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The Geometry and Topology of 3-Sasakian Manifolds

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In recent years quaternionic Kähler and hyperkähler manifolds have received a great deal of well-deserved attention. They appear, often unexpectedly yet naturally, in many different areas of mathematics and mathematical physics. This attention has resulted in the rapid development into the rich theory of quaternionic manifolds that exists today. It has even been argued that these recent advances in quaternionic geometry vindicate Hamilton's conviction that the algebra of quaternions should play an important role in mathematical physics [At, Hi1].

The purpose of this paper is to describe in detail the geometry and topology of a class of Riemannian Einstein manifolds that are closely related to both hyperkähler and quaternionic Kähler manifolds. These manifolds, known as manifolds with a Sasakian 3-structure, first appeared in a paper by Kuo in 1970 [Ku]. Incidentally, Kuo's paper appeared a few years before Ishihara [I1] and Calabi [Cal] introduced the now commonly accepted terms "quaternionic Kähler" and "hyperkähler", respectively, into the differential geometry vocabulary. We shall refer to manifolds with a Sasakian 3-structure as 3-Sasakian manifolds. Sasakian structures historically grew out of research in contact manifolds and were studied extensively in the 1960's especially by the Japanese school (See [YK] and references therein). In 1970 three more papers, [KuTach], [TachYu], and [Tan1], were published in the Japanese literature discussing 3-Sasakian structures. These structures were then vigorously studied by Japanese mathematicians from 1970-1975, culminating with an important paper of Konishi in 1975 [Kon] which shows the existence of a 3-Sasakian structure on a certain principal $SO(3)$ bundle over any quaternionic Kähler manifold of positive scalar curvature. Earlier on, in 1973 Ishihara [I2] had shown that if the distribution formed by the three Killing vector fields which define the Sasakian 3-structure is regular then the space of leaves is a quaternionic Kähler manifold. This then led Ishihara to his foundational work on quaternionic Kähler manifolds [I1]. It is notable that in this

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early period the only examples of 3-Sasakian manifolds that were given were those of constant curvature, namely the spheres S^{4k-1}, the real projective spaces $\mathbb{RP}^{4k-1}$, and spherical space forms in dimension three [Sas1]. Even though Konishi's result mentioned above combined with the earlier work of Wolf [Wo] on the classification of homogeneous quaternionic Kähler manifolds of positive scalar curvature give many new homogeneous examples, no further work on 3-Sasakian manifolds was done until very recently in [BGM1], and in [FK] for dimension 7.

Unlike the current intense interest in quaternionic Kähler and hyperkähler structures, Sasakian 3-structures appear to have been largely neglected in recent years. For example, in Besse's comprehensive book on Einstein manifolds [Bes], there is an entire chapter devoted to quaternionic Kähler and hyperkähler manifolds. By contrast, there is no explicit mention of 3-Sasakian manifolds which appear only implicitly and very briefly as examples of homogeneous Einstein spaces. For instance, Besse proves the existence of two Einstein metrics on a certain principal $SO(3)$ bundle Y over any quaternionic Kähler manifold with positive scalar curvature [Bes: Proposition 14.85]. The manifold Y with one of these two metrics is an example of a 3-Sasakian manifold, and the fibration is the one given in the lower right hand corner of diagram 0.1. Although this result was discovered by Konishi [Ko] almost 20 years ago, Besse did not seem to be aware of this fact.

We were led to study this geometry because it appears as a natural object in a new quotient construction for certain hyperkähler manifolds [BGM1]. We found that 3-Sasakian manifolds provided a natural piece of a puzzle that links together three other different geometric structures. In particular, for any quaternionic Kähler manifold M of positive scalar curvature there exists a commutative diagram

0.1
$$
\begin{tikzcd}
& \mathcal{U} \arrow[dl, "\mathbb{C}^*/\mathbb{Z}_2"'] \arrow[dd, "\mathbb{H}^*/\mathbb{Z}_2"] & \\
\mathcal{Z} \arrow[dr, "\mathbb{CP}^1"'] & & \mathcal{S} \arrow[ul, "\iota"'] \arrow[dl, "\mathbb{RP}^3"] \\
& M &
\end{tikzcd}
$$

where $\mathcal{U}$ is hyperkähler (the Swann bundle associated to M [Sw]), $\mathcal{Z}$ is Kähler-Einstein (the twistor space associated to M [Sal1]), and $\mathcal{S}$ is 3-Sasakian (the Konishi bundle associated to M [Kon]). The map $\iota : \mathcal{S} \hookrightarrow \mathcal{U}$ is the inclusion of a level set of a natural real valued function while all the other maps in diagram 0.1 are fibrations where we have denoted each map by its associated fiber. Furthermore, both $\mathcal{Z}$ and $\mathcal{S}$ are compact, of positive scalar curvature, and $\mathcal{S}$ is a principal circle bundle over $\mathcal{Z}$. It is important to realize that all four geometries in diagram 0.1 are Einstein. The important observation, due to Kashiwada [Ka], that 3-Sasakian manifolds are always Einstein spaces and the relationship of $\mathcal{S}$ to the other quaternionic geometries appearing in diagram 0.1 motivated our efforts to study and understand these spaces.

After investigating this intriguing geometry we have arrived at the compelling conclusion that 3-Sasakian manifolds are by no means less interesting than their hyperkähler or quaternionic Kähler counterparts. In fact, a case can be made that they are even richer and more interesting. As we have already mentioned there is always at least one 3-Sasakian manifold associated with every quaternionic Kähler manifold of positive scalar curvature. However, as we pointed out in [BGM1], 3-Sasakian manifolds are much more plentiful than quaternionic Kähler manifolds of positive scalar curvature. In all but five quaternionic dimensions there are only 3 explicitly known examples of compact quaternionic Kähler manifolds of positive scalar curvature (in dimension 1 there are only two such examples whereas in dimensions 7, 10, 16, and 28 there are four). Moreover, all of these examples are symmetric spaces and can be found in Wolf's classification [Wo]. It is also known that in quaternionic dimensions 1 and 2 there are no others [Hi2], [PoSal]. By contrast, a 3-Sasakian manifold must be of real dimension $4k+3$ and in each such allowable dimension we were surprised to find infinitely many examples of compact 3-Sasakian manifolds. In addition, as we shall show below, these examples range through infinitely many distinct homotopy types in every dimension. Moreover, in dimension 7, we found countable families of *strongly inhomogeneous* 3-Sasakian manifolds, that is manifolds which are not homotopy equivalent to any compact Riemannian homogeneous space. These examples, which are discussed in detail in the later part of this paper, are, to the best of our knowledge, the only examples of complete inhomogeneous Einstein manifolds of positive scalar curvature. Most importantly, as we explain below, in order to recapture the close relationship between these new 3-Sasakian examples and quaternionic Kähler geometry, one must generalize diagram 0.1 to allow the base space M to be a quaternionic Kähler *orbifold*.

We now explain how this paper is organized and highlight some of our main results. In section one we introduce some basic facts about Riemannian foliations and Riemannian orbifolds which are needed to generalize diagram 0.1 and other results to the orbifold category. Section two begins with the definition of a Sasakian 3-structure on a Riemannian manifold and continues with a brief discussion of some classical results for such structures. In particular, we recall both Ishihara's [I2] and Konishi's [Kon] constructions, as well as Kashiwada's [Ka] observation that every 3-Sasakian manifold is necessarily Einstein. At this point we make a fundamental extension of these known results to the case of orbifold fibrations by proving the following theorem.

Theorem A: *Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian manifold of dimension $4n+3$ such that the Killing vector fields ξ^a are complete for $a = 1, 2, 3$. Then*

(i) $(\mathcal{S}, g, \xi^a)$ is an Einstein manifold of positive scalar curvature equal to $2(2n+1)(4n+3)$.

(ii) The metric g is bundle-like with respect to the foliation $\mathcal{F}$, defined by $\{\xi^a\}_{a=1,2,3}$.

(iii) Each leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ is a 3-dimensional homogeneous spherical space form.

(iv) The space of leaves $\mathcal{S}/\mathcal{F}$ is a quaternionic Kähler orbifold of dimension $4n$ with positive scalar curvature equal to $16n(n+2)$.

Hence, every complete 3-Sasakian manifold is compact with finite fundamental group and diameter less than or equal to π.

Thus, we show both that every complete 3-Sasakian manifold is necessarily of positive scalar curvature and that it must also fiber, in the orbifold sense, over a compact quaternionic Kähler orbifold. This observation, first hinted at in [BGM1], turns out to be crucial, both in terms of describing the geometry of such spaces and constructing non-trivial examples.

In section three we prove the converse of the result presented in [BGM1], where we showed that 3-Sasakian manifolds occur naturally as level sets of the hyperkähler potential function ν on a certain hyperkähler manifold. Here we show that every 3-Sasakian manifold can be embedded in a hyperkähler manifold as the level set of such a hyperkähler potential. More precisely we prove:

Theorem B: *Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a complete 3-Sasakian manifold. Then the product manifold $M = \mathcal{S} \times \mathbb{R}^+$ with the cone metric $g_M = dr^2 + r^2 g_{\mathcal{S}}$ is hyperkähler so that there is a commutative diagram of (orbifold) fibrations*

0.2

$$\begin{tikzcd}
& \mathcal{S} \times \mathbb{R}^+ \arrow[dl] \arrow[dd] \arrow[dr] & \\
\mathcal{Z} \arrow[dr] & & \mathcal{S} \arrow[dl] \\
& \mathcal{O} &
\end{tikzcd}$$

where $\mathcal{O}$ is a quaternionic Kähler orbifold.

Thus, Theorem B shows that every 3-Sasakian manifold comes from the hyperkähler quotient construction given in [BGM1]. Moreover, Theorem B and our results of later sections can be used to give many new examples of compact hypercomplex manifolds. This is Corollary 3.11.

In section four we classify all 3-Sasakian homogeneous spaces, that is 3-Sasakian manifolds with transitive action of the group of automorphisms of the Sasakian 3-structure. Combining Wolf's [Wo] classification with results of Ishihara [I2], Tanno [Tan1], and Theorem A we prove the following theorem.

Theorem C: *Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian homogeneous space. Then $\mathcal{S}$ is precisely one of the following homogeneous spaces:*

$$\frac{Sp(n)}{Sp(n-1)} \simeq S^{4n-1}, \qquad \frac{Sp(n)}{Sp(n-1)\times \mathbb{Z}_2} \simeq \mathbb{RP}^{4n-1},$$

$$\frac{SU(m)}{S\big(U(m-2)\times U(1)\big)}, \qquad \frac{SO(k)}{SO(k-4)\times Sp(1)},$$

$$\frac{G_2}{Sp(1)}, \qquad \frac{F_4}{Sp(3)}, \qquad \frac{E_6}{SU(6)}, \qquad \frac{E_7}{\mathrm{Spin}(12)}, \qquad \frac{E_8}{E_7}.$$

Here $n \geq 1$, $Sp(0)$ denotes the identity group, $m \geq 3$, and $k \geq 7$. Furthermore, the fiber F over the Wolf space is $Sp(1)$ in only one case which occurs precisely when $(\mathcal{S}, g, \xi^a)$ is simply connected with constant curvature; that is, when $\mathcal{S} = S^{4n-1}$. In all other cases $F = SO(3)$.

The classification of 3-Sasakian homogeneous spaces is an expected consequence of the combined work of Wolf and Ishihara. However, Theorem A provides the key ingredient in the proof. The metrics on all these cosets spaces are Einstein and they were considered in this context in Besse. However, with the exception of the constant curvature case, these are not the normal homogeneous metrics, as they are not naturally reductive and thus are not obtained from the bi-invariant metric on G by Riemannian submersion.

The key technique for constructing new examples of 3-Sasakian manifolds is the reduction procedure described in section five. Explicitly, we prove

THEOREM D: *Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a 3-Sasakian manifold with a compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries. Let $\mu_{\mathcal{S}}$ be the corresponding 3-Sasakian moment map and assume both that 0 is a regular value of $\mu_{\mathcal{S}}$ and that G acts freely on the submanifold $\mu_{\mathcal{S}}^{-1}(0)$. Furthermore, let*

$$\iota : \mu_{\mathcal{S}}^{-1}(0) \longrightarrow \mathcal{S}$$

and

$$\pi : \mu_{\mathcal{S}}^{-1}(0) \longrightarrow \mu_{\mathcal{S}}^{-1}(0)/G$$

denote the corresponding embedding and submersion. Then

$$(\check{\mathcal{S}} = \mu_{\mathcal{S}}^{-1}(0)/G, \check{g}_{\mathcal{S}}, \check{\xi}^a)$$

is a smooth 3-Sasakian manifold of dimension $4(n - \dim \mathfrak{g}) - 1$ with metric $\check{g}_{\mathcal{S}}$ and Sasakian vector fields $\check{\xi}^a$ determined uniquely by the two conditions

$$\iota^* g_{\mathcal{S}} \;=\; \pi^* \check{g}_{\mathcal{S}}$$

and

$$\pi_*(\xi^a \,|\, \mu_{\mathcal{S}}^{-1}(0)) \;=\; \check{\xi}^a.$$

Theorem D is then used in the next two sections to obtain explicit new families of 3-Sasakian manifolds. First, in section six we give an explicit description of the Riemannian metic for the 3-Sasakian structure on the coset spaces

$$\frac{U(n)}{U(n-2)\times U(1)} \quad \text{and} \quad \frac{SO(n)}{SO(n-4)\times Sp(1)}.$$

These spaces are obtained from Theorem D as 3-Sasakian quotients of the unit sphere S^{4n-1} with its canonical round metric by actions of $U(1) \subset Sp(n)$ and $Sp(1) \subset Sp(n)$, respectively. Here the group $Sp(n)$ is the automorphism group of the 3-Sasakian structure on S^{4n-1}. Moreover, this detailed discussion for $\dfrac{U(n)}{U(n-2)\times U(1)}$ sets the notation for an important deformation construction given in section seven. We do not know of any such explicit description of the remaining five exceptional examples given at the end of Theorem C, but in this case the Konishi bundle construction gives the existence of the 3-Sasakian metric on G/L_1.

Next, in section seven we present a deformation theory associated to the 3-Sasakian homogeneous manifold $\frac{U(n)}{U(n-2)\times U(1)}$ which we use to generate our most important families of inhomogeneous 3-Sasakian manifolds in every allowable dimension. The point is that, in the smooth category under the assumption of positive scalar curvature, the quaternionic Kähler quotient construction given in [GL] is extremely rigid. The only know quotients which can occur as the base space in diagram 0.1 are the quotients of the quaternionic projective spaces $\mathbb{HP}^{n-1}$ by $U(1)$ and $Sp(1)$. These two quotients produce the complex and real Wolf spaces $X(n-2)$ and $Y(n-4)$, respectively. They lift to the quotients of the Konishi bundle, giving the two 3-Sasakian quotients of S^{4n-1} mentioned above and discussed in section six. However, these two examples are not the only quotients of S^{4n-1} that yield complete 3-Sasakian manifolds. The $U(1)$-quotient at this level actually has infinitely many discrete deformations of the standard homogeneous example. Every deformation produces a smooth 3-Sasakian manifold. All these quotients project to orbifold quotients considered in [GL]. This is why the generalization from diagram 0.1 to diagram 0.2 given in [BGM1] is so important. In particular, we prove

THEOREM E: *Let $\mathbf{p}=(p_1,\ldots,p_n)\in\mathbb{Z}^n_+$ be an n-tuple of pairwise relatively prime positive integers. Let $\mathcal{S}(\mathbf{p})$ be the quotient of the complex Stiefel manifold $V^{\mathbb{C}}_{n,2}$ by a free circle action, or equivalently, the left-right quotient of the unitary group $U(n)$ by $H\subset G^2=G_L\times G_R$, where $H=U(1)\times U(n-2)$ and the action is given by the formula*

$$\mathbb{W}\xrightarrow{(\tau,\mathbb{B})}\begin{pmatrix}\tau^{p_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \tau^{p_n}\end{pmatrix}\mathbb{W}\begin{pmatrix}\mathbb{I}_2 & \mathbb{O}\\ \mathbb{O} & \mathbb{B}\end{pmatrix}.$$

Here $\mathbb{W}\in U(n)$ and $(\tau,\mathbb{B})\in U(1)\times U(n-2)$. Then $\mathcal{S}(\mathbf{p})$ is a compact, simply connected $(4n-5)$-dimensional 3-Sasakian manifold whose integral cohomology ring $H^*\big(\mathcal{S}(\mathbf{p}),\mathbb{Z}\big)$ is generated by two classes

$$b_2\in H^2\big(\mathcal{S}(\mathbf{p}),\mathbb{Z}\big)\quad\text{and}\quad f_{2n-1}\in H^{2n-1}\big(\mathcal{S}(\mathbf{p}),\mathbb{Z}\big)$$

which satisfy the following relations:

$$\sigma_{n-1}(\mathbf{p})b_2^{n-1}=0,\quad b_2^n=0,\quad f_{2n-1}^2=0,\quad f_{2n-1}b_2^{n-1}=0.$$

Here $\sigma_{n-1}(\mathbf{p})=\sum_{j=1}^n p_1\cdots\hat{p_j}\cdots p_n$ is the $(n-1)^{st}$ elementary symmetric polynomial in $\mathbf{p}$.

The computation of the integral cohomology ring of $\mathcal{S}(\mathbf{p})$, which is based on techniques developed by Eschenburg [Esch], is presented in section eight. Notice that Theorem D immediately implies that in every dimension of the form $(4k-5)$ for $k\geq 3$ there are infinitely many distinct homotopy types of complete 3-Sasakian manifolds. Theorem C implies that all of them, except in the case when $\mathbf{p}=(1,\ldots,1)$, are inhomogeneous Einstein manifolds. Moreover, it turns out that some of our examples are not even homotopy equivalent to any compact homogeneous Riemannian space. We prove this result in section nine, where we consider the case of $n=3$ in more detail. At this point we rewrite the 3-Sasakian 7-manifold $\mathcal{S}(p_1,p_2,p_3)$ as a left-right quotient of $SU(3)$ by a free circle action. We also describe the relation of our spaces $\mathcal{S}(p_1,p_2,p_3)$ to the bi-quotients of $SU(3)$ considered by Eschenburg in [Esch]. Most importantly, combining a result of Eschenburg and Theorem E we have the following theorem.

THEOREM F: *If $\sigma_2(\mathbf{p}) = p_1p_2+p_2p_3+p_3p_1 \equiv 2 \pmod 3$ then $\mathcal{S}(p_1,p_2,p_3)$ is strongly inhomogeneous; that is, it is not homotopy equivalent to any compact Riemannian homogeneous space.*

In particular, for any odd $c>0$, the manifold $\mathcal{S}(c,c+1,c+2)$ satisfies the condition that $\sigma_2(\mathbf{p}) \equiv 2 \pmod 3$. Thus, there exists a countable family of strongly inhomogeneous Einstein spaces of positive scalar curvature. To our knowledge, this family (and many other similar $\mathcal{S}(p_1,p_2,p_3)$ families) are the only known examples of strongly inhomogeneous compact Einstein manifolds of positive scalar curvature. Furthermore, we show that as c tends to infinity, $\mathcal{S}(c,c+1,c+2)$ approaches $SU(3)/U(1)$ with its homogeneous Sasakian 3-structure in the Cheeger ρ^*-topology [Ch1,Ch2]. This, on the other hand, implies that, for large c, our 3-Sasakian manifolds $\mathcal{S}(c,c+1,c+2)$ admit metrics of positive sectional curvature. Finally, in the last section we briefly discuss some open problems arising from our investigations.

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§1. Orbifolds and Riemannian Foliations.

In this section we review some important properties of both orbifolds and related Riemannian foliations. Roughly speaking orbifolds are like differentiable manifolds except that instead of being modelled on $\mathbb{R}^n$ they locally look like $\mathbb{R}^n/\Gamma$, where Γ is a discrete group of diffeomorphisms of $\mathbb{R}^n$. This idea was first introduced by Satake [Sat] and he called these spaces V-manifolds. They also became known as Satake manifolds or orbifolds. We will use the term orbifold which has gained recent acceptance in the literature. Orbifolds appear naturally as the space of leaves of certain nicely behaved Riemannian foliations. In this paper we will not be concerned with the most general type of orbifold nor the most general type of foliation, but rather only with those orbifolds $\mathcal{O}$ that arise as the quotient space of a locally free action of a compact Lie group G on a smooth manifold M (in our case G will be either $SU(2)$ or $SO(3)$). In this case the fundamental vector fields of the action of G on M define a foliation $\mathcal{F}$ on M and the space of leaves $M/\mathcal{F}$ has the structure of an orbifold. Thus, the smooth manifold M can be viewed as a desingularization of the orbifold $M/\mathcal{F}$. More generally, the leaf space $M/\mathcal{F}$ of any Riemannian foliation $(M,\mathcal{F},g)$ with compact leaves is an orbifold. Satake's original article [Sat] is a good reference for the theory of orbifolds and the books of Molino [Mo] and Reinhart [Rei] are good references for the theory of Riemannian foliations.

Following Satake [Sat] and Molino [Mo] we define orbifolds and smooth maps between them. Let $\mathcal{O}$ be a second countable Hausdorff space, and $U \subset \mathcal{O}$ an open set. A *local uniformizing system* (l.u.s.) for U is a triple $\{\tilde{U}, \Gamma, \pi\}$ where $\tilde{U} \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, Γ is a finite group of diffeomorphisms on $\tilde{U}$, and $\pi : \tilde{U} \to U$ is a continuous map satisfying

(i) $\pi \circ \sigma = \pi$ for all $\sigma \in \Gamma$

(ii) π induces a homeomorphism $\phi : U \to \tilde{U}/\Gamma$.

The pair (U, ϕ) is called a local *chart* of $\mathcal{O}$.

In particular, let $\mathcal{O} = \tilde{U}/\Gamma$ and $V \subset \tilde{U}/\Gamma$ any open set. Then $\{\tilde{V}, \Gamma, \pi\}$ is a local uniformizing system for V with $\tilde{V} = \{p \in \tilde{U} \mid \tilde{\pi}(p) \in V\}$ and $\pi : \tilde{V} \to V$ equal to the restriction to $\tilde{V}$ of the natural projection $\tilde{\pi} : \tilde{U} \to \tilde{U}/\Gamma$. In this case the local chart ϕ is the restriction of the identity map. Now consider open sets $\tilde{U} \subset \mathbb{R}^n$ and $\tilde{U}' \subset \mathbb{R}^m$ together with finite groups of diffeomorphisms Γ and Γ' acting on $\tilde{U}$ and $\tilde{U}'$, respectively. We say that a continuous map $f : \tilde{U}/\Gamma \to \tilde{U}'/\Gamma'$ is *smooth* if f lifts locally to a smooth map, that is, for every point $p \in U/\Gamma$ there are neighborhoods $V \subset \tilde{U}/\Gamma$ of p and $V' \subset \tilde{U}'/\Gamma'$ of $f(p)$, local uniformizing systems $\{\tilde{U}, \Gamma, \pi\}$ and $\{\tilde{U}', \Gamma', \pi'\}$ for V and V', respectively, and a smooth map $\tilde{f} : \tilde{V} \to \tilde{V}'$ such that the diagram

$$
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\tilde{f}} & \tilde{V}' \\
\Big\downarrow \pi & & \Big\downarrow \pi' \\
V & \xrightarrow{f} & V'
\end{array}
\tag{1.1}
$$

commutes. The *rank* of the smooth map f is defined to be the rank of $\tilde{f}$. The notions of orbifold immersions, submersions, diffeomorphisms, etc., are defined in a similar manner.

Notice that a smooth map $f : \tilde{U}/\Gamma \to \tilde{U}'/\Gamma'$ defines a group homomorphism $\Gamma \to \Gamma'$ as follows: Let $\sigma \in \Gamma$ then by the commutativity of diagram 1.1 $\tilde{f}(\sigma(p))$ lies in the fiber $\pi'^{-1}(f(\pi(p)) \simeq \Gamma'$. Hence, there is a unique $\sigma' \in \Gamma'$ such that $\tilde{f} \circ \sigma = \sigma' \circ \tilde{f}$. One easily checks that the map $\sigma \mapsto \sigma'$ is a homomorphism. If f is a diffeomorphism onto its image $f(\tilde{U}/\Gamma) \subset \tilde{U}'/\Gamma$, then the map $\sigma \mapsto \sigma'$ is a group monomorphism. In particular, if $\tilde{f}$ covers the identity map $f = id$ then Satake [Sat] calls $\tilde{f}$ an *injection*.

Definition 1.2: *Let $\mathcal{O}$ be a second countable Hausdorff space. A smooth orbifold atlas for $\mathcal{O}$ is a cover $\{U_i\}$ of $\mathcal{O}$ by open sets U_i with a local uniformizing system $\{\tilde{U}_i, \Gamma_i, \pi_i\}$ for each U_i such that the homeomorphisms ϕ_i satisfy the condition that*

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

is a diffeomorphism for each i, j with $U_i \cap U_j \neq \emptyset$. Then $\mathcal{O}$ together with a maximal orbifold atlas is called an orbifold.

A point of $p \in \mathcal{O}$ is *regular* if p has a neighborhood U that has a local uniformizing system with $\Gamma = \mathrm{id}$. Otherwise p is called *singular*. The regular points of $\mathcal{O}$ form a dense open set.

Remark 1.3: Satake's original definition included the requirement that the fixed point set of any of the finite groups Γ_i be of codimension at least 2. This restrictive definition is not theoretically convenient at this point. For example, Molino's theorem stated below (see 1.8) does not hold for Satake's less general definition of orbifold. However, the orbifolds that we construct in this paper do have singular sets of codimension greater than or equal two.

Satake then generalizes such standard notions of differential geometry as bundles, differential forms, Riemannian metrics, etc., to the orbifold category (see [Sat] for further details). As mentioned above our interest in orbifolds stems from the fact that they occur naturally as the leaf space of certain Riemannian foliations.

Recall that a foliation $\mathcal{F}$ on a manifold M is given by an integrable subbundle $\mathcal{V}$ of the tangent bundle TM, that is, a subbundle whose smooth sections form a Lie subalgebra of the Lie algebra of smooth vector fields on M. There is an exact sequence of vector bundles

1.4 $$0 \longrightarrow \mathcal{V} \longrightarrow TM \longrightarrow TM/\mathcal{V} \longrightarrow 0.$$

A Riemannian metric g on M splits this exact sequence to yield the Whitney sum

$$TM = \mathcal{V} \oplus \mathcal{H}$$

defining the *horizontal* subbundle $\mathcal{H}$.. The integrable subbundle $\mathcal{V}$ is called the *vertical* subbundle. In general, and in our case in particular, the horizontal subbundle $\mathcal{H}$ is not integrable. A Riemannian manifold (M, g) together with a foliation $\mathcal{F}$ on M is called a *foliated Riemannian manifold* and is denoted by $(M, g, \mathcal{F})$. The following definition is due to Reinhardt.

Definition 1.5: [Rei2] *Let $(M, \mathcal{F}, g)$ be a foliated Riemannian manifold. The metric g is said to be bundle-like if for any horizontal vector fields X, Y in the normalizer of $\mathcal{V}$ under the Lie bracket, the equation*

$$Vg(X, Y) = 0$$

holds for any vertical vector field V.

Remarks 1.6:

1. Vector fields belonging to the normalizer of $\mathcal{V}$ are often called *foliate* [Mo].
2. Functions that are annihilated by all vertical vector fields, as in Definition 1.5 above, are known as *basic* [Mo].
3. Definition 1.5 is equivalent to the condition that the horizontal distribution $\mathcal{H}$ be totally geodesic (see [Mo] or [Rei1] for details).

We shall make use of the following lemma:

Lemma 1.7: *Let $(M, g, \mathcal{F})$ be a foliated Riemannian manifold, and suppose that the vertical distribution $\mathcal{V}$ is spanned by Killing vector fields. Then g is bundle-like.*

Proof: It is enough to show that the condition in definition 1.5 holds when V is any Killing vector field. Let X, Y be horizontal vector fields on M belonging to the normalizer of $\mathcal{V}$, then we have

$$Vg(X, Y) = (\mathcal{L}_V g)(X, Y) + g([V, X], Y) + g(X, [V, Y]) = 0$$

where $\mathcal{L}_V$ denotes the Lie derivative with respect to V. The first term vanishes since V is a Killing vector field. The two remaining terms vanish since X and Y are horizontal and the terms in brackets are vertical. ∎

The following result, which is given in Molino, is fundamental to our work:

THEOREM 1.8: [Mo: Proposition 3.7] *Let $(M, \mathcal{F}, g)$ be a Riemannian foliation of codimension q with compact leaves and bundle-like metric g. Then the space of leaves $M/\mathcal{F}$ admits the structure of a q dimensional orbifold such that the natural projection $\pi : M \longrightarrow M/\mathcal{F}$ is an orbifold submersion.*

Another important concept in the theory of foliations is that of the leaf holonomy group. This group is a certain image of the fundamental group of a leaf in the local group of germs of diffeomorphisms of a transverse submanifold to the leaf. It measures how transversals change as one moves along a loop in the leaf. In our situation we have the following:

PROPOSITION 1.9: *Let $(M, \mathcal{F}, g)$ be a Riemannian foliation with compact leaves and bundle like metric as in theorem 1.8. The dense open set $(M/\mathcal{F})^o$ of regular points of $M/\mathcal{F}$ is precisely the set of leaves with trivial holonomy and there is a unique Riemannian metric $\breve{g}$ on $(M/\mathcal{F})^o$ such that the natural projection $\pi : M \longrightarrow M/\mathcal{F}$ restricts to a locally trivial Riemannian submersion on $\pi^{-1}((M/\mathcal{F})^o)$. At a singular point of $M/\mathcal{F}$ the finite group Γ is precisely the holonomy of the leaf. In particular, if there are no singular points, the projection $\pi : M \longrightarrow M/\mathcal{F}$ is a locally trivial Riemannian fibration.*

PROOF: Except for the local triviality statement, this follows from Molino [Mo: §3.6]. The local triviality is a consequence of the Ehresmann fibration theorem. ∎

Proposition 1.9 allows one to talk about the quotient $M/calf$ as a Riemmanian orbifold. The metric $\breve{g}$ is defined only on the dense open set $(M/\mathcal{F})^o$; however, the transverse part g_T of the metric g is a metic on the horizontal distribution $\mathcal{H}$ which satisfies

$$\pi^*\breve{g} \;=\; g_T$$

on the points of $\pi^{-1}((M/\mathcal{F})^o)$. Thus, g_T can be interpreted as describing the metric on the whole orbifold including its singular locus. Accordingly all of O'Neill's standard formulae for Riemannian submersions hold for these more general orbifold Riemannian submersions (see [Rei1: page 160]). Hence, we shall freely apply these well-known formulae, given for example in Besse [Bes], to this more general setting of orbifold Riemannian submersions.

§2. Some Old and New Results on 3-Sasakian Manifolds

In this section we review some known results about Riemanian manifolds which admit 3-Sasakian structures and then give an important generalization. Following Ishihara and Konishi [IKon], we begin by recalling the definitions of Sasakian and 3-Sasakian structures on a Riemannian manifold.

DEFINITION 2.1: *Let $(\mathcal{S}, g)$ be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. Then $(\mathcal{S}, g)$ has a Sasakian structure if there exists a Killing vector field ξ of unit length on $\mathcal{S}$ so that the tensor field Φ of type $(1,1)$, defined by*

$$(i) \qquad \Phi \;=\; \nabla\xi$$

satisfies the condition

$$(ii) \qquad (\nabla_X \Phi)(Y) \;=\; \eta(Y)X - g(X,Y)\xi,$$

for any pair of vector fields X and Y on $\mathcal{S}$. Here η denotes the 1-form dual to ξ with respect to g, i.e. $g(Y,\xi) = \eta(Y)$ for any vector field Y, and satifies an equation dual to (i), namely,

$$(iii) \qquad (\nabla_X \eta)(Y) \;=\; g(Y, \Phi X).$$

We write (Φ, ξ, η) to denote the specific Sasakian structure on $(\mathcal{S}, g)$ and sometimes refer to $\mathcal{S}$ with such a structure as a Sasakian manifold.

It is straightforward to verify that the following equations hold.

PROPOSITION 2.2: *Let $(\mathcal{S}, g, \xi)$ be a Sasakian manifold and X and Y any pair of vector fields on $\mathcal{S}$. Then*

$$(i) \qquad \Phi \circ \Phi(Y) \;=\; -Y + \eta(Y)\xi,$$

$$(ii) \qquad \Phi\xi \;=\; 0,$$

$$(iii) \qquad \eta(\Phi Y) \;=\; 0,$$

$$(iv) \qquad g(X, \Phi Y) + g(\Phi X, Y) \;=\; 0,$$

$$(v) \qquad g(\Phi Y, \Phi Z) \;=\; g(Y,Z) - \eta(Y)\eta(Z),$$

$$(vi) \qquad d\eta(Y,Z) \;=\; 2g(\Phi Y, Z).$$

Furthermore, the Nijenhuis torsion tensor

$$N_\Phi(Y,Z) \;=\; [\Phi Y, \Phi Z] + \Phi^2[Y,Z] - \Phi[Y, \Phi Z] - \Phi[\Phi Y, Z],$$

of Φ satisfies

$$(vii) \qquad N_\Phi(Y,Z) \;=\; d\eta(Y,Z) \otimes \xi.$$

We now define our main objects of interest.

DEFINITION 2.3: *Let $(\mathcal{S}, g)$ be a Riemannian manifold that admits three distinct Sasakian structures $\{\Phi^a, \xi^a, \eta^a\}_{a=1,2,3}$ such that*

$$(i) \qquad g(\xi^a, \xi^b) \;=\; \delta^{ab}$$

and

$$(ii) \qquad [\xi^a, \xi^b] = 2\epsilon^{abc}\xi^c$$

for $a, b, c = 1, 2, 3$. *Then* $(\mathcal{S}, g)$ *is a* 3*-Sasakian manifold with Sasakian* 3*-structure* $(\mathcal{S}, g, \xi^a)$.

It follows directly from the definition that every 3-Sasakian manifold admits a local action of either $Sp(1)$ or $SO(3)$ as local isometries, and if the vector fields ξ^a are complete these are global isometries. We refer to this action as the *standard* $Sp(1)$ action on $\mathcal{S}$. In the remainder of this paper we shall assume that the vector fields ξ^a are complete. This structure has several important implications. First, it is not difficult to verify that the following relations between the Sasakian structures hold:

2.4
$$\begin{aligned} \eta^a(\xi^b) &= \delta^{ab}, \\ \Phi^a\xi^b &= -\epsilon^{abc}\xi^c, \\ \Phi^a \circ \Phi^b - \xi^a \otimes \eta^b &= -\epsilon^{abc}\Phi^c - \delta^{ab}\mathrm{id}. \end{aligned}$$

The following result is well-known:

THEOREM 2.5: *Every* 3*-Sasakian manifold* $(\mathcal{S}, g, \xi^a)$ *has dimension* $4n+3$ *and defines a Riemannian foliation* $(\mathcal{S}, \mathcal{F})$ *of codimension* $4n$ *with totally geodesic leaves of constant curvature* 1. *Furthermore* $(\mathcal{S}, g, \xi^a)$ *is an Einstein manifold.*

The first result is due to Kuo and Tachibana [KuTach] and the important observation that every 3-Sasakian manifold is Einstein is due to Kashiwada [Ka]. For more general almost contact 3-structures Kuo [Ku] has proven:

THEOREM 2.6: [Kuo] *The structure group of any manifold with an almost contact* 3*-structure is reducible to* $Sp(n) \times \mathbb{I}_3$ *where* $\mathbb{I}_3$ *denotes the three by three identity matrix.*

Kuo's theorem has an important corollary:

COROLLARY 2.7: *Every* 3*-Sasakian manifold is a spin manifold.*

We shall also deduce Corollary 2.7 by using the embedding techniques given in the next section.

In addition, using harmonic theory on compact Sasakian manifolds, Kuo [Ku] has also shown that on a compact $4n+3$-dimensional 3-Sasakian manifold $\mathcal{S}$ the i^{th} Betti number, $b_i(\mathcal{S})$, must be of the form $4q$ whenever i is odd and $i < 2n+2$.

Much of the previous work on 3-Sasakian manifolds has concentrated on the regular case when $(\mathcal{S}, g, \xi^a)$ is the total space of a Riemannian submersion [IKon,I2]. The following is a theorem of Ishihara [I2].

THEOREM 2.8: [I2] *Let* $(\mathcal{S}, g, \xi^a)$ *be a* 3*-Sasakian manifold such that the space of leaves* $\mathcal{S}/\mathcal{F}$ *is a Riemannian manifold and the natural projection*

$$\pi : (\mathcal{S}, g) \longrightarrow (\mathcal{S}/\mathcal{F}, \breve{g})$$

is a Riemannian submersion. Then $(\mathcal{S}/\mathcal{F}, \breve{g})$ *is a quaternionic Kähler manifold.*

A converse of this theorem was obtained by Konishi [Kon].

THEOREM 2.9: [Kon] *Let $(M, \breve{g})$ be a quaternionic Kähler manifold with positive scalar curvature. Then there is a principal $SO(3)$ bundle $\mathcal{S}$ over M whose total space admits a metric g with an associated 3-Sasakian structure.*

Konishi also considers the case when the quaternionic Kähler manifold has negative scalar curvature. This gives a Sasakian 3-structure on $\mathcal{S}$ with indefinite signature $(3, 4n)$. We shall not consider this case. In the positive scalar curvature case, there is an obstruction to lifting the $SO(3)$ bundle to an $SU(2)$ bundle. This obstruction is the Marchiafava-Romani class ϵ [MR] of the quaternionic Kähler manifold M, and a result of Salamon [Sal1] says that if the quaternionic Kähler manifold is complete with positive scalar curvature then ϵ vanishes if and only if $M = \mathbb{HP}^n$. This result does not hold in the case of quaternionic Kähler orbifolds, nor if the completeness assumption is dropped.

We generalize these results as follows:

THEOREM 2.10: *Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian manifold of dimension $4n+3$ such that the Killing vector fields ξ^a are complete for $a = 1, 2, 3$. Then*

(i) $(\mathcal{S}, g, \xi^a)$ is an Einstein manifold of positive scalar curvature equal to $2(2n+1)(4n+3)$.

(ii) The metric g is bundle-like with respect to the foliation $\mathcal{F}$.

(iii) Each leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ is 3-dimensional homogeneous spherical space form.

(iv) The space of leaves $\mathcal{S}/\mathcal{F}$ is a quaternionic Kähler orbifold of dimension $4n$ with positive scalar curvature equal to $16n(n+2)$.

Hence, every complete 3-Sasakian manifold is compact with finite fundamental group and diameter less than or equal to π.

PROOF: The last statement is a direct consequence of the first statement and Myers' theorem [Mi]. Next we prove the first statement. Since the vector fields ξ^a are Killing vector fields, the metric g is bundle-like by lemma 1.7. Furthermore, since these vector fields are complete the foliation $\mathcal{F}$ has compact leaves. So, by Molino's theorem 1.8, the space of leaves $\mathcal{S}/\mathcal{F}$ is an orbifold $\mathcal{O}$ of dimension $4n$, and by proposition 1.9 the natural projection $\pi : \mathcal{S} \to \mathcal{S}/\mathcal{F}$ is a Riemannian submersion on the dense open set of regular leaves. The fact that the space of leaves $\mathcal{S}/\mathcal{F}$ has a quaternionic Kähler structure follows from Ishihara's theorem 2.8 applied to the dense open set of leaves. Now the dimension of each leaf is three, and all leaves are totally geodesic of constant curvature 1 by 2.5. In particular, O'Neill's tensor field T vanishes, and so for each $a = 1, 2, 3$ the tensor field Φ^a restricted to any leaf $\mathcal{L}$ defines a Sasakian structure there. Hence, $(g, \Phi^a, \xi^a)_{a=1,2,3}$ restricted to $\mathcal{L}$ makes $\mathcal{L}$ a 3-Sasakian manifold of dimension 3. These were classified by Sasaki [Sas1], and it follows that each leaf is a 3-dimensional homogeneous spherical space form with scalar curvature 6. This proves everything except for the last statement about the scalar curvature.

To compute the scalar curvature we first determine the Einstein constant λ of g. Using proposition 1.9 we can apply [Bes: 9.62] to the dense open set of regular points to give

$$\lambda \;=\; \frac{1}{3}(6+\mid A\mid^2), \tag{2.11}$$

where A is O'Neill's tensor. In particular, λ is positive and it remains to compute the O'Neill tensor A to determine the scalar curvature explicitly. Again proposition 1.9 shows that $\pi : \mathcal{S}\longrightarrow\mathcal{O}$ is a locally trivial orbifold bundle. The generic fibres have the form $SU(2)/\Gamma$ where Γ is a discrete subgroup of $SU(2)$. In the case that $\Gamma = \mathrm{id}$ or $\mathbb{Z}_2$, $\mathcal{S}$ is a principal orbifold bundle with group $SU(2)$ or $SO(3)$, respectively. Otherwise $\mathcal{S}$ is an associated bundle. In either case we show that the three 1-forms η^a for $a = 1,2,3$ define the components of a connection in the orbifold bundle $\mathcal{S}$, and that we can compute the tensor field A from the curvature of this connection. We have

LEMMA 2.12: *The three 1-forms η^a with $a = 1,2,3$ are the components of a connection 1-form in the orbifold bundle $\mathcal{S}$.*

PROOF: Choosing a basis e_a where $a = 1,2,3$ for the Lie algebra $\mathfrak{sp}(1)$ and defining $\eta = \eta^a e_a$ we obtain a Lie algebra valued 1-form which annihilates the distribution $\mathcal{H}$ orthogonal to $\mathcal{V}$ with respect to the metric g. To see that the horizontal distribution $\mathcal{H}$ is equivariant and thus defines a connection on $\mathcal{S}$ with connection form η, we use Proposition 2.2 (vi) to compute

$$\begin{aligned} 0 \;&=\; 2g(\Phi^a\xi^b, X) \;=\; d\eta^a(\xi^b, X) \\ &=\; \xi^b\eta^a(X) - X\eta^a(\xi^b) - \eta^a([\xi^b, X]) \\ &=\; \eta^a([\xi^b, X]). \end{aligned}$$

Here X is a horizontal vector field and $1 \leq a, b \leq 3$. But this implies that $[\xi^a, X]$ is horizontal for all $a = 1,2,3$. ∎

LEMMA 2.13: *For any pair of horizontal vector fields X and Y on $\mathcal{S}$*

(i) $A_X Y \;=\; \sum_{a=1}^3 g(\Phi^a X, Y)\xi^a.$

(ii) $A_X \xi^a \;=\; \Phi^a X.$

PROOF: (i) follows from Proposition 2.2 (vi) and the fact that if Ω is the curvature two form of the principal connection η, then [Bes: 9.54] shows that

$$A_X Y \;=\; -\frac{1}{2}\theta^{-1}\Omega(X,Y),$$

where $\theta : \mathcal{V}\longrightarrow\mathfrak{sp}(1)$ denotes the isomorphism between the vertical vector space $\mathcal{V}$ at a point of $\mathcal{S}$ and the Lie algebra $\mathfrak{sp}(1)$. (ii) now follows from equation (i) and [Bes: 9.21d]. ∎

Returning to computation of the scalar curvature in the proof of theorem 2.10, we let X_i for $1 \leq i \leq 4n$ denote a local orthonormal basis of the horizontal distribution $\mathcal{H}$, and compute using lemma 2.13, Proposition 2.2 (vi), and [Bes: 9.33a], viz.

2.14
$$\begin{aligned} | A |^2 &= \sum_{i=1}^{4n} g(A_{X_i}, A_{X_i}) = \sum_{i=1}^{4n}\sum_{a=1}^{3} g(A_{X_i}\xi^a, A_{X_i}\xi^a) \\ &= \sum_{i=1}^{4n}\sum_{a=1}^{3} g(\Phi^a X_i, \Phi^a X_i) = \sum_{i=1}^{4n}\sum_{a=1}^{3} g(X_i, X_i) = 12n. \end{aligned}$$

Substituting equation 2.14 into equation 2.11 and using the relation between the scalar curvature and the Einstein constant establishes Theorem 2.10 part (iv). ∎

REMARK 2.15: The homogeneous spherical space forms in dimension 3 are well known. They are $Sp(1)/\Gamma$ where Γ is one of the finite subgroups of $Sp(1)$, namely:

(1) $\Gamma = \mathrm{id}$.

(2) $\Gamma = \mathbb{Z}_m$ the cyclic group of order m.

(3) $\Gamma = \mathbb{D}^*_m$ a binary dihedral group with m is an integer greater than 2.

(4) $\Gamma = \mathbb{T}^*$ the binary tetrahedral group.

(5) $\Gamma = \mathbb{O}^*$ the binary octahedral group.

(6) $\Gamma = \mathbb{I}^*$ the binary ocosahedral group.

Thus Theorem 2.10 part (iii) shows that every leaf of any 3-Sasakian manifold is of the form $Sp(1)/\Gamma$ where Γ is one of the groups listed above.

An interesting corollary of Theorem 2.10 and a theorem of Bérard-Bergery [BéBer], which rescales the metric along the fibres, is

COROLLARY 2.16: *Every complete 3-Sasakian manifold has two distinct Einstein metrics of positive scalar curvature. The first is given in Theorem 2.10 part (i) and the second Einstein metric has scalar curvature*

$$2(2n+1)(4n+9) - \frac{12n}{2n+3}.$$

By distinct here we mean nonhomothetic.

PROOF: The first statement follows from theorem 2.10 and [Bes: 9.73]. To compute the scalar curvature for the second Einstein metric we use [Bes: 9.70d] and [Bes: 9.74]. ∎

REMARK 2.17: Given any Einstein metric we can easily obtain a one parameter family of Einstein metrics by scaling the metric. The scale factor, however, for the 3-Sasakian metric g is fixed by the 3-Sasakian structure. This is not the case for the second Einstein metric. Perhaps a more meaningful invariant for the second Einstein metric is not its scalar curvature, but the ratio of the scalar curvature of the second (non 3-Sasakian) Einstein metric to the scalar curvature of the first (3-Sasakian) Einstein metric. This ratio is given by

2.18
$$1 + \frac{6(n+1)}{(2n+3)(2n+1)}.$$

Now let $\mathcal{O}$ be any quaternionic Kähler orbifold of positive scalar curvature. In general Konishi's principal $SO(3)$ bundle over the dense open set of regular points of $\mathcal{O}$ extends to an orbifold bundle over $\mathcal{O}$ whose total space is an orbifold, but not necessarily a smooth manifold. We will say that the quaternionic Kähler orbifold is a *good* orbifold if the total space $\mathcal{S}$ of the principal $SO(3)$ bundle over $\mathcal{O}$ is a smooth manifold. We have the following corollary of our main theorem.

COROLLARY 2.19: *There is a one-to-one correspondence (up to covering) between simply connected 3-Sasakian manifolds of dimension $4n+3$ and good quaternionic Kähler orbifolds of dimension $4n$ with positive scalar curvature equal to $16n(n+2)$.*

REMARK 2.20: A complete quaternionic Kähler manifold M of positive scalar curvature is necessarily simply connected [Sal1]. This follows from a theorem of Kobayashi [Kob] which says that any complete Kähler manifold with positive definite Ricci curvature is simply connected, and a theorem of Salamon [Sal1] saying that the twistor space of a quaternionic Kähler manifold with positive scalar curvature is Kähler-Einstein with positive scalar curvature. It would be interesting to see whether this result generalizes to the case of quaternionic Kähler orbifolds.

Finally, we give some general results concerning the curvature of any 3-Sasakian manifold. Since the curvature of any Riemannian manifold is completely determined by its sectional curvature and the sectional curvature of any Sasakian manifold [YK] is completely determined by the Φ-sectional curvature, we essentially give the latter.

PROPOSITION 2.21: *Let $(\mathcal{S},g,\xi^a)$ be a 3-Sasakian manifold and let K and $\check{K}$ denote the sectional curvatures of g and its transverse component $\check{g}$, respectively. Then if X is any horizontal vector field of unit length on $\mathcal{S}$, we have*

(i) $K(\xi^a,\xi^b)=1$ *where* $a+1\equiv b \pmod 3$.

(ii)) $K(X,\xi^a)=1$.

(iii) $K(X,\Phi^aX)=\check{K}(X,\Phi^aX)-3$.

PROOF: (i) follows from theorem 2.5, but is also easy to compute directly. Next, notice that proposition 1.9 implies that we can use [Bes: 9.29] applied to the dense open set of regular points of $\mathcal{S}/\mathcal{F}$ and that the equations in proposition 2.2 show that if X is horizontal of unit length, then the set $\{X,\Phi^1X,\Phi^2X,\Phi^3X\}$ is an orthonormal 4-frame. Thus, (ii) follows from [Bes: 9.29b] and part (ii) of lemma 2.13. Finally part (i) of lemma 2.13 implies $A_X\Phi^aX=\xi^a$ and thus (iii) follows from this fact and [Bes: 9.29c]. ∎

Thus the local geometry of any 3-Sasakian manifold determines and is determined by that of the associated good quaternionic Kähler orbifold.

§3. An Embedding Theorem for 3-Sasakian Manifolds

In [BGM1] we showed how certain 3-Sasakian manifolds naturally arise as the level sets of hyperkähler manifolds with certain additional properties. In this section we prove a converse to this result by embedding every 3-Sasakian manifold $(\mathcal{S},g,\xi^a)$ in a hyperkähler manifold. To begin we recall

THEOREM 3.1: [BGM1] *Let G be either $Sp(1)$ or $SO(3)$ and let M be a hyperkähler manifold admitting a locally free isometric action of G permuting the complex structures on M. Then there is an $Sp(1)$ invariant function ν and an obstruction section ϕ of the fourth order symmetric product of the spin bundle S^4H on M. If this obstruction section ϕ is constant on M then each level set of ν admits a 3-Sasakian structure.*

To construct the desired hyperkähler manifold associated to $(\mathcal{S}, g, \xi^a)$ notice that the cartesian product manifold $\mathcal{S} \times \mathbb{R}^+$ has a natural $Sp(1)$ action defined to be the standard $Sp(1)$ action on $\mathcal{S}$ and the trivial action on $\mathbb{R}^+$.

THEOREM 3.2: *Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a complete 3-Sasakian manifold. Then the product manifold $M = \mathcal{S} \times \mathbb{R}^+$ with the cone metric*

3.3
$$g_M = dr^2 + r^2 g_{\mathcal{S}}$$

is hyperkähler in such a way that the obstruction section ϕ associated to the natural $Sp(1)$ action is constant. Moreover, $\mathcal{S}\times\mathbb{R}^+$ is not complete with respect to metric g_M and cannot be completed by filling in the cone point unless $\mathcal{S} = S^{4n+3}$ with its standard Sasakian 3-structure.

PROOF: We construct a hyperkähler structure on $M = \mathcal{S} \times \mathbb{R}^+$ as follows: Let $\Psi = r\frac{\partial}{\partial r}$ denote the Euler field on M and for each $a = 1, 2, 3$ define smooth sections I^a of $End\, TM$ by the formulae

3.4
$$\begin{aligned} I^a Y &= -\Phi^a Y + \eta^a(Y)\Psi \\ I^a\Psi &= -\xi^a. \end{aligned}$$

Here Y is any vector field on M that is tangent to $\mathcal{S}$. Now the action of $Sp(1)$ on $M = \mathcal{S} \times \mathbb{R}^+$ extends the ξ^a to vector fields on M and, by abuse of notation, we let η^a denote the 1-forms η^a on $\mathcal{S}$ pulled back to M. Hence, on $T\mathcal{S}$ we have

$$\begin{aligned} I^a \circ I^b &= I^a \circ (-\Phi^b + \Psi \otimes \eta^b) \\ &= -(-\Phi^a + \Psi \otimes \eta^a) \circ \Phi^b + I^a\Psi \otimes \eta^b \\ &= \Phi^a\Phi^b + \epsilon^{abc}\Psi \otimes \eta^c - \xi^a \otimes \eta^b \\ &= -\epsilon^{abc}\Phi^c - \delta^{ab}\mathrm{id} + \epsilon^{abc}\Psi \otimes \eta^c \\ &= \epsilon^{abc} I^c - \delta^{ab}\mathrm{id} \end{aligned}$$

where we have used 2.2 and the easily verified identity $\eta^a \circ \Phi^b = -\epsilon^{abc}\eta^c$. Additionally, in the normal direction to $\mathcal{S}$ we have

$$I^a \circ I^b\Psi \;=\; -I^a\xi^b \;=\; \Phi^a\xi^b - \eta^a(\xi^b)\Psi \;=\; -\epsilon^{abc}\xi^c - \delta^{ab}\Psi \;=\; \epsilon^{abc}I^c\Psi - \delta^{ab}\Psi.$$

Thus, the I^a's form an almost quaternionic structure on M. Furthermore, if X and Y are tangent to $\mathcal{S}$ then equation 2.2.v shows that

$$\begin{aligned} g_M(I^aX, I^aY) &= g_M(-\Phi^aX + \eta^a(X)\Psi, -\Phi^aY + \eta^a(Y)\Psi) \\ &= g_M(\Phi^aX, \Phi^aY) + \eta^a(X)\eta^a(Y)g_M(\Psi,\Psi) \\ &= r^2 g_{\mathcal{S}}(\Phi^aX, \Phi^aY) + \eta^a(X)\eta^a(Y)r^2 \\ &= r^2 g_{\mathcal{S}}(X,Y) - r^2\eta^a(X)\eta^a(Y) + r^2\eta^a(X)\eta^a(Y) \\ &= r^2 g_{\mathcal{S}}(X,Y) \;=\; g_M(X,Y), \end{aligned}$$

whereas, in the normal direction,

$$g_M(I^a\Psi, I^a\Psi) \;=\; g_M(\xi^a, \xi^a) \;=\; r^2 \;=\; g_M(\Psi, \Psi).$$

Hence, g_M is almost hyperhermitian.

To prove that (M, g_M) is hyperkähler we show that the complex structures I^a are parallel, i.e., that $\nabla I^a = 0$. This then implies that the almost complex structures I^a are integrable and that Hermitian structure (M, g_M, I^a) is Kähler for each $a = 1, 2, 3$. We begin by computing the second fundamental form of the embedding $\S \hookrightarrow M$ obtained as the level set $r = 1$. Actually, it is just as easy to compute the second fundamental form for the family of embeddings determined by arbitrary r.

Lemma 3.5: *Let S be a Riemannian manifold of dimension n, and $M = S \times \mathbb{R}^+$ the cone on S with cone metric given by 3.3. Then the second fundamental form s of the embedding $S \hookrightarrow M$ as the level set for any fixed nonzero r is given by $s(X, Y) = -g_S(X, Y)\Psi$. Hence, the embedding is totally umbilical.*

Proof: Let $\{\theta^i\}$ denote a local orthonormal coframe for the metric g_S on S, then we obtain a local orthonormal coframe $\{\phi^\mu\}$ for the cone metric g_M on M by setting

3.6 $$\phi^i = r\theta^i \qquad \phi^0 = dr,$$

where $1 \leq i \leq n + 1$. The first Cartan structure equations for M are

3.7 $$d\phi^\mu + \omega^\mu_\nu \wedge \phi^\nu \;=\; 0.$$

Here ω^μ_ν denotes the connection 1-forms with respect to the Levi-Civita connection ∇^M on M, i.e., if $\{X_\mu\}$ denote the local orthonormal frame on M dual to $\{\phi^\mu\}$, then for any vector field X on M,

$$\nabla^M_X X_\mu \;=\; \omega^\nu_\mu(X) X_\nu.$$

Similarly, the structure equations on S are

3.8 $$d\theta^i + \omega^i_j \wedge \theta^j \;=\; 0.$$

This together with 3.6 implies that

$$\omega^i_0 \;=\; \theta^i.$$

The lemma now follows from this formula and Gauss' formula

$$\nabla^M_X Y \;=\; \nabla^S_X Y + s(X, Y),$$

where X and Y are both vector fields on M that are tangent to S. ∎

Returning to the proof of theorem 3.2, we next show that $\nabla^M I^a = 0$. First let X and Y be vector fields on M that are both tangent to $\mathcal{S}$. Then, using equation 3.4, Gauss' formula, and lemma 3.5, we have

$$\begin{aligned}
(\nabla^M_X I^a)(Y) \;&=\; \nabla^M(I^a Y) - I^a \nabla^M_X Y \\
&=\; \nabla^M_X(-\Phi^a Y + \eta^a(Y)\Psi) - I^a(\nabla^{\mathcal{S}}_X Y - g_{\mathcal{S}}(X, Y)\Psi) \\
&=\; -\nabla^{\mathcal{S}}_X(\Phi^a Y) + g_{\mathcal{S}}(X, \Phi^a Y)\Psi + X\eta^a(Y)\Psi + \eta^a(Y)\nabla^M\Psi \\
&+\; \Phi^a(\nabla^{\mathcal{S}}_X Y) - \eta^a(\nabla^{\mathcal{S}}_X Y)\Psi + g_{\mathcal{S}}(X, Y) I^a\Psi.
\end{aligned}$$

Now Weingarten's equation and lemma 3.5 implies that $\nabla^M_X\Psi = X$. Thus, using 2.1 and 2.2, the equation above becomes

$$\begin{aligned} &-(\nabla^{\mathcal{S}}_X\Phi^a)(Y) - \Phi^a(\nabla^{\mathcal{S}}_X Y) - g_{\mathcal{S}}(\Phi^a X, Y)\Psi + X\eta^a(Y)\Psi \\ &+\eta^a(Y)X + \Phi^a(\nabla^{\mathcal{S}}_X Y) - \eta^a(\nabla^{\mathcal{S}}_X Y)\Psi - g_{\mathcal{S}}(X,Y)\xi^a. \end{aligned} \tag{3.9}$$

But equation 3.9 can be rewritten as

$$[X(\eta^a(Y)) - (\nabla^{\mathcal{S}}_X\eta^a)(Y) - \eta^a(\nabla^{\mathcal{S}}_X Y)]\Psi + [-(\nabla^{\mathcal{S}}_X\Phi^a)(Y) + \eta^a(Y)X - g_{\mathcal{S}}(X,Y)\xi^a].$$

Clearly, the first term in brackets vanishes, and 2.1.ii implies that the second term in brackets also vanishes. This shows that I^a is parallel when X and Y are both tangent to $\mathcal{S}$. Similar computations show that

$$\begin{aligned} (\nabla^M_X I^a)(\Psi) &= \nabla^M_X(I^a\Psi) - I^a(\nabla^M_X\Psi) \\ &= -\nabla^M_X\xi^a - I^aX \\ &= -\nabla^{\mathcal{S}}_X\xi^a + g_{\mathcal{S}}(X,\xi^a)\Psi + \Phi^aX - \eta^a(X)\Psi \\ &= 0. \end{aligned}$$

Finally, we note from the proof of lemma 3.5 that the connection 1-forms ω^μ_ν have no dr component. This implies that $\nabla^M_\Psi Y = 0$ for any vector field Y on M, and hence, that $\nabla^M_\Psi I^a = 0$. This completes the proof that (M, g_M) is hyperkähler.

To compute the obstruction section ϕ defined in [BGM1] notice that equations 2.4 and 3.4 give

$$I^a\xi^b = \epsilon^{abc}\xi^c + \delta^{ab}\Psi. \tag{3.10}$$

Comparing this with equation 2.16 of [BGM1] does indeed show that the obstruction section ϕ is constant. This proves the first statement of theorem 3.2. The second statement follows from a result of Yano [Y] as pointed out in [BGM1]. ∎

Remarks 3.11:

1. The proof given here that I^a is parallel with respect to the Levi-Civita connection ∇^M shows that any Sasakian manifold embeds into a Kähler manifold with a cone metric. This is a previously known result of Tashiro [Tas].
2. Using the second Cartan structure equations it is easy to show that any cone metric g_M is Einstein if and only if $g_{\mathcal{S}}$ is Einstein. In particular, the second Einstein metric on our 3-Sasakian manifold $\mathcal{S}$ gives an Einstein metric on M with positive scalar curvature. Of course, the 3-Sasakian Einstein metric on $\mathcal{S}$ induces a Ricci flat metric on M, as it must since M is hyperkähler.
3. Corollary 2.6 also follows from theorem 3.2.

Finally, we can use Theorem 3.2 to give a generalization of the standard Hopf surface construction which we can then use to construct many new compact hypercomplex manifolds. Consider the manifold $\mathcal{S}\times S^1$ obtained from $\mathcal{S}\times\mathbb{R}^+$ as the quotient by the multiplicative action of $\mathbb{Z}$ on $\mathbb{R}^+$ generated by $r\mapsto ar$ where $a\neq 1$ is a fixed positive real number.

COROLLARY 3.12: *Let $\mathcal{S}$ be a complete 3-Sasakian manifold, then the manifold $\mathcal{S}\times S^1$ constructed above has a naturally induced hypercomplex structure. In fact, the product metric is locally conformally hyperkähler.*

PROOF: The Euler vector field Ψ passes to the quotient manifold and generates a circle action on S^1. Thus, it follows from equation 3.4 that the I^a define a hypercomplex structure on $\mathcal{S}\times S^1$. Setting $r=e^u$ we see that the product metric

$$du^2+g_{\mathcal{S}}$$

is conformally equivalent to the cone metric from equation 3.3 restricted to an open set in the fundamental domain of the multiplicative action given above. ∎

Combining Corollary 3.12 with the results of sections four and five give explicit examples of homogeneous hypercomplex manifolds while combining Corollary 3.12 with the results of section six give, for each $n\geq 2$, infinitely many homotopically distinct, non-homogeneous, $4n$-dimensional hypercomplex manifolds (4-dimensional hypercomplex manifolds were classified in [Boy]). Of course, these manifolds described by Corollary 3.12 are not simply connected. However, Joyce [Joy] noticed that by twisting an associated space with a certain circle bundle one can obtain simply connected hypercomplex manifolds. Thus, twisting the manifolds constructed in sections 4 and 6 give explicit examples of homogeneous hypercomplex manifolds. Moreover, Joyce's twisting construction generalizes to the orbifold category to give, together with the results of section 7, infinitely many homotopically distinct, non-homogeneous, simply connected, hypercomplex manifolds in dimension $8n-4$ for $n\geq 2$. These manifolds are analyzed in [BGM2].

§4. The Classification of Homogeneous 3-Sasakian Structures.

In this section we classify 3-Sasakian homogeneous spaces. To begin notice that the Killing vector fields ξ^1, ξ^2, and ξ^3 which give a Riemannian manifold $(\mathcal{S},g)$ a 3-Sasakian structure generate non-trivial isometries. Thus, every 3-Sasakian manifold $(\mathcal{S},g,\xi^a)$ has a nontrivial isometry group $I(\mathcal{S},g)$. Let $I_0(\mathcal{S},g)$ denote the subgroup of $I(\mathcal{S},g)$ consisting of those isometries that leave the tensor fields Φ^a invariant for all $a=1,2,3$. We refer to elements of $I_0(\mathcal{S},g)$ as 3-Sasakian isometries. The following theorem was proven by Tanno.

THEOREM 4.1: [Tan1] *Let $(\mathcal{S},g,\xi^a)$ be a complete 3-Sasakian manifold which is not of constant curvature. Then*

$$dim\ I(\mathcal{S},g)\ =\ dim\ I_0(\mathcal{S},g)+3.$$

Furthermore, the Killing vector fields ξ^a generate the three dimensional subspace of isometries that are not 3-Sasakian isometries. Let $\mathfrak{i}$ and $\mathfrak{i}_0$ denote the Lie algebras of $I(\mathcal{S},g)$ and $I_0(\mathcal{S},g)$, respectively. Hence, Tanno's theorem says that if $(\mathcal{S},g)$ is not of constant curvature, then

4.2 $$\mathfrak{i}=\mathfrak{i}_0+\mathfrak{sp}(1),$$

where + indicates vector space direct sum. However, more is true, namely

LEMMA 4.3: *The direct sum in equation 4.2 is a direct sum of Lie algebras, i.e.,*

$$\mathfrak{i} = \mathfrak{i}_0 \oplus \mathfrak{sp}(1).$$

This lemma is implicit in Tanno's work, although he never stated it explicitly there. It follows immediately from

LEMMA 4.4: *Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian manifold and $X \in \mathfrak{i}$ be a Killing vector field on $\mathcal{S}$. Let $\mathcal{L}_X$ denote the Lie derivative with respect to X. Then the following conditions are equivalent*

(i) $\mathcal{L}_X \Phi^a = 0, \quad a = 1, 2, 3,$

(ii) $\mathcal{L}_X \eta^a = 0, \quad a = 1, 2, 3,$

(iii) $\mathcal{L}_X \xi^a = 0, \quad a = 1, 2, 3.$

Furthermore, if any (hence, all) of the conditions above is satisfied, then for any vector field Y on $\mathcal{S}$ we have

(iv) $X\eta^a(Y) = \eta^a([X, Y]).$

PROOF: Let $X \in \mathfrak{i}$, then it follows from equation 2.2.vi and the definition of the Nijenhuis tensor that the following three conditions are equivalent

$$\mathcal{L}_X \Phi^a \;=\; 0, \qquad \mathcal{L}_X N_{\Phi^a} \;=\; 0, \qquad \mathcal{L}_X d\eta^a \;=\; 0.$$

and so $X \in \mathfrak{i}_0$ if any one of the these conditions is satisfied. Thus, equation 2.2.vii implies that

$$0 \;=\; \mathcal{L}_X N_{\Phi^a} \;=\; \mathcal{L}_X (d\eta^a \otimes \xi^a) \;=\; (\mathcal{L}_X d\eta^a) \otimes \xi^a + d\eta^a \otimes (\mathcal{L}_X \xi^a) \;=\; d\eta^a \otimes (\mathcal{L}_X \xi^a).$$

This shows that (i) implies (iii). But since η^a is dual to ξ^a through the metric g and X is a Killing vector field, (iii) holds if and only if (ii) holds. Next we show that (iii) implies (i): Since any infinitesimal isometry is an infinitesimal affine transformation with respect to the Levi-Civita connection, we have for any vector field Y

$$\mathcal{L}_X(\Phi^a Y) = \mathcal{L}_X \nabla_Y \xi^a = \nabla_Y \mathcal{L}_X \xi^a + \nabla_{[X,Y]} \xi^a = \Phi^a [X, Y].$$

But the left hand side is

$$(\mathcal{L}_X \Phi^a)(Y) + \Phi^a [X, Y]$$

which proves (i). Finally (iv) follows easily from (ii). ∎

Notice that any of the first three conditions in Lemma 4.4 can be used to describe the Lie subalgebra $\mathfrak{i}_0 \in \mathfrak{i}$. Moreover, the equivalence of conditions (iii) and (i) says that the Lie algebra $\mathfrak{c}(\mathfrak{sp}(1))$ of the centralizer of $Sp(1)$ in $I(\mathcal{S}, g)$ is precisely $\mathfrak{i}_0$. Globally, on the group level we have:

PROPOSITION 4.5: *Let $(\mathcal{S}, g, \xi^a)$ be a complete 3-Sasakian manifold. Then both the isometry groups $I(\mathcal{S}, g)$ and $I_0(\mathcal{S}, g)$ are compact. Furthermore, if $(\mathcal{S}, g, \xi^a)$ is not of constant* *curvature then either $I(\mathcal{S},g)=I_0(\mathcal{S},g)\times Sp(1)$ or $I(\mathcal{S},g)=I_0(\mathcal{S},g)\times SO(3)$. Finally, if $(\mathcal{S},g,\xi^a)$ does have constant curvature then $I(\mathcal{S},g)$ strictly contains either $I_0(\mathcal{S},g)\times Sp(1)$ or $I_0(\mathcal{S},g)\times SO(3)$ as a proper subgroup and $I_0(\mathcal{S},g)$ is the centralizer of $Sp(1)$ or $SO(3)$.*

PROOF: The first assertion follows from theorem 2.10 and a standard result of Myers and Steenrod (cf. [Bes]). Next, since $I_0(\mathcal{S},g)$, $Sp(1)$, and $SO(3)$ are all compact, the direct sum on the Lie algebra level given in lemma 4.3 also gives a direct product of Lie groups. The last assertion follows immediately from lemma 4.4 ∎

We are particularly interested in the case of a transitive isometry group.

DEFINITION 4.6: *A 3-Sasakian homogeneous space is a 3-Sasakian manifold $(\mathcal{S},g,\xi^a)$ on which $I_0(\mathcal{S},g)$ acts transitively.*

PROPOSITION 4.7: *Let $(\mathcal{S},g,\xi^a)$ be a 3-Sasakian homogeneous space. Then all leaves are diffeomorphic and $\mathcal{S}/\mathcal{F}$ is a quaternionic Kähler manifold where the natural projection $\pi:\mathcal{S}\rightarrow\mathcal{S}/\mathcal{F}$ is a locally trivial Riemannian fibration. Furthermore, $I_0(\mathcal{S},g)$ acts transitively on the space of leaves $\mathcal{S}/\mathcal{F}$.*

PROOF: Let $\psi: I_0(\mathcal{S},g)\times\mathcal{S}\rightarrow\mathcal{S}$ denote the action map so that, for each $a\in I_0(\mathcal{S},g)$, $\psi_a=\psi(a,\cdot)$ is a diffeomorphism of $\mathcal{S}$ to itself. Proposition 4.4 implies that the isometry group $I(\mathcal{S},g)$ contains $I_0(\mathcal{S},g)\times Sp(1)$ where either $Sp(1)$ acts effectively or its $\mathbb{Z}_2$ quotient $SO(3)\simeq Sp(1)/\mathbb{Z}_2$ acts effectively. Since the Killing vector fields ξ^a for $a=1,2,3$ are both the infinitesimal generators of the group $Sp(1)$ and a basis for the vertical distribution $\mathcal{V}$, it follows that $Sp(1)$ acts transitively on each leaf with isotropy subgroup of a point some finite subgroup $\Gamma\subset Sp(1)$. Now let p_1 and p_2 be any two points of $\mathcal{S}$ and let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the corresponding leaves throught p_1 and p_2, respectively. Since $I_0(\mathcal{S},g)$ acts transitively on $\mathcal{S}$, there exists an $a\in I_0(\mathcal{S},g)$ such that $\psi_a(p_1)=p_2$. Now ψ_a restricted to $\mathcal{L}_1$ maps $\mathcal{L}_1$ diffeomorphically onto its image, and, since the $Sp(1)$ factor acts transitively on each leaf and commutes with $I_0(\mathcal{S},g)$, the image of ψ_a lies in $\mathcal{L}_2$. But the same holds for the inverse map $\psi_{a^{-1}}$ with $\mathcal{L}_1$ and $\mathcal{L}_2$ interchanged, so the leaves must be diffeomorphic. Thus, the leaf holonomy is trivial and $\pi:\mathcal{S}\rightarrow\mathcal{S}/\mathcal{F}$ is a locally trivial Riemannian fibration by proposition 1.9. The fact that the space of leaves $\mathcal{S}/\mathcal{F}$ is a quaternionic Kähler manifold now follows from Ishihara's theorem 2.8. Finally, the constructions above shows directly that $I_0(\mathcal{S},g)$ acts transitively on $\mathcal{S}/\mathcal{F}$. ∎

The following proposition is now immediate from proposition 4.6 and theorem 2.10.iii.

PROPOSITION 4.8: *Let $(\mathcal{S},g,\xi^a)$ be a 3-Sasakian homogeneous space. Then $\mathcal{S}$ is the total space of a locally trivial Riemannian fibration over a quaternionic Kähler homogeneous space M of positive scalar curvature (i.e., a Wolf space) with fibre $F=Sp(1)/\Gamma$ where Γ is one of the finite subgroups of $Sp(1)$ (cf. 2.15).*

While this proposition enumerates a complete list of possibilities for all the 3-Sasakian homogeneous spaces we now show that not all of them actually arise. The following classification theorem is the main result of this section.

THEOREM 4.9: *Let $(\mathcal{S},g,\xi^a)$ be a 3-Sasakian homogeneous space. Then $\mathcal{S}$ is precisely one of the following homogeneous spaces:*

$$\frac{Sp(n)}{Sp(n-1)}\ \simeq\ S^{4n-1},\qquad \frac{Sp(n)}{Sp(n-1)\times\mathbb{Z}_2}\ \simeq\ \mathbb{RP}^{4n-1},$$

$$\frac{SU(m)}{S\big(U(m-2)\times U(1)\big)}, \qquad \frac{SO(k)}{SO(k-4)\times Sp(1)},$$

$$\frac{G_2}{Sp(1)}, \qquad \frac{F_4}{Sp(3)}, \qquad \frac{E_6}{SU(6)}, \qquad \frac{E_7}{\mathrm{Spin}(12)}, \qquad \frac{E_8}{E_7}.$$

Here $n \geq 1$, $Sp(0)$ denotes the trivial group, $m \geq 3$, and $k \geq 7$. Furthermore, the fiber F over the Wolf space is $Sp(1)$ in only one case which occurs precisely when $(\mathcal{S}, g, \xi^a)$ is simply connected with constant curvature; that is, when $\mathcal{S} = S^{4n-1}$. In all other cases $F = SO(3)$.

PROOF: If $\mathcal{S}$ is a 3-Sasakian homogeneous manifold then each fibre must be a 3-Sasakian homogeneous 3-manifold. But the fibres are all of the form $Sp(1)/\Gamma$ where Γ is a finite subgroup of $Sp(1)$. These space forms are both homogeneous and 3-Sasakian [Sas]; however, they are *not* 3-Sasakian homogeneous unless $\Gamma = \mathrm{id}$ or $\mathbb{Z}_2$. To see this notice that there are two equivalent 3-Sasakian structures on $Sp(1) \simeq S^3$ both with the constant curvature bi-invariant metric. One 3-Sasakian structure is obtained from the right invariant vector fields on $Sp(1)$ while the other structure comes from the left invariant vector fields. Consider the right invariant structure. Then to obtain a compatible 3-Sasakian homogeneous structure Γ must act on $Sp(1)$ from the left. But if Γ is neither the identity subgroup nor $\mathbb{Z}_2$ then Γ is not in the center of $Sp(1)$. Hence, the centralizer of Γ in $Sp(1)$ is a proper subgroup of $Sp(1)$. So its dimension is less than two, and thus cannot act transitively on $Sp(1)$. This proves that the fibre is either $Sp(1)$ or $Sp(1)/\mathbb{Z}_2 \simeq SO(3)$.

It follow from this fact and proposition 4.7 that $\mathcal{S}$ is a principal $Sp(1)$ or $SO(3)$ bundle over a Wolf space $\mathcal{W}$. The Wolf spaces are well known [Wo1] to have the form $G/L_1 \cdot Sp(1)$ where G is a simple compact Lie group and L_1 (Wolf's notation) is a certain subgroup of G. Wolf showed that each homogeneous quaternionic Kähler manifold $\mathcal{W} = G/L_1 \cdot Sp(1)$ is the base space of an S^2 bundle whose total space is one of the homogeneous complex contact manifolds $\mathcal{Z} \simeq G/L_1 \cdot S^1$ (since identified as the twistor space of $\mathcal{W}$) which were classified by Boothby [Boo]. Let $\mathfrak{g}^{\mathbb{C}}$ denote the complexification of the Lie algebra $\mathfrak{g}$ of G. Swann [Sw] has identified the total space of the dual of the contact line bundle on $\mathcal{Z}$ with the highest root nilpotent adjoint orbit $\mathcal{N}$ in $\mathfrak{g}^{\mathbb{C}}$. The nilpotent orbits $\mathcal{N}$ are well known [Kr] to have a hyperkähler structure and Swann has further identified $\mathcal{N}$ with his $\mathbb{H}^*/\mathbb{Z}_2$ bundle $\mathcal{U}(\mathcal{W})$. It follows from [BGM1: Proposition 4.21] (see also theorem 3.1) that the level set $\nu^{-1}(1/2)$ of the hyperkähler potential has a Sasakian 3-structure. This level set is easily identified with $S \simeq G/L_1$. It is a principal $SO(3)$ bundle over $\mathcal{W}$ and a principal S^1 bundle over $\mathcal{Z}$. Furthermore, as explained in the remark about the Marchiafava-Romani class made after Theorem 2.9 the only time that this $SO(3)$ bundle lifts to a $Sp(1)$ bundle is when the base space is $\mathbb{HP}^{n-1}$. The theorem now follows from the classification of Wolf spaces [Wo1] (cf. [Bes: pg. 409]) or the classification of homogeneous complex contact manifolds [Boo]. ∎

Using a similar result for Wolf spaces or homogeneous complex contact manifolds we have the following immediate corollary.

COROLLARY 4.10: *There is a one-to-one correspondence between the simple Lie algebras and the simply connected 3-Sasakian homogeneous manifolds.*

As mentioned above and used in the proof of Theorem 4.8, Sasaki [Sas], classified 3 dimensional 3-Sasakian manifolds. They are precisely the homogeneous spherical space forms $Sp(1)/\Gamma$ where the finite subgroups Γ are listed explicitly in remark 2.15. However, as we have just seen they are not 3-Sasakian homogeneous manifolds unless $\Gamma = \mathrm{id}$ or $\mathbb{Z}_2$. Sasaki asked the natural question: Which spherical space forms in dimension $4n-1$ admit a 3-Sasakian structure? We do not solve this problem here, but only mention that Sasaki also noticed that taking the quotient of the diagonal embedding $\Gamma \to Sp(n+1)$ gives a 3-Sasakian manifold $\Gamma\backslash S^{4n-1}$. In this case $I_0(\mathcal{S}, g) = Sp(n+1)$ acts on the left where the sphere S^{4n-1} is represented by a quaternionic valued column vector of unit length. The infinitesimal isometries which generate multiplication by a unit quaternion on each component from the right then give S^{4n-1} and hence $\Gamma\backslash S^{4n-1}$ a 3-Sasakian structure. However, as the homogeneous structure and 3-Sasakian structure are not compatible, $\Gamma\backslash S^{4n-1}$ is not a 3-Sasakian homogeneous manifold. This construction of 3-Sasakian manifolds appears to be special to the spheres. If one attempts a similar procedure for the other homogeneous spaces, one obtains a double coset space $\Gamma\backslash G/L_1$ which, in general, is an orbifold.

§5. 3-Sasakian Reduction

In this section we give a general 3-Sasakian reduction procedure which constructs new 3-Sasakian manifolds from a given 3-Sasakian manifold with a non-trivial 3-Sasakian isometry group. In section 6 we apply this technique to explicitly construct the Riemannian metrics for the 3-Sasakian homogeneous manifolds arising from the simple classical Lie algebras. Then, in section 7, we apply this same technique to explicitly construct new infinite families of homotopy distinct, non-homogeneous, 3-Sasakian manifolds.

The key to this construction is the quaternionic reduction of 3-Sasakian manifolds constructed in [BGM1]. Actually, this is a reduction that is associated with a quadruple of spaces, namely, the quaternionic Kähler space, the corresponding twistor space, Swann bundle, and the 3-Sasakian Konishi bundle. It incorporates the quaternionic Kähler, twistor space, and hyperkähler reductions as well as the 3-Sasakian reduction presented in this section. For example, diagram 6.1 given in the next section pictorially represents how all these various reductions follow from the flat hyperkähler metric on $\mathbb{H}^n \setminus \{0\}$ in the homogeneous case.

To begin let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a 3-Sasakian manifold with a nontrivial group $I_0(\mathcal{S}, g_{\mathcal{S}})$ of 3-Sasakian isometries. By the embedding Theorem 3.1, $M = \mathcal{S} \times \mathbb{R}^+$ is a hyperkähler manifold with respect to the cone metric g_M given in equation 3.2. The isometry group $I_0(\mathcal{S}, g_{\mathcal{S}})$ extends to a group $I_0(M, g_M) \cong I_0(\mathcal{S}, g_{\mathcal{S}})$ of isometries on M by defining each element to act trivially on $\mathbb{R}^+$. Furthermore, it follows easily from the definition of the complex structures I^a given in equation 3.3 that these isometries $I_0(M, g_M)$ are hyperkähler; that is, they preserve the hyperkähler structure on M. Recall [HKLR] shows that any subgroup $G \subset I_0(M, g_M)$ gives rise to a hyperkähler moment map

$$\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3,$$

where $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^*$ is its dual. Thus, we can define a 3-Sasakian moment map

5.1 $$\mu_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

by restriction $\mu_{\mathcal{S}} = \mu \,|\, \mathcal{S}$. We denote the components of $\mu_{\mathcal{S}}$ with respect to the standard basis of $\mathbb{R}^3$, which we have identified with the imaginary quaternions, by $\mu^a_{\mathcal{S}}$. Recall that ordinarily moment maps determined by Abelian group actions (in particular, those associated to 1-parameter groups) are only specified up to an arbitrary constant. This is not the case for 3-Sasakian moment maps since we require that the group $Sp(1)$ generated by the Sasakian vector fields ξ^a acts on the level sets of $\mu_{\mathcal{S}}$. However, we shall see that 3-Sasakian moment maps are given by a particularly simple expression.

PROPOSITION 5.2: *Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries. Let τ be an element of the Lie algebra $\mathfrak{g}$ of G and let X^τ denote the corresponding infinitesimal isometry. Then there is a unique 3-Sasakian moment map $\mu_{\mathcal{S}}$ such that the zero set $\mu_{\mathcal{S}}^{-1}(0)$ is invariant under the group $Sp(1)$ generated by the vector fields ξ^a. This moment map is given by*

5.3 $$< \mu^a_{\mathcal{S}}, \tau > = \frac{1}{2}\eta^a(X^\tau).$$

Furthermore, the zero set $\mu_{\mathcal{S}}^{-1}(0)$ is G invariant.

PROOF: Using the embedding Theorem 3.1 we can define the 2-forms $\omega^a_{\mathcal{S}}$ on $\mathcal{S}$ as the restriction of the hyperkähler 2-forms ω^a. Then any 3-Sasakian moment map $\mu^a_{\mathcal{S}}(\tau)$ determined by $\tau \in \mathfrak{g}$ satisfies

$$2d\mu^a_{\mathcal{S}}(\tau) \;=\; 2X^\tau \rfloor \omega^a_{\mathcal{S}} \;=\; -X^\tau \rfloor d\eta^a.$$

As X^τ is a 3-Sasakian infinitesimal isometry, lemma 4.3 implies that

$$0 \;=\; L^\tau_X \eta^a \;=\; d(X^\tau \rfloor \eta^a) + X^\tau \rfloor d\eta^a,$$

and thus that

$$d\big(2 < \mu^a_{\mathcal{S}}, \tau > - \eta^a(X^\tau)\big) \;=\; 0.$$

Hence, locally we have

5.4 $$2 < \mu^a_{\mathcal{S}}, \tau > = \eta^a(X^\tau) + C^a_\tau$$

for some locally defined constants C^a_τ. Now the Lie bracket relations appearing in definition 2.3 and lemma 2.12 imply that

$$L_{\xi^b}\eta^a \;=\; -2\epsilon^{abc}\eta^c.$$

Using this equation and 5.4 we can compute the Lie derivative of the moment map to obtain

$$L_{\xi^b} < \mu^a_{\mathcal{S}}, \tau > = -2\epsilon^{abc} < \mu^c_{\mathcal{S}}, \tau > + \epsilon^{abc} C^c_\tau.$$

It follows that $\mu_{\mathcal{S}}^{-1}(0)$ is invariant under the group generated by ξ^a if and only if the constants C^a_τ vanish. So locally 5.4 becomes 5.3, and locally the moment map $\mu^a_{\mathcal{S}}(\tau)$ is clearly unique. But the functions $\eta^a(X^\tau)$ are globally defined on $\mathcal{S}$ so equation 5.3 must hold globally.

To prove the last statement we can work infinitesimally since G is compact. Let $\zeta,\tau\in\mathfrak{g}$, then by 4.4.iv we have

5.5 $$X^\tau\eta^a(X^\zeta) \;=\; \eta^a([X^\tau,X^\zeta]).$$

Since the bracket in the last term is in $\mathfrak{g}$ this term vanishes on the zero set $\mu_{\mathcal{S}}^{-1}(0)$ which proves the G invariance. ∎

Henceforth by the 3-Sasakian moment map, we shall mean the moment map $\mu_{\mathcal{S}}$ determined in Proposition 5.2. The embedding Theorem 3.1, Proposition 5.2, and the results of [BGM1] now imply the following fact.

THEOREM 5.6: *Let $(\mathcal{S},g_{\mathcal{S}},\xi^a)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries. Let $\mu_{\mathcal{S}}$ be the corresponding 3-Sasakian moment map and assume both that 0 is a regular value of $\mu_{\mathcal{S}}$ and that G acts freely on the submanifold $\mu_{\mathcal{S}}^{-1}(0)$. Furthermore, let*

$$\iota:\mu_{\mathcal{S}}^{-1}(0)\longrightarrow\mathcal{S}$$

and

$$\pi:\mu_{\mathcal{S}}^{-1}(0)\longrightarrow\mu_{\mathcal{S}}^{-1}(0)/G$$

denote the corresponding embedding and submersion. Then

$$(\check{\mathcal{S}}=\mu_{\mathcal{S}}^{-1}(0)/G,\check{g}_{\mathcal{S}},\check{\xi}^a)$$

is a smooth 3-Sasakian manifold of dimension $4(n-\dim\mathfrak{g})-1$ with metric $\check{g}_{\mathcal{S}}$ and Sasakian vector fields $\check{\xi}^a$ determined uniquely by the two conditions

$$\iota^*g_{\mathcal{S}} \;=\; \pi^*\check{g}_{\mathcal{S}}$$

and

$$\pi_*(\xi^a\,|\,\mu_{\mathcal{S}}^{-1}(0)) \;=\; \check{\xi}^a.$$

Next we have the following fact concerning 3-Sasakian isometries:

PROPOSITION 5.7: *Assume that the hypothesis of Theorem 5.6 holds. In addition assume that $(\mathcal{S},g_{\mathcal{S}})$ is complete and hence compact. Let $C(G)\subset I_0(\mathcal{S},g_{\mathcal{S}})$ denote the centralizer of G in $I_0(\mathcal{S},g_{\mathcal{S}})$ and let $C_0(G)$ denote the subgroup of $C(G)$ given by the connected component of the identity. Then $C_0(G)$ acts on the submanifold $\mu_{\mathcal{S}}^{-1}(0)$ as isometries with respect to the restricted metric $\iota^*g_{\mathcal{S}}$ and the 3-Sasakian isometry group $I_0(\check{\mathcal{S}},\check{g}_{\mathcal{S}})$ of the quotient $(\check{\mathcal{S}},\check{g}_{\mathcal{S}})$ determined in Theorem 5.6 contains an isomorphic copy of $C_0(G)$. Furthermore, if $C_0(G)$ acts transitively on $\check{\mathcal{S}}$, then $\check{\mathcal{S}}$ is a 3-Sasakian homogeneous space.*

PROOF: By Proposition 4.5 $I_0(\mathcal{S},g_{\mathcal{S}})$ is compact and connected, so it suffices to prove the corresponding result on the Lie algebra level. Let $\mathfrak{i}_0(\mathcal{S},g_{\mathcal{S}})$, $\mathfrak{g}$, and $\mathfrak{c}(\mathfrak{g})$ denote the Lie algebras of $I_0(\mathcal{S},g_{\mathcal{S}})$, G, and $C_0(G)$ respectively. For any $x\in\mathfrak{i}_0(\mathcal{S},g_{\mathcal{S}})$ we let X^x denote the corresponding vector field on $\mathcal{S}$. Then lemma 4.4 implies that for any $y\in\mathfrak{c}(\mathfrak{g})$ and for all $\tau\in\mathfrak{g}$ we have

$$X^y\eta^a(X^\tau) \;=\; \eta^a([X^y,X^\xi]) \;=\; 0.$$

Hence, $C_0(G)$ acts on the zero set $\mu_{\mathcal{S}}^{-1}(0)$. Furthermore, this action is an isometry on $\mu_{\mathcal{S}}^{-1}(0)$ since the metric is the restricted metric and $C_0(G) \subset I_0(\mathcal{S}, g_{\mathcal{S}})$. This proves the first statement.

Next, by Proposition 5.2, G acts by isometries on $\mu_{\mathcal{S}}^{-1}(0)$ so the action of $C_0(G)$ on $\mu_{\mathcal{S}}^{-1}(0)$ passes to an action of $C_0(G)$ on the quotient $\check{\mathcal{S}} = \mu_{\mathcal{S}}^{-1}(0)/G$. It is easy to check that $C_0(G)$ acts as 3-Sasakian isometries on $(\check{\mathcal{S}}, \check{g}_{\mathcal{S}}, \check{\xi}^a)$. Notice here that if G is commutative then $G \subset C_0(G)$, and so we do not require that $I_0(\check{\mathcal{S}}, \check{g}_{\mathcal{S}})$ acts effectively. ∎

§6. 3-Sasakian Reduction and the Classical 3-Sasakian Homogeneous Metrics

In this section we give a general 3-Sasakian reduction procedure and then apply this technique to explicitly construct the Riemannian metrics for the 3-Sasakian homogeneous manifolds arising from the simple classical Lie algebras. These metrics are precisely the ones associated to the three infinite families appearing in Theorem 4.6. We used this technique in [BGM1] to show that these homogeneous spaces admit a Sasakian 3-structure; however, this general 3-Sasakian reduction construction was not formulated there and, except for the trivial case of the S^{4n-1} sphere, the Riemannian metrics were not explicitly given. Recall that the unit sphere S^{4n-1} with its canonical round metric g_{can} is the simplest example of a 3-Sasakian manifold and that the quaternionic Hopf fibration exhibits this sphere as the total space projecting to the quaternionic projective space $\mathbb{HP}^{n-1}$ with fibre $Sp(1)$. This is a locally trivial Riemannian fibration where base space $\mathbb{HP}^{n-1}$ has its standard quaternionic Kähler metric. This example is quoted in almost every article on 3-Sasakian geometry. It is important to notice that the canonical round metric on S^{4n-1} is *not* the standard homogeneous metric on the homogeneous space $Sp(n)/Sp(n-1)$ with respect to the reductive decomposition $\mathfrak{sp}(n) \simeq \mathfrak{sp}(n-1) + \mathfrak{m}$. While it is, of course, the standard homogeneous metric with respect to the naturally reductive decomposition of the orthogonal Lie algebra, $\mathfrak{o}(4n) \simeq \mathfrak{o}(4n-1) + \mathfrak{m}$ this is quite special to the sphere and orthogonal group. As we shall see, in general the 3-Sasakian metrics given in Theorem 4.6 are not naturally reductive with respect to any reductive decomposition.

The key to the construction of the 3-Sasakian homogeneous metrics for the classical Lie groups as well as to the general 3-Sasakian reduction scheme is the quaternionic reduction of 3-Sasakian manifolds constructed in [BGM1]. Actually, this is a reduction that is associated with a quadruple of spaces, namely, the quaternionic Kähler space, the corresponding twistor space, Swann bundle, and the 3-Sasakian Konishi bundle. It incorporates hyperkähler, quaternionic Kähler and twistor space reductions beginning with the flat hyperkähler metric on $\mathbb{H}^n \setminus \{0\}$. Here we are giving the reduction for homogeneous quotients and not the most general orbifold reduction [BGM1]. This homogeneous reduction can be represented schematically by the following diagram:

6.1
$$
\begin{tikzcd}
 & \mathbb{H}^n \setminus \{0\} \arrow[dl, "\mathbb{R}^+"'] \arrow[dr, "\mathbb{C}^*"] \arrow[dd, "\mathbb{H}^*"] \arrow[rrr, "Q(\mathbb{F})"] & & & \mathcal{N} \arrow[dl, "t(\mathbb{C})"'] \arrow[dr, "t(\mathbb{R})"] \arrow[dd, "t(\mathbb{H})"] & \\
S^{4n-1} \arrow[dr] & & \mathbb{CP}^{2n-1} \arrow[dl] \arrow[r, "Q_2(\mathbb{F})"] & \mathcal{Z} \arrow[dr] & & \mathcal{S} \arrow[dl] \\
 & \mathbb{HP}^{n-1} \arrow[rrr, "Q_2(\mathbb{F})"] & & & \mathcal{W} &
\end{tikzcd}
$$

Here

a. $\mathbb{F}$ denotes any of the three (skew) fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

b. $\mathbb{F}^*$ denotes the group of nonzero elements of $\mathbb{F}$.

c. $\mathbb{R}^+$, the positive reals, is the component of $\mathbb{R}^*$ connected to the identity.

d. $Q(\mathbb{F}) \subset \mathbb{F}^*$ denotes the subgroup of $\mathbb{F}^*$ consisting of elements of norm one. Explicitly the groups $Q(\mathbb{F})$ are $Q(\mathbb{R}) = \mathbb{Z}_2$, $Q(\mathbb{C}) = U(1)$, and $Q(\mathbb{H}) = Sp(1)$, respectively.

e. $Q_2(\mathbb{F}) = Q(\mathbb{F})/\mathbb{Z}_2$.

f. $t(\mathbb{F}) = \mathbb{F}^*/\mathbb{Z}_2$.

g. $n \geq 1 + [\mathbb{F} : \mathbb{R}]$ where $[\mathbb{F} : \mathbb{R}]$ is the dimension of $\mathbb{F}$ over $\mathbb{R}$.

We now give a general 3-Sasakian reduction theorem. Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a 3-Sasakian manifold with a nontrivial group $I_0(\mathcal{S}, g_{\mathcal{S}})$ of 3-Sasakian isometries. By the embedding Theorem 3.1, $M = \mathcal{S} \times \mathbb{R}^+$ is a hyperkähler manifold with respect to the cone metric g_M given in equation 3.2. The isometry group $I_0(\mathcal{S}, g_{\mathcal{S}})$ extends to a group $I_0(M, g_M) \cong I_0(\mathcal{S}, g_{\mathcal{S}})$ of isometries on M by defining each element to act trivially on $\mathbb{R}^+$. Furthermore, it follows easily from the definition of the complex structures I^a given in equation 3.3 that these isometries $I_0(M, g_M)$ are hyperkähler; that is, they preserve the hyperkähler structure on M. Recall [HKLR] shows that any subgroup $G \subset I_0(M, g_M)$ gives rise to a hyperkähler moment map

$$\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

where $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^*$ is its dual. Thus, we can define a 3-Sasakian moment map

6.2
$$\mu_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

by restriction $\mu_{\mathcal{S}} = \mu \,|\, \mathcal{S}$. We denote the components of $\mu_{\mathcal{S}}$ with respect to the standard basis of $\mathbb{R}^3$, which we have identified with the imaginary quaternions, by $\mu^a_{\mathcal{S}}$. Recall that ordinarily moment maps determined by Abelian group actions (in particular, those associated to 1-parameter groups) are only specified up to an arbitrary constant. This is not the case for 3-Sasakian moment maps since we require that the group $Sp(1)$ generated by the Sasakian vector fields ξ^a acts on the level sets of $\mu_{\mathcal{S}}$. However, we shall see that 3-Sasakian moment maps are given by a particularly simple expression.

Proposition 6.3: *Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a 3-Sasakian manifold with a compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries. Let τ be an element of the Lie algebra $\mathfrak{g}$ of G and let X^τ denote the corresponding infinitesimal isometry. Then there is a unique 3-Sasakian moment map $\mu_{\mathcal{S}}$ such that the zero set $\mu_{\mathcal{S}}^{-1}(0)$ is invariant under the group $Sp(1)$ generated by the vector fields ξ^a. This moment map is given by*

6.4 $$< \mu^a_{\mathcal{S}}, \tau > = \frac{1}{2}\eta^a(X^\tau).$$

Furthermore, the zero set $\mu_{\mathcal{S}}^{-1}(0)$ is G invariant.

Proof: Using the embedding Theorem 3.1 we can define the 2-forms $\omega^a_{\mathcal{S}}$ on $\mathcal{S}$ as the restriction of the hyperkähler 2-forms ω^a. Then any 3-Sasakian moment map $\mu^a_{\mathcal{S}}(\tau)$ determined by $\tau \in \mathfrak{g}$ satisfies

$$2d\mu^a_{\mathcal{S}}(\tau) \;=\; 2X^\tau \rfloor \omega^a_{\mathcal{S}} \;=\; -X^\tau \rfloor d\eta^a.$$

As X^τ is a 3-Sasakian infinitesimal isometry, lemma 4.3 implies that

$$0 \;=\; L^\tau_X \eta^a \;=\; d(X^\tau \rfloor \eta^a) + X^\tau \rfloor d\eta^a,$$

and thus that

$$d\big(2 < \mu^a_{\mathcal{S}}, \tau > -\eta^a(X^\tau)\big) \;=\; 0.$$

Hence, locally we have

6.5 $$2 < \mu^a_{\mathcal{S}}, \tau > = \eta^a(X^\tau) + C^a_\tau$$

for some locally defined constants C^a_τ. Now the Lie bracket relations appearing in definition 2.3 and lemma 2.12 imply that

$$L_{\xi^b}\eta^a \;=\; -2\epsilon^{abc}\eta^c.$$

Using this equation and 6.5 we can compute the Lie derivative of the moment map to obtain

$$L_{\xi^b} < \mu^a_{\mathcal{S}}, \tau > = -2\epsilon^{abc} < \mu^c_{\mathcal{S}}, \tau > + \epsilon^{abc} C^c_\tau.$$

It follows that $\mu_{\mathcal{S}}^{-1}(0)$ is invariant under the group generated by ξ^a if and only if the constants C^a_τ vanish. So locally 6.5 becomes 6.4, and locally the moment map $\mu^a_{\mathcal{S}}(\tau)$ is clearly unique. But the functions $\eta^a(X^\tau)$ are globally defined on $\mathcal{S}$ so equation 6.4 must hold globally.

To prove the last statement we can work infinitesimally since G is compact. Let $\zeta, \tau \in \mathfrak{g}$, then by 4.4.iv we have

6.6 $$X^\tau \eta^a(X^\zeta) \;=\; \eta^a([X^\tau, X^\zeta]).$$

Since the bracket in the last term is in $\mathfrak{g}$ this term vanishes on the zero set $\mu_{\mathcal{S}}^{-1}(0)$ which proves the G invariance. ∎

Henceforth by the 3-Sasakian moment map, we shall mean the moment map $\mu_{\mathcal{S}}$ determined in Proposition 6.3. The embedding Theorem 3.1, Proposition 6.3, and the results of [BGM1] now imply the following fact.

THEOREM 6.7: *Let $(\mathcal{S}, g_{\mathcal{S}}, \xi^a)$ be a 3-Sasakian manifold with a compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries. Let $\mu_{\mathcal{S}}$ be the corresponding 3-Sasakian moment map and assume both that 0 is a regular value of $\mu_{\mathcal{S}}$ and that G acts freely on the submanifold $\mu_{\mathcal{S}}^{-1}(0)$. Furthermore, let*

$$\iota : \mu_{\mathcal{S}}^{-1}(0) \longrightarrow \mathcal{S}$$

and

$$\pi : \mu_{\mathcal{S}}^{-1}(0) \longrightarrow \mu_{\mathcal{S}}^{-1}(0)/G$$

denote the corresponding embedding and submersion. Then

$$(\check{\mathcal{S}} = \mu_{\mathcal{S}}^{-1}(0)/G, \check{g}_{\mathcal{S}}, \check{\xi}^a)$$

is a smooth 3-Sasakian manifold of dimension $4(n - \dim \mathfrak{g}) - 1$ with metric $\check{g}_{\mathcal{S}}$ and Sasakian vector fields $\check{\xi}^a$ determined uniquely by the two conditions

$$\iota^* g_{\mathcal{S}} \;=\; \pi^* \check{g}_{\mathcal{S}}$$

and

$$\pi_*(\xi^a \,|\, \mu_{\mathcal{S}}^{-1}(0)) \;=\; \check{\xi}^a.$$

Next we have the following fact concerning 3-Sasakian isometries:

PROPOSITION 6.8: *Assume that the hypothesis of Theorem 6.7 holds. In addition assume that $(\mathcal{S}, g_{\mathcal{S}})$ is complete and hence compact. Let $C(G) \subset I_0(\mathcal{S}, g_{\mathcal{S}})$ denote the centralizer of G in $I_0(\mathcal{S}, g_{\mathcal{S}})$. Then $C(G)$ acts on the submanifold $\mu_{\mathcal{S}}^{-1}(0)$ as isometries with respect to the restricted metric $\iota^* g_{\mathcal{S}}$ and the 3-Sasakian isometry group $I_0(\check{\mathcal{S}}, \check{g}_{\mathcal{S}})$ of the quotient $(\check{\mathcal{S}}, \check{g}_{\mathcal{S}})$ determined in Theorem 6.7 contains an isomorphic copy of $C(G)$. Furthermore, if $C(G)$ acts transitively on $\check{\mathcal{S}}$, then $\check{\mathcal{S}}$ is a 3-Sasakian homogeneous space.*

PROOF: By Proposition 4.5 $I_0(\mathcal{S}, g_{\mathcal{S}})$ is compact, so it suffices to prove the corresponding result on the Lie algebra level. Let $\mathfrak{i}_0(\mathcal{S}, g_{\mathcal{S}})$, $\mathfrak{g}$, and $\mathfrak{c}(\mathfrak{g})$ denote the Lie algebras of $I_0(\mathcal{S}, g_{\mathcal{S}})$, G, and $C(G)$, respectively. For any $x \in \mathfrak{i}_0(\mathcal{S}, g_{\mathcal{S}})$ we let X^x denote the corresponding vector field on $\mathcal{S}$. Then lemma 4.4 implies that for any $y \in \mathfrak{c}(\mathfrak{g})$ and for all $\tau \in \mathfrak{g}$ we have

$$X^y \eta^a(X^\tau) \;=\; \eta^a([X^y, X^\xi]) \;=\; 0.$$

Hence, $C(G)$ acts on the zero set $\mu_{\mathcal{S}}^{-1}(0)$. Furthermore, this action is an isometry on $\mu_{\mathcal{S}}^{-1}(0)$ since the metric is the restricted metric and $C(G) \subset I_0(\mathcal{S}, g_{\mathcal{S}})$. This proves the first statement.

Next, by Proposition 6.3, G acts by isometries on $\mu_{\mathcal{S}}^{-1}(0)$ so the action of $C(G)$ on $\mu_{\mathcal{S}}^{-1}(0)$ passes to an action of $C(G)$ on the quotient $\check{\mathcal{S}} = \mu_{\mathcal{S}}^{-1}(0)/G$. It is easy to check that $C(G)$ acts as 3-Sasakian isometries on $(\check{\mathcal{S}}, \check{g}_{\mathcal{S}}, \check{\xi}^a)$. Notice here that if G is commutative then $G \subset C(G)$, and so we do not require that $I_0(\check{\mathcal{S}}, \check{g}_{\mathcal{S}})$ acts effectively. ∎

We now apply the reduction Theorem 6.7 to the unit round sphere S^{4n-1} to obtain a description of the classical 3-Sasakian homogeneous spaces. First we set some conventions. We describe the unit sphere S^{4n-1} by its embedding in flat space and we represent an element

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{H}^n$$

as a column vector. The quaternionic components of this vector are denoted by $\mathbf{u}^0$ for the real component and by $\mathbf{u}^a$ for the three imaginary components. Then the quotients by the groups $\mathbb{R}^+$, $\mathbb{C}^*$, and $\mathbb{H}^*$ in the left most diagram in 6.1 are given by right scalar multiplication, i.e., $\mathbf{u} \mapsto \mathbf{u}q$ where $q \in \mathbb{R}^+$, $q \in \mathbb{C}^*$, and $q \in \mathbb{H}^*$, respectively. In particular, the infinitesimal generators of the subgroup $Sp(1) \subset \mathbb{H}^*$ acting from the right are the defining vector fields ξ^a for the Sasakian 3-structure. These vector fields are given explicitly by

6.9
$$\xi^a_r = \mathbf{u}^0 \cdot \frac{\partial}{\partial \mathbf{u}^a} - \mathbf{u}^a \cdot \frac{\partial}{\partial \mathbf{u}^0} - \epsilon^{abc}\mathbf{u}^b \cdot \frac{\partial}{\partial \mathbf{u}^c}$$

where the dot indicates sum over the vector components u_i and the subscript r means that these vector fields are the generators of the right action.

The group $Q(\mathbb{F}) \subset Sp(1)$ used for the reduction procedure is then given by left scalar multiplication; i.e., $\mathbf{u} \mapsto \sigma\mathbf{u}$ for $\sigma \in Q(\mathbb{F})$. The non-commutativity of the quaternions distinguishes these two actions. Notice, however, that any $r \in \mathbb{R}$ commutes with any $\mathbf{u} \in \mathbb{H}^n$ and this gives rise to the $\mathbb{Z}_2$ factor that appears in the reduction. Our choice of hyperkähler structure on $\mathbb{H}^n\backslash\{0\}$, and hence the 3-Sasakian structure on S^{4n-1}, is such that the left action preserves the hyperkähler structure and hence the 3-Sasakian structure. Notice here that the corresponding induced left actions on the quotients $\mathbb{CP}^{2n-1}$ and $\mathbb{HP}^{n-1}$ preserve the corresponding complex contact and quaternionic Kähler structures, respectively. The infinitesimal generators of the group $Q(\mathbb{H}) = Sp(1)$ acting from the left are

6.10
$$\xi^a_l = \mathbf{u}^0 \cdot \frac{\partial}{\partial \mathbf{u}^a} - \mathbf{u}^a \cdot \frac{\partial}{\partial \mathbf{u}^0} + \epsilon^{abc}\mathbf{u}^b \cdot \frac{\partial}{\partial \mathbf{u}^c}.$$

Our first task in the reduction procedure is to find the zero set of the moment map. We identify the imaginary quaternions $\mathbb{R}^3$ with the Lie algebra $\mathfrak{sp}(1)$ in equation 6.2 and let $\mathfrak{q}(\mathbb{F})$ denote the Lie algebra of $Q(\mathbb{F})$. Then equation 6.2 becomes

6.11
$$\mu_{\mathcal{S}} : S^{4n-1} \longrightarrow \mathfrak{q}(\mathbb{F})^* \otimes \mathfrak{sp}(1).$$

Now $\mathfrak{q}(\mathbb{F})$ can be identified with the pure imaginary elements in the field $\mathbb{F}$. Notice that $\mathfrak{q}(\mathbb{R}) = 0$, so that $\mu_{\mathcal{S}}$ is the zero map and $\mu_{\mathcal{S}}^{-1}(0)$ is the entire sphere S^{4n-1} in the real case. Thus, it is convenient to make the following definition.

Definition 6.12: *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then $N(\mathbb{F})$ is the zero set $\mu_{\mathcal{S}}^{-1}(0)$.*

Next, we need to recall some facts about Stiefel manifolds. Let $\mathbb{F}^n$ denote the n-dimensional vector space over $\mathbb{F}$ with its natural inner product which we denote by $\bar{\mathbf{u}} \cdot \mathbf{v}$, where $\mathbf{u} \mapsto \bar{\mathbf{u}}$ denotes conjugation in $\mathbb{F}$ on each component. Let $U(n,\mathbb{F})$ denote the subgroup of $GL(n,\mathbb{F})$ that preserves this inner product so $U(n,\mathbb{R}) = O(n)$, $U(n,\mathbb{C}) = U(n)$, and $U(n,\mathbb{H}) = Sp(n)$. Now let $V^{\mathbb{F}}_{n,k}$ denote the Stiefel manifold $\mathbb{F}$-orthonormal k dimensional frames in $\mathbb{F}^n$. It is convenient to introduce the notion of an "opposite field" $\mathbb{F}^{op}$ as follows $\mathbb{R}^{op} = \mathbb{H}$, $\mathbb{C}^{op} = \mathbb{C}$, and $\mathbb{H}^{op} = \mathbb{R}$. The Stiefel manifolds that appear in our 3-Sasakian reduction are $V^{\mathbb{F}^{op}}_{n,[\mathbb{F}:\mathbb{R}]}$. As usual these can be represented in matrix terminology as follows. Let $\mathcal{M}_{n,k}(\mathbb{F})$ denote the n by k matrices over $\mathbb{F}$, then

6.13 $$V^{\mathbb{F}^{op}}_{n,[\mathbb{F}:\mathbb{R}]} \;=\; \{\mathbb{A} \in \mathcal{M}_{n,[\mathbb{F}:\mathbb{R}]}(\mathbb{F}^{op}) \mid \mathbb{A}^*\mathbb{A} = \mathbb{I}_{[\mathbb{F}:\mathbb{R}]}\},$$

where $*$ denotes transpose together with conjugation in $\mathbb{F}$ and $\mathbb{I}_k$ denotes the k by k identity matrix. There is a natural Riemannian metric on $V^{\mathbb{F}^{op}}_{n,[\mathbb{F}:\mathbb{R}]}$ given by restricting the flat metric

6.14 $$h \;=\; \mathrm{tr}(d\mathbb{A}^* \cdot d\mathbb{A})$$

on $\mathcal{M}_{n,[\mathbb{F}:\mathbb{R}]}(\mathbb{F}^{op})$ to $V^{\mathbb{F}^{op}}_{n,[\mathbb{F}:\mathbb{R}]}$. We denote this restricted metric by h_1. The Stiefel manifold $V^{\mathbb{F}^{op}}_{n,[\mathbb{F}:\mathbb{R}]}$ with this Riemannian metric is a homogeneous Riemannian manifold with homogeneous structure given by

6.15 $$\frac{U(n,\mathbb{F}^{op})}{U(n-[\mathbb{F}:\mathbb{R}],\mathbb{F}^{op})}.$$

Proposition 6.16: *Under a rescaling $N(\mathbb{F})$ is precisely the Stiefel manifold $V^{\mathbb{F}^{op}}_{n,[\mathbb{F}:\mathbb{R}]}$. Thus,*

$$\iota : N(\mathbb{F}) \;\hookrightarrow\; S^{4n-1}$$

is a smooth compact submanifold of dimension $4n+2-3[\mathbb{F}:\mathbb{R}]$ on which $Q(\mathbb{F})$ acts freely. Furthermore, the Riemannian metrics are related by the equation

$$\iota^* g_{can} \;=\; \frac{1}{[\mathbb{F}:\mathbb{R}]} h_1.$$

Proof: As mentioned above when $\mathbb{F} = \mathbb{R}$ we have $N(\mathbb{F}) = S^{4n-1}$ which is just the Stiefel manifold $V^{\mathbb{H}}_{n,1}$. For the remaining two cases we compute the moment map 6.4. First, let $\mathbb{F} = \mathbb{H}$. Proposition 6.3 shows to compute $\mu_{\mathcal{S}}$ we need the 1-forms η^a which we can easily obtain from the flat space metric g_0 and the Sasakian vector fields ξ^a_r given explicitly by 6.9 as

6.17 $$\eta^a \;=\; \mathbf{u}^0 \cdot d\mathbf{u}^a - \mathbf{u}^a \cdot d\mathbf{u}^0 - \epsilon^{abc}\mathbf{u}^b \cdot d\mathbf{u}^c.$$

The vector fields that generate the left action of $Sp(1) = Q(\mathbb{H})$ are given by the ξ^a_l in equation 6.10. Thus, equation 6.4 and a straightforward computation shows that

6.18 $$< \mu^a_{\mathcal{S}}, \xi^b_l > = \frac{(3\mathbf{u}^0 \cdot \mathbf{u}^0 - \mathbf{u}^c \cdot \mathbf{u}^c)}{6}\delta^{ab} + \epsilon^{abc}\mathbf{u}^c \cdot \mathbf{u}^0 + (\mathbf{u}^a \cdot \mathbf{u}^b - \mathbf{u}^c \cdot \mathbf{u}^c\frac{\delta^{ab}}{3}).$$

The three terms in this equation correspond to the decomposition of

$$\mathfrak{sp}(1)^* \otimes \mathfrak{sp}(1) \simeq \mathfrak{sp}(1) \otimes \mathfrak{sp}(1)$$

into irreducible representations under the action of $Sp(1)$, namely the identity representation, the adjoint representation $\mathfrak{sp}(1)$, and the 5-dimensional representation of three by three traceless symmetric matrices over $\mathbb{R}$. Hence, the zero set of the moment map is determined precisely by the vanishing of the components of the moment map in each of these three irreducible representations. It is directly to check that this implies that the quaternionic components $(\mathbf{u}^0, \mathbf{u}^a)$ are mutually orthogonal and satisfy

$$\mathbf{u}^0 \cdot \mathbf{u}^0 \;=\; \mathbf{u}^1 \cdot \mathbf{u}^1 \;=\; \mathbf{u}^2 \cdot \mathbf{u}^2 \;=\; \mathbf{u}^3 \cdot \mathbf{u}^3 \;=\; \frac{1}{4}.$$

Thus, a simple scale transformation identifies $N(\mathbb{H})$ with the Stiefel manifold $V^{\mathbb{R}}_{n,4}$ in 6.13 and Riemannian metrics are related by the factor of $1/4$.

Finally, when $\mathbb{F} = \mathbb{C}$, the group $Q(\mathbb{C})$ is a $U(1)$ subgroup of $Sp(1)$. This corresponds to stabilizing one component in the imaginary quaternions $\mathbb{R}^3$, say the i component (for example $b = 1$ in equation 6.18). Then the components of the moment map $\mu_{\mathcal{S}}$ are

6.19
$$\begin{aligned} 2 < \mu^1_{\mathcal{S}}, \xi^1_l > &= \mathbf{u}^0 \cdot \mathbf{u}^0 + \mathbf{u}^1 \cdot \mathbf{u}^1 - \mathbf{u}^2 \cdot \mathbf{u}^2 - \mathbf{u}^3 \cdot \mathbf{u}^3 \\ < \mu^2_{\mathcal{S}}, \xi^1_l > &= \mathbf{u}^2 \cdot \mathbf{u}^1 - \mathbf{u}^3 \cdot \mathbf{u}^0 \\ < \mu^3_{\mathcal{S}}, \xi^1_l > &= \mathbf{u}^3 \cdot \mathbf{u}^1 + \mathbf{u}^2 \cdot \mathbf{u}^0. \end{aligned}$$

Defining the complex vectors $\mathbf{z}^1 = \mathbf{u}^0 + i\mathbf{u}^1$ and $\mathbf{z}^2 = \mathbf{u}^2 + i\mathbf{u}^3$ permits us to rewrite $\mathbf{u} = \mathbf{z}^1 + j\bar{\mathbf{z}}^2$. The vanishing of the moment map in 6.19 gives the conditions

$$\bar{\mathbf{z}}^1 \cdot \mathbf{z}^1 \;=\; \bar{\mathbf{z}}^2 \cdot \mathbf{z}^2 \;=\; \frac{1}{2}$$

and

$$\bar{\mathbf{z}}^2 \cdot \mathbf{z}^1 \;=\; 0.$$

Again a simple scale transformation identifies $N(\mathbb{C})$ with the complex Stiefel manifold $V^{\mathbb{C}}_{n,2}$ with the Riemannian metrics correspondingly related.

In each case the Stiefel manifolds are known to be smooth compact submanifolds of S^{4n-1} of dimension $4n + 2 - 3[\mathbb{F} : \mathbb{R}]$. Lastly, it is easy check that the left action of $Q(\mathbb{F})$ is free in each case. ∎

The free action of $Q(\mathbb{F})$ on $N(\mathbb{F})$ makes $N(\mathbb{F})$ a principal $Q(\mathbb{F})$ bundle over the quotient $N(\mathbb{F})/Q(\mathbb{F})$. Moreover, given the metric $\iota^* g_{can}$ on $N(\mathbb{F})$ there is a unique Riemannian metric $\breve{g}$ on $N(\mathbb{F})/Q(\mathbb{F})$ such that

$$\pi : N(\mathbb{F}) \longrightarrow N(\mathbb{F})/Q(\mathbb{F})$$

is a Riemannian submersion. The group $I(S^{4n-1}, g_{can})$ of 3-Sasakian isometries acting on S^{4n-1} is precisely $Sp(n)$ acting from the left and we have

LEMMA 6.20: *The centralizer $C(Q(\mathbb{F}))$ of $Q(\mathbb{F})$ in $Sp(n)$ in precisely $U(n,\mathbb{F}^{op})$.*

PROOF: Of course $Sp(n) \subset \mathcal{M}_{n,n}(\mathbb{H}) \simeq \mathcal{M}_{n,n}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H}$. By $\mathbb{R}$ linearity it suffices to prove the result on a simple element $A \otimes q \in \mathcal{M}_{n,n}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H}$. But $Q(\mathbb{F}) \subset \mathbb{H}$ is a subgroup for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and, in every case, is embedded in $Sp(n)$ as the diagonal embedding $\sigma \mapsto \sigma \mathbb{I}_n$. So finding $C(Q\mathbb{F}))$ amounts to finding the centralizer $C_{Q(\mathbb{F})}(\mathbb{H})$ of $Q(\mathbb{F})$ in $\mathbb{H}$. But it is direct to check that $C_{Q(\mathbb{F})}(\mathbb{H}) = \mathbb{F}^{op}$. So $A \otimes q$ commutes with $\sigma \mathbb{I}_n$ for all $\sigma \in Q(\mathbb{F})$ if and only if $q \in \mathbb{F}^{op}$. ∎

Thus, Lemma 6.20, Theorem 6.7, Proposition 6.8, equation 6.15, and Proposition 6.16 imply the following fact.

THEOREM 6.21: *With the restrictions on n given in Theorem 4.7, the Riemannian manifold $(N(\mathbb{F})/Q(\mathbb{F}), \breve{g})$ is one of the classical 3-Sasakian homogeneous manifolds*

$$\frac{U(n,\mathbb{F}^{op})}{U(n-[\mathbb{F}:\mathbb{R}],\mathbb{F}^{op}) \times Q(\mathbb{F})}$$

listed in Theorem 4.7. Furthermore, the metric $\breve{g}$ is given explicitly by

$$\breve{g} \;=\; \frac{1}{[\mathbb{F}:\mathbb{R}]}\pi_* h_1.$$

REMARKS 6.22:

1. Except for the real case $\mathbb{F} = \mathbb{R}$, when the reduction is just given by taking a $\mathbb{Z}_2$ quotient, the metric $\breve{g}$ is not naturally reductive with respect to the homogeneous space structure and hence $\breve{g}$ is not the standard homogeneous metric on $\mathcal{S}$. In fact, the standard homogeneous metric for $\mathbb{F} = \mathbb{C}, \mathbb{H}$ is not Einstein [Bes].
2. There are some duplications in low dimensions due to the following isomorphisms of the classical Lie groups $SO(5) \simeq Sp(2)/\mathbb{Z}_2$ and $SO(6) \simeq SU(4)/\mathbb{Z}_2$.

   Notice that we have the following sequence of fibrations:

$$\begin{array}{ccccccc}
Q(\mathbb{F}) & \hookrightarrow & \frac{U(n,\mathbb{F}^{op})}{U(n-[\mathbb{F}:\mathbb{R}],\mathbb{F}^{op})} & \simeq & (N(\mathbb{F}), \iota^* g_{can}) & \stackrel{\iota}{\hookrightarrow} & (S^{4n-1}, g_{can}) \\
 & & \big\downarrow \pi & & & & \\
SO(3) & \hookrightarrow & \frac{U(n,\mathbb{F}^{op})}{U(n-[\mathbb{F}:\mathbb{R}],\mathbb{F}^{op}) \times Q(\mathbb{F})} & \simeq & (\mathcal{S}, \breve{g}) & & , \\
 & & \big\downarrow \pi_0 & & & & \\
 & & \frac{U(n,\mathbb{F}^{op})}{U(n-[\mathbb{F}:\mathbb{R}],\mathbb{F}^{op}) \times Q(\mathbb{F}) \cdot Sp(1)} & \simeq & (\mathrm{Gr}^{\mathbb{F}^{op}}_{n,[\mathbb{F}:\mathbb{R}]}, \breve{\breve{g}}) & &
\end{array} \tag{6.22}$$

where $(\mathrm{Gr}^{\mathbb{F}^{op}}_{n,[\mathbb{F}:\mathbb{R}]}, \breve{\breve{g}})$ denotes the corresponding Wolf space Grassmannian with its quaternionic Kähler metric.

Thus far we have not been able to explicitly obtain the metrics in the cases of the exceptional groups appearing in Theorem 4.7 by a reduction procedure from the canonical unit sphere S^{4n-1}. Nevertheless, Theorem 4.7 guarantees the existence of corresponding 3-Sasakian homogeneous metrics. As in the classical case they can not be naturally reductive. This follows from the fact that naturally reductive homogeneous metrics of a compact Lie group have non-negative sectional curvature (cf. [Bes: 9.87]). But then Proposition 2.21 implies that the Wolf space $\mathcal{W}$ with its symmetric quaternionic Kähler metric would have positive sectional curvature greater than or equal to 3. For $n \geq 3$ a theorem of Berger ([Bes: 14.43]) implies that $\mathcal{W} = \mathbb{HP}^{n-1}$. If $n = 2$ then $\mathcal{W} = \mathbb{CP}^2$ with its Fubini-Study metric which has positive sectional curvature. However, with our normalization one can check that the sectional curvature takes on all values between 2 and 5. Summarizing we have

PROPOSITION 6.23: *Let $(\mathcal{S}, g)$ be a 3-Sasakian homogeneous manifold which is not of constant curvature. Then the metric g is not naturally reductive with respect to its homogeneous structure.*

To end this section we show how embeddings of spheres in spheres give rise to similar embeddings of 3-Sasakian manifolds into 3-Sasakian manifolds. Consider the embeddings $\psi^k : S^{4n-1} \to S^{4n+3}$ defined by

6.24
$$\psi^k(\mathbf{u}) = (u_1, \cdots, u_{k-1}, 0, u_{k+1}, \cdots, u_n).$$

Here it is understood that for $k = 1$ or n, u_1 or u_n is set equal to 0, respectively. Let $\theta_q : S^{4n-1} \to S^{4n-1}$ denote the action map for each $q \in Q(\mathbb{F})$ and for any positive integer n. Then the following diagram commutes

6.25
$$\begin{tikzcd} S^{4n-1} \arrow[r, "\psi_k"] \arrow[d, "\theta_q"] & S^{4n+3} \arrow[d, "\theta_q"] \\ S^{4n-1} \arrow[r, "\psi_k"] & S^{4n+3}. \end{tikzcd}$$

Now consider the Stiefel manifold $N^{l(n)}(\mathbb{F})$ where $l(n) = \dim N(\mathbb{F}) = 4n + 2 - 3[\mathbb{F} : \mathbb{R}]$ indicates its dimension. Since the group $Q(\mathbb{F})$ acts on the corresponding submanifolds $N^{l(n)}(\mathbb{F}) \subset S^{4n-1}$ and $N^{l(n+1)}(\mathbb{F}) \subset S^{4n+3}$, there is a similar diagram with the spheres replaced by the submanifolds $N^{l(n)}(\mathbb{F})$ and $N^{l(n+1)}(\mathbb{F})$, and corresponding map ψ_k is obtained by restriction. Again the commutivity of the inclusions ψ_k with the action of $Q(\mathbb{F})$ guarentees that there are well defined inclusions $\check{\psi}_k$ on the quotients, which by Theorem 6.21 are the classical 3-Sasakian homogeneous spaces. Thus, for each $k = 1, \cdots, n$ we have embeddings

6.26
$$\frac{U(n, \mathbb{F}^{op})}{U(n - [\mathbb{F} : \mathbb{R}], \mathbb{F}^{op}) \times Q(\mathbb{F})} \xrightarrow{\check{\psi}_k} \frac{U(n+1, \mathbb{F}^{op})}{U(n+1 - [\mathbb{F} : \mathbb{R}], \mathbb{F}^{op}) \times Q(\mathbb{F})}.$$

Furthermore, the maps $\check{\psi}_k$ are 3-Sasakian in the sense that, for each k, $\check{\psi}$ is an isometry with respect to the corresponding 3-Sasakian metrics and that the corresponding

3-Sasakian vector fields ξ^a are $\check{\psi}_k$ related. As with spheres one can iterate this procedure to obtain nested embeddings of the corresponding 3-Sasakian homogeneous spaces. Also there is nothing special about the homogeneous examples except that there are sequences of them labelled by n. One can obtain a sequence of embedded 3-Sasakian manifolds by 3-Sasakian reduction from another sequence of 3-Sasakian manifolds as long as the embedding maps commute with (intertwine) the actions. A non-homogeneous example will be given at the end of the next section.

There are also corresponding embeddings for the Wolf spaces, together with a commutative diagram of fibrations. Finally, an analogous diagram for the inhomogeneous case is given in the next section.

§7. Reductions by Deformed Circle Actions

In this section we apply the 3-Sasakian reduction technique described in the previous section to a "deformed" circle action giving rise to new infinite families of homotopy distinct 3-Sasakian manifolds. These manifolds can be thought of as "discrete deformations" of the 3-Sasakian homogeneous manifolds $\frac{U(n)}{U(n-2)\times U(1)}$ obtained from the circle action whose moment map is given by equation 6.17. The idea is quite simple. Instead of considering a circle S^1 embedded in $Sp(n)=I_0(S^{4n-1},g_{can})$ as a diagonal subgroup with equal weights, we now consider the most general circle subgroup of the maximal torus of $Sp(n)$ embedded diagonally but with unequal weights. By applying the 3-Sasakian reduction method described in the previous section to this general circle action, we are able to construct new 3-Sasakian manifolds in all dimensions $4n-1$ for $n\geq 2$. In the next section we will study the topology of these manifolds and in section 9 we will give a more detailed analysis of both the geometry and the topology in dimension 7.

We begin by considering a maximal torus T^n of $Sp(n)=I_0(S^{4n-1},g_{can})$. Up to conjugacy T^n is unique and can be taken to act on S^{4n-1} as the subgroup of norm preserving diagonal matrices acting on the quaternionic coordinates $\mathbf{u}$ of $\mathbb{H}^n$. Here, as in the paragraph preceding 6.9, we view $\mathbf{u}$ as a column vector and the action is given by matrix multiplication from the left. Explicitly, we have that the action

$$\tilde{\theta}:T^n\times S^{4n-1}\longrightarrow S^{4n-1}$$

is given by

7.1
$$\tilde{\theta}(\mathbf{t},\mathbf{u}) \;=\; (e^{2\pi it_1}u_1,\cdots,e^{2\pi it_n}u_n)$$

where $t_j\in\mathbb{R}$ and $u_j\in\mathbb{H}$ denote the j^{th} component of $\mathbf{t}$ and $\mathbf{u}$, respectively.

Now consider a sequence $\mathbf{p}=(p_1,\cdots,p_n)$ of nonzero integers. For each $\mathbf{p}\in(\mathbb{Z}^*)^n$, we can define a "general" circle subgroup $U(1)_{\mathbf{p}}\subset T^n$ by setting

$$t_i \;=\; p_it$$

for each $1\leq i\leq n$ in 7.1, where $t\in\mathbb{R}$. Then the action $\tilde{\theta}$ restricts to the circle action

$$\theta_{\mathbf{p}}:S^1\times S^{4n-1}\longrightarrow S^{4n-1}$$

given by

$$\theta_{\mathbf{p}}(t,\mathbf{u}) \;=\; (e^{2\pi i p_1 t}u_1,\cdots,e^{2\pi i p_n t}u_n). \tag{7.2}$$

Notice that the case $\mathbf{p}=\mathbf{1}=(1,\cdots,1)$ is precisely the circle action of the previous section.

Next we compute the moment map $\mu_{\mathcal{S}}(\mathbf{p}):S^{4n-1}{\longrightarrow}i\mathbb{R}^3$ associated to the circle action 7.2. Here we identify the Lie algebra $\mathfrak{u}(1)$ with the pure imaginary numbers $i\mathbb{R}$, and we shall write $\mu_{\mathcal{S}}(\mathbf{p})(\mathbf{u})$ for $<\mu_{\mathcal{S}}(\mathbf{p})(\mathbf{u}),i>$. The infinitesimal generators for the action 7.1 of the maximal torus T^n are given by

$$H_j \;=\; \Bigl(u_j^0\frac{\partial}{\partial u_j^1}-u_j^1\frac{\partial}{\partial u_j^0}+u_j^2\frac{\partial}{\partial u_j^3}-u_j^3\frac{\partial}{\partial u_j^2}\Bigr), \tag{7.3a}$$

where there is no sum on the repeated index j. The vector field corresponding to the action of the circle subgroup $U(1)_{\mathbf{p}}$ is

$$\xi_l(\mathbf{p}) \;=\; \sum_{j=1}^n p_j\Bigl(u_j^0\frac{\partial}{\partial u_j^1}-u_j^1\frac{\partial}{\partial u_j^0}+u_j^2\frac{\partial}{\partial u_j^3}-u_j^3\frac{\partial}{\partial u_j^2}\Bigr) \;=\; \sum_{j=1}^n p_jH_j. \tag{7.3b}$$

The moment map can now be obtained from 7.3b, 6.4, and 6.15. This computation yields the following lemma.

LEMMA 7.4: *The components of the moment map $\mu_{\mathcal{S}}(\mathbf{p})$ of the circle action given by 7.2 are*

$$\begin{aligned}
2\mu^1_{\mathcal{S}}(\mathbf{p})(\mathbf{u}) \;&=\; -i\sum_{j=1}^n p_j(u_j^0u_j^0+u_j^1u_j^1-u_j^2u_j^2-u_j^3u_j^3)\\
\mu^2_{\mathcal{S}}(\mathbf{p})(\mathbf{u}) \;&=\; -i\sum_{j=1}^n p_j(u_j^1u_j^2-u_j^3u_j^0)\\
\mu^3_{\mathcal{S}}(\mathbf{p})(\mathbf{u}) \;&=\; -i\sum_{j=1}^n p_j(u_j^0u_j^2+u_j^1u_j^3).
\end{aligned}$$

Notice that these equations specialize to equations 6.17 in the case that $\mathbf{p}=\mathbf{1}$. That is, $\mu_{\mathcal{S}}(\mathbf{1})$ is precisely $\mu_{\mathcal{S}}$ of section 6 when $\mathbb{F}=\mathbb{C}$. Just as in the homogeneous case considered in the last section the zero set of this moment map is a fundamental object of interest.

DEFINITION 7.5: $N(\mathbf{p}) \;=\; \mu_{\mathcal{S}}(\mathbf{p})^{-1}(0)$.

PROPOSITION 7.6: *For each $\mathbf{p}\in(\mathbb{Z}^*)^n$ the zero set $N(\mathbf{p})$ of the moment map $\mu_{\mathcal{S}}(\mathbf{p})$ is diffeomorphic to the complex Stiefel manifold $V^{\mathbb{C}}_{n,2}$. Thus, $N(\mathbf{p})$ is a smooth compact submanifold of S^{4n-1} of dimension $4n-4$. Furthermore, the circle action $\theta_{\mathbf{p}}$ on S^{4n-1} restricts to a circle action on $N(\mathbf{p})$ which is free if the absolute values $|p_j|$ of the components of $\mathbf{p}$ are pairwise relatively prime.*

PROOF: For each $\mathbf{p}\in(\mathbb{Z}^*)^n$ we define a linear map $T_{\mathbf{p}}:\mathbb{H}^n{\longrightarrow}\mathbb{H}^n$ by the equation

$$T_{\mathbf{p}}\mathbf{u} \;=\; (\sqrt{|p_1|}u_1,\cdots,\sqrt{|p_n|}u_n). \tag{7.7}$$

Clearly $T_{\mathbf{p}}$ is an isomorphism of quaternionic vector spaces for each $\mathbf{p} \in (\mathbb{Z}^*)^n$; however, it is norm preserving only if $\mid p_j \mid = 1$ for all $j = 1, \cdots, n$. Thus, we define maps $\phi_{\mathbf{p}} : S^{4n-1} \rightarrow S^{4n-1}$ by

$$\phi_{\mathbf{p}}(\mathbf{u}) = \frac{T_{\mathbf{p}}\mathbf{u}}{\| T_{\mathbf{p}}\mathbf{u} \|} \tag{7.8}$$

where $\| \cdot \|$ denotes the standard norm in $\mathbb{H}^n$. The map $\phi_{\mathbf{p}}$ is clearly a diffeomorphism for each $\mathbf{p} \in (\mathbb{Z}^*)^n$. Let $\mu_{\mathcal{S}}(\mathbf{1})$ and $\mu(\mathbf{1})$ denote the 3-Sasakian and hyperkähler moment maps, respectively, of the homogeneous case given explicitly by equation 6.17. Then using lemma 7.4, equations 6.17, 7.7, and 7.8, and the fact that the moment maps are homogeneous functions of degree 2 in the variables $\mathbf{u}$ we have the string of equalities

$$\begin{aligned} \mu_{\mathcal{S}}(\mathbf{p})(\mathbf{u}) &= \mu(\mathbf{1})(T_{\mathbf{p}}\mathbf{u}) \\ &= \mu(\mathbf{1})(\| T_{\mathbf{p}}\mathbf{u} \| \, \phi_{\mathbf{p}}(\mathbf{u})) \\ &= \| T_{\mathbf{p}}\mathbf{u} \|^2 \, \mu_{\mathcal{S}}(\mathbf{1})(\phi_{\mathbf{p}}(\mathbf{u})). \end{aligned} \tag{7.9}$$

Hence, $\mathbf{u} \in N(\mathbf{p})$ if and only if $\phi_{\mathbf{p}}(\mathbf{u}) \in N(\mathbb{C})$. Thus, the diffeomorphism $\phi_{\mathbf{p}}$ restricts to a diffeomorphism $\phi_{\mathbf{p}} : N(\mathbf{p}) \rightarrow N(\mathbb{C})$. But recall that the identification of $N(\mathbb{C})$ with $V^{\mathbb{C}}_{n,2}$ in Proposition 6.14 requires a dilation by a factor of $\sqrt{2}$. Composing $\phi_{\mathbf{p}}$ with this dilation gives the required diffeomorphism.

Since the circle action $\theta_{\mathbf{p}}$ given by 7.2 is a linear map on $\mathbb{H}^n$, it clearly restricts to the zero set $N(\mathbf{p})$. Now assume that the integers $|p_j|$ are pairwise relatively prime. It follows from equation 7.2 that the only fixed points on the S^{4n-1} can occur along a quaternionic coordinate axis, say the k^{th} axis given by $u_j = 0$ for all $j \neq k$. In this case the isotropy subgroup of any such point is the cyclic group $\mathbb{Z}_{p_k}$. It is convenient to recall the complex coordinates $\mathbf{z}^1 = \mathbf{u}^0 + i\mathbf{u}^1$ and $\mathbf{z}^2 = \mathbf{u}^2 + i\mathbf{u}^3$ introduce after equations 6.17. In terms of these coordinates the moment map $\mu_{\mathcal{S}}(\mathbf{p})$ takes the form

$$\begin{aligned} 2\mu^1_{\mathcal{S}}(\mathbf{p})(\mathbf{z}^1, \mathbf{z}^2) &= -i\sum_{j=1}^{n} p_j(|z^1_j| - |z^2_k|) = 0 \\ \mu^-_{\mathcal{S}}(\mathbf{p})(\mathbf{z}^1, \mathbf{z}^2) &= -i\sum_{j=1}^{n} p_j \bar{z}^2_j z^1_j \end{aligned} \tag{7.10}$$

where $\mu^-_{\mathcal{S}} = \mu^2_{\mathcal{S}} - i\mu^3_{\mathcal{S}}$. Hence, the vanishing of the moment map restricted to the k^{th} quaternionic coordinate axis takes the form

$$p_k \, |z^1_k| = p_k \, |z^2_k| \quad \text{and} \quad p_k \bar{z}^2_k z^1_k = 0.$$

These two equations imply that $z^1_k = z^2_k = 0$ which cannot happen on $N(\mathbf{p}) \subset S^{4n-1}$. Thus, the circle action $\theta_{\mathbf{p}}$ is free on $N(\mathbf{p})$. ∎

Remark 7.11: Notice that there is nothing in the proof that the zero sets $N(\mathbf{p})$ are compact submanifolds of S^{4n-1} diffeomorphic to the Stiefel manifolds $V^{\mathbb{C}}_{n,2}$ that prohibits

$\mathbf{p}$ from being any real vector in $(\mathbb{R}^*)^n$. In this sense the $N(\mathbf{p})$ can be thought of as smooth deformations of $V^{\mathbb{C}}_{n,2}$. Of course, for general $\mathbf{p} \in (\mathbb{R}^*)^n$ the quotients defined below in Definition 7.15 will not be manifolds. Nevertheless, we can think of 3-Sasakian manifolds $\mathcal{S}(\mathbf{p})$ defined below as "discrete" deformations of the 3-Sasakian homogeneous manifold $\mathcal{S}(\mathbf{1})$.

Let $\iota_{\mathbf{p}} : N(\mathbf{p}) \hookrightarrow S^{4n-1}$ denote the embedding given by the zero set of the moment map $\mu_{\mathcal{S}}(\mathbf{p})$. We define a Riemannian metric $g(\mathbf{p})$ on $N(\mathbf{p})$ by restricting the canonical metric on S^{4n-1}, that is $g(\mathbf{p}) = \iota^*_{\mathbf{p}} g_{can}$. We shall make use of the following simple observation: Any infinitesimal isometry (Killing vector field) on (S^{4n-1}, g_{can}) that has the property that when it is restricted to $N(\mathbf{p})$ it is tangent to $N(\mathbf{p})$ is also an infinitesimal isometry of $(N(\mathbf{p}), g(\mathbf{p}))$. In particular, the 3-Sasakian vector fields ξ^a_r and the infinitesimal generator $\xi_l(\mathbf{p})$ of the circle group $U(1)_{\mathbf{p}}$ satisfy this property by Proposition 6.3. More generally, it follows from equation 6.6 that any Killing vector field in $\mathfrak{c}(\mathfrak{u}(1)_{\mathbf{p}})$ also satisfies this property. In particular, the maximal torus T^n acts as isometries on $(N(\mathbf{p}), g(\mathbf{p}))$. Let W_n denote the Weyl group of $Sp(n)$. W_n is isomorphic to the semidirect product $(\mathbb{Z}_2)^n \rtimes \Sigma_n$ where Σ_n is the symmetric group on n letters, and W_n acts on the Lie algebra $\mathfrak{t}_n$ of the maximal torus by permutations and sign changes. Notice that $(\mathbb{Z}_2)^n \rtimes \Sigma_n$ also acts naturally on $(\mathbb{Z}^*)^n$ by permutations and sign changes. We shall identify this group with the Weyl group and denote this action by $\mathbf{p} \mapsto w\mathbf{p}$ for $w \in W_n$. Now the Weyl group W_n can be realized as a subgroup of $Sp(n) = I_0(S^{4n-1}, g_{can})$ by the following action on the quaternionic coordinates $\mathbf{u} \in \mathbb{H}^n$: The symmetric group acts by permutations of the vector components $(u_1, \cdots, u_n) \in \mathbb{H}^n$. The l^{th} reflection of W_n acts by sending the l^{th} component u_l of $\mathbf{u}$ to ju_l and leaving all other components fixed. Taking the direct product of these actions gives an action of W_n on $S^{4n-1} \times (\mathbb{Z}_2)^n$. It is now easy to check that

Proposition 7.12: *The action of W_n on $S^{4n-1} \times (\mathbb{Z}^*)^n$ described above induces an isometry between $(N(w\mathbf{p}), g(w\mathbf{p}))$ and $(N(\mathbf{p}), g(\mathbf{p}))$ which preserves the 3-Sasakian vector fields ξ^a_r.*

It follows from this proposition that without loss of generality we can take the integers p_j to be positive integers and order $\mathbf{p} = (p_1, \cdots, p_n)$ such that $p_1 \leq p_2 \leq \cdots \leq p_n$. Henceforth, we shall assume this to be the case unless otherwise specified.

Proposition 7.13: *Let $\mathbf{p} = (p_1, \cdots, p_n)$ be an n-tuple of pairwise relatively prime positive integers, and let k be the number of 1's in $\mathbf{p}$. Then the centralizer $C(U(1)_{\mathbf{p}})$ of $U(1)_{\mathbf{p}}$ in $Sp(n)$ is $U(k) \times T^{n-k}$.*

Proof: An argument similar to that given in the proof of Lemma 6.20 shows that $C(U(1)_{\mathbf{p}})$ must lie in $U(n)$. It is then a standard computation to check that the centralizer is as stated. ∎

Let $F : V^{\mathbb{C}}_{n,2} \to N(\mathbf{p})$ denote the diffeomorphism of Proposition 7.6, and consider the metric $F^* g(\mathbf{p})$ on $V^{\mathbb{C}}_{n,2}$. Then the following corollary follows immediately from Proposition 7.13.

Corollary 7.14: *Let $\mathbf{p} \neq \mathbf{1}$. Then the metric $F^* g(\mathbf{p})$ is not $U(n)$ invariant; hence, it is not homothetic to the homogeneous metric $\frac{1}{2} h_1$ of Proposition 6.16.*

We now come to our major objects of study.

DEFINITION 7.15: *Let* $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ *be an* n*-tuple of relatively prime ordered positive integers. Then*

$$(\mathcal{S}(\mathbf{p}), \breve{g}(\mathbf{p}))$$

is the Riemannian manifold $N(\mathbf{p})/U(1)_{\mathbf{p}}$ *with the unique Riemannian metric* $\breve{g}(\mathbf{p})$ *that makes* $\pi : N(\mathbf{p}) \to N(\mathbf{p})/U(1)_{\mathbf{p}}$ *a Riemannian submersion.*

The following theorem is a direct corollary of Theorems 2.10, 6.7, and Corollary 2.16.

THEOREM 7.16: *For each* n*-tuple* $\mathbf{p}$ *of ordered relatively prime positive integers, the Riemannian manifold* $(\mathcal{S}(\mathbf{p}), \breve{g}(\mathbf{p}))$ *is a compact* 3*-Sasakian manifold; hence, it is a compact Einstein manifold of positive scalar curvature equal to* $2(2n-3)(4n-5)$*. The space of leaves* $\mathcal{S}(\mathbf{p})/\mathcal{F}$ *is a compact* $4(n-2)$*-dimensional quaternionic Kähler orbifold* $\mathcal{O}(\mathbf{p})$ *of scalar curvature equal to* $16n(n-2)$*. Furthermore,* $\mathcal{S}(\mathbf{p})$ *has a second Einstein metric nonhomothetic to* $\breve{g}(\mathbf{p})$ *with positive scalar curvature equal to* $1 + \frac{6(n-1)}{(2n-1)(2n-3)}$ *times the scalar curvature of* $\breve{g}(\mathbf{p})$.

Our next task is to understand the manifolds $\mathcal{S}(\mathbf{p})$ group theoretically. Consider the subgroup $U(2) \times U(n-2)$ of $U(n)$ given by the block diagonal matrices of the form

7.17
$$\begin{pmatrix} \mathbb{A} & 0 \\ 0 & \mathbb{B} \end{pmatrix},$$

where $\mathbb{A} \in U(2)$ and $\mathbb{B} \in U(n-2)$. The Stiefel manifold $V^{\mathbb{C}}_{n,2}$ is the homogeneous space $U(n)/U(n-2)$ where the subgroup $U(n-2)$ is obtained by setting $\mathbb{A} = \mathbb{I}_2$, the 2 by 2 identity matrix, in 7.17, and the multiplication is from the right. The generators of the $SU(2)$ subgroup of $U(2)$ obtained by taking $\mathbb{A} \in SU(2)$ pass to the quotient and can be identified with the 3-Sasakian vector fields ξ^a_r in equation 6.9 restricted to the submanifold $V^{\mathbb{C}}_{n,2}$. The subgroup $U(1)_{\mathbf{p}}$ acting from the left then shows

PROPOSITION 7.18: *The* 3*-Sasakian manifold* $\mathcal{S}(\mathbf{p})$ *can be identified with the double coset space*

$$U(1)_{\mathbf{p}} \backslash U(n)/U(n-2).$$

If $\mathbf{p} = \mathbf{1}$ *then the subgroup* $U(1)_{\mathbf{p}}$ *is central and the the* 3*-Sasakian manifold* $\mathcal{S}(\mathbf{1})$ *is the homogeneous space given in Theorem 6.21 when* $\mathbb{F} = \mathbb{C}$.

REMARK 7.19: Notice that for any $\mathbf{p} \neq \mathbf{1}$, it follows from Proposition 7.13 that $\mathcal{S}(\mathbf{p})$ is not homogeneous with respect to the $U(n)$ action described above. Actually, in section 9 we shall prove a much stronger result in dimension 7 for a certain infinite subset of the $\mathbf{p}$; namely, that those $\mathcal{S}(\mathbf{p})$ are not homotopy equivalent to any homogeneous space.

Our construction of the 3-Sasakian manifolds $\mathcal{S}(\mathbf{p})$ can be summarized in the following diagram

7.20
$$\begin{tikzcd}
U(1)_{\mathbf{p}} \arrow[r, hook] & (V^{\mathbb{C}}_{n,2}, F^*g(\mathbf{p})) \arrow[r, "F", "\simeq"'] \arrow[d, "\pi'_{\mathbf{p}}"] & (N(\mathbf{p}), g(\mathbf{p})) \arrow[r, hook, "\iota_{\mathbf{p}}"] \arrow[dl, "\pi_{\mathbf{p}}"] & (S^{4n-1}, g_{can}) \\
& (\mathcal{S}(\mathbf{p}), \breve{g}(\mathbf{p})) \arrow[d, "\pi_0"] & & \\
& \mathcal{O}(\mathbf{p}) \arrow[r, phantom, "="] & \mathcal{S}(\mathbf{p})/\mathcal{F} &
\end{tikzcd}$$

Here π_0 is the orbifold projection onto the quaternionic Kähler orbifold $\mathcal{O}(\mathbf{p}) = \mathcal{S}(\mathbf{p})/\mathcal{F}$, and $\pi'_{\mathbf{p}}$ is the unique Riemannian submersion map that makes the triangle commute. The quaternionic Kähler metric $\check{\breve{g}}(\mathbf{p})$ is isometric to the associated transverse metric $\breve{g}_T(\mathbf{p})$ to the bundle-like metric $\breve{g}(\mathbf{p})$ on $\mathcal{S}$.

Proposition 7.21: *Let $\mathbf{p} = (p_1, \cdots, p_n)$ be an n-tuple of positive pairwise relatively prime integers, so that $\mathcal{S}(\mathbf{p})$ is a smooth compact 3-Sasakian manifold. A generic leaf of the foliation $\mathcal{F}$ on $\mathcal{S}(\mathbf{p})$ is isomorphic to:*

(i) $SO(3)$ if all p_j are odd,

(ii) $Sp(1)$ otherwise.

The isotropy subgroup of any singular leaf is a cyclic subgroup of the circle group $U(1) \subset Sp(1)_r$ corresponding to the complex direction i.

Proof: The proof of the first statement amounts to whether or not the central $\mathbb{Z}_2$ in $Sp(1)$ lies in the circle group $U(1)_{\mathbf{p}}$. The conditions for this are that for all $i, j = 1, \cdots, n$ there exist positive integers k_i, k_j such that

7.22
$$\frac{2k_i + 1}{p_i} = \frac{2k_j + 1}{p_j}.$$

Now take $j > i$, then $p_j \geq p_i$ and equality holds if and only if $p_i = p_j = 1$. Equation 7.22 becomes

7.23
$$p_j - p_i = 2(k_j p_i - k_i p_j)$$

whose left hand side is even if p_j are odd for all j. On the other hand, the condition that the p_j's be pairwise relatively prime implies that at most one p_j is even, so if the p_j are not all odd equation 7.23 cannot hold over the integers.

To prove the second statement we compute first on S^{4n-1}. The condition for the existence of fixed points is that

7.24
$$e^{2\pi i p_l t} u_l \sigma = u_l$$

for some $\sigma \in Sp(1)$ and each $1 \leq l \leq n$. Recall from the proof of Proposition 7.6 that on $N(\mathbf{p})$ at least two such u_l must be nonvanishing. In terms of the complex coordinates of equation 7.10 these conditions are

7.25 $$e^{2\pi i p_l t} z_l^1 \sigma \;=\; z_l^1, \quad \text{and} \quad e^{-2\pi i p_l t} z_l^2 \sigma \;=\; z_l^2,$$

where not all z's vanish. This implies that σ must be of the form $e^{2\pi i s}$ for some real number s; that is, $\sigma \in U(1)$ corresponding to the complex direction i. But the isotropy subgroup of a leaf must be of the form given in 2.14 and the only such groups that are subgroups of a circle are the cyclic groups. ∎

The singular locus $\Sigma(\mathbf{p}) \subset \mathcal{O}(\mathbf{p})$ of the $4(n-2)$-dimensional quaternionic Kähler orbifold $\mathcal{O}(\mathbf{p})$ can be rather complicated depending on the choice of $\mathbf{p}$. We shall describe $\Sigma(\mathbf{p})$ in detail, when $n = 3$, in the Section 9. Here, we make the following two observations. First, notice that $\mathcal{O}(\mathbf{p})$ is a leaf space of a Seifert fibration [OWag]. Recall that the group $Sp(1)_r$ generated by the 3-Sasakian vector fields ξ^a_r acts as isometries on $(N(\mathbf{p}), g(\mathbf{p}))$. Moreover, since $Sp(1)_r$ acts freely on S^{4n-1}, it acts freely on the submanifold $N(\mathbf{p})$. Thus, $N(\mathbf{p})$ is a principal $Sp(1)$ bundle over its quotient $N(\mathbf{p})/Sp(1)$. Now define $M(\mathbf{p}) = N(\mathbf{p})/Sp(1)_r$, then

Proposition 7.26: *The group $U(1)_{\mathbf{p}}$ acts locally freely on $M(\mathbf{p})$, and thus defines a Seifert fibration $\pi_s : M(\mathbf{p}) \to \mathcal{O}(\mathbf{p})$ over the quaternionic Kähler orbifold $\mathcal{O}(\mathbf{p})$. Furthermore, $M(\mathbf{p})$ is a submanifold of quaternionic projective space $\mathbb{HP}^{n-1}$ and is diffeomorphic to the homogeneous space*

$$\frac{U(n)}{U(n-2)\times SU(2)}.$$

Proof: First we can check as in the proof of Proposition 7.20 above that the circle group $U(1)_{\mathbf{p}}$ acts locally freely on $M(\mathbf{p})$. The remainder of the proof then follows from Proposition 7.6 and the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{S}(\mathbf{p}) & \stackrel{\pi_{\mathbf{p}}}{\leftarrow} & N(\mathbf{p}) & \hookrightarrow & S^{4n-1} \\
\Big\downarrow{\scriptstyle \pi_0} & & \downarrow & & \downarrow \\
\mathcal{O}(\mathbf{p}) & \stackrel{\pi_s}{\leftarrow} & M(\mathbf{p}) & \hookrightarrow & \mathbb{HP}^{n-1}.
\end{array}$$

∎

The embedding $M(\mathbf{p}) \hookrightarrow \mathbb{HP}^{n-1}$ is realized as the inclusion of the zero set of the quaternionic Kähler moment map of Galicki and Lawson [GL]. Also when $n = 3$ it is easy to see that $M(\mathbf{p}) \simeq S^5$. We discuss this case in section 9. Finally, we give an inhomogeneous analog of the 3-Sasakian embeddings described in diagram 6.26 for the homogeneous case. First, we define maps $\rho_k : \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n-1}$ by

7.27 $$\rho_k(\mathbf{p}) \;=\; (p_1, \ldots, \widehat{p_k}, \ldots, p_n),$$

where $1 \leq k \leq n$ and the $\widehat{\ }$ means that we have deleted that integer. The embeddings defined in 6.24 intertwine the actions of the circle groups $U(1)_{\mathbf{p}}$ and $U(1)_{\rho_k(\mathbf{p})}$ as follows

7.28
$$\begin{array}{ccc} S^{4n-1} & \xrightarrow{\psi_k} & S^{4n+3} \\ \Big\downarrow \theta_{\rho_k(\mathbf{p})} & & \Big\downarrow \theta_{\mathbf{p}} \\ S^{4n-1} & \xrightarrow{\psi_k} & S^{4n+3}. \end{array}$$

Thus, the analysis at the end of section 6 gives the following commutative diagram

7.29
$$\begin{array}{ccc} \mathcal{S}(\rho_k(\mathbf{p})) & \xrightarrow{\check{\psi}_k} & \mathcal{S}(\mathbf{p}) \\ \Big\downarrow \pi_0 & & \Big\downarrow \pi_0 \\ \mathcal{O}(\psi_k(\mathbf{p})) & \xrightarrow{\check{\check{\psi}}_k} & \mathcal{O}(\mathbf{p}). \end{array}$$

The corresponding dimensions should be kept in mind here. For example, $\mathcal{S}(\mathbf{p})$ has dimension $4n-5$ whereas $\mathcal{S}(\rho_k(\mathbf{p}))$ has dimension $4n-9$. Diagram 7.29 can be used to simplify the analysis of the singular locus $\Sigma(\mathbf{p})$ of the orbifold $\mathcal{O}(\mathbf{p})$. In particular, there are cases when $\Sigma(\mathbf{p}) = \bigcup_k \Sigma_k$, where Σ_k are singular sets of orbifolds $\mathcal{O}(\psi_k(\mathbf{p}))$.

In the next section we compute the cohomology ring of the manifolds $\mathcal{S}(\mathbf{p})$.

§8. The Integral Cohomology Ring of $\mathcal{S}(\mathbf{p})$

We now prove the following theorem

THEOREM 8.1: *Let $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n_+$ be any n-tuple of pairwise relatively prime positive integers; that is, $\gcd(p_i, p_j) = 1$ for $1 \leq i < j \leq n$. Then the 3-Sasakian manifold $\mathcal{S}(\mathbf{p})$ is a compact, simply connected $(4n-5)$-dimensional manifold whose integral cohomology ring $H^*\big(\mathcal{S}(\mathbf{p}), \mathbb{Z}\big)$ is generated by two classes*

$$b_2 \in H^2\big(\mathcal{S}(\mathbf{p}), \mathbb{Z}\big) \quad \text{and} \quad f_{2n-1} \in H^{2n-1}\big(\mathcal{S}(\mathbf{p}), \mathbb{Z}\big)$$

which satisfy the following relations:

$$\sigma_{n-1}(\mathbf{p}) b_2^{n-1} = 0, \quad b_2^n = 0, \quad f_{2n-1}^2 = 0, \quad f_{2n-1} b_2^{n-1} = 0.$$

Here $\sigma_{n-1}(\mathbf{p}) = \sum_{j=1}^n p_1 \cdots \hat{p_j} \cdots p_n$ is the $(n-1)^{st}$ elementary symmetric polynomial in $\mathbf{p}$.

COROLLARY 8.2: *As abelian groups*

$$H^i\big(\mathcal{S}(\mathbf{p}), \mathbb{Z}\big) = \begin{cases} 0 & \text{when } i = 1, 3, \ldots, 2n-3, 2n, 2n+2, \ldots, 4n-6, \\ \mathbb{Z} & \text{when } i = 0, 2, 4, \ldots, 2n-4, 2n-1, 2n+1, \ldots, 4n-5, \\ \mathbb{Z}_{\sigma_{n-1}(\mathbf{p})} & \text{when } i = 2n-2. \end{cases}$$

This immediately gives the following

COROLLARY 8.3: *There are infinitely many non-homotopy equivalent simply-connected compact 3-Sasakian manifolds in dimension* $4n-5$ *for every* $n \geq 3$.

PROOF: Let $\mathbf{p}(n,d) = (1,\ldots,1,d)$. Then $\mathcal{S}(\mathbf{p}(n,d))$ is a simply-connected, compact, $4n-5$ dimensional 3-Sasakian manifold with $H^{2n-2}\big(\mathcal{S}(\mathbf{p}),\mathbb{Z}\big) \cong \mathbb{Z}_{(n-1)d+1}$. ∎

Recall that from Definition 7.15 and Proposition 7.6 we have the fibration

8.4
$$\begin{array}{ccc} U(1)_{\mathbf{p}} & \longrightarrow & \frac{U(n)}{U(n-2)} \\ & & \Big\downarrow \pi \\ & & \mathcal{S}(\mathbf{p}). \end{array}$$

The long exact sequence in homotopy then implies that $\mathcal{S}(\mathbf{p})$ is simply connected and that $\pi_2(\mathcal{S}(\mathbf{p})) = H_2(\mathcal{S}(\mathbf{p})) = \mathbb{Z}$. Furthermore, since the Stiefel manifold $V^{\mathbb{C}}_{n,2} = \frac{U(n)}{U(n-2)}$ is $2n-4$ connected, it follows from the Serre spectral sequence for 8.4 that

$$H^i\big(\mathcal{S}(\mathbf{p}),\mathbb{Z}\big) = \begin{cases} 0 & \text{when } i = 1,3,\ldots,2n-5, \\ \mathbb{Z} & \text{when } i = 0,2,4,\ldots,2n-4. \end{cases}$$

But, as $\mathcal{S}(\mathbf{p})$ is oriented and compact, Poincaré duality then shows that

$$H^i\big(\mathcal{S}(\mathbf{p}),\mathbb{Z}\big) = \begin{cases} 0 & \text{when } i = 2n,2n+2,\ldots,4n-6, \\ \mathbb{Z} & \text{when } i = 2n-1,2n+1,\ldots,4n-5. \end{cases}$$

Thus simple considerations applied to 8.4 directly compute all the cohomology groups with the exceptions of the two groups in dimension $2n-3$ and $2n-2$. Up to this point the answer is independent of $\mathbf{p}$. However, it is not easy to see how to use the Serre spectral sequence associated to the fibration 8.4 to compute the two key groups $H^{2n-3}(\mathcal{S}(\mathbf{p});\mathbb{Z})$ and $H^{2n-2}(\mathcal{S}(\mathbf{p});\mathbb{Z})$.

Thus, to prove theorem 8.1 we use a spectral sequence argument of Eschenburg [Esch] which exploits ideas of Borel [Bor1]. To begin let M be a compact manifold and U a compact Lie Group that acts freely on M. Further assume that the cohomology rings of both M and U are known and one wants to compute the cohomology of M/U. Rather than using the principal U bundle

8.5
$$U \longrightarrow M \xrightarrow{\pi} M/U$$

analogous to 8.4 above, Borel replaced M and M/U by homotopy equivalent models so as to construct a fibration whose Serre spectral sequence is easier to analyze. More precisely, let $U \to E_U \to B_U$ be the universal classifying space bundle for U. Then M is homotopy equivalent to $E_U \times M$ as E_U is contractible and M/U is homotopy equivalent to

$$M//U \;=\; E_U \times_U M$$

where U acts diagonally on M and E_U. The point is that the fibration

$$M \longrightarrow M/U \longrightarrow B_U$$

which classifies 8.5 has a good homotopy model

8.6 $$E_U \times M \longrightarrow M//U \stackrel{\pi}{\longrightarrow} B_U$$

whose associated Serre spectral sequence is easier to work with than the one associated to the fibration 8.5.

Eschenburg [Esch] showed that when $M = G$ is a compact Lie group and U is a subgroup of $G \times G$ acting on G by left and right multiplication then it is possible to compute the differentials in the Serre spectral sequence associated to 8.6. Notice that while U is a subgroup of $G \times G$ and acts freely on G it is not necessary that U be a subgroup of G. He then made explicit calculations when $G = SU(3)$ and $U = U(1) \times U(1)$. We can use his methods here as Proposition 7.18 shows that

$$\mathcal{S}(\mathbf{p}) \;=\; U(1)_{\mathbf{p}} \backslash U(n)/U(n-2)$$

is such a space. That is, setting $M = G = U(n)$ and $U = U(1)_{\mathbf{p}} \times U(n-2)$ the computations necessary to prove theorem 8.1 follow directly from the methods of [Esch: §3]. Since these computations are essential to prove corollary 8.3 (and are the first step in understanding the possible homotopy types of the $\mathcal{S}(\mathbf{p})$), we have included sufficient detail in order to make the discussion here self-contained.

Eschenburg notes that, as U is a subgroup of $G \times G$, one can use E_{G^2} for E_U and thus construct a bundle map

8.7 $$\begin{array}{ccccccc} & & & G & \cong & G \\ & & & \downarrow & & \downarrow \\ G/U \simeq G//U & = & & E_{G^2} \times_U G & \stackrel{\hat{\rho}}{\longrightarrow} & E_{G^2} \times_{G^2} G \\ & & & \downarrow p & & \downarrow p' \\ & & & B_U & \stackrel{\rho}{\longrightarrow} & B_{G^2} \end{array}$$

which is an equivalence on the fibres. Next he points out that the bundle in the right hand vertical column of 8.7 is easily seen to be isomorphic to the bundle

8.8 $$G \longrightarrow B_G \stackrel{\Delta}{\longrightarrow} B_{G^2}$$

where $\Delta = B\delta$ is induced by the diagonal map $\delta : G \rightarrow G^2$. Here the map on the total spaces

$$E_{G^2}/\delta(G) \rightarrow E_{G^2} \times_{G^2} G$$

is given by $\delta Gx \mapsto G^2(x,1)$ for all $x \in E_{G^2}$.

It is well-known [Bor2] that

$$H^*(U(n);\mathbb{Z}) \;=\; E[e_1, e_3, \ldots, e_{2n-1}]$$

is an exterior algebra on generators e_{2i-1} of dimension $2i-1$ for $2 \le i \le n$. In fact, $H^*(U(n);\mathbb{Z})$ is a connected, associative, coassociative, commutative, cocommutative, finite dimensional Hopf algebra. Next,

$$H^*(B_{U(n)};\mathbb{Z}) \;\cong\; \mathbb{Z}[c_1, c_2, \ldots, c_n]$$

where c_i is the restriction of the i^{th} universal Chern class to $U(n)$ and is thus of dimension $2i$. Furthermore, the differentials in the cohomology Serre spectral sequence associated to the universal bundle $U(n) \longrightarrow EU(n) \longrightarrow BU(n)$ are generated by the transgressive differentials

$$d_{2i-1}(e_{2i-1}) \;=\; c_i.$$

As $B_{G^2} = B_G \times B_G$ we have

8.9 $$H^*(B_{U(n)^2};\mathbb{Z}) \;\cong\; \mathbb{Z}[a_1, \ldots, a_n, b_1, \ldots, b_n]$$

where $a_i = c_i \otimes 1$ and $b_i = 1 \otimes c_i$.

The Serre spectral sequence associated to 8.8 (for $G = U(n)$) has

$$E_2^{*,*}(\Delta) \;\cong\; H^*(B_{U(n)^2};\mathbb{Z}) \otimes H^*(U(n);\mathbb{Z})$$

as the cohomology of the base and fibre are torsion free. Let

$$k_j^* : H^*(B_{U(n)^2};\mathbb{Z}) \longrightarrow E_j^{*,0}(\Delta)$$

denote the natural projection of the $E_2^{*,0}(\Delta)$ term along the base.

PROPOSITION 8.10: [Bor1]

$$\Delta^* \;=\; k_\infty : H^*(B_{U(n)^2};\mathbb{Z}) \longrightarrow E_\infty^{*,0}(\Delta) \;\subset\; H^*(U(n);\mathbb{Z}).$$

Using this proposition of Borel and the fact that Δ^* induces the cup product in $H^*(B_G)$ so

$$\Delta^*(u \otimes 1) \;=\; \Delta^*(1 \otimes u) \;=\; u,$$

Eschenburg computed the differentials in $E_j^{*,*}(\Delta)$. Actually, he makes the computation explicitly when $G = SU(3)$ but correctly points out that it is direct to generalize his computations and obtain

LEMMA 8.11: [Esch] *For all $U(n)$ with $n \ge 3$ the differentials*

$$d_j : E_j^{s,t}(\Delta) \longrightarrow E_j^{s+j,t-j+1}(\Delta)$$

in the cohomology spectral sequence $E_j^{*,*}(\Delta)$ *converging to* $H^*(B_{U(n)};\mathbb{Z})$ *are generated by*

1. $d_j(e_{2i-1}) = 0$ *for* $j \le i$.
2. $d_{2i}(e_{2i-1}) = \pm k_{2i}(a_i - b_i)$ *for* $1 \le i \le n$.

Up to bundle isomorphism we may replace 8.8 for the vertical right hand column in 8.7 to obtain the commutative map of fibrations

8.12
$$\begin{tikzcd}
 & U(n) \arrow[d] \arrow[r, phantom, "\cong"] & U(n) \arrow[d] \\
\mathcal{S}(\mathbf{p}) \simeq & U(1)_{\mathbf{p}}\backslash U(n)/U(n-2) \arrow[r, "\hat{\rho}"] \arrow[d, "p"] & B_{U(n)} \arrow[d, "\Delta"] \\
 & B_{U(1)_{\mathbf{p}}\times U(n-2)} \arrow[r, "\rho"] & B_{U(n)^2}
\end{tikzcd}$$

which is an equivalence on the fibres. Thus, by naturality of the Serre spectral sequence, we have

LEMMA 8.13: *For all* $U(n)$ *with* $n \ge 3$ *the differentials*

$$d_j : E_j^{s,t}(p) \longrightarrow E_j^{s+j,t-j+1}(p)$$

in the cohomology spectral sequence $E_j^{*,*}(p)$ *converging to* $H^*(\mathcal{S}(\mathbf{p});\mathbb{Z})$ *are generated by*

1. $d_j(e_{2i-1}) = 0$ *for* $j \le i$.
2. $d_{2i}(e_{2i-1}) = \pm k_{2i}\rho^*(a_i - b_i)$ *for* $1 \le i \le n$.

Once again,

$$k_j^* : H^*(B_{U(1)_{\mathbf{p}}\times U(n-2)};\mathbb{Z}) \longrightarrow E_j^{*,0}(p)$$

denotes the natural projection of the $E_2^{*,0}(p)$ *term along the base in the spectral sequence.*

PROOF: That these differentials exist follows directly from naturality. Moreover, the identity map gives an isomorphism $E_2^{0,*}(\Delta) \overset{id}{\cong} E_2^{0,*}(\pi)$ along the fibres and the differentials in the first cohomology spectral sequence are all transgressively generated. ∎

With these preliminaries established we are now able to prove theorem 8.1.

PROOF OF THEOREM 8.1: Once we compute

$$\rho^* : H^*(B_{U(n)^2};\mathbb{Z}) \longrightarrow H^*(B_{U(1)_{\mathbf{p}}};\mathbb{Z}) \otimes H^*(B_{U(n-2)};\mathbb{Z})$$

we can apply lemma 8.13 to compute the differentials in the Serre spectral sequence converging to $H^*(\mathcal{S}(\mathbf{p});\mathbb{Z})$. Recall from Proposition 7.18 that $\mathcal{S}(\mathbf{p})$ is a double coset space. We now describe the action in more detail. We begin with the inclusion

$$U(1)_{\mathbf{p}} \times U(n-2) \longrightarrow U(n) \times U(n).$$

which is the product of the composition mapping

8.14 $$U(1) \xrightarrow{\Delta_{\mathbf{p}}} T^n \longrightarrow U(n)$$

on the first factor and the natural inclusion

8.15 $$U(n-2) \xrightarrow{j_n} U(n)$$

on the second factor. Here the maximal torus T^n includes as diagonal matrices into $U(n)$ in the standard way, and the maps $\Delta_{\mathbf{p}}$ and j_n are given by

$$\Delta_{\mathbf{p}}(t) \;=\; \begin{pmatrix} e^{2\pi i p_1 t} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & e^{2\pi i p_n t} \end{pmatrix} \quad \text{and} \quad j_n(\mathbb{B}) \;=\; \begin{pmatrix} \mathbb{I}_2 & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix}.$$

Then the action of $U(1)_{\mathbf{p}} \times U(n-2)$ on $U(n)$ is given by the map sending $\mathbb{W} \in U(n)$ to $\Delta_{\mathbf{p}}(t)\mathbb{W}j_n(\mathbb{B})$.

We now compute in cohomology. The inclusion map 8.15 implies that

8.16 $$\rho^*(b_i) \;=\; \rho^*(1 \otimes c_i) \;=\; 1 \otimes y_i$$

for $1 \leq i \leq n-2$. Here the classes c_i and y_i are the i^{th} Chern classes in $H^*(BU(n);\mathbb{Z})$ and $H^*(BU(n-2);\mathbb{Z})$, respectively and b_i is defined in equation 8.9.

Next, recall that if $T^n \;=\; U(1) \times U(1) \times \cdots \times U(1)$ is a maximal torus in $U(n)$ then the map in cohomology induced by the natural inclusion $\iota_n : T^n \rightarrow U(n)$

8.17 $$\begin{array}{ccc} H^*(B_{U(n)};\mathbb{Z}) & \xrightarrow{\iota_n^*} & H^*(T^n;\mathbb{Z}) \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ \mathbb{Z}[c_1, c_2 \ldots, c_n] & \xrightarrow{\iota_n^*} & \mathbb{Z}[x_1, x_2, \ldots, x_n] \end{array}$$

is an injection. Here $x_i \in H^2(B_{U(1)} = \mathbb{CP}^\infty;\mathbb{Z})$ is the two dimensional generator of the i^{th} factor. In fact, ι_n^* is an isomorphism

8.18 $$\begin{aligned} \iota_n^* : \mathbb{Z}[c_1, \ldots, c_n] &\xrightarrow{\cong} \mathbb{Z}[x_1, \ldots, x_n]^{\Sigma_n} \\ &\cong \;\; \mathbb{Z}[\sigma_1(\mathbf{x}), \ldots, \sigma_n(\mathbf{x})] \;\subset\; \mathbb{Z}[x_1, x_2, \ldots, x_n] \end{aligned}$$

onto the polynomial subalgebra of Σ_n invariant polynomials which are freely generated by the elementary symmetric functions in the x_i's.

Returning to the composition 8.14 notice that if $x \in H^2(B_{U(1)_{\mathbf{p}}};\mathbb{Z})$ is the two dimensional generator then

8.19 $$\Delta^*_{\mathbf{p}}(x_i) \; = \; p_i x$$

for $1 \leq i \leq n$. This immediately implies that for $1 \leq i \leq n$

8.20 $$\rho^*(a_i) \; = \; \rho^*(c_i \otimes 1) \; = \; \sigma_i(\mathbf{p})x^{2i} \otimes 1$$

where $\sigma_i(\mathbf{p})$ is the i^{th} elementary symmetric function of the coordinates of $\mathbf{p}$ and a_i is defined in equation 8.9.

Finally, the $E_2^{*,*}(\mathcal{S}(\mathbf{p}))$ term for the spectral sequence converging to $H^*(\mathcal{S}(\mathbf{p});\mathbb{Z})$ is isomorphic to

8.21 $$\mathbb{Z}[x \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_{n-2}] \otimes E[e_1, \ldots, e_{2n-1}].$$

Equations 8.19, 8.20, and lemma 8.13 imply there are differentials

1. $d_j(e_{2i-1}) = 0$ for $j \leq i$.
2. $d_{2i}(e_{2i-1}) = \pm k_{2i}(\sigma_i(\mathbf{p})x^{2i} \otimes 1 - 1 \otimes y_i)$ for $1 \leq i \leq n-2$.
3. $d_{2i}(e_{2i-1}) = \pm k_{2i}(\sigma_i(\mathbf{p})x^{2i} \otimes 1)$ for $n-1 \leq i \leq n$.

A direct calculation using items 1. and 2. shows that

8.22 $$E^{*,*}_{2n-3} \; \cong \; \mathbb{Z}[x \otimes 1] \otimes E[e_{2n-3}, e_{2n-1}]$$

where the classes on the right hand side are understood to be the E_{2n-3} level equivalence classes. Theorem 8.1 now follows by using the differentials in item 3. and the fact that $\sigma_{n-1}(\mathbf{p})$ and $\sigma_n(\mathbf{p})$ are relatively prime. ∎

As pointed out in the introduction, Corollary 8.3 shows that there are infinitely many distinct homotopy types for the $\mathcal{S}(\mathbf{p})$ in every dimension $4n-5$ for $n \geq 3$. Of course, two CW complexes may have isomorphic cohomology rings and still not be homotopy equivalent as Whitehead's theorem requires the existence of a continuous map on the space level inducing the isomorphism. Here the invariant $\sigma_{n-1}(\mathbf{p})$, which is the order of the torsion group $H^{2n-2}(\mathcal{S}(\mathbf{p});\mathbb{Z})$, merely determines the first non-trivial attaching map in a CW decomposition for $\mathcal{S}(\mathbf{p})$.

§9. Strongly Inhomogeneous Einstein Spaces

We now consider the 7-dimensional 3-Sasakian manifolds $\mathcal{S}(p_1,p_2,p_3)$ in more detail. This case, when $n=3$, is special as $V^{\mathbb{C}}_{3,2} \simeq SU(3)$ which will let us write $\mathcal{S}(\mathbf{p})$ as a quotient of $SU(3)$ by a certain circle action. This alternative description permits us to analyze the associated leaf orbifold space appearing in diagram 7.12 and Proposition 7.26. Finally, we show, in a very precise way, just how far some of our examples are from homogenous spaces.

We begin by considering the diffeomorphism $\alpha : SU(3) \times U(1) \to U(3)$ defined by

9.1 $$\alpha(\mathbb{A}, \tau) \;=\; \mathbb{A}_\tau,$$

where $\mathbb{A} \in SU(3)$, $\tau \in U(1)$, and $\mathbb{A}_\tau$ is a matrix obtained from $\mathbb{A}$ by multiplying the 3^{rd} column by τ. In other words the map α is given by the following composition

9.2 $$SU(3) \times U(1) \longrightarrow U(3) \times U(3) \stackrel{\mu}{\longrightarrow} U(3),$$

where the first arrow is the natural inclusion on the first factor and the inclusion j_3 given by 8.15, when $n = 3$, on the second factor. The second map μ is the group multiplication in $U(3)$. Here the map j_3 is given explicitly by

$$j_3(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau \end{pmatrix}.$$

Notice that α is not a group homomorphism; however, $U(1)$ acts on $SU(3) \times U(1)$ by multiplication in the second factor, and on $U(3)$ by the inclusion j_3 followed by right multiplication. Furthermore, α intertwines these two actions, that is we have a commutative diagram:

9.3 $$\begin{array}{ccc} SU(3) \times U(1) & \stackrel{\alpha}{\longrightarrow} & U(3) \\ \downarrow & & \downarrow \\ SU(3) \times U(1) & \stackrel{\alpha}{\longrightarrow} & U(3). \end{array}$$

Thus, α induces a map of homogeneous spaces

9.4 $$\hat{\alpha} : SU(3) \longrightarrow \frac{U(3)}{U(1)} \;=\; V^{\mathbb{C}}_{3,2},$$

that is, $\hat{\alpha}$ is a diffeomorphism satisfying

$$\hat{\alpha}(\mathbb{D}\mathbb{A}) \;=\; \mathbb{D}\hat{\alpha}(\mathbb{A})$$

for $\mathbb{D} \in SU(3)$. Explicitly, $\hat{\alpha}$ is the composition of the natural inclusion $SU(3) \to U(3)$ followed by the natural projection $U(3) \to U(3)/j_3(U(1))$. To describe the inverse map we recall that any $\mathbb{B} \in U(n)$ can be viewed as n column vectors in $\mathbb{C}^n$ that are mutually orthogonal with respect to the standard Hermitian inner product in $\mathbb{C}^n$. Thus, writing any $\mathbb{B} \in U(3)$ in terms of its column vectors, $\mathbb{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, the map $\hat{\alpha}^{-1}$ is given by

9.5 $$\hat{\alpha}^{-1}\big(\mathbb{B}j_3(U(1))\big) \;=\; \left(\frac{\mathbf{b}_1}{\| \mathbf{b}_1 \|}, \frac{\mathbf{b}_2}{\| \mathbf{b}_2 \|}, \frac{\mathbf{b}_3}{\| \mathbf{b}_3 \|}\right).$$

It is easy to check that this is independent of the representative of the coset.

Now consider the circle subgroup $U(1)_{\mathbf{p}} \subset T^3$ of the maximal torus of $U(3)$ with the conventions adopted in section 7; in particular, $\mathbf{p} = (p_1, p_2, p_3)$ and the action on the zero set $N(\mathbf{p}) = N(p_1, p_2, p_3)$ is given by restricting the action $\tau_{\mathbf{p}}$ given in equation 7.2. We have a diffeomorphism given by the composition

9.6
$$G : N(p_1, p_2, p_3) \xrightarrow{F^{-1}} V^{\mathbb{C}}_{3,2} \xrightarrow{\hat{\alpha}^{-1}} SU(3),$$

where F is the diffeomorphism given in Proposition 7.6. Thus, the free circle action $\theta_{\mathbf{p}}$ on $N(\mathbf{p})$ induces a free circle action $\vartheta_{\mathbf{p}}$ on $SU(3)$. A straightforward computation shows that

9.7
$$\vartheta_{\mathbf{p}}(\tau, \mathbb{A}) = \begin{pmatrix} \tau^{p_1} & 0 & 0 \\ 0 & \tau^{p_2} & 0 \\ 0 & 0 & \tau^{p_3} \end{pmatrix} \mathbb{A} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau^{-(p_1+p_2+p_3)} \end{pmatrix},$$

where $\mathbf{p} = (p_1, p_2, p_3)$. Notice that the image of $U(1)_{\mathbf{p}}$ under $\vartheta_{\mathbf{p}}$ is a subgroup of $S(U(3) \times U(3))$ acting on $SU(3)$ by left-right multiplication. We shall denote the quotient of $SU(3)$ by this circle action by $\mathcal{T}(p_1, p_2, p_3)$. Thus, we get an isomorphism of 3-Sasakian manifolds

9.8
$$\mathcal{S}(p_1, p_2, p_3) = \frac{N(p_1, p_2, p_3)}{U(1)_{\mathbf{p}}} \overset{G}{\simeq} \frac{SU(3)}{U(1)_{\mathbf{p}}} = \mathcal{T}(p_1, p_2, p_3),$$

where G is the diffeomorphism given in equation 9.6. The metric on $SU(3)$ is obtained by pulling back the metric on $N(p_1, p_2, p_3)$, namely $\hat{\alpha}^*(F^* g(\mathbf{p}))$, and the metric on $\mathcal{T}(p_1, p_2, p_3)$ is $(G^{-1})^* \breve{g}(\mathbf{p})$. Thus, $(\mathcal{T}(p_1, p_2, p_3), (G^{-1})^* \breve{g}(\mathbf{p}))$ is an isometric model for $(\mathcal{S}(p_1, p_2, p_3), \breve{g}(\mathbf{p}))$. In addition, it is direct to verify that the corresponding 3-Sasakian vector fields are G-related. Explicitly, the action of the group $Sp(1)$ generated by the 3-Sasakian vector fields on $\mathcal{T}(p_1, p_2, p_3)$ is that induced by the action of $Sp(1) \simeq SU(2)$ on $SU(3)$ given by

9.9
$$\mathbb{A} \xrightarrow{(\epsilon,\sigma)} \mathbb{A} \begin{pmatrix} \epsilon & \bar{\sigma} & 0 \\ -\sigma & \bar{\epsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{where } \begin{pmatrix} \epsilon & \bar{\sigma} \\ -\sigma & \bar{\epsilon} \end{pmatrix} \in SU(2)$$

We shall now see why this alternative model is so useful. To begin recall the manifold $M(\mathbf{p}) = N(\mathbf{p})/Sp(1)_r$ appearing in Proposition 7.24. Here $M(\mathbf{p})$ is actually isomorphic to S^5 under the isomorphism above as the quotient $SU(3)$ by $SU(2)$ with action given by equation 9.9. The circle action $\vartheta_{\mathbf{p}}$ on $SU(3)$ given in equation 9.7 commutes with the $SU(2)$ action given in equation 9.9, and thus passes to act on the quotient S^5. However, this action is, in general, not free but only locally free. There is a commutative diagram

9.10
$$\begin{array}{ccc} SU(3) & \longrightarrow & SU(3)/U(1)_{\mathbf{p}} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \pi_0} \\ S^5 & \xrightarrow{\pi_s} & \mathcal{O}(p_1, p_2, p_3), \end{array}$$

where the maps π_0 and π_s are orbifold submersions. In fact, π_s is a Seifert fibration. Now according to Proposition 7.21 the generic fibre of π_0 is

1. $SO(3)$ if $\sigma_1(\mathbf{p}) = p_1 + p_2 + p_3$ is odd.
2. $Sp(1)$ if $\sigma_1(\mathbf{p})$ is even.

Thus, we need to distinquish the two cases. We have

PROPOSITION 9.11: *Let (p_1, p_2, p_3) be a triple of ordered pairwise relatively prime positive integers. Then the quaternionic Kähler orbifolds appearing in diagram 9.10 are:*

(i) $\mathcal{O}(p_1, p_2, p_3) \;=\; \mathbb{CP}^2(p_1 + p_2, p_1 + p_3, p_2 + p_3)$ *when $p_1 + p_2 + p_3$ is even.*

(ii) $\mathcal{O}(p_1, p_2, p_3) \;=\; \mathbb{CP}^2\left(\frac{p_1+p_2}{2}, \frac{p_1+p_3}{2}, \frac{p_2+p_3}{2}\right)$ *when $p_1 + p_2 + p_3$ is odd.*

Here the terms on the right are weighted projective spaces.

PROOF: The circle action 9.7 on $SU(3)$ passes to the action on the quotient S^5 given by

9.12 $$(z_1, z_2, z_3) \;\mapsto\; (e^{-2\pi i(p_2+p_3)t} z_1, e^{-2\pi i(p_1+p_3)t} z_2, , e^{-2\pi i(p_1+p_2)t} z_3).$$

For each triple (p_1, p_2, p_3) the quotient space of S^5 by this action is known to be a weighted projective spaces (cf. [GL]). If $p_1 + p_2 + p_3$ is even, then precisely one of the p_j is even, so two of the sums $p_i + p_j$ are odd and one is even; whereas, if $p_1 + p_2 + p_3$ is odd, all the p_j's are odd and so all the sums $p_i + p_j$ are even. ∎

Next we analyse the singular locus $\Sigma(p_1, p_2, p_3)$ of the orbifold $\mathcal{S}(p_1, p_2, p_3)$. This is a straigtforward exercise using the action 9.12. Again we distinquish two case:

CASE 1: Let $p_1 + p_2 + p_3$ be even. There are two possibilities for $\Sigma(p_1, p_2, p_3)$:

1. Three isolated points. This occurs when the entries in the triple $(p_1{+}p_2, p_1{+}p_3, p_2{+}p_3)$ are pairwise relatively prime.
2. A single copy of S^2 and an isolated point. This occurs when two of the $p_i + p_j$ have a common factor.

CASE 2: Let $p_1 + p_2 + p_3$ be odd. Then there are four possibilities for $\Sigma(p_1, p_2, p_3)$:

1. The empty set. This occurs when $\mathbf{p} = (1, 1, 1)$ and corresponds to the regular case when $\mathcal{S}(1, 1, 1)$ is homogenous and fibres over the standard $\mathbb{CP}^2$.
2. Three isolated points. This occurs when the entries in the triple $(p_1{+}p_2, p_1{+}p_3, p_2{+}p_3)$ are pairwise relatively prime modulo 2.
3. A single copy of S^2. This occurs when $\mathbf{p} = (1, 1, 2k + 1)$.
4. A single S^2 with an additional isolated orbifold point. This occurs in all the other cases not covered in items 1,2, and 3.

Thus we see that the orbifold locus $\Sigma(p_1, p_2, p_3)$ is either empty (only in the one regular case), or it consists of either three isolated orbifold points, a 2-sphere with an additional isolated orbifold point, or a single 2-sphere.

Summarizing the results of this section with the results of previous sections implies the following theorem.

Theorem 9.13: *Let (p_1, p_2, p_3) be a triple of ordered pairwise relatively prime positive integers. Then the manifold $\mathcal{T}(p_1, p_2, p_3)$ defined in equation 9.8 is isometric to the 3-Sasakian manifold $\mathcal{S}(p_1, p_2, p_3)$ constructed in section 7. Therefore $\mathcal{T}(p_1, p_2, p_3)$ admits a 3-Sasakian structure with a non-homogeneous Einstein metric with scalar curvature equal to 42. The leaf space $\mathcal{O}(p_1, p_2, p_3)$ of the associated 3-Sasakian foliation is an orbifold which is smoothly equivalent to the weighted projective space*

(i) $\mathbb{CP}^2(p_1+p_2, p_1+p_3, p_2+p_3)$ *when* $p_1+p_2+p_3$ *is even.*

(ii) $\mathbb{CP}^2\left(\frac{p_1+p_2}{2}, \frac{p_1+p_3}{2}, \frac{p_2+p_3}{2}\right)$ *when* $p_1+p_2+p_3$ *is odd.*

Here the base orbifold has the quaternionic Kähler orbifold metric (with a fixed scale) constructed by Galicki and Lawson [GL], and the singular locus $\Sigma(p_1, p_2, p_3)$ that is described above.

Although we know from Corollary 7.14 that the 3-Sasakian structure on $\mathcal{T}(p_1, p_2, p_3)$ is not homogeneous (except in the case when $p_1 = p_2 = p_3 = 1$) one can still ask if $\mathcal{T}(p_1, p_2, p_3)$ is diffeomorphic or homeomorphic or even homotopy equivalent to a homogeneous space. To answer this question it is worthwhile to notice that there is a very close connection between our 7-manifolds $\mathcal{T}(p_1, p_2, p_3)$ and the construction of Eschenburg [Esch] which motivated the cohomology calculations described in section 8. More precisely, Eschenburg considers quotients of $SU(3)$ by free circle subgroups of

$$T^1_{klmn} \subset SU(3)_L \times SU(3)_R \simeq SU(3)^2.$$

It is easy to see that all such circle actions are given by the following left-right quotients

9.14
$$\mathbb{A} \xrightarrow{\tau} \begin{pmatrix} \tau^k & 0 & 0 \\ 0 & \tau^l & 0 \\ 0 & 0 & \tau^{-(k+l)} \end{pmatrix} \mathbb{A} \begin{pmatrix} \tau^m & 0 & 0 \\ 0 & \tau^n & 0 \\ 0 & 0 & \tau^{-(m+n)} \end{pmatrix},$$

where $\mathbb{A} \in SU(3)$, $\tau \in S^1$, and the quadruple of integers (k, l, m, n) must satisfy some additional conditions in order that T^1_{klmn} acts freely (see [Esch; prop 21] for the precise constraints). When the action is free, Eschenburg denoted the quotient space by M_{klmn}. Now, M_{klmn} is a simply connected, compact 7-manifold and using the ideas described in section 8, Eschenburg computed $H^*(M_{klmn}; \mathbb{Z})$ as a graded ring. Not surprisingly, the result is strikely similar to Theorem 8.1.

Theorem 9.15: [Esch] *As a graded ring $H^*(M_{klmn}; \mathbb{Z})$ is generated by two classes*

$$b_2 \in H^2(M_{klmn}, \mathbb{Z}) \quad \text{and} \quad f_5 \in H^5(M_{klmn}, \mathbb{Z})$$

which satisfy the following relations

$$rb_2^2 = 0, \quad b_2^3 = 0, \quad f_5^2 = 0, \quad f_5 b_2^2 = 0.$$

Here $r = |k^2 + l^2 + kl - (m^2 + n^2 + mn)|$.

Eschenburg's M_{klmn} manifolds are related to several other manifolds of general interest. For example, when $m = n = 0$, Eschenburg's construction recovers the homogeneous

Aloff-Wallach spaces M_{kl} extensively studied from many different points of view [AW, KrSt1, KrSt2, KrSt3, Wan, WanZi]. Most interesting to us here is that a straightforward computation shows that if $p_1+p_2+p_3 \equiv 0 \pmod 3$ then the $T^1(p_1,p_2,p_3)$ circle action given in equation 9.7 can be rewritten as an Eshenburg action in 9.14 with the following relations between p_1,p_2,p_3 and k,l,m,n:

9.16
$$\begin{aligned} k &= \frac{1}{3}(2p_1-p_2-p_3), \\ l &= \frac{1}{3}(2p_2-p_3-p_1), \\ m = n &= \frac{1}{3}(p_1+p_2+p_3). \end{aligned}$$

Consequently, we have

PROPOSITION 9.17: *If $p_1+p_2+p_3 \equiv 0 \pmod 3$ then the 3-Sasakian manifold $\mathcal{T}(p_1,p_2,p_3)$ is diffeomorphic to the Eschenburg manifold M_{klmn} where (k,l,m,n) is determined by equation 9.16.*

However, the M_{klmn} Riemannian manifolds with the metrics constructed by Eshenburg are *not* 3-Sasakian manifolds so the diffeomorphisms in Proposition 9.17 are not isometries (except, of course, in the homogeneous case when $\mathbf{p}=(1,1,1)$).

Most importantly, in his paper Eschenburg introduces the concept of a *strongly inhomogeneous* space. Such a space is a compact topological space which is not homotopy equivalent to any compact Riemannian homogeneous space. He then proves that his manifolds M_{klmn} are strongly inhomogeneous if $r \equiv 2 \pmod 3$. Recall here that r is the order of the finite cyclic group $H^4(M_{klmn};\mathbb{Z})$. Actually, Eschenburg's proof shows much more. He assumes that M is a compact, closed, oriented connected 7-dimensional smooth manifold such that

9.18
$$\pi_1(M)=0, \quad \pi_2(M)=\mathbb{Z}, \quad \pi_3(M)=\mathbb{Z}, \quad \pi_4(M)=0,$$

and then he completely classifies the homogeneous spaces with these properties. In particular, he deduces that if $H^4(M;\mathbb{Z})$ is a finite cyclic group of order r then $r \not\equiv 2 \pmod 3$. Thus, the actual proof appearing in [Esch: §4] implies

THEOREM 9.19: *If $\sigma_2(\mathbf{p})=p_1p_2+p_2p_3+p_3p_1 \equiv 2 \pmod 3$ then $\mathcal{T}(p_1,p_2,p_3)$, and hence $\mathcal{S}(p_1,p_2,p_3)$ are strongly inhomogeneous.*

To see that this result is not vacuous notice that

$$\sigma_2(c,c+1,c+2) \;=\; 3c^2+6c+2 \equiv 2 \pmod 3$$

for all odd integers c so we have produced an infinite family of 3-Sasakian strongly inhomogeneous 7-manifolds. Moreover, this family exhibits a rather interesting limiting behavior as c grows. In order to explain this precisely we need to recall that Cheeger [Che1], [Ch2] has constructed a distance $\rho^*\big((M,g),(M',g')\big)$ between two compact n-dimensional Riemannian manifolds. Considering the convergence in the ρ^*-topology we get the following

THEOREM 9.20: *Let c be any positive odd integer. Then there is a sequence*

$$\{\mathcal{S}(c,c+1,c+2)\}_{c=1}^{\infty}$$

of manifolds such that no element $\mathcal{S}(c,c+1,c+2)$ is homotopy equivalent to any compact homogeneous space. Furthermore, this sequence converges in the Cheeger ρ^*-topology to the homogeneous space $\mathcal{S}(1,1,1)$ with respect to the Aloff-Wallach metrics.

PROOF: Let c be any positive odd integer. Then the triple $(c,c+1,c+2)$ consists of pairwise relatively prime integers. We have just seen that $\sigma_2(\mathbf{p}) = 3c^2+6c+2 \equiv 2 \pmod 3$ so each $\mathcal{S}(c,c+1,c+2)$ is strongly inhomogeneous. Thus $\{\mathcal{S}(c,c+1,c+2)\}_c$ is a sequence of simply connected, strongly inhomogeneous, 3-Sasakian manifolds of distinct homotopy types. Furthermore, since

$$p_1+p_2+p_3 \;=\; 3c+3 \;\equiv\; 0 \pmod 3$$

Proposition 9.17 implies there is a smooth equivalence

$$\mathcal{S}(c,c+1,c+2) \;\simeq\; M_{-1,0,c+1,c+1}.$$

But Eschenburg shows that the curvatures of the elements in the smoothly equivalent sequence $\{M_{(-1,0,c+1,c+1)}\}_c$ converge to the curvature of the homogeneous Wallach space $M_{1,1}$ [Esch: Proposition 22]. Finally, Wang and Ziller [WanZi: Proposition 4.3] observed that the curvature convergence considered by Eschenberg is equivalent to convergence in the Cheeger distance ρ^*. The theorem now follows from the simple observation that $M_{1,1} \simeq \mathcal{S}(1,1,1)$. ∎

Notice that the leaf space associated to the 3-Sasakian foliation of $\mathcal{S}(c,c+1,c+2)$ is the weighted projective space $\mathbb{CP}^2(2c+1,2c+2,2c+3)$, which is a 4-dimensional quaternionic Kähler orbifold with exactly 3 disjoint isolated points. Theorem 9.20 is a much stronger result than the statement that the Einstein metric compatible with the 3-Sasakian structure on $\mathcal{S}(c,c+1,c+2)$ fails to be homogeneous. This weaker fact is a simple implication of the theory of homogeneous 3-Sasakian structures presented in section 4 and was, in fact, already mentioned in [BGM1]. To our knowledge the 3-Sasakian manifolds appearing in Theorem 9.20 are the first examples of compact, strongly inhomogeneous, Einstein manifolds of positive scalar curvature.

Theorem 9.20 has an important corollary:

COROLLARY 9.21: *For all sufficiently large odd positive integers c, the manifolds*

$$\mathcal{S}(c,c+1,c+2)$$

admit metrics of positive sectional curvature.

These metrics are obtained as left-right quotients of $SU(3)$ induced by submersions from a special Riemannian metric g on $SU(3)$ such that g is $SU(3)_L \times H_R$-invariant. Here $H = U(2) \subset SU(3)$ is the canonical imbedding, and the quotient M_{pq} has strictly positive curvature for arbitrary p and q. As shown in [AW], all such metrics are given by the following scalar product in the Lie algebra $\mathfrak{su}(3)$:

9.22 $$< X,Y > = \; B(X,Y) + tB(X_H,Y_H),$$

where $t \in (-1,0) \cup (0,1/3)$, B is an Ad-invariant scalar inner product and X_H is the orthogonal projection of X to $H = U(2)$.

§10. Concluding Remarks

The geometry of the Einstein manifolds considered in the last three sections is quite rich. In particular, it is natural to ask about relationships between $S(\mathbf{p})$ and $S(\mathbf{p}')$ under the various types of equivalence: homotopy type, homeomorphism, diffeomorphism, and isometry. The first problem is to classify the 3-Sasakian manifolds $\mathcal{S}(\mathbf{p})$ up to homotopy type. In dimension seven, theorem 8.1 implies that, as a CW complex,

$$\mathcal{S}(p_1,p_2,p_3) \;=\; \big([(S^2 \vee S^3) \cup_f e^4] \cup_g e^5\big) \cup_h e^7$$

and the order of the torsion group $H^4(\mathcal{S}(\mathbf{p});\mathbb{Z})$, determines the first non-trivial attaching map

$$[f] = (1, \sigma_2(p_1,p_2,p_3)) \;\in\; \pi_3(S^2 \vee S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Similarly, the other attaching maps g and h are functions of (p_1,p_2,p_3) which must be determined explicitly to classify the $\mathcal{S}(p_1,p_2,p_3)$ spaces up to homotopy type. We note that a similar question remains unanswered even in the case of the homogeneous Aloff-Walach spaces $M_{k,l}$ [AlWal].

Next it would be very interesting to classify the spaces $S(\mathbf{p})$ up to homeomorphism and diffeomorphism type. Again, in dimension seven, this can be done in principle by computing the Kreck and Stolz invariants [KreSt1, KreSt2]. We recall that Kreck and Stolz studied smooth compact closed oriented seven manifolds M^7 whose integral cohomology ring is generated by two classes $b_2 \in H^2(M^7,\mathbb{Z})$ and $f_5 \in H^5(M^7,\mathbb{Z})$ subject to the relations:

$$i(M^7)b_2^2 = 0, \quad b_2^3 = 0, \quad f_5^2 = 0, \quad f_5 b_2^2 = 0.$$

Here $i(M^7) \in \mathbb{Z}$. Kreck and Stolz associate to each such M^7 three homeomorphism invariants

$$\bar{s}_i(M^7) \;\in\; \mathbb{Q}/\mathbb{Z} \quad \text{for } i = 1,2,3$$

and three diffeomorphism invariants

$$s_i(M^7) \;\in\; \mathbb{Q}/\mathbb{Z} \quad \text{for } i = 1,2,3$$

which taken together completely determine the homeomorphism and diffeomorphism type of M^7. As an application of the general theory they compute these invariants [KreSt1, KreSt2] for the Wang-Ziller spaces $W_{k,l} = (S^3 \times S^5)/T_{kl}$ [WanZi], and for the Aloff-Wallach spaces $M_{k,l}$. In particular, they discovered that there are examples of Aloff-Wallach spaces which are homeomorphic but not diffeomorphic [KreSt2].

Thus, Theorem 8.1 implies that the Kreck-Stoltz invariants exist and determine the homeomorphism and diffeomorphism type of the $\mathcal{S}(p_1,p_2,p_3)$ spaces. Once again, it is necessary to compute $s_i(\mathcal{S}(p_1,p_2,p_3))$ and $\bar{s}_i(\mathcal{S}(p_1,p_2,p_3))$ as explicit functions of (p_1,p_2,p_3) to classify these manifolds. Given the parallels between our manifolds as bi-quotients of $SU(3)$ by circle subgroups of $S(U(3) \times U(3))$ and the Aloff-Wallach spaces, it is reasonable to also expect examples that are homeomorphic but not diffeomorphic in our case. Notice that, in some sense, our examples are more plentiful in that every odd integer is realized as the order of $H^4(\mathcal{S}(p_1,p_2,p_3);\mathbb{Z})$ for some $\mathbf{p}$. Also it is worth noting that the two sequences of 3-Sasakian 7-manifolds

$$\big\{\mathcal{S}(c,c+1,c+2)\big\}_{c\in\mathbb{Z}_{\rm odd}} \qquad \text{and} \qquad \big\{\mathcal{S}(1,1,\frac{(3c^2+6c+1)}{2})\big\}_{c\in\mathbb{Z}_{\rm odd}},$$

are indistinguishable in terms of their integral cohomology rings. Yet geometrically, for each fixed c, the corresponding pairs are very different. For instance, the spaces of leaves are not the same, and the 3-Sasakian metrics are clearly not isometric. Also, as c tends to infinity in the ρ^* Cheeger distance, it is clear that the limits are different. There are many more sub-families in $\mathcal{S}(p_1,p_2,p_3)$ with similar interesting properties. All this makes the homeomorphism and diffeomorphism classification problem for these spaces even more intriguing.

Finally, it seems likely that for $\mathbf{p},\mathbf{p}'\in(\mathbb{Z}^*)^n$ the Riemannian manifolds $(S(\mathbf{p}),g(\mathbf{p}))$ and $(S(\mathbf{p}'),g(\mathbf{p}'))$ are isometric if and only if there is an element w in the Weyl group W_n of $Sp(n)$ such that $\mathbf{p}'=w\mathbf{p}$.

More recently, Kreck and Stoltz [KS3] considered a $\mathbb{Q}$-valued invariant $s(M,g)$ for positive scalar curvature metrics on closed $(4k-1)$-dimensional spin manifolds M with vanishing real Pontriagin classes. They computed this invariant for the Wang-Ziller spaces $W_{k,l}$ and for the Aloff-Wallach spaces $M_{k,l}$. They showed that there are manifolds in the $W_{k,l}$ family for which the moduli spaces $\mathcal{R}^+_{Ric}(M)/Diff(M)$ of Riemannian metrics of positive Ricci curvature have infinitely many components. Furthermore, they showed that there are manifolds in the $M_{k,l}$ family for which the moduli space $\mathcal{R}^+_{sec}(M)/Diff(M)$ of metrics of positive sectional curvature is not connected. Each $\mathcal{S}(p_1,p_2,p_3)$ manifold admits two Einstein metrics of positive scalar curvature. These are in the same component of the moduli space $\mathcal{R}^+_{scal}(M)/Diff(M)$. As shown in section 9, some of the $\mathcal{S}(p_1,p_2,p_3)$ spaces admit metrics of positive sectional curvature as well. In principle, one could compute the $|s|$-invariant of Kreck and Stoltz [KreSt3] for our spaces. It would be interesting to see if there are manifolds (perhaps in the $\mathcal{S}(c,c+1,c+2)$ family?) for which the moduli space of positive sectional curvature metrics $\mathcal{R}^+_{sec}(M)/Diff(M)$ has more than one component.

Lastly, how much of the discussion above for the 7-dimensional case extends to the general $(4n-5)$-dimensional situation? For example, should one expect some strongly inhomogeneous Einstein manifolds among $\mathcal{S}(\mathbf{p})$? Is it possible that $\mathcal{S}(\mathbf{p})$ admit metrics of positive sectional curvature? This last question seems particularly intriguing as the only known simply connected manifolds of dimension >24 admitting such metrics are spheres and projective spaces over $\mathbb{C}$ and $\mathbb{H}$.

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