $N(\mathbf{a},\mathbf{b})$ is precisely the zero set of a 3-Sasakian moment map $\mu : S^{4k+7}\longrightarrow \mathfrak{t}_k^*\otimes\mathbb{R}^3$ corresponding to the $k$-torus action 2.1. So by the reduction theorem [BGM2] the quotient $\mathcal{S}(\mathbf{a},\mathbf{b})=\mu^{-1}(0)/T^k$ is a 3-Sasakian 7-manifold, and hence, is Einstein of positive scalar curvature. The 3-Sasakian manifolds described in [BGM2,BGM3,BGM4] correspond to the case $k=1$.

The crucial point is to show that $b_2(\mathcal{S}(\mathbf{a},\mathbf{b}))=k$. This is done by constructing a stratification of $\mathcal{S}(\mathbf{a},\mathbf{b})$ related to the stratification by orbit types of its isometry group. The maximal torus $T^{k+2}$ of the group $Sp(k+2)$ of 3-Sasakian isometries of $S^{4k+2}$ centralizes the $k$-torus $T^k$ described in 2.1. Thus, the 3-Sasakian manifold $\mathcal{S}(\mathbf{a},\mathbf{b})$ has a $T^2$ as 3-Sasakian isometries. This together with the $Sp(1)$ isometries of any 3-Sasakian manifold gives a five dimensional isometry group $T^2\times Sp(1)$. One can then analyze the fixed point sets under $T^2\times Sp(1)$ and its subgroups. This together with known results about cohomogeneity two manifolds [Bre] are used to show that the image of the natural quotient projection is a closed $(k+2)$-gon in $\mathbb{R}^2$. The generic stratum consists of either $T^2\times S^3$ or $T^k\times SO(3)$ over the interior of the $(k+2)$-gon. There are two other strata, one lying over the edges of the $(k+2)$-gon and the other over the vertices. The first of these has codimension one and is the disjoint union of $k+2$ copies of the product of circles with lens spaces over an interval. The other stratum, which is of codimension two, consists of the disjoint union of $k+2$ copies of lens spaces (not necessarily the same). One then uses a Leray spectral sequence together with the fact that odd Betti numbers vanish below the middle dimension on any 3-Sasakian manifold [GS] to give the desired result.

## 4. Relationship with Other Geometries

It is known [BGM1,BGM2] that every 3-Sasakian manifold has two distinct homothety classes of Einstein metrics only one of which is 3-Sasakian. Furthermore, in dimension 7 both of these metrics have weak $G_2$ holonomy [GS,FKMS]. Thus, Theorem A implies

Corollary F: *There exist 7-manifolds with arbitrary second Betti number having metrics of weak $G_2$ holonomy.*

In [BG] it was shown that the twistor space of any 3-Sasakian manifold has the structure of a $\mathbb{Q}$-factorial Fano variety. Thus, results of [BG] and Theorem A give:

Corollary F: *There exist $\mathbb{Q}$-factorial Fano 3-folds $X$ with $b_2(X)=l$ for any positive integer $l$. Furthermore, $X$ has both a complex contact structure and a Kähler-Einstein metric.*

As mentioned in the introduction this result contrasts sharply with the smooth case where Mori and Mukai [MM] tell us that $b_2\leq 10$. There is a well-known relationship [BGM1,BG] between 3-Sasakian geometry on the one hand and both quaternionic Kähler geometry of positive scalar curvature and Fano contact geometry on the other (Here $l = k+1$ for the $l$ in Corollary F and $k$ in Theorem A). But in general this relationship involves Riemannian metrics with orbifold singularities for both the quaternionic Kähler and Fano geometries. It is the existence of these singularities that allow the violation of finiteness, as well as the violation of the Betti number bound. In the smooth case LeBrun's $b_2 \leq 1$ result for quaternionic Kähler manifolds $M$ is proved by using a theorem of Wiśniewski [Wi] on the twistor space $\mathcal{Z}$ of $M$ which is a Fano manifold with a complex contact structure. The existence of such a contact structure implies that the index of the anti-canonical divisor be large and Wiśniewski severely limits the possibilities. However, Wiśniewski's theorem fails in the orbifold category since both the contact divisor and the anticanonical divisor are now $\mathbb{Q}$-divisors, and the singularity index can be arbitrarily high.

By an analysis similar to that described in section 3 one can obtain quaternionic Kähler orbifolds $\mathcal{O}$ of positive scalar curvature with arbitrary second Betti number. In dimension four, these spaces are compact, self-dual, Einstein orbifolds. Thus we have

THEOREM G: *Let $\mathcal{O}(\mathbf{a},\mathbf{b})$ be the compact, self-dual, Einstein orbifold associated to the 7-dimensional 3-Sasakian manifold $\mathcal{S}(\mathbf{a},\mathbf{b})$ given in Theorem B. Then*

$$b_2(\mathcal{O}(\mathbf{a},\mathbf{b})) = b_2(\mathcal{S}(\mathbf{a},\mathbf{b})) = k.$$

*Hence, there are compact, self-dual, Einstein orbifolds of positive scalar curvature with arbitrary second Betti number.*

Again we mention the constrast with LeBrun's result in the smooth case. The orbifolds $\mathcal{O}(\mathbf{a},\mathbf{b})$ were first studied in [GN] and later in [BGM1]. They give a generalization of the self-dual Einstein metrics that can be introduced on the weighted complex projective plane [GL,BGM2]. A result analogous to Corollary B also holds for the orbifolds $\mathcal{O}(\mathbf{a},\mathbf{b})$.


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