EINSTEIN METRICS ON EXOTIC SPHERES IN DIMENSIONS 7,11 AND 15

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Abstract. In a recent article the first three authors proved that in dimension $4m+1$ all homotopy spheres that bound parallelizable manifolds admit Einstein metrics of positive scalar curvature which, in fact, are Sasakian-Einstein. They also conjectured that all such homotopy spheres in dimension $4m-1, m \geq 2$ admit Sasakian-Einstein metrics [BGK03], and proved this for the simplest case, namely dimension 7. In this paper we describe computer programs that show that this conjecture is also true for 11-spheres and 15-spheres. Moreover, a program is given that determines the partition of the 8610 deformation classes of Sasakian-Einstein metrics into the 28 distinct oriented diffomorphism types in dimension 7.

1. Introduction

In a recent article the first three authors gave a method for constructing Einstein metrics of positive scalar curvature on odd-dimensional homotopy spheres [BGK03]. By Kervaire and Milnor [KM63] and Smale [Sma62], for each $n \geq 5$, differentiable homotopy spheres of dimension n form an Abelian group Θ_n, where the group operation is connected sum. Θ_n has a subgroup bP_{n+1} consisting of those homotopy n-spheres which bound parallelizable manifolds V_{n+1}. Kervaire and Milnor [KM63] proved that $bP_{2m+1} = 0$ for $m \geq 1$, $bP_{4m+2} = 0$, or $\mathbb{Z}_2$ and is $\mathbb{Z}_2$ if $4m+2 \neq 2^i - 2$ for any $i \geq 3$. The most interesting groups are bP_{4m} for $m \geq 2$. These are cyclic of order

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1}-1) \text{ numerator } \left(\frac{4B_m}{m}\right),$$

where B_m is the m-th Bernoulli number. Thus, for example $|bP_8| = 28, |bP_{12}| = 992, |bP_{16}| = 8128$ and $|bP_{20}| = 130,816$. In the first two cases these include all exotic spheres. The correspondence is given by

$$KM : \Sigma \mapsto \tfrac{1}{8}\tau(V_{4m}(\Sigma)) \mod |bP_{4m}|,$$

where $V_{4m}(\Sigma)$ is any parallelizable manifold bounding Σ and τ is its signature. Let Σ_i denote the exotic sphere with $KM(\Sigma_i) = i$.

In [BGK03] the authors proposed the following:

Conjecture 1. *The construction of* [BGK03] *yields Einstein metrics on every exotic sphere that bounds a parallelizable manifold.*

The construction is described in sections 2–3. The method gives Einstein metrics whose isometry group is one dimensional and they are even Sasakian-Einstein.

In [BGK03] the conjecture was shown to be true in dimensions $4m+1$. In dimension 7 we were also able to verify it; the relevant signature calculations were carried out by a computer.

The main aim of this paper is to provide more evidence for our conjecture by demonstrating that it is true in dimensions 11 and 15 as well. More precisely we show

Theorem 2. *Every homotopy sphere $\Sigma_i \in bP_{12}$ and $\Sigma_i \in bP_{16}$ admits at least one Einstein metric.*

We also give a complete enumeration of all oriented diffeomorphism types in dimension 7, namely,

Theorem 3. *In dimension 7, Σ_i admits at least n_i inequivalent deformation classes of Einstein metrics, where* $(n_1,\dots,n_{28}) = (376, 336, 260, 294, 231, 284, 322, 402, 317, 309, 252, 304, 258, 390, 409, 352, 226, 260, 243, 309, 292, 452, 307, 298, 230, 307, 264, 353)$, *giving a total of* 8610 *cases.*

Actually for dimensions 11 and 15, just as in dimension 7, we do get several deformation types, but the signature was computed only for a sample of all cases. For instance, in dimension 15 our method gives at least 10^{50} deformation classes of Einstein metrics on all homotopy 15-spheres, and even their complete enumeration is impossible with the current programs and facilities.

2. Brieskorn-Pham Singularities and Their Links

For $\mathbf{a} = (a_1,\dots,a_m) \in \mathbb{Z}_+^m$ set $F_{\mathbf{a}}(\mathbf{z}) := \sum_{i=1}^m z_i^{a_i}$. Consider a Brieskorn–Pham singularity

$$Y(\mathbf{a}) := (F_{\mathbf{a}}(\mathbf{z}) = 0) \subset \mathbb{C}^m, \quad \text{and its link} \quad L(\mathbf{a}) := Y(\mathbf{a}) \cap S^{2m-1}(1).$$

Set $C = \mathrm{lcm}(a_i : i = 1,\dots,m)$. Both $Y(\mathbf{a})$ and $L(\mathbf{a})$ are invariant under the $\mathbb{C}^*$-action

$$(z_1,\dots,z_m) \mapsto (\lambda^{C/a_1} z_1,\dots,\lambda^{C/a_m} z_m).$$

If we denote $\mathbf{w} = (w_1,\dots,w_m) = (C/a_1,\dots,C/a_m)$, then $F_{\mathbf{a}}$ is a weighted homogeneous polynomial on $\mathbb{C}^m$ with weight $\mathbf{w}$ and degree C, i.e.,

$$F_{\mathbf{a}}(\lambda^{w_1} z_1,\dots,\lambda^{w_m} z_m) = \lambda^C F_{\mathbf{a}}(z_1,\dots,z_m).$$

Consider the orbit spaces: $X^{orb}(\mathbf{a}) := Y(\mathbf{a}) \setminus \{0\}/\mathbb{C}^*$ and the weighted projective space $\mathbb{P}(\mathbf{w}) := (\mathbb{C}^m \setminus \{0\}/\mathbb{C}^*)$. We get a commutative diagram

$$\begin{tikzcd} L(\mathbf{a}) \arrow[r] \arrow[d, "\pi"] & S^{2m-1} \arrow[d] \\ X^{orb}(\mathbf{a}) \arrow[r] & \mathbb{P}(\mathbf{w}). \end{tikzcd}$$

It is known that the sphere S^{2m-1} can be given a Sasakian structure with respect to the projection $S^{2m-1} \longrightarrow \mathbb{P}(\mathbf{w})$ associated to the characteristic foliation [YK84]. In such a case the embedding $L(\mathbf{a}) \longrightarrow S^{2m-1}$ is Sasakian and $X^{orb}(\mathbf{a})$ is the horizontal space of the characteristic foliation of the link $L(\mathbf{a})$ [BG01].

3. Orbifolds and Einstein Metrics

Let $C^j = \mathrm{lcm}(a_i : i \neq j)$, $\quad b_j = \gcd(a_j, C^j)$ $\quad$ and $\quad d_j = a_j/b_j$. The following result was established in [BGK03].

Theorem 4. *The orbifold $X^{orb}(\mathbf{a}) = Y(\mathbf{a})/\mathbb{C}^*$ is Fano and has a Kähler-Einstein metric if*

(1) $1 < \sum_{i=1}^m \frac{1}{a_i}$,
(2) $\sum_{i=1}^m \frac{1}{a_i} < 1 + \frac{m-1}{m-2} \min_i\{\frac{1}{a_i}\}$, *and*
(3) $\sum_{i=1}^m \frac{1}{a_i} < 1 + \frac{m-1}{m-2} \min_{i,j}\{\frac{1}{b_i b_j}\}$.

In this case the link $L(\mathbf{a})$ admits a Sasakian-Einstein metric with one-dimensional isometry group.

The first inequality is necessary for $X^{orb}(\mathbf{a})$ to be Fano. Hence, it is also necessary for the link $L(\mathbf{a})$ to admit any Sasakian-Einstein structure. The second inequality is necessary for our algebraic approach to Kähler-Einstein metrics to work, while the third inequality is most likely an artifice of our estimates. Hopefully, it is not needed at all. We should reiterate that the failure of our method does not imply that $X^{orb}(\mathbf{a})$ cannot admit a positive Kähler-Einstein metric as long as $X^{orb}(\mathbf{a})$ is Fano.

For any $m \geq 3$ there are infinitely many m-tuples satisfying the conditions of Theorem 4. For example, we can take $\mathbf{a} = (m-1,\dots,m-1,k)$, where $\gcd(m-1,k) = 1$ and $k > (m-1)(m-2)$.

However, in this paper we are only interested in the case when the link $L(\mathbf{a})$ is a homotopy sphere and, as we shall see, $L(m-1,\ldots,m-1,k)$ is not.

4. Homotopy Spheres as Brieskorn-Pham Links

To every m-tuple $\mathbf{a}$ one can associate a graph $G(\mathbf{a})$ whose m vertices are labeled by $a_1,\cdots,a_m$. Two vertices a_i and a_j are connected if and only if $\gcd(a_i,a_j)>1$. Let C_{ev} denote the connected component of $G(\mathbf{a})$ determined by the even integers. Note that all even vertices belong to C_{ev}, but C_{ev} may contain odd vertices as well. Brieskorn shows that:

Theorem 5. [Bri66] *The link $L(\mathbf{a})$ (with $m>3$) is a homotopy sphere if and only if either of the following hold:*

(1) *$G(\mathbf{a})$ contains at least two isolated points, or*
(2) *$G(\mathbf{a})$ contains one odd isolated point and C_{ev} has an odd number of vertices and for any distinct $a_i,a_j\in C_{ev}$, $\gcd(a_i,a_j)=2$.*

We observe that, in each dimension, there are only finitely many m-tuples that yield homotopy spheres and satisfy the conditions (4.1–3). For that we introduce

Example 6 (Euclid's or Sylvester's sequence)**.** (See [GKP89, Sec.4.3] or [Slo03, A000058].)

Consider the sequence defined by the recursion relation

$$c_{k+1}=c_1\cdots c_k+1=c_k^2-c_k+1$$

beginning with $c_1=2$. We call this sequence the *extremal sequence.* It starts as

$$2,3,7,43,1807,3263443,10650056950807,\ldots,$$

and it is easy to see (cf. [GKP89, 4.17]) that

$$\sum_{i=1}^{m}\frac{1}{c_i}=1-\frac{1}{c_{m+1}-1}=1-\frac{1}{c_1\cdots c_m}.$$

In [Sou03] it was proved that if the sum of reciprocals of m natural numbers is less than 1, then it is at most $1-1/(c_{m+1}-1)$. Thus, in this sense the sequence $\{c_i\}$ is extremal.

We use the sequence c_i to show that the number of m-tuples that yield homotopy spheres and satisfy the conditions (4.1–3) is finite. Without loss of generality we shall assume that the exponents are arranged in the non-decreasing order.

Proposition 7. *Assume that $\mathbf{a}\in\mathbb{Z}_+^m$ satisfies the conditions (4.1–2) and (5). Then $a_k<(m-k+1)(c_k-1)$, for $k=1,\ldots,m-1$ and $a_m<\frac{m!}{m-2}(c_m-1)$. In particular, the number of such m-tuples is finite for each $m>3$.*

Proof. Step 1. We first observe that $\sum_{i=1}^{m-2}\frac{1}{a_i}<1$. For otherwise we would have

$$\frac{1}{a_{m-1}}+\frac{1}{a_m}<\frac{m-1}{m-2}\cdot\frac{1}{a_m}<\frac{2}{a_m},$$

which is impossible.

Step 2. Now, assume that $\sum_{i=1}^{k}\frac{1}{a_i}<1$. Then it is also $\le 1-1/(c_{k+1}-1)$. The remaining $m-k$ reciprocals must sum to more than $1/(c_{k+1}-1)$, hence we obtain that $a_{k+1}\le(m-k)(c_{k+1}-1)$. By Step 1 this takes care of all a_i for $i\le m-1$ and also of a_m if $\sum_{i=1}^{m-1}\frac{1}{a_i}<1$.

Thus we are left with Step 3: $\sum_{i=1}^{m-1}\frac{1}{a_i}\ge 1$. If equality holds there is no bound for a_m; however, in this case $L(\mathbf{a})$ is not a homotopy sphere, since Theorem 5 says that at least one of the $a_1,\ldots,a_{m-1}$

(or half of it) is relatively prime to the others, and this implies that we cannot get an integer as a sum of reciprocals. Otherwise we have

$$\sum_{i=1}^{m-1}\frac{1}{a_i} > 1+\frac{1}{a_1\cdots a_{m-1}} \geq 1+\frac{1}{m!\cdot c_1\cdots c_{m-1}} = 1+\frac{1}{m!\cdot(c_m-1)}.$$

Thus we obtain that

$$1+\frac{1}{m!\cdot(c_m-1)}+\frac{1}{a_m} < \sum_{i=1}^{m}\frac{1}{a_i} \leq 1+\frac{m-1}{m-2}\cdot\frac{1}{a_m}.$$

Comparing the two ends gives that

$$a_m < \frac{m!}{m-2}(c_m-1).$$

□

We wrote a simple program which we call `candidates.c`[1]. This is a C code which enumerates all ordered m-tuples satisfying the conditions (4.1–3) and (5.1) or (5.2) in any given range $a_i^{\min}\leq a_i\leq a_i^{\max}$, $i=1,\ldots,m$, with the condition that $a_1^{\min}\leq\cdots\leq a_m^{\min}$. In principle, for any $m\geq 4$, the program can be used to enumerate **all** m-tuples of this type. However, this is not feasible already for $m=7$. On the other hand, the program has the flexibility to "hunt" for such m-tuples in any specified region of the integral lattice defined by Proposition 7.

5. Diffeomorphism Types – Brieskorn, Zagier, and Hirzebruch

By Theorem 5, we know when $L(\mathbf{a})$ is a homotopy sphere. We now would like to be able to determine the diffeomorphism types of various links. In this article, we are only interested in the case when $m=2k+1$.

In this case, the diffeomorphism type of a homotopy sphere $L(\mathbf{a})\in bP_{2m-2}$ is determined [KM63] by the signature $\tau(M)$ of a parallelizable manifold M whose boundary is $\Sigma_{\mathbf{a}}^{2m-3}$. By the Milnor Fibration Theorem [Mil68] we can take M to be the Milnor fiber $M_{\mathbf{a}}^{2m-2}$ which, for links of isolated singularities coming from weighted homogeneous polynomials is diffeomorphic to the hypersurface $\{\mathbf{z}\in\mathbb{C}^m \mid F(z_1,\cdots,z_m)=1\}$.

Brieskorn shows that the signature of $M_{\mathbf{a}}^{2m-2}$ can be written combinatorially as

5.1
$$\begin{aligned}\tau(M_{\mathbf{a}}^{4k}) &= \#\Big\{\mathbf{x}\in\mathbb{Z}^{2k+1} \mid 0<x_i<a_i \text{ and } 0<\sum_{j=0}^{2k}\frac{x_i}{a_i}<1 \mod 2\Big\}\\ &\quad -\#\Big\{\mathbf{x}\in\mathbb{Z}^{2k+1} \mid 0<x_i<a_i \text{ and } 1<\sum_{j=0}^{2k}\frac{x_i}{a_i}<2 \mod 2\Big\},\end{aligned}$$

where $m=2k+1$.

Using a formula of Eisenstein, Zagier (cf. [Hir71]) has rewritten this formula as:

5.2
$$\tau(M_{\mathbf{a}}^{4k}) = \frac{(-1)^k}{N}\sum_{j=0}^{N-1}\cot\frac{\pi(2j+1)}{2N}\cot\frac{\pi(2j+1)}{2a_0}\cdots\cot\frac{\pi(2j+1)}{2a_{2k}},$$

where N is any common multiple of the a_i's. Both formulas are quite well suited to computer use. We wrote a second C code which we call `sig.c`. For any m-tuple with $m=2k+1=5,7,9$ `sig.c` computes the signature $\tau(\mathbf{a}):=\tau(M_{\mathbf{a}}^{4k})$ and the diffeomorphism type of the link using either of the above formulas. Furthermore, one can use `sig.c` to compute signature and diffeomorphism type of

[1]The codes `candidates.c` and `sig.c` as well as all the relevant data files mentioned later can be downloaded at http://www.math.unm.edu/~galicki/papers/codes.html.

a single m-tuple, or one can select an arbitrary set of m-tuples $\mathcal{I}$ and compute the signature and diffeomorphism type associated to every m-tuple $\mathbf{a} \in \mathcal{I}$. One last feature of `sig.c` is that, provided an appropriate option is chosen, the program will start computing diffeomorphism type $g(\mathbf{a})$ of each m-tuple $\mathbf{a} \in \mathcal{I}$ until it finds all possible oriented diffeomorphism types in bP_{2m-2} after which it stops.

6. The proofs

Proof. [**Theorem 3**] In dimension 7 the `candidates.c` can be run in the maximal range specified by Proposition 7. The result is exactly 8610 solutions. These solutions become an input data file $\mathcal{I}$ for the signature computation using `sig.c` with either Brieskorn or Zagier formula. In the case of 5-tuples the choice is not important. The signature computation takes a couple of hours on a Pentium 4 processor and the result is a list of 8610 5-tuples $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$ each with a number $g(\mathbf{a}) \in \mathbb{Z}_{28}$ which determines the oriented diffeomorphism type of $L(\mathbf{a})$. The results are contained in the output file `7spheres.txt`. This file can be easily sorted grouping 5-tuples with the same $g(\mathbf{a})$ and we get the result described in Theorem 3. □

Proof. [**Theorem 2**] In dimension 11 the `candidates.c` cannot be run in the maximal range of Proposition 7. The complete enumeration would take too long a time. Instead, the code `candidates.c` is used to select 7-tuples in a specified range. This will become an input file $\mathcal{I}$ for the subsequent signature computation. One important point in selecting $\mathcal{I}$ is that $C = \mathrm{lcm}(a_1, \ldots, a_7)$ should not be too large. The time of every individual signature computation with `sig.c` is approximately linear in C. Another relevant point is that $bP_{12} = \mathbb{Z}_{992}$ so that $|\mathcal{I}|$ should be sufficiently large. For example, we can ask `candidates.c` to search for 7-tuples in the following range: $2 \le a_1 \le 6$, $3 \le a_2 \le 11$ and $i + 1 \le a_i \le 30$ for $i = 3, 4, 5, 6, 7$. This guarantees a relatively small $C < 66 \cdot 30^5$ for all solutions and $|\mathcal{I}| = 21,535$. One should point out that there is nothing special about the choice of $\mathcal{I}$ – other choices can be equally successful in yielding the desired result. We now want to determine if we find all $g(\mathbf{a}) \in \mathbb{Z}_{992}$ among $\mathbf{a} \in \mathcal{I}$. This is done by feeding each 7-tuple $\mathbf{a} \in \mathcal{I}$ into `sig.c` with the following option: the program will calculate the signature $\tau(\mathbf{a})$ and the diffeomorphism type $g(\mathbf{a})$ of each 7-tuple $\mathbf{a} \in \mathcal{I}$ in the order specified by $\mathcal{I}$. Any 7-tuple $\mathbf{a} \in \mathcal{I}$ with a diffeomorphism type $g(\mathbf{a})$ not previously found gets automatically recorded into the output file $\mathcal{A}$. Once the program finds all 992 oriented diffeomorphism types it stops. The output file contains a subset of the original input file (hopefully) containing exactly 992 7-tuples. All this work can done on a single PC with a Pentium 4 processor. An example of an output file $\mathcal{A}$ called `11spheres.txt` can be found at the URL mentioned in the earlier footnote. We needed approximately 9000 7-tuples to find the 992 necessary to prove Theorem 2.

In dimension 15 we repeat the steps outlined in the 11-dimensional case. Selecting appropriately large data file with `candidates.c` is not a problem. This can be done on a single PC. Given that $bP_{16} \simeq \mathbb{Z}_{8128}$ one needs an input file $\mathcal{I}$ with about 80K 9-tuples for the signature computation with `sig.c`. More challenging problem has to do with computing signature for these 9-tuples. To minimize computing time some care should be given to how $\mathcal{I}$ is selected. The length of a single computation varies depending on (1) $\mathbf{a} = (a_1, \ldots, a_9)$ itself; (2) the formula used for the signature computation; (3) the processor's speed. Also, this is an easy parallelization task because it consists of tens of thousand runs which are almost completely independent of each other. The only co-ordination that is required is to stop the process when all $g(\mathbf{a}) \in \mathbb{Z}_{8128}$ are found.

We actually generated two sets $\mathcal{I}_1, \mathcal{I}_2$ for the signature calculation. The first set $\mathcal{I}_1$ was was created by appropriately restricting the size of all exponents. After the calculations for $\mathcal{I}_1$ were completed a second set $\mathcal{I}_2$ was chosen to select 9-tuples with a restricted upper bound on $C = \mathrm{lcm}(a_1, \ldots, a_9)$. We first used the Zagier formula 5.2 to calculate the signature $\tau(\mathbf{a})$ of the 9-tuples in the selected input files $\mathcal{I}_1, \mathcal{I}_2$. Zagier's formula was chosen as this calculation is much faster for most individual

$\mathbf{a}$'s. Exactly how much faster depends on the ratio $a_1 \cdots a_9/C$. If $C = a_1 \cdots a_9$ then the Brieskorn formual 5.1 is slightly faster. On the other hand, the problem with using the Zagier formula for very large C is that there is a large round-off error on Intel x86 processors even at maximum precision. When C is of the order of 10^9 this error becomes large enough that $g(\mathbf{a})$ is sometimes calculated incorrectly. While this was not an issue for all 5-tuples and also for carefully selected 7-tuples the case of 15-spheres was more of a problem. Instead of forcing the program to do a better round-off error control with the Zagier option, we decided to do the first calculation with the Zagier formula and then verify all signature calculations for the candidate solution with the Brieskorn formula 5.1. By its nature, this formula does not have any round-off error. At the end we actually generated two disjoint sets of 9-tuples. One is contained in the file `15spheresA.txt`. The other one is in `15spheresB.txt`. □

The Zagier calculation on the first set $\mathcal{I}_1$ was done at the University of Melbourne on an IBM eServer 1350 which is a cluster of 48 2.4GHz Intel Xeon processors. The calculation leading to the data set `15spheresA.txt` took approximately 9500 hours of processor time and tested nearly 70,000 candidates. The Brieskorn verification was performed on `15spheresA.txt` at the University of New Mexico High Performance Computer Center on a 256 node cluster of 733Mhz processors. This required 80,000 hours of processor time. In the case of one 9-tuple the code calculating with the Zagier formula yielded the wrong answer: $g_Z(3,4,8,8,9,43,83,85,97) = 3323$ while $g_B(3,4,8,8,9,43,83,85,97) = 3322$ is correct. Note that for this particular example $C_{\mathbf{a}} = 2,118,701,160$. It is in the 10^9 range where `sig.c` becomes unreliable with the Zagier option. An additional search for a 9-tuple with that particular oriented diffeomorphism type was performed so that `15spheresA.txt` actually contains the full set of 8128 examples. We replaced it with $\mathbf{a} = (6,6,6,6,6,10,25,59,73)$ which came out of $\mathcal{I}_2$. Note that here $C_{\mathbf{a}} = 646,050$ which is smaller by 3 orders of magnitude.

Realizing that one can do much better by a careful selection of candidates with low $C_{\mathbf{a}} = \mathrm{lcm}(a_1, \ldots, a_9)$ we used `candidates.c` to select more "efficient" input data set $\mathcal{I}_2$. As a result it was possible to obtain all 8128 distinct $g(\mathbf{a})$'s calculating with the Zagier option in only about 160 hours on the University of Melbourne facility. A very significant improvement indeed. That second calculation generated `15spheresB.txt`. The Brieskorn verification was performed on `15spheresB.txt` at the University of Melbourne facility and it took only 1700 hours. No errors were found in the Zagier calculation which is no surprise: a typical $C_{\mathbf{a}}$ for 9-tuples of $\mathcal{I}_2$ was about 3 orders of magnitude lower.

Note that one can easily improve the "least one" statement of Theorem 2 by repeating the same calculation with several disjoint input files $\mathcal{I}$. It is a simple exercise to do it for 7-tuples and much more time consuming in the case of 9-tuples. For 9-tuples we actually showed that there are at least two Sasakian-Einstein metrics on each homotopy sphere $\sigma_i^{15} \in bP_{16}$ as the lists `15spheresA.txt` and `15spheresB.txt` are disjoint.

On the other hand, to calculate signatures of all candidate 7-tuples and 9-tuples to get the statement similar to the one expressed in Theorem 3 would take thousands of years with the present technology.

Remark 8. It is clear that our approach breaks down for $(2n+1)$-tuples, where n is "large enough". What is exactly "large enough" depends on several factors. Our rough estimate indicates that assuming the same facilities and the same codes are used it would take about 100 years to do the same calculation for 19-spheres. No doubt the `sig.c` code can be improved to calculate faster. On the other hand an average C for 11-tuples will be at least 10^3 larger that in the 9-tuple case. In addition, $bP_{20} = 130,816$ is much bigger. Taking these two factors into account, a calculation for 19-spheres would take about 10^4 times longer that a similar calculation for 15-spheres.

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