New Einstein Metrics on $8\#(S^2 \times S^3)$

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Abstract: We show that $\#8(S^2\times S^3)$ admits two 8-dimensional complex family of inequivalent non-regular Sasakian-Einstein structures. These are the first known non-regular Sasakian-Einstein metrics on this 5-manifold.

Introduction

Recently, the authors [BG2] introduced a new method for showing the existence of Sasakian-Einstein metrics on compact simply connected 5-dimensional spin manifolds. This method was based on work of Demailly and Kollár [DK] who gave sufficient algebraic conditions on log del Pezzo surfaces anticanonically embedded in weighted projective spaces to guarantee the existence of a Kähler-Einstein orbifold metric. This, in turn enabled us to construct Sasakian-Einstein metrics on certain S^1 V-bundles over these log del Pezzo surfaces. One could then use known monodromy techniques on the links of isolated hypersurface singularities together with a classification result of Smale [Sm] to identify the 5-manifold. The Demailly and Kollár methods were further developed by Johnson and Kollár [JK] where a computer code was written to solve the algebraic equations. The authors in collaboration with M. Nakamaye [BGN1,BGN2] were then able to construct many Sasakian-Einstein metrics on certain connected sums of $S^2 \times S^3$ as well as modify the Johnson-Kollár computer code to handle more general log del Pezzo surfaces with higher Fano index. The original Johnson-Kollár list contained several examples where the existence of a Kähler-Einstein metric was still in question. One of these was treated in [BGN2] while two more were handled recently by C. Araujo [Ar]. It is the purpose of this note to show that the two new anticanonically embedded log del Pezzo surfaces shown in [Ar] to admit Kähler-Einstein metrics can be used to construct families of new Sasakian-Einstein metrics on $8\#(S^2 \times S^3)$.

It is well-known [FK, BFGK] that $8\#(S^2 \times S^3)$ admits regular Sasakian-Einstein metrics; in fact, it admits an 8 complex dimensional family of Sasakian-Einstein metrics [BGN1]. But up until now it was not known whether there are any non-regular Sasakian-Einstein metrics on $8\#(S^2 \times S^3)$. In this note we prove the following:

Theorem A: *The 5-manifold $8\#(S^2 \times S^3)$ admits two families of non-regular Sasakian-Einstein metrics. Each family depends on 8 complex parameters. Hence, $8\#(S^2 \times S^3)$ admits 3 distinct 8 complex parameter families of Sasakian-Einstein metrics, one regular and two non-regular families. These metrics are all inequivalent as Riemannian metrics.*

1. Sasakian Structures on Links of Isolated Hypersurface Singularities

Although the purpose of this note is to describe the Sasakian-Einstein geometry of $8\#(S^2 \times S^3)$ we shall begin with a very brief summary of the Sasakian and Sasakian-Einstein geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. For more details we refer the reader to [BG2, BGN1, BGN2]. Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action $\mathbb{C}^*_{\mathbf{w}}$ given by $(z_0,\ldots,z_n) \mapsto (\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n)$, where the *weights* w_j are positive integers. It is convenient to view

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1.1 $$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n),$$

where $d \in \mathbb{Z}^+$ is the degree of f. We are interested in the *weighted affine cone* C_f defined by the equation $f(z_0, \ldots, z_n) = 0$. We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity, in fact the only singularity, of f. Then the link L_f defined by

1.2 $$L_f = C_f \cap S^{2n+1},$$

where

$$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} | \sum_{j=0}^{n} |z_j|^2 = 1\}$$

is the unit sphere in $\mathbb{C}^{n+1}$, is a smooth manifold of dimension $2n-1$. Furthermore, it is well-known [Mil] that the link L_f is $(n-2)$-connected.

Recall that a Sasakian structure consists of a quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ where g is a Riemannian metric, ξ is a unit length Killing vector field, η is a contact 1-form such that ξ is its Reeb vector field, and Φ is a $(1,1)$ tensor field which annihilates ξ and describes an integrable complex structure on the contact vector bundle $\mathcal{D} = \ker \eta$. On S^{2n+1} there is a well-known "weighted" Sasakian structure $\mathcal{S}_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$, where the vector field $\xi_{\mathbf{w}}$ is the infinitesimal generator of the circle subgroup $S^1_{\mathbf{w}} \subset \mathbb{C}^*_{\mathbf{w}}$. This Sasakian structure on S^{2n+1} induces a Sasakian structure, also denoted by $\mathcal{S}_{\mathbf{w}}$, on the link L_f. (See [YK, BG2, BGN1] for details. The quotient space $\mathcal{Z}_f$ of S^{2n+1} by $S^1_{\mathbf{w}}$, or equivalently the space of leaves of the characteristic foliation $\mathcal{F}_\xi$ of $\mathcal{S}_{\mathbf{w}}$, is a compact Kähler orbifold which is a projective algebraic variety embedded in the weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, w_1, \cdots, w_n)$, in such a way that there is a commutative diagram

$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 V-bundles and orbifold Riemannian submersions.

We are interested in deformations of the Sasakian structure that leaves the Reeb vector field invariant. Such deformations of a given Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ are obtained by adding to η a continuous one parameter family of basic1-forms ζ_t. We require that the 1-form $\eta_t = \eta + \zeta_t$ satisfy the conditions

1.5 $$\eta_0 = \eta, \qquad \zeta_0 = 0, \qquad \eta_t \wedge (d\eta_t)^n \neq 0 \ \ \forall \ \ t \in [0,1].$$

Since ζ_t is basic ξ is the Reeb (characteristic) vector field associated to η_t for all t. This gives rise to a Sasakian structure $\mathcal{S}_t = (\xi, \eta_t, \Phi_t, g_t)$ for all $t \in [0,1]$ that has the same underlying contact structure and the same characteristic foliation. In general these structures are inequivalent and the moduli space of Sasakian structures having the same characteristic vector field is infinite dimensional.

Suppose now we have a link L_f with a given Sasakian structure (ξ,η,Φ,g). When can we find a 1-form ζ such that the deformed structure $(\xi,\eta+\zeta,\Phi',g')$ is Sasakian-Einstein? This is a Sasakian version [BGN3] of the Calabi problem and its solution is equivalent to solving the corresponding Calabi problem on the space of leaves $\mathcal{Z}_f$. Since a Sasakian-Einstein manifold necessarily has positive Ricci tensor, its Sasakian structure is necessarily positive. This also implies that the Kähler structure on $\mathcal{Z}_f$ be positive, i.e. $c_1(\mathcal{Z}_f)$ can be represented by a positive definite $(1,1)$ form. In this case there are well-known obstructions to solving to solving the Calabi problem. These obstructions for finding a solution to the Monge-Ampere equations involve the non-triviality of certain *multiplier ideal sheaves* [Na, DK] associated with effective anti-canonical $\mathbb{Q}$-divisors on the space of leaves $\mathcal{Z}_f$. Consequently, if one can show that these multiplier ideal sheaves coincide with the full structure sheaf, one obtains the existence of a positive Kähler-Einstein metric on $\mathcal{Z}_f$ and hence, a Sasakian-Einstein metric on L_f.

2. The Construction

The proof of Theorem A is based first on previous work of the authors (and M. Nakamaye) [BG1,BGN1], which in turn is based on the work of Demailly and Kollár [DK] and Johnson and Kollár [JK], and second on the recent work of C. Araujo [Ar]. Indeed, in [Ar] it is shown that the two log del Pezzo surfaces $\mathcal{Z}_{10}\subset\mathbb{P}(1,2,3,5)$ and $\mathcal{Z}_{15}\subset\mathbb{P}(1,3,5,7)$ admit Kähler-Einstein metrics. The weighted homogeneous polynomials f_{10} and f_{15} describing these log del Pezzo surfaces consist of 17 and 19 monomials, respectively. It is assumed that the coefficients of the monomials are such that the del Pezzo surface is quasi-smooth. Moreover, the coefficient of the $z_1z_2z_3$ term in the $\mathcal{Z}_{15}$ surface is non-vanishing, for otherwise it is not known whether $\mathcal{Z}_{15}$ admits a Kähler-Einstein metric [Ar]. These log del Pezzo surfaces are orbifolds and the total space of the circle V-bundles over them are the links of isolated hypersurface singularities in $\mathbb{C}^4$ described by the corresponding weighted homogeneous polynomial. Combining our previous work [BG1,BG2] with Theorem 4.1 of [Ar] gives

LEMMA 2.1: *The total spaces L_{10} and L_{15} of the V-bundles over $\mathcal{Z}_{10}$ and $\mathcal{Z}_{15}$, respectively whose first Chern classes are represented by the Kähler form on the corresponding log del Pezzo surface admit Sasakian-Einstein metrics.*

Next we need to identify the links L_{10} and L_{15}. First we mention that L_{10} and L_{15} are inequivalent as links. This follows by computing their Milnor numbers

$$\mu=\mu(L_d)=\prod_{i=0}^{4}\Big(\frac{d}{w_i}-1\Big).$$

We find $\mu(L_{10})=84$ and $\mu(L_{15})=128$.

LEMMA 2.2: *The links L_{10} and L_{15} are diffeomorphic to* $8\#(S^2\times S^3)$.

PROOF: The result was stated without proof in [BGN1]. First we notice that in both cases the weights are pairwise relatively prime, so the hypersurfaces $\mathcal{Z}_{10}$ and $\mathcal{Z}_{15}$ are well-formed. Thus, by Lemma 5.8 of [BG2] there is no torsion in $H_2(L,\mathbb{Z})$. Moreover, it is well known that the links of an isolated hypersurface are always simply connected in this dimension. Then since by Lemma 1 L_{10} and L_{15} admit a Sasakian-Einstein metric, it follows from Corollary 2.1.6 of [BG1] they are both spin. It then follows from a well known theorem of Smale [Sm] that both L_{10} and L_{15} must be of the form $k\#(S^2\times S^3)$ for some positive integer k. To find k we compute the second Betti number by the method of Milnor and Orlik [MO]. We give the details for the case L_{15} as L_{10} is similar. Let $\Delta(t)$ denote the Alexander polynomial of the link L_{15}, and let Div $\Delta(t)$ denote its divisor in the group ring $\mathbb{Z}[\mathbb{C}^*]$. Then Div $\Delta(t)$ takes the form

$$\text{Div }\Delta(t)=1+\sum_i a_i\Lambda_i$$

for some $a_i\in\mathbb{Z}$ where $\Lambda_i=\text{Div }(t^i-1)$. Then the second Betti number is given by

$$b_2(L_f)=1+\sum_i a_i.$$

The Milnor-Orlik procedure gives

$$\begin{aligned}\text{Div }\Delta(t)=&(\Lambda_{15}-1)(\frac{\Lambda_{15}}{7}-1)(\Lambda_5-1)(\Lambda_3-1)\\ =&(\Lambda_{15}+1)(\Lambda_{15}-\Lambda_5-\Lambda_3+1)\\ =&9\Lambda_{15}-\Lambda_5-\Lambda_3+1.\end{aligned}$$

Thus, $k=b_2(L_{15})=8$. Here we have used the relations $\Lambda_a\Lambda_b=\gcd(a,b)\Lambda_{lcm(a,b)}$. ∎

3. Moduli of Sasakian-Einstein Structures on $8\#(S^2\times S^3)$

Next we determine the effective parameters in the families of Sasakian-Einstein metrics. Recall that $\mathbb{P}(\mathbf{w})$ can be defined as a scheme $\text{Proj}(S(\mathbf{w}))$, where

$$S(\mathbf{w})=\bigoplus_d S^d(\mathbf{w})=\mathbb{C}[z_0,z_1,z_2,z_3].$$

The ring of polynomials $\mathbb{C}[z_0,z_1,z_2,z_3]$ is graded with grading defined by the weights $\mathbf{w}=(w_1,w_1,w_2,w_3)$. As a projective variety we can embed $\mathbb{P}(\mathbf{w})\subset\mathbb{CP}^N$ and then the group $\mathfrak{G}_{\mathbf{w}}$ is a subgroup of $PGL(N,\mathbb{C})$. Note that $\mathbb{P}(\mathbf{w})$ is a toric variety and we can describe its complex automorphism group $\mathfrak{G}_{\mathbf{w}}$ explicitly as follows: Let $\mathbf{w}=(w_0,w_1,w_2,w_3)$ be ordered with $w_0\le w_1\le w_2\le w_3$. We consider the group $G(\mathbf{w})$ of automorphisms of the graded ring $S(\mathbf{w})$ defined on generators by

$$\varphi_{\mathbf{w}}\begin{pmatrix}z_0\\ z_1\\ z_2\\ z_3\end{pmatrix}=\begin{pmatrix}f_0^{(w_0)}(z_0,z_1,z_2,z_3)\\ f_1^{(w_1)}(z_0,z_1,z_2,z_3)\\ f_2^{(w_2)}(z_0,z_1,z_2,z_3)\\ f_3^{(w_3)}(z_0,z_1,z_2,z_3)\end{pmatrix},$$

where $f_i^{(w_i)}(z_0,z_1,z_2,z_3)$ is an arbitrary weighted homogeneous polynomial of degree w_i in (z_0,z_1,z_2,z_3). This is a finite dimensional Lie group and it is a subgroup of $GL(N,\mathbb{C})$. Projectivising, we get $\mathfrak{G}_{\mathbf{w}}=\mathbb{P}_{\mathbb{C}}(G(\mathbf{w}))$.

Note that when $\mathbf{w}=(1,1,1,1)$ then $G(\mathbf{w})=GL(4,\mathbb{C})$ and when $\mathbf{w}=(1,1,1,k)$ then $G(\mathbf{w})$ contains $GL(3,\mathbb{C})$. Other than these case three weights are never the same if $\mathbb{P}(\mathbf{w})$ is well-formed. If two weights coincide then $G(\mathbf{w})$ contains $GL(2,\mathbb{C})$ as a subgroup. Finally, when all weights are distinct we can write

$$\varphi_{\mathbf{w}}\begin{pmatrix} z_0\\ z_1\\ z_2\\ z_3\end{pmatrix}=\begin{pmatrix} a_0z_0\\ a_1z_1+f_1^{(w_1)}(z_0)\\ a_2z_2+f_2^{(w_2)}(z_0,z_1)\\ a_3z_3+f_3^{(w_3)}(z_0,z_1,z_2)\end{pmatrix}$$

where $(a_0,a_1,a_2,a_3)\in(\mathbb{C}^*)^4$ and $f_i^{(w_i)},\ \ i=1,2,3$ are weighted homogeneous polynomials of degree w_i. The simplest situation occurs when $f_1=f_2=f_3$ are forced to vanish. Then $\mathfrak{G}_{\mathbf{w}}=(\mathbb{C}^*)^3$ is the smallest it can possibly be as $\mathbb{P}(\mathbf{w})$ is toric.

Let $S^d_{\mathbf{w}}\subset S(\mathbf{w})$ be the vector subspace spanned by all monomials in (z_0,z_1,z_2,z_3) of degree $d=|w|-I$, and let $\hat{S}^d(\mathbf{w})\subset S^d(\mathbf{w})$ denote subset all quasi-smooth elements. Then we define $m^d_{\mathbf{w}}$ to be the dimension of the subspace generated by $\hat{S}^d_{\mathbf{w}}$. Now the automorphism group $G(\mathbf{w})$ acts on $S^d_{\mathbf{w}}$ leaving the subset $\hat{S}^d(\mathbf{w})$ of quasi-smooth polynomials invariant. Thus, for each log del Pezzo surface we define the moduli space

$$\mathcal{M}^d_{\mathbf{w}}=\hat{S}^d_{\mathbf{w}}/G(\mathbf{w})=\mathbb{P}(\hat{S}^d_{\mathbf{w}})/\mathfrak{G}_{\mathbf{w}},$$

with $n^d_{\mathbf{w}}=\dim(\mathcal{M}^d_{\mathbf{w}})$. Now there is an injective map

$$\mathcal{M}^d_{\mathbf{w}}\longrightarrow\mathcal{M}^{\mathbb{C}}(\mathcal{Z}_{\mathbf{w}}),$$

and each element in $\mathcal{M}^d_{\mathbf{w}}$ corresponds to a unique homothety class of Kähler-Einstein metrics modulo $\mathfrak{G}_{\mathbf{w}}$ and hence, to a unique Sasakian-Einstein structure on the corresponding 5-manifold modulo the group $\mathfrak{G}_{\mathbf{w}}$ acting as CR automorphisms.

Let us now consider our $\mathcal{Z}_{10}$. It is easy to see that $S^{10}(1,2,3,5)$ is isomorphic to $\mathbb{C}^{20}$ and it is spanned by the monomials $z_3^2,\ z_1z_2z_3,\ z_1^2z_2^2,\ z_1^5,\ z_0z_2^3,\ z_0z_1^2z_3,\ z_0z_1^3z_2,\ z_0^2z_2z_3,$ $z_0^2z_1z_2^2,\ z_0^2z_1^4,\ z_0^3z_1z_3,\ z_0^3z_1^2z_2,\ z_0^4z_2^2,\ z_0^4z_1^3,\ z_0^5z_3,\ z_0^5z_1z_2,\ z_0^6z_1^2,\ z_0^7z_2,\ z_0^8z_1,\ z_0^{10}$. We take the open submanifold $\hat{S}^{10}(1,2,3,5)\subset S^{10}(1,2,3,5)$. This is acted on by the complex automorphism group, namely the group $G(1,2,3,5)$ generated by

$$\varphi_{\mathbf{w}}\begin{pmatrix} z_0\\ z_1\\ z_2\\ z_3\end{pmatrix}=\begin{pmatrix} a_0^1z_0\\ a_1^1z_1+a_1^2z_0^2\\ a_2^1z_2+a_2^2z_0^2+a_2^3z_0z_1\\ a_3^1z_3+a_3^2z_0^5+a_3^3z_0^3z_1+a_3^3z_0^2z_2+a_3^4z_0z_1^2+a_3^5z_2z_1\end{pmatrix},$$

where $a_i^1\in\mathbb{C}^*$ and all other coefficients are in $\mathbb{C}$. $G(1,2,3,5)$ is a 12-dimensional complex Lie group acting on the open submanifold $\hat{S}^{10}(1,2,3,5)\subset S^{10}(1,2,3,5)\approx\mathbb{C}^{20}$. It follows that the quotient is an 8-dimensional complex manifold which by the Bando-Mabuchi Theorem [BM] is the moduli space of positive Kähler-Einstein metrics on the underlying compact orbifold $\mathcal{Z}_{10}$.

Similarly for $\mathcal{Z}_{15}$ it is easy to see that $S^{15}(1,3,5,7)$ is isomorphic to $\mathbb{C}^{19}$ and that it is spanned by the monomials $z_2^3,\ z_1z_2z_3,\ z_1^5,\ z_0z_3^2,\ z_0z_1^3z_2,\ z_0^2z_1z_2^2,\ z_0^2z_1^2z_3,\ z_0^3z_2z_3,\ z_0^3z_1^4,$

$z_0^4z_1^2z_2,\ z_0^5z_2^2,\ z_0^5z_1z_3,\ z_0^6z_1^3,\ z_0^7z_1z_2,\ z_0^8z_3,\ z_0^9z_1^2,\ z_0^{10}z_2,\ z_0^{12}z_1,\ z_0^{15}$. The open submanifold $\hat{S}^{15}(1,3,5,7)\subset S^{15}(1,3,5,7)$ of quasi-smooth elements is under the action of the complex automorphism group, namely the group $G(1,3,5,7)$ generated by

$$\varphi_{\mathbf{w}}\begin{pmatrix} z_0\\ z_1\\ z_2\\ z_3\end{pmatrix} = \begin{pmatrix} a_0^1z_0\\ a_1^1z_1+a_1^2z_0^3\\ a_2^1z_2+a_2^2z_0^5+a_2^3z_0^2z_1\\ a_3^1z_3+a_3^2z_0^7+a_3^3z_1z_0^4+a_3^4z_2^2z_0^2+a_3^5z_1^2z_0\end{pmatrix},$$

where $a_i^1\in\mathbb{C}^*$ and all other coefficients are in $\mathbb{C}$. $G(1,3,5,7)$ is a 11-dimensional complex Lie group acting on the open submanifold $\hat{S}^{15}(1,3,5,7)\subset S^{15}(1,3,5,7)\approx\mathbb{C}^{19}$. It follows that the quotient is an 8-dimensional complex manifold. However, because of the condition in [Ar] that the coefficient of the term $z_1z_2z_3$ in the polynomial f_{15} be non-vanishing, it is not known whether all the elements of $\hat{S}^{15}(1,3,5,7)$ admit a Kähler-Einstein metric. Nevertheless, it is shown in [Ar] that there is a positive Kähler-Einstein metric on the open submanifold $\tilde{S}^{15}(1,3,5,7)$ defined by demanding that the coefficient of the $z_1z_2z_3$ term is not zero. It then follows from [BGN1] that the moduli space has complex dimension 8.

Thus for both f_{10} and f_{15} we obtain an 8-complex dimensional family of Sasakian-Einstein metrics on $8\#(S^2\times S^3)$. So the moduli space of Sasakian-Einstein metrics on $8\#(S^2\times S^3)$ has at least three 8-complex dimensional families of Sasakian-Einstein metrics one of which is regular and the other two non-regular. In a forthcoming work we show that these three families belong to distinct components. It also follows as in [BGN1] that the metrics are inequivalent as Riemannian metrics. This completes the proof of Theorem A. ∎

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