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On the Geometry of Sasakian-Einstein 5-Manifolds

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Abstract: On simply connected five manifolds Sasakian-Einstein metrics coincide with Riemannian metrics admitting real Killing spinors which are of great interest as models of near horizon geometry for three-brane solutions in superstring theory [KW]. We expand on the recent work of Demailly and Kollár [DK] and Johnson and Kollár [JK1] who give methods for constructing Kähler-Einstein metrics on log del Pezzo surfaces. By [BG1] circle V-bundles over log del Pezzo surfaces with Kähler-Einstein metrics have Sasakian-Einstein metrics on the total space of the bundle. Here these simply connected 5-manifolds arise as links of isolated hypersurface singularities which by the well known work of Smale [Sm] together with [BG3] must be diffeomorphic to $S^5\#l(S^2\times S^3)$. More precisely, using methods from Mori theory in algebraic geometry we prove the existence of 14 inequivalent Sasakian-Einstein structures on $S^2\times S^3$ and infinite families of such structures on $\#l(S^2\times S^3)$ with $2\leq l\leq 7$. We also discuss the moduli problem for these Sasakian-Einstein structures.

0. Introduction

Surprisingly little is known about complete Einstein metrics on compact 5-manifolds. Until now one could list only two constructions of such metrics. The more recent one was developed by Böhm [Bö] who obtained cohomogeneity one complete Einstein metrics on S^5 and $S^2\times S^3$. On the other hand, the well-known irreducible homogeneous Einstein metrics on these two spaces are the simplest examples of what is called a Sasakian-Einstein structure. Roughly, a Sasakian structure on a manifold is a metric contact structure whose Reeb vector field is a unit Killing field and whose transverse geometry is Kähler. In fact, any complex surface whose metric is Kähler-Einstein and of positive scalar curvature admits a unique simply connected circle bundle which is canonically Sasakian-Einstein. Since all such del Pezzo surfaces $\mathcal{Z}$ admitting Kähler-Einstein metrics are known [Siu,Ti1-3, TY] one gets the list and the classification statement of the first theorem of the introduction. Purely in the context of Einstein metrics this construction is due to Kobayashi [Be] whereas the realization that $\mathcal{S}$ is actually a 5-manifold with real Killing spinors and, hence, Sasakian-Einstein space came later in the work of Friedrich and Kath [FK]. More generally, the importance of Killing spinors was already realized in 1980 in the context

During the preparation of this work the first two authors were partially supported by NSF grant DMS-9970904, and third author by NSF grant DMS-0070190. 2000 Mathematics Classification: 53C25, 53C12, 14E30 of the spectrum of that Dirac operator [F]. A classification of Riemannian manifolds admitting real Killing spinors was finally obtained by Bär [Bä] who observed that the real Killing spinors on M correspond to the parallel spinors on $\mathcal{C}(M) = (\mathbb{R}^+ \times M, dt^2 + t^2 g)$ the metric cone on M. This simple fact allowed for an elegant formulation of the problem in the powerful language of holonomy groups.

The physicists' interest in five and seven-dimensional manifolds admitting real Killing spinors dates back to the early eighties, where Kaluza-Klein models played a central role in the supergravity theory. Today we witness a renewed interest in these manifolds in the context of p-brane solutions in superstring theory. These so-called p-branes, "near the horizon" are modeled by the pseudo-Riemannian geometry of the product $\mathrm{adS}_{p+2} \times M$, where adS_{p+2} is the $(p+2)$-dimensional anti-de-Sitter space (a Lorentzian version of a space of constant sectional curvature) and (M, g) is a Riemannian manifold of dimension $d = D - p - 2$. Here D is the dimension of the original supersymmetric theory. In the most interesting cases of M2-branes, M5-branes, and D3-branes D equals to either 11 (Mp-branes of M-theory) or 10 (Dp-branes in type IIA or type IIB string theory). String theorists are particularly interested in those vacua of the form $\mathrm{adS}_{p+2} \times M$ that preserve some residual supersymmetry. It turns out that this requirement imposes constraints on the geometry of the Einstein manifold M which is forced to admit real Killing spinors. Depending on the dimension d, the possible geometries of M are as follows: nearly Kähler for $d = 6$, weak G_2 holonomy for $d = 7$, Sasakian-Einstein for $d = 2k + 1$, and 3-Sasakian for $d = 4k + 3$ [AFHS, MP]. Furthermore, given a p-brane solution of the above type, the interpolation between $\mathrm{adS}_{p+2} \times M$ and $\mathbb{R}^{p,1} \times \mathcal{C}(M)$ leads to a conjectured duality between supersymmetric background of the form $\mathrm{adS}_{p+2} \times M$ and a $(p+1)$-dimensional superconformal field theory of n coincident p-branes located at the conical singularity of the $\mathbb{R}^{p,1} \times \mathcal{C}(M)$ vacuum. This is a generalized version of the Maldacena's conjecture.

In the case of D3-branes of string theory the relevant near horizon geometry is that of $\mathrm{adS}_5 \times M$, where M is a Sasakian-Einstein 5-manifold. The D3-brane solution interpolates between $\mathrm{adS}_5 \times M$ and $\mathbb{R}^{3,1} \times \mathcal{C}(M)$, where the cone $\mathcal{C}(M)$ is a Calabi-Yau threefold. In its original version the Maladacena conjecture (also known as AdS/CFT duality) states that the 't Hooft large n limit of $N = 4$ supersymmetric Yang-Mills theory with gauge group $SU(n)$ is dual to type IIB superstring theory on $\mathrm{adS}_5 \times S^5$ [Ma, Wi]. This conjecture was further examined by Klebanov and Witten [KW] for the type IIB theory on $\mathrm{adS}_5 \times T^{1,1}$, where $T^{1,1}$ is the other homogeneous Sasakian-Einstein 5-manifold $T^{1,1} = S^2 \times S^3$ and the Calabi-Yau 3-fold $\mathcal{C}(T^{1,1})$ is simply the quadric cone in $\mathbb{C}^4$. Using the well-known fact that $\mathcal{C}(T^{1,1})$ is a Kähler quotient of $\mathbb{C}^4$ (or, equivalently, that $S^2 \times S^3$ is a Sasakian-Einstein quotient of S^7), a dual super Yang-Mills theory was proposed, representing D3-branes at the conical singularities. In the framework of D3-branes and the AdS/CFT duality the question of what are all the possible near horizon geometries M and $\mathcal{C}(M)$ might be of importance. Until this year, the only known examples of such geometries were few and they were exactly the circle bundles over the del Pezzo surfaces (see the comment of Yau in [Y]). This has drastically changed now as we shall show in our paper.

Recently Demailly and Kollár have developed some new techniques to study the existence of Kähler-Einstein metrics on compact Fano orbifolds [DK], and applied their methods to prove the existence of Kähler-Einstein metrics on three log del Pezzo surfaces. Following this, two of the present authors [BG3] showed how one can use the results of [DK] to obtain new Sasakian-Einstein 5-manifolds. Later Johnson and Kollár [JK1] discovered a more efficient way to treat the algebraic equation involved, and with the aid of a computer program gave many more examples of log del Pezzo surfaces in weighted projective 3-spaces, including one infinite series example. All of the [JK1] examples have Fano index equal to one. In the current paper we extend the results of [JK1] to the case of higher index as well as use these results together with the results of [JK1] to construct a plethora of Sasakian-Einstein metrics in dimension five. This is accomplished by realizing the compact simply connected 5-manifolds as links of isolated hypersurface singularities given by weighted homogeneous polynomials in $\mathbb{C}^4$. The projectivization then gives the log del Pezzo surfaces $\mathcal{Z}$ and proving the existence of a Kähler-Einstein metric $\mathcal{Z}$ is then tantamount to proving the existence of an Sasakian-Einstein metric on the 5-manifold [BG1]. It turns out that from the work of Smale [Sm] one can deduce that these 5-manifolds are all of the form $\mathcal{S}_l = \#l(S^2 \times S^3)$ for $1 \leq l \leq 7$. We also show that the moduli of Sasakian-Einstein structures on $\#l(S^2 \times S^3)$ is exceedingly rich. Recall [BG1] that a Sasakian manifold is *quasi-regular* if the leaves of the foliation generated by the unit Killing field ξ are all circles, and it is *regular* if this foliation is a locally trivial fibration. In the latter case the space of leaves $\mathcal{Z}$ is a smooth Kähler manifold while generally in the former $\mathcal{Z}$ has orbifold singularities. First it is known [FK,BG1] that the regular Sasakian-Einstein structures on $\#l(S^2 \times S^3)$ satisfy

THEOREM [FK,BG1]: *Let $\mathcal{S}_l = S^5\#l(S^2 \times S^3)$.*

1) *For each $l = 0, 1, 3, 4$, there is precisely one regular Sasakian-Einstein structure on $\mathcal{S}_l$.*

2) *For each $5 \leq l \leq 8$ there is a $2(l-4)$ complex parameter family of inequivalent regular Sasakian-Einstein structures on $\mathcal{S}_l$.*

3) *For $l = 2$ or $l \geq 9$ there are no regular Sasakian-Einstein structures on $\mathcal{S}_l$.*

The first non-regular Sasakian-Einstein structures on any 5-manifold were given in [BG3]. Here the two of the present authors showed the existence of two inequivalent non-regular Sasakian-Einstein structures on $S^2 \times S^3$, and one non-regular Sasakian-Einstein structure on $\#2(S^2 \times S^3)$. The latter was the first example of any Sasakian-Einstein metric on $\#2(S^2 \times S^3)$. In the present work we use the methods presented in [BG3] as well as those in [JK1] to greatly expand the list of non-regular Sasakian-Einstein structures on simply connected 5-manifolds. Explicitly we prove the following:

THEOREM A:

1) *$S^2 \times S^3$ admits 14 inequivalent non-regular Sasakian-Einstein structures.*

2) *$\#2(S^2 \times S^3)$ admits two distinct 1-complex parameter families plus 21 inequivalent non-regular Sasakian-Einstein structures.*

3) *$\#3(S^2 \times S^3)$ admits two distinct 2-complex parameter families, and four distinct 1-complex parameter families, and one countably infinite family of inequivalent non-regular Sasakian-Einstein structures.*

4) *$\#4(S^2 \times S^3)$ admits two distinct 3-parameter complex families, one 2-complex parameter family, two countably infinite complex families, and two distinct inequivalent non-regular Sasakian-Einstein structures.*

5) $\#5(S^2 \times S^3)$ *admits one 4-complex parameter family, one 3-complex parameter family, and two countably infinite families of inequivalent non-regular Sasakian-Einstein structures.*

6) $\#6(S^2 \times S^3)$ *admits one 5-complex parameter family, one 3-complex parameter family, and two countably infinite families of inequivalent non-regular Sasakian-Einstein structures.*

7) $\#7(S^2 \times S^3)$ *admits a countably infinite series of 5-complex parameter families of inequivalent non-regular Sasakian-Einstein structures.*

Furthermore, inequivalent Sasakian structures correspond to inequivalent Riemannian metrics.

In light of these results an outstanding question is:

QUESTION: *Are there any Sasakian-Einstein structures on* $\#l(S^2 \times S^3)$ *for* $l \geq 9$*?*

It is well-known for the regular case that the second Betti number of any smooth del Pezzo surface is less than or equal to 9. However, in stark contrast to this one can construct log del Pezzo surfaces with any desired second Betti number, but as shown in Theorem 4.5 below the methods used in this paper for proving the existence of Kähler-Einstein metrics apply only to log del Pezzo surfaces with second Betti number less than or equal to 10. Indeed, we shall show in a forthcoming work that our methods prove the existence of a 7-complex parameter family of Kähler-Einstein metrics on a log del Pezzo surface whose second Betti number is 10; hence, proving the existence of a 7-complex parameter family of Sasakian-Einstein structures on $\#9(S^2 \times S^3)$.

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1. The Transverse Geometry of a Sasakian Manifold

In this section we study the transverse geometry of the Riemannian foliation $\mathcal{F}_\xi$ of a Sasakian manifold M. Good references for the transverse geometry of foliations are [Ton] and [Mol]. First we recall some basic notation and definitions. Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection of g, and let $R(X, Y)$ denote the Riemann curvature tensor of ∇.

DEFINITION-PROPOSITION 1.1: *A Riemannian manifold* (M, g) *is called a Sasakian manifold if any one, hence all, of the following equivalent conditions hold:*

(i) There exists a Killing vector field ξ *of unit length on* M *so that the tensor field* Φ *of*

type $(1,1)$, *defined by* $\Phi(X) \;=\; \nabla_X \xi$, *satisfies the condition*

$$(\nabla_X \Phi)(Y) \;=\; g(\xi, Y)X - g(X,Y)\xi$$

for any pair of vector fields X *and* Y *on* M.

(ii) There exists a Killing vector field ξ *of unit length on* M *so that the Riemann curvature satisfies the condition*

$$R(X,\xi)Y \;=\; g(\xi, Y)X - g(X,Y)\xi,$$

for any pair of vector fields X *and* Y *on* M.

(iii) The metric cone on M, $(C(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ *is Kähler with Kähler form* $d(r\eta)$.

We shall refer to the quadruple (ξ, η, Φ, g) as a *Sasakian structure* on M. Associated to any Sasakian structure is the *characteristic foliation* $\mathcal{F}_\xi$ which is the one dimensional foliation generated by the unit vector field ξ, and whose leaves are geodesics with respect to the Sasakian metric g which is bundle-like with respect to $\mathcal{F}_\xi$.

Let (ξ, η, Φ, g) be a Sasakian structure on M, and consider the contact subbundle $\mathcal{D} = \ker\, \eta$. There is an orthogonal splitting of the tangent bundle as

1.2 $$TM = \mathcal{D} \oplus L_\xi,$$

where L_ξ is the trivial line bundle generated by the Reeb vector field ξ. The contact subbundle $\mathcal{D}$ is just the normal bundle to the characteristic foliation $\mathcal{F}_\xi$ generated by ξ. It is naturally endowed with both a complex structure $J = \Phi|\mathcal{D}$ and a symplectic structure $d\eta$. Hence, $(\mathcal{D}, J, d\eta)$ gives M a *transverse Kähler* structure with Kähler form $d\eta$ and metric $g_{\mathcal{D}}$ defined by

1.3 $$g_{\mathcal{D}}(X,Y) = d\eta(X, JY)$$

which is related to the Sasakian metric g by

$$g = g_{\mathcal{D}} \oplus \eta \otimes \eta.$$

Recall [Ton] that a smooth p-form α on M is called *basic* if

1.4 $$\xi \rfloor \alpha = 0, \qquad \mathcal{L}_\xi \alpha = 0,$$

and we let Λ^p_B denote the sheaf of germs of basic p-forms on M, snd by Ω^p_B the set of global sections of Λ^p_B on M. The sheaf Λ^p_B is a module under the ring, Λ^0_B, of germs of smooth basic functions on M. We let $C^\infty_B(M) = \Omega^0_B$ denote global sections of Λ^0_B, i.e. the ring of smooth basic functions on M. Since exterior differentiation preserves basic forms we get a de Rham complex

1.5 $$\cdots \longrightarrow \Omega^p_B \xrightarrow{d} \Omega^{p+1}_B \longrightarrow \cdots$$

whose cohomology $H^*_B(\mathcal{F}_\xi)$ is called the *basic cohomology* of $(M,\mathcal{F}_\xi)$. The basic cohomology ring $H^*_B(\mathcal{F}_\xi)$ is an invariant of the foliation $\mathcal{F}_\xi$ and hence, of the Sasakian structure on M. It is related to the ordinary de Rham cohomology $H^*(M,\mathbb{R})$ by the long exact sequence [Ton]

$$1.6 \qquad \cdots \longrightarrow H^p_B(\mathcal{F}_\xi) \longrightarrow H^p(M,\mathbb{R}) \xrightarrow{j_p} H^{p-1}_B(\mathcal{F}_\xi) \xrightarrow{\delta} H^{p+1}_B(\mathcal{F}_\xi) \longrightarrow \cdots$$

where δ is the connecting homomorphism given by $\delta[\alpha]_B = [d\eta\wedge\alpha]_B = [d\eta]_B \cup [\alpha]_B$, and j_p is the composition of the map induced by $\xi\rfloor$ with the well known isomorphism $H^r(M,\mathbb{R}) \approx H^r(M,\mathbb{R})^{S^1}$ where $H^r(M,\mathbb{R})^{S^1}$ is the S^1-invariant cohomology defined from the S^1-invariant r-forms $\Omega^r(M)^{S^1}$. Here we denote cohomology classes in $H^p_B(\mathcal{F}_\xi)$ by $[\cdot]_B$ in order to distinquish them from the ordinary cohomology classes. We also note that $d\eta$ is basic even though η is not.

Next we exploit the fact that the transverse geometry is Kähler [ElK]. Let $\mathcal{D}_\mathbb{C}$ denote the complexification of $\mathcal{D}$, and decompose it into its eigenspaces with respect to J, that is, $\mathcal{D}_\mathbb{C} = \mathcal{D}^{1,0}\oplus\mathcal{D}^{0,1}$. Similarly, we get a splitting of the complexification of the sheaf Λ^1_B of basic one forms on M, namely

$$\Lambda^1_B\otimes\mathbb{C} = \Lambda^{1,0}_B\oplus\Lambda^{0,1}_B.$$

We let $\Lambda^{p,q}_B$ denote the sheaf of germs of basic forms of type (p,q), and as in the usual case there is a splitting

$$1.7 \qquad \Lambda^r_B\otimes\mathbb{C} = \bigoplus_{p+q=r}\Lambda^{p,q}_B,$$

as well as the *basic Dolbeault complex*

$$1.8 \qquad 0\longrightarrow\Lambda^{p,0}_B \xrightarrow{\bar\partial} \Lambda^{p,1}_B \xrightarrow{\bar\partial} \cdots \longrightarrow \Lambda^{p,n}_B \longrightarrow 0,$$

together with its *basic Dolbeault cohomology groups* $H^{p,q}_B(\mathcal{F}_\xi)$. Most of the usual results about Kähler geometry carry over to transverse Kähler geometry [ElK]. In particular, we have the following Proposition which follows from transverse Kähler geometry:

PROPOSITION 1.9: *Let (M,ξ,η,Φ,g) be a compact Sasakian manifold of dimension $2n+1$. Then we have*

(1) $H^{n,n}_B(\mathcal{F}_\xi)\approx H^{2n}_B(\mathcal{F}_\xi)\approx\mathbb{R}$.

(2) The class $[d\eta]_B\in H^{1,1}_B(\mathcal{F}_\xi)$ *is nontrivial.*

(3) $H^{p,p}_B(\mathcal{F}_\xi)>0$.

(4) $H^{2p+1}_B(\mathcal{F}_\xi)$ *has even dimension.*

(5) $H^1(M,\mathbb{R})\approx H^1_B(\mathcal{F}_\xi)$.

(6) $H^r_B(\mathcal{F}_\xi)=\bigoplus_{p+q=r}H^{p,q}_B(\mathcal{F}_\xi)$.

(7) Complex conjugation induces an anti-linear isomorphism $H^{p,q}_B(\mathcal{F}_\xi)\approx H^{q,p}_B(\mathcal{F}_\xi)$.

(8) If ω *is a closed real* $(1,1)$ *form such that* $\omega\in[d\eta]_B$*, then there exists a smooth basic function* ϕ *such that* $\omega=d\eta+i\partial\bar\partial\phi$.

Proof: (1): The exact sequence 1.6 gives an isomorphism $H^{2n+1}(M)\overset{j_{2n+1}}{\approx}H^{2n}_B(\mathcal{F}_\xi)$. (2): $d\eta$ is a (1,1) form by 1.3, and since $\eta\wedge(d\eta)^n$ is a volume form on M, the isomorphism j_{2n+1} implies that $d\eta$ defines a nontrivial class in $H^2_B(\mathcal{F}_\xi)$. (3) follows since $[d\eta]_B$ cups to a non-trivial element in $H^{n,n}_B(\mathcal{F}_\xi)$. (4): This follows from transverse Hodge theory in the usual way [ElK].

To prove (5) we notice that the beginning of the exact sequence 1.6 is

$$0\longrightarrow H^1_B(\mathcal{F}_\xi)\longrightarrow H^1(M,\mathbb{R})\xrightarrow{j_1}\mathbb{R}\xrightarrow{\delta_0}H^2_B(\mathcal{F}_\xi).$$

By (2) δ_0 is injective, so $\{0\}=\ker\delta_0=\operatorname{im}j_1$; hence, j_1 is the zero map. (6) and (7) follow from (1) and Theorem 3.4.6 of [ElK], and (8) is Proposition 3.5.1 of [ElK]. ∎

Next we define the *transverse Ricci tensor* Ric^T_g of g to be the Ricci tensor of $g_{\mathcal{D}}$. It is related to the Ricci tensor Ric_g of g by

1.10a $$\mathrm{Ric}^T_g=\mathrm{Ric}_g|_{\mathcal{D}\times\mathcal{D}}+2g|_{\mathcal{D}\times\mathcal{D}}.$$

Similarly, we define the *Ricci form* ρ_g and *transverse Ricci form* ρ^T_g by

1.10b $$\rho_g(X,Y)=\mathrm{Ric}_g(X,\Phi Y),\qquad \rho^T_g(X,Y)=\mathrm{Ric}^T_g(X,\Phi Y)$$

for X,Y smooth sections of $\mathcal{D}$. It is easy to check that these are anti-symmetric of type $(1,1)$ and are related by

1.10c $$\rho^T_g=\rho_g+2d\eta.$$

Now the contact subbundle $\mathcal{D}$ is a complex vector bundle and thus has a first Chern class $c_1(\mathcal{D})\in H^2(M,\mathbb{Z})$. Consider the long exact sequence 1.6 together with the natural map $H^2(M,\mathbb{Z})\longrightarrow H^2(M,\mathbb{R})$ whose kernel is the torsion part of $H^2(M,\mathbb{Z})$. From (5) of Proposition 1.9 we have

1.11 $$\begin{tikzcd} & & H^2(M,\mathbb{Z}) \arrow[d] & & \\ 0 \arrow[r] & \mathbb{R} \arrow[r, "\delta"] & H^2_B(\mathcal{F}_\xi) \arrow[r, "\iota_*"] & H^2(M,\mathbb{R}) \arrow[r] & H^1(M,\mathbb{R}) \arrow[r] & \cdots. \end{tikzcd}$$

As in 1.6 the map δ is given by $\delta(c)=c[d\eta]$ where $c\in\mathbb{R}$. Now on a Sasakian manifold the vector bundle $\mathcal{D}^{1,0}$ is holomorphic with respect to the CR-structure, so we can compute the free part of $c_1(\mathcal{D})=c_1(\mathcal{D}^{1,0})$ from the transverse Kähler geometry in the usual way. That is $c_1(\mathcal{D})$ can be represented by a basic real closed $(1,1)$-form ρ_B. The class $c_1^B=[\rho_B]\in H^2_B(\mathcal{F}_\xi)$ is independent of the transverse metric and basic connection used to compute it, and depends only on the foliated manifold $(M,\mathcal{F}_\xi)$ with its CR-structure. It is described in [ElK] and called the *basic first Chern class of* $\mathcal{D}$ there. Alternatively, we can think of c_1^B as the negative of the first Chern class of the "transverse canonical bundle" $K=(\Lambda^{1,0}_B)^n$ of M.

We now consider deformations of the Sasakian structure which fix the basic first Chern class c_1^B. We do this by considering deformations that fix the foliation $\mathcal{F}_\xi$, in fact fix the characteristic vector field ξ. Let η_t be a continuous one parameter family of real1-forms obtained by adding to η a continuous family of basic 1-forms ζ_t so that $\eta_t=\eta+\zeta_t$ satisfies the conditions

1.12 $$\eta_0=\eta,\qquad \zeta_0=0,\qquad \eta_t\wedge(d\eta_t)^n\neq 0\ \ \forall\ \ t\in[0,1].$$

This last non-degeneracy condition implies that η_t is a contact form on M for all $t\in[0,1]$. Then by Gray's Stability Theorem η_t belongs to the same underlying contact structure as η. Moreover, since ζ_t is basic ξ is the Reeb (characteristic) vector field associated to η_t for all t. Now let us define

1.13 $$\begin{aligned}\Phi_t&=\Phi-\xi\otimes\zeta_t\circ\Phi,\\ g_t&=d\eta_t\circ(\mathrm{id}\otimes\Phi_t)+\eta_t\otimes\eta_t.\end{aligned}$$

In [BG3] it was proved that for each $t\in[0,1]$, (ξ,η_t,Φ_t,g_t) defines a Sasakian structure on M associated to the foliation $\mathcal{F}_\xi$ and belonging to the same underlying contact structure as η. These new Sasakian structures correspond to a different splitting of the tangent bundle, namely

1.14 $$TM=L_\xi\oplus\mathcal{D}_t$$

which is orthogonal with respect to the metric g_t, where $\mathcal{D}_t=\ker\,\eta_t$.

Given a Sasakian structure (ξ,η,Φ,g) on a manifold M, we define $\mathfrak{F}(\xi)$ to be the family of all Sasakian structures obtained by the deformations above. We are now ready for

DEFINITION 1.15: *Two Sasakian structures $\mathcal{S}=(\xi,\eta,\Phi,g)$ and $\mathcal{S}'=(\xi',\eta',\Phi',g')$ on M are said to be homologous if $\xi'=\xi$ and $[d\eta]_B=[d\eta']_B\in H^{1,1}_B(\mathcal{F})$. In this case we also say that the Sasakian metrics g and g' are homologous.*

Notice that if $L_{\xi'}=L_\xi$ we can always scale the Sasakian structure by choosing $\xi'=\xi$, so there is no loss in generality by choosing the characterstic vector fields to coincide. We have

PROPOSITION 1.16: *Any two Sasakian structures in $\mathfrak{F}(\xi)$ are homologous and the class $c_1^B\in H^{1,1}_B(\mathcal{F}_\xi)\subset H^2_B(\mathcal{F}_\xi)$ depends only on the family $\mathfrak{F}(\xi)$.*

PROOF: The first statement follows from (8) of Proposition 1.9, while the second statement follows from the fact that for all $t \in [0,1]$ the complex vector bundles $(\mathcal{D}_t, \Phi_t)$ are isomorphic. The isomorphism between $\mathcal{D}$ and $\mathcal{D}_t$ is given by the map

$$\mathbb{I} - \xi \otimes \zeta_t : TM \longrightarrow TM,$$

and the induced map on the exterior bundle $\Lambda\mathcal{D}$ is

$$\mathbb{I} - \zeta_t \otimes \xi$$

which is the identity on basic forms. ∎

Now we wish to consider the Sasakian analogue of the Calabi problem. Its solution essentially follows from the 'transverse Yau Theorem' given by El Kacimi-Alaoui [ElK]:

THEOREM 1.17[ElK]: *If c_1^B is represented by a real basic $(1,1)$ form ρ^T, then it is the Ricci curvature form of a unique transverse Kähler form ω^T in the same basic cohomology class as $d\eta$.*

This is given in local CR-coordinates $(z_i, \bar{z}_i, x)$ on M by solving the "transverse Monge-Ampere equation"

1.18 $$\frac{\det(g^T_{i\bar{j}} + \phi_{i\bar{j}})}{\det(g^T_{i\bar{j}})} = e^{-k\phi+F}, \quad g^T_{i\bar{j}} + \phi_{i\bar{j}} > 0,$$

for the $k = 0$ case. Here g^T is the transverse metric, ϕ and F are real basic functions, and $\phi_{i\bar{j}}$ are the components of $i\partial\bar{\partial}\phi = d\zeta_t$ with respect to the transverse coordinates $(z_i, \bar{z}_{\bar{j}})$. Then by 1.10c this translates for Sasakian geometry to:

THEREOM 1.19: *Let (M, ξ, η, Φ, g) be a Sasakian manifold whose basic first Chern class c_1^B is represented by the real basic $(1,1)$ form ρ, then there is a unique Sasakian structure $(\xi, \eta_1, \Phi_1, g_1) \in \mathfrak{F}(\xi)$ homologous to (ξ, η, Φ, g) such that $\rho_{g_1} = \rho - 2d\eta_1$ is the Ricci form of g_1, and $\eta_1 = \eta + \zeta_1$, with $\zeta_1 = \frac{1}{2}d^c\phi$. The metric g_1 and endomorphism Φ_1 are then given by 1.12.*

Notice that the Ricci forms are related by

1.20 $$\rho = \rho_g + \frac{1}{2}dd^c(F - \phi).$$

Next we discuss a positivity requirement on the basic Chern class of a compact Sasakian manifold.

DEFINITION 1.21: *A Sasakian manifold M is said to be positive if its basic first Chern class c_1^B can be represented by a positive definite $(1,1)$-form.*

Notice that a positive Sasakian manifold does not necessarily have a metric with positive Ricci curvature; however, Theorem 1.19 does imply:

Proposition 1.22: *A Sasakian manifold (M, g, ξ, η, Φ) is positive if and only if there is a Sasakian metric g' homologous to g whose Ricci curvature satisfies the bound $Ric_{g'} > -2$. In particular, a complete positive Sasakian manifold is compact with finite fundamental group.*

Since a Sasakian-Einstein manifold necessarily has positive Ricci tensor, its Sasakian structure is necessarily positive. We are interested in sufficient conditions on a Sasakian manifold that guarentee the existence of a Sasakian-Einstein structure. These conditions are algebraic geometric in nature, so we need to impose another condition on the Sasakian structure, namely that it is quasi-regular. Recall that the *toral rank* or just *rank* defined [BG2] is the dimension of the closure of the characteristic foliation and denoted by $\mathrm{rk}(M)$. For a Sasakian manifold M^{2n+1} of dimension $2n+1$ we have $1 \leq \mathrm{rk}(M) \leq n+1$. The case $\mathrm{rk}(M) = 1$ corresponds to the quasi-regular case, and if $\mathrm{rk}(M) > 1$ there are infinitely many rank one structures that are close in an appropriate sense [BG2]. In the remainder of this paper we are essentially interested in the rank one case only. Now the aforementioned algebraic geometric conditions which imply the existence of a Kähler-Einstein metric are described in the works of Nadel [Na] and Demailly and Kollár [DK]. The main point is that the obstructions for finding a solution to the Monge-Ampere equations 1.18 involves the non-triviality of certain *multiplier ideal sheaves* associated with effective canonical $\mathbb{Q}$-divisors on the space of leaves $\mathcal{Z}$. Consequently, if one can show that these multiplier ideal sheaves coincide with the full structure sheaf, one obtains the existence of a Kähler-Einstein metric. Equivalently, and this is the approach of Johnson and Kollár [JK1], this is given in the language of Mori theory by the appellation *Kawamata log terminal*, whose precise technical definition is given in 2.3 below. It is important to realize that these conditions are only *sufficient conditions* for the existence of Kähler-Einstein metrics. Indeed, they do not hold in the case of complex projective space. We can easily reformulate the Demailly-Kollár result in terms of Sasakian geometry as

Theorem 1.23: *Let $(\mathcal{S}, g, \xi)$ be a rank 1 positive Sasakian manifold and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation, and $K_{\mathcal{Z}}$ be its canonical bundle. Suppose further that for some $\epsilon > 0$ and every effective $\mathbb{Q}$-divisor D on $\mathcal{Z}$ numerically equivalent to $-K_{\mathcal{Z}}$ the pair $(\mathcal{Z}, \frac{n+\epsilon}{n+1}D)$ is klt. Then there is a Sasakian-Einstein metric g' on M homologous to g.*

Proof: By Theorem 2.1 of [BG1] $\mathcal{Z}$ is a $\mathbb{Q}$-factorial Fano variety,and by Demailly and Kollár [DK] as stated in Theorem 7 of [JK1] $\mathcal{Z}$ has a Kähler-Einstein metric. Thus, again by (iv) of Theorem 2.1 of [BG1] $\mathcal{S}$ has a Sasakian-Einstein metric. ∎

Finally we recall [BG1] that the *orbifold Fano index* or just *index* of a positive rank 1 Sasakian manifold $\mathcal{S}$ is the largest positive integer I such that $\frac{1}{I}K_{\mathcal{Z}}$ is an element of $\mathrm{Pic}^{orb}(\mathcal{Z})$ where $K_{\mathcal{Z}}$ is the canonical V-bundle on $\mathcal{Z}$. Thus, the index is just the generalization to the orbifold category of the usual notion of Fano index; however, it is important to notice that the index I is an invariant of the Sasakian structure on $\mathcal{S}$; in fact, we have

Proposition 1.25: *The orbifold Fano index I is an invariant of the family of Sasakian structures $\mathfrak{F}(\xi)$.*

2. The Existence of Kähler-Einstein Metrics

In this section we quickly review the basic definitions and theorems from minimal model theory; all of this can be found in [KMM]. Throughout this section X will denote a normal projective variety defined over an algebraically closed field k of characteristic zero. By a variety we mean an integral separated scheme of finite type over k. Since X is normal, $\mathrm{codim}(X_{\mathrm{sing}}, X) \geq 2$, where X_{sing} denotes the singular locus of X. The sheaf of Kähler differentials Ω_X is locally free of rank $\dim X$ on $X_{\mathrm{reg}} := X \backslash X_{\mathrm{sing}}$. This gives a well-defined invertible sheaf $\bigwedge^{\dim X} \Omega_{X_{\mathrm{reg}}}$ on X_{reg}.

Definition 2.1: *A canonical divisor K_X is any Weil divisor with*

$$\mathcal{O}_{X_{\mathrm{reg}}}(K_X) \simeq \bigwedge^{\dim X} \Omega_{X_{\mathrm{reg}}}.$$

K_X is well defined modulo linear equivalence since $\mathrm{codim}(X_{\mathrm{sing}}, X) \geq 2$. Because singularities are essential in the study of minimal models, it is important to note the distinction between Weil divisors and Cartier divisors.

Definition 2.2: *A Weil divisor $D \subset X$ is called $\mathbb{Q}$–Cartier if some integral multiple mD is a Cartier divisor, i.e. if $\mathcal{O}_X(mD)$ is invertible. We say that X is $\mathbb{Q}$–factorial (or that X has at worst $\mathbb{Q}$–factorial singularities) if all Weil divisors are $\mathbb{Q}$–Cartier.*

There is a certain class of singularities called "log–terminal" which play an especially prominent role in the minimal model program.

Definition 2.3: *Let X be a normal variety and let $\Delta = \sum a_i E_i$ be an effective $\mathbb{Q}$–Weil divisor such that $K_X + \Delta$ is $\mathbb{Q}$–Cartier. Assume $0 \leq a_i < 1$ and that the E_i are all distinct. Then the pair (X, Δ) has only log–terminal singularities (also called klt or Kawamata log-terminal) if there exists a resolution of singularities $\pi : Y \to X$ such that the union of the π–exceptional locus and $\pi^{-1}(\cup E_i)$ is a normal crossing divisor and*

$$K_Y \equiv \pi^*(K_X + \Delta) + \sum a_i F_i$$

with $a_i > -1$ for all π–exceptional F_i (here $\equiv$ denotes numerical equivalence). If $a_i \geq -1$ then the pair (X, Δ) is said to have log–canonical singularities. In the case where $\Delta = 0$, X is said to have has log–terminal (or log–canonical) singularities. When $\Delta = 0$ and $a_i > 0$ (resp. $a_i \geq 0$) X is said to have terminal singularities (resp. canonical singularities).

Note that Kollár and Mori, [KM] Definition 2.34, give a different definition of klt singularities, one which does not require the "boundary divisor" Δ to be effective. However, in applications one almost always needs the effectivity hypothesis in order to apply the machinery of the minimal model program. All varieties we encounter will have mild quotient singularities and hence will be $\mathbb{Q}$–factorial.

Definition 2.4: *Suppose L is a line bundle on a variety X. We write*

$$\mathrm{BS}(L) = \{x \in X : s(x) = 0 \ \text{ for all sections } s \in H^0(X, L)\}$$

for the base locus of L. If $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf then we will denote by $\mathrm{BS}(L \otimes \mathcal{I})$ the base locus of those sections $\sigma \in H^0(X, L)$ whose zero locus contains the zero locus of $\mathcal{I}$.

We will be interested in the case where X is a projective surface with mild quotient singularities. We will assume that $-K_X$ is an ample $\mathbb{Q}$–Cartier divisor and we will be interested in showing that for some $\epsilon > 0$ and *every* effective $\mathbb{Q}$–divisor $D \in |-K_X|$, the pair $(X, \frac{2+\epsilon}{3}D)$ is klt. In particular this is certainly true if the pair (X, D) is log–canonical for all choices of D. Checking whether or not (X, D) is log–canonical will be intricate at singular points of X and we begin by reducing this condition to a much simpler one on a smooth finite cover of X.

We begin by noting that by [KM] Proposition 4.18, the variety X itself is klt at a singular point x if and only if X has a quotient singularity at x. In order to check whether or not the pair (X, D) is klt at x we need to consider a local cover

$$\pi : \left(\mathbb{C}^2, 0\right) \to (X, x),$$

where the map π is given by taking the quotient via a finite group action. We let $U \subset \mathbb{C}^2$ denote an open subset on which π is defined and let

$$p : S \to U$$

be the blow–up of U at 0 with exceptional divisor E. We will show that if $(S, p_*^{-1}(\pi^*D)+E)$ is log–canonical near E then the pair (X, D) is log–canonical at x. Shokurov's inversion of adjunction ([KM] Theorem 5.50) states that the pair $(S, p_*^{-1}(\pi^*D) + E)$ is log–canonical near E if $(E, p_*^{-1}\pi^*(D)|E)$ is log–canonical. But $E \simeq \mathbb{P}^1$ and so this is an easy condition to check, satisfied provided the round–up of $p_*^{-1}\pi^*(D)|E$ is a sum of distinct points, each with multiplicity one.

By [KM] Lemma 2.30 and Proposition 5.20.4 together with the inversion of adjunction, one derives

Lemma 2.5: *Suppose that* $\mathrm{mult}_0(\pi^*D) \leq 2$ *and that* $(E, p_*^{-1}\pi^*D + E)$ *is log–canonical. Then* (X, D) *is log–canonical at* x.

We sketch here the details of Lemma 2.5. Suppose that $(S, p_*^{-1}(\pi^*D) + E)$ is log–canonical near E and consider the following commutative diagram:

$$\begin{tikzcd} & Y \arrow[dl, "q"'] \arrow[dr, "q'"] \arrow[dd, "\mu"] & \\ S \arrow[dr, "p"'] & & X'. \arrow[dl, "p'"] \\ & U & \end{tikzcd}$$

Here we choose μ so that $\mu^{-1}(\pi^*D) \cup \mathrm{Exc}(\mu)$ is a divisor with normal crossings. Suppose we write

2.6 $$K_Y \equiv q^*(K_S + p_*^{-1}(\pi^*D) + E) + \sum_{i=1}^{s} a_i E_i.$$

Then $(S, p_*^{-1}(\pi^* D)+E)$ is log–canonical near E provided $a_i \geq -1$ for all E_i contracted by q. We also have, using $q^* K_S - \mu^* K_U = q^* E$,

$$
\begin{aligned}
K_Y &\equiv \mu^*(K_U + \pi^*(D)) + \sum_{j=1}^{s} b_j E_j \\
&\equiv q^*(K_S - E + p^* \pi^* D) + \sum_{j=1}^{s} b_j E_j \\
&\equiv q^*(K_S + p_*^{-1}(\pi^* D) + E) - (2 - \mathrm{mult}_0(\pi^* D)) q^* E + \sum_{j=1}^{s} b_j E_j. \qquad (2.7)
\end{aligned}
$$

Combining 2.6 and 2.7 shows that $b_i \geq a_i$ for all i, provided that $\mathrm{mult}_0(\pi^* D) \leq 2$. Thus, whenever $\mathrm{mult}_0(\pi^* D) \leq 2$, $(U, \pi^* D)$ is log–canonical at 0 provided $(S, p_*^{-1}(\pi^* D) + E)$ is log–canonical near E.

Next we need to relate the pair (X, D) at x to the pair $(U, \pi^* D)$ at 0. For this, we consider the commutative diagram

$$
\begin{tikzcd}
 & W \arrow[dl, "g"'] \arrow[dr, "h"] & \\
U \arrow[dr, "\pi"'] & & \tilde{X}. \arrow[dl, "f"] \\
 & X &
\end{tikzcd}
$$

Write

$$
\begin{aligned}
K_W &= g^* K_U + \sum_{j=1}^{s} b_j G_j, \\
K_{\tilde{X}} &= f^* K_X + \sum_{i=1}^{r} a_i F_i.
\end{aligned}
$$

Using the logarithmic ramification formula ([Ka] Lemma 1.6) we find

$$
\sum_{j=1}^{s} (1 + b_j) G_j = h^* \left(\sum_{i=1}^{r} (1 + a_i) F_i \right) + R, \qquad (2.8)
$$

where all divisorial components of R are contracted by h. We now remove the boundary divisor from both sides of 2.8 giving

$$
\sum_{j=1}^{s} (1 + b_j) G_j - g^* \pi^* D = h^* \left(\sum_{i=1}^{r} (1 + a_i) F_i - f^* D \right) + R. \qquad (2.9)
$$

We rewrite 2.9 as

2.10 $$\sum_{j=1}^{s'} \beta_j G'_j = h^*\left(\sum_{i=1}^{r'} \alpha_i F'_i\right) + R.$$

Suppose now that F is one of the F'_i in 2.10 which is f–exceptional and let α_F denote the corresponding discrepancy. Let G denote a divisor on W such that $h : G \to F$ is finite, with discrepancy β_G. Then 2.10 says that $\alpha_F \geq -1$ provided $\beta_G \geq -1$. But $(U, \pi^* D)$ is log–canonical near E if and only if $\beta_j \geq -1$ for all G_j contracted by g. Thus $\alpha_F \geq -1$ for all F contracted by f and this shows that (X, D) is log–canonical at x.

We now give some general criteria for establishing whether or not (X, D) is log–canonical at a point x.

LEMMA 2.11: *Suppose X is a normal projective surface and D an effective $\mathbb{Q}$–divisor on X. Suppose moreover that $x \in X_{\text{reg}}$. Then the pair (X, D) is klt at x provided $A \cdot D < a$ for some divisor A such that the linear series $|A \otimes m_x^a|$ has only isolated base points.*

PROOF: Suppose $x \in \text{supp}(D)$. Choose a representative $E \in |A \otimes m_x^a|$ such that $E \cap \text{supp}(D)$ is proper. Then we have

$$i_x(D, E) \leq D \cdot E < a.$$

On the other hand

$$i_x(D, E) \geq \text{mult}_x(D)\text{mult}_x(E) \geq a(\text{mult}_x(D)).$$

Thus we have

$$\text{mult}_x(D) < 1$$

and Lemma 2.11 follows from [KM] Theorem 4.5 (1). ∎

Note that Lemma 2.11 throws away a lot of information because the estimate

$$D \cdot E \geq i_x(D, E)$$

can be very crude; to give a more precise estimate, however, requires information about the specific geometry of X. We will also need a slight refinement of Lemma 2.11 to deal with the singular points of X.

LEMMA 2.12: *Suppose X is a normal projective surface with a quotient singularity of index m at x. Suppose D is an effective $\mathbb{Q}$–divisor and A a divisor on X such that $|A \otimes m_x^a|$ has only isolated base points for some positive integer a. Then (X, D) is klt at x provided $A \cdot D < \frac{a}{m}$.*

PROOF: Choose a general divisor $E \in |A \otimes m_x^a|$ so that $E \cap D$ is proper. Choose a local resolution $\pi : Y \to X$ of X at x with $\pi^{-1}(x) = y$. Then

$$\text{mult}_y(\pi^* D) \leq \frac{mD \cdot A}{a}.$$

Thus, by hypothesis $\mathrm{mult}_y(\pi^*D) < 1$ so that (Y, π^*D) is klt at y. By Lemma 2.11 this implies that the pair (X, D) is klt at x. ∎

3. The Sasakian Geometry of Links of Weighted Homogeneous Polynomials

In this section we discuss the Sasakian geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action given by $(z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n)$, where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w} \in (\mathbb{Z}^+)^{n+1}$, and we shall assume that they are ordered $w_0 \leq w_1 \leq \cdots \leq w_n$ and that $\gcd(w_0, \ldots, w_n) = 1$. Let f be a quasi-homogeneous polynomial, that is $f \in \mathbb{C}[z_0, \ldots, z_n]$ and satisfies

3.1
$$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n),$$

where $d \in \mathbb{Z}^+$ is the degree of f. We are interested in the *weighted affine cone* C_f defined by the equation $f(z_0, \ldots, z_n) = 0$. We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity, in fact the only singularity, of f. Then the link L_f defined by

3.2
$$L_f = C_f \cap S^{2n+1},$$

where

$$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} | \sum_{j=0}^{n} |z_j|^2 = 1\}$$

is the unit sphere in $\mathbb{C}^{n+1}$, is a smooth manifold of dimension $2n-1$. Furthermore, it is well-known [Mil] that the link L_f is $(n-2)$-connected.

On S^{2n+1} there is a well-known [YK] "weighted" Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which in the standard coordinates $\{z_j = x_j + iy_j\}_{j=0}^{n}$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is determined by

$$\eta_{\mathbf{w}} = \frac{\sum_{i=0}^{n}(x_i dy_i - y_i dx_i)}{\sum_{i=0}^{n} w_i(x_i^2 + y_i^2)}, \qquad \xi_{\mathbf{w}} = \sum_{i=0}^{n} w_i(x_i \partial_{y_i} - y_i \partial_{x_i}),$$

and the standard Sasakian structure (ξ, η, Φ, g) on S^{2n+1}. The embedding $L_f \hookrightarrow S^{2n+1}$ induces a Sasakian structure on L_f [BG3].

Given a sequence $\mathbf{w} = (w_0, \ldots, w_n)$ of ordered positive integers one can form the graded polynomial ring $S(\mathbf{w}) = \mathbb{C}[z_0, \ldots, z_n]$, where z_i has grading or *weight* w_i. The weighted projective space [Dol, Fle] $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$ is defined to be the scheme $\mathrm{Proj}(S(\mathbf{w}))$. It is the quotient space $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*(\mathbf{w})$, where $\mathbb{C}^*(\mathbf{w})$ is the weighted action defined in 3.1, or equivalently, $\mathbb{P}(\mathbf{w})$ is the quotient of the weighted Sasakian sphere $S^{2n+1}_{\mathbf{w}} = (S^{2n+1}, \xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ by the weighted circle action $S^1(\mathbf{w})$ generated by $\xi_{\mathbf{w}}$. As such $\mathbb{P}(\mathbf{w})$ is also a compact complex orbifold with an induced Kähler structure. We have from [BG3]

Theorem 3.3: *The quadruple $(\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$ gives L_f a quasi-regular Sasakian structure such that there is a commutative diagram*

$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 V-bundles and orbifold Riemannian submersions. Moreover, if Z_f is Fano, L_f is the total space of the principal S^1 V-bundle over the orbifold $\mathcal{Z}_f$ whose first Chern class in $H^2_{orb}(\mathcal{Z}_f,\mathbb{Z})$ is $c_1(\mathcal{Z}_f)/I$, where I is the index.

We should also mention that $c_1(\mathcal{Z}_f)$ pulls back to the basic first Chern class $c_1^B\in H^2_B(\mathcal{F}_{\xi_{\mathbf{w}}})$ and $\eta_{\mathbf{w}}$ is the connection in this V-bundle whose curvature is $d\eta=\frac{2ni}{I}\pi^*\omega_{\mathbf{w}}$, where $\omega_{\mathbf{w}}$ is the Kähler form on $\mathcal{Z}_f$.

Now conditions on the weights that guarantee that the hypersurface $C_f\subset\mathbb{C}^{n+1}$ have only an isolated singularity at the origin are well-known [Fle,JK1]. These conditions become more complicated as the dimension increases [Fle,JK2]; however, in this paper we will only be interested in the $n=3$ case of hypersurfaces in a weighted complex projective 3-space. These conditions, known as *quasi-smoothness* conditions gaurentee that $\mathcal{Z}_f$ is smooth in the orbifold sense, that is, at a vertex $P_i\in\mathbb{P}(\mathbf{w})$ the preimage of $\mathcal{Z}_f$ in the orbifold chart of $\mathbb{P}(\mathbf{w})$ is smooth. It is easy to see that one can formulate all these conditions as follows [Fle,JK1]:

Quasi-Smoothness Conditions 3.4:

I. For each $i=0,\cdots,3$ there is a j and a monomial $z_i^{m_i}z_j\in\mathcal{O}(d)$. Here $j=i$ is possible.

II. If $\gcd(w_i,w_j)>1$ then there is a monomial $z_i^{b_i}z_j^{b_j}\in\mathcal{O}(d)$.

III. For every i,j either there is a monomial $z_i^{b_i}z_j^{b_j}\in\mathcal{O}(d)$, or there are monomials $z_i^{c_i}z_j^{c_j}z_k$ and $z_i^{d_i}z_j^{d_j}z_l\in\mathcal{O}(d)$ with $\{k,l\}\neq\{i,j\}$.

In condition I the $i=j$ case corresponds to the case when $\mathcal{Z}_f$ does not pass through the point P_i. The second condition is equivalent to $\mathcal{Z}_f$ not containing any of the singular lines in $\mathbb{P}(\mathbf{w})$. If $\mathcal{Z}_f$ contains a coordinate axis (say $z_j=z_j=0$) then the condition III forces $\mathcal{Z}_f$ to be smooth along it, except possibly at the vertices.

There is another condition apart from quasi-smoothness that assures us that the adjunction theory behaves correctly, and that $\mathbb{P}(\mathbf{w})$ does not have any orbifold singularities of codimension 1. It is [Dol,Fle]

Well-formedness Condition 3.5

IV. For any triple $i, j, k \neq$, we have $\gcd(w_i, w_j, w_k) = 1$.

Condition IV guarantees that the canonical V-bundle $K_{\mathcal{Z}}$ is determined in terms of the degree and index by

3.6
$$K_{\mathcal{Z}} \simeq \mathcal{O}(-I) = \mathcal{O}(d - |\mathbf{w}|),$$

where $|\mathbf{w}| = \sum_i w_i$.

Finally we end this section by giving a corollary of Lemma 2.12 which, by Theorem 1.23, gives sufficient conditions to ensure that $\mathcal{Z}_f$ admits a Kähler-Einstein metric.

COROLLARY 3.7: *Let $\mathcal{Z}_f \subset \mathbb{P}(w_0, w_1, w_2, w_3)$ be a hypersurface of degree d in weighted projective space with $\mathbf{w} = (w_0, w_1, w_2, w_3)$ well-formed. Let $\Delta \in |-\alpha K_{\mathcal{Z}_f}|$ be an effective representative for some rational $\alpha > 0$. Writing $K_{\mathcal{Z}_f} = \mathcal{O}_{\mathcal{Z}_f}(-I)$, suppose $x \in \mathcal{Z}_f$ is a point at which $\mathbb{P}(\mathbf{w})$ has a singularity of order ℓ_x. If $\mathrm{BS}(\mathcal{O}_{\mathcal{Z}_f}(w_0) \otimes \mathcal{I}_x)$ does not contain any component of Δ then $(\mathcal{Z}_f, \Delta)$ is klt at x provided*

$$\alpha \ell_x d I < w_1 w_2 w_3.$$

If $\mathrm{BS}(\mathcal{O}_{\mathcal{Z}_f}(w_0 w_1) \otimes \mathcal{I}_x)$ does not contain Δ then $(\mathcal{Z}_f, \Delta)$ is klt at x provided

$$\alpha \ell_x d I < w_0 w_2 w_3.$$

In general, unless $\mathcal{Z}_f$ is the hypersurface $\{z_0 = 0\}$, one always has that $(\mathcal{Z}_f, \Delta)$ is klt at x if

$$\alpha \ell_x d I < w_0 w_1 w_3.$$

PROOF: This corollary follows immediately from Lemma 2.12 by looking at the linear systems $|\mathcal{O}_{\mathcal{Z}_f}(w_0)|, |\mathcal{O}_X(w_0 w_1)|$, and $|\mathcal{O}_{\mathcal{Z}_f}(w_0 w_1 w_2)|$ respectively. For example, taking $A = |\mathcal{O}_{\mathcal{Z}_f}(w_0 w_1)|$ in Lemma 2.12, we find that we can take $a = w_0$ since one of the sections $z_0^{w_1}$, $z_1^{w_0}$ will not vanish along Δ by hypothesis and $w_0 \leq w_1$. Then Lemma 2.12 states that $(\mathcal{Z}_f, \alpha\Delta)$ is klt at x provided $A \cdot \alpha\Delta < \frac{w_0}{\ell_X}$. But

$$A \cdot \alpha\Delta = \frac{w_0 w_1 d \alpha I}{w_0 w_1 w_2 w_3} = \frac{d\alpha I}{w_2 w_3}.$$

The middle formula follows immediately and the others are obtained similarly. ∎

4. The Algebraic Equations and Their Solutions

We want to repeat the analysis of [JK1] for the case of an arbitrary index I. Following their approach we consider the linear system

4.1
$$m_i w_i + w_{j(i)} = d = w_0 + w_1 + w_2 + w_3 - I$$

where m_i is a positive integer and both i and $j(i)$ are integers ranging over $0, 1, 2, 3$. Note we can have $j(i) = i$. This is an obvious translation of the quasi-smoothness condition 3.4.I

LEMMA 4.2: *Assuming conditions 3.4.I and 3.4.IV, the following bounds hold*

(1) $1 \leq m_3 \leq 2$.

(2) Either $2 \leq m_2 \leq 4$, *or* $2w_0 \leq I$.

(3) Either $2 \leq m_1 \leq 10$, *or* $2I = w_0 + w_1$, *or* $2I \geq 3w_0$, *or condition 3.4.II is violated.*

PROOF: Since the weights are ordered $w_0 \leq w_1 \leq w_2 \leq w_3$, (1) follows immediately from the linear system with $i = 3$. To prove (2) we consider $i = 1$ in the system 4.1. There are two cases: $w_{j(2)} = w_3$ and $w_{j(2)} \neq w_3$. If $w_{j(2)} = w_3$ we immediately have

$$m_2 w_2 = w_0 + w_1 + w_2 - I \leq 3w_2 - I$$

which implies $m_2 \leq 2$.

Now consider $w_{j(2)} \neq w_3$, then there are two subcases depending on whether $m_3 = 2$ or 1. If $m_3 = 2$ our system gives

$$w_0 + w_1 + w_2 = w_3 + w_{j(3)} + I,$$

and here there are two subcases giving:

1. If $w_{j(3)} = w_3$, then $2w_3 < 3w_2$.

2. If $w_{j(3)} \neq w_3$, then $w_3 < 2w_2$,

and in both cases we have $w_3 < 2w_2$. Thus since $w_{j(2)} \neq w_3$, we get

$$m_2 w_2 \leq 2w_2 + w_3 - I < 4w_2 - I$$

implying $m_2 \leq 3$.

Finally we consider the case $m_3 = 1$. Here we obtain

4.3 $$w_{j(3)} + I = w_0 + w_1 + w_2.$$

If $w_3 = w_{j(3)} = w_0 + w_1 + w_2 - I$, then $d = 2w_3$, so $m_2 w_2 + w_{j(2)} = 2w_0 + 2w_1 + 2w_2 - 2I$. This gives

$$m_2 w_2 \leq 5w_2 - 2I,$$

implying that $m_2 \leq 4$, whereas if $w_{j(3)} \neq w_3$, then we have $I = w_i + w_j$ for some distinct $i, j = 0, 1, 2$. We have thus shown that either $1 \leq m_2 \leq 4$ or $I = w_i + w_j$ for some distinct $i, j = 0, 1, 2$. It is now easy to see that if $m_2 = 1$ then we must have $2w_0 \leq I$. Thus, in all cases (2) of the lemma holds.

Finally we consider (3). We proceed along the lines of (2). If possible we solve the system 4.1 for $i = 2, 3$ for w_3 and w_2; however, there are exceptional cases, that is, values of m_3 and m_2 such w_3 and/or w_2 are free. We shall show that in these cases when m_1 is unbounded that either $2I \geq 3w_0$, or $2I = w_0 + w_1$, or condition 3.4.II is violated. Moreover, when we can solve for w_2 and w_3 and substitute into 4.1 with $i = 1$ we obtain an equation for m_1 in terms of m_2, m_3, w_1, w_2, w_3. This gives a bound for m_1 which is certainly maximal for $I = 1$ (in fact for $I = 0$) and one can see that $m_1 \leq 10$ as in [JK]. To see the lower bound on m_1 we show that if $m_1 = 1$ then $I \geq 2w_0$. There are two cases to consider, $w_{j(1)} = w_1$ and $w_{j(1)} \neq w_1$, and one easily sees that both cases give the desired estimate.

We now consider the exceptional cases. For the $i = 3$ equation of the system 4.1 this occurs for $m_3 = 1$, and one easily sees that there are 3 exceptional cases all which imply the estimate $I \geq 2w_0$. Now assuming that w_3 is determined in terms of the remaining weights and index, we consider the $i = 2$ equation in 4.1. There are 4 subcases. Since $m_2 > 1$ the case $w_{j(2)} = w_3$ gives no exceptional solution. Now if $w_{j(2)} = w_2$ we have $m_2 w_2 = w_0 + w_1 + w_3 - I$ and we must plug in the possibilities for w_3 determined by the $i = 3$ equation. But since all of these possibilities are of the form $w_3 = aw_2 + b$ with $a = 0$, $a = \frac{1}{2}$ or $a = 1$, we see that there are no exceptional solutions in this case either.

Next we put $w_{j(2)} = w_1$, and consider the 5 possibilities for w_3. The first is $w_3 = w_0 + w_1 + w_2 - I$, and this leads to the exceptional case with $m_2 = 2$ and $2I = 2w_0 + w_1 \geq 3w_0$. The next case with $2w_3 = w_0 + w_1 + w_2 - I$ is not exceptional, and the third with $w_3 = w_0 + w_1 - I$ again gives $2I \geq 3w_0$. The fourth case $w_3 = w_0 + w_2 - I$, however, gives the exceptional case $m_2 = 2$ and $I = w_0$, and this requires a bit more analysis. Here we have that $w_3 = w_2$ and so the weights are (I, w_1, w_2, w_2) with degree $d = 2w_2 + w_1$. But this fails the quasi-smoothness condition (2) of section 3 the w_2 must divide d, and this can occur only if $w_2 = w_1$ which violates the well-formedness condition. Finally the fifth subcase with $w_3 = w_1 + w_2 - I$ leads to $m_2 = 2$ and $2I = w_0 + w_1$.

Finally we have the case $w_{j(2)} = w_0$. The first possibility for w_3 gives an exceptional case with $2I \geq 3w_0$, while the next two do not lead to any exceptional cases. Subcase 4, however, with $w_3 = w_0 + w_2 - I$ give $m_2 = 2$ and $2I = w_0 + w_1$. This implies the inequalities $w_0 \leq I \leq w_1$ and the equation $w_3 = w_2 - w_1 + I$ which violate the order $w_2 \leq w_3$ unless $w_3 = w_2$ and $I = w_1 = w_0$. But in this case the degree is $d = 2w_2 + I$, and condition 3.4.II then implies that $w_2 = w_3 = I$ which contradicts the well-formedness condition 3.4.IV. The last possibility for w_3 is $w_3 = w_1 + w_2 - I$ and this leads to the exceptional case $m_2 = 2$ and $I = w_1$. But then we have $w_3 = w_2$ and $d = 2w_2 + w_0$ which violates condition 3.4.II above again since w_2 cannot divide w_0. ∎

Remarks 4.4: There are solutions with arbitrary m_1 that are quasi-smooth, but our lemma implies that these must satisfy $2I \geq 3w_0$ or $2I = w_0 + w_1$. As we shall see in the next section, in both cases the sufficient condition for the existence of a Kähler-Einstein metric fails. Examples of such a series are: $(1, 1, k, k)$ of degree $2k$ and index 2, $(1, 2, 2k + 1, 2k + 1)$ of degree $2(2k + 1)$ and index 3, and $(2, 2, 2k + 1, 2k + 1)$ of degree $2(2k+1)$ and index 4, as well as the general series $(I - n, I + n, w, w + n)$ of degree $2w + n + I$ and index I. There are also double series such as $(1, 1, m, m + k)$ of degree $2m + k$ and index 2.

We have modified the computer program of [JK1] to solve the system 4.1 for any index I as well as to discard some solutions that are not quasi-smooth. In light of Remarks 4.4, we shall now consider all solutions for which $1 \leq I \leq 10$, $2I < 3w_0$, and $2I \neq w_0 + w_1$. The last column of the tables indicate whether the klt condition (which implies that $\mathcal{Z}_{\mathbf{w}}$ admits a Kähler-Einstein metric) holds (Y) or is unknown (?). The computer search gives the following complete list:

THEOREM 4.5: *Let $\mathcal{Z}_{\mathbf{w}}$ be a well-formed quasi-smooth log del Pezzo surface of index $I \leq 10$ and degree d embedded in the weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, w_1, w_2, w_3)$. If $2I \geq 3w_0$ or $2I = w_0 + w_1$ then for every $\epsilon > 0$ there exists a divisor $D \in |-K_{\mathcal{Z}_{\mathbf{w}}}|$ such that $(\mathcal{Z}_{\mathbf{w}}, \frac{2+\epsilon}{3}D)$ is not klt. If $2I < 3w_0$ and $2I \neq w_0 + w_1$ then $\mathcal{Z}_{\mathbf{w}}$ must belong to one of the following cases:*

(1) If $I = 1$ [JK1] then $\mathcal{Z}_{\mathbf{w}}$ is either one of the series solution below

$\mathbf{w}$	d	b_2	K-E
$(2, 2k+1, 2k+1, 4k+1)$	$8k+4$	8	Y

or it is one of the 19 sporadic solutions listed in Table 1.

(2) If $I = 2$ then $\mathcal{Z}_{\mathbf{w}}$ is one of the 6 infinite series solutions with $\mathbf{w}$ and d given by:

$\mathbf{w}$	d	b_2	K-E
$(4, 2k+1, 2k+1, 4k)$	$8k+4$	7	Y
$(3, 3k+1, 6k+1, 9k+3)$	$18k+6$	6	Y
$(3, 3k+1, 6k+1, 9k)$	$18k+3$	5	?
$(3, 3k, 3k+1, 3k+1)$	$9k+3$	7	Y
$(3, 3k+1, 3k+2, 3k+2)$	$9k+6$	5	Y
$(4, 2k+1, 4k+2, 6k+1)$	$12k+6$	6	?

or it is one of the 25 sporadic solutions listed in Table 1.

(3) If $I = 3$ then $\mathcal{Z}_{\mathbf{w}}$ is one of the 7 sporadic solutions listed in Table 1.

(4) If $I = 4$ then $\mathcal{Z}_{\mathbf{w}}$ is one of the 3 infinite series with $\mathbf{w}$ and d given by:

$\mathbf{w}$	d	b_2	K-E
$(6, 6k+3, 6k+5, 6k+5)$	$18k+15$	5	Y
$(6, 6k+5, 12k+8, 18k+9)$	$36k+24$	3	?
$(6, 6k+5, 12k+8, 18k+15)$	$36k+30$	4	Y

or it is one of the 10 sporadic solutions of Table 1.

(5) If $I = 5$ then $\mathcal{Z}_{\mathbf{w}}$ is one of the 3 sporadic solutions listed in Table 2.

(4) If $I = 6$ then $\mathcal{Z}_{\mathbf{w}}$ is one of the 2 infinite series given by:

$\mathbf{w}$	d	b_2	*K-E*
$(8, 4k+1, 4k+3, 4k+5)$	$12k+11$	*3*	*?*
$(9, 3k+2, 3k+5, 6k+1)$	$12k+11$	*3*	*?*

or it is one of the 3 sporadic solutions of Table 1.

(5) *If* $I = 7, 8, 9, 10$ *then* $\mathcal{Z}_{\mathbf{w}}$ *is one of the 6 sporadic solutions listed in Table 1.*

Table 1. Sporadic Examples of $\mathcal{Z}_{\mathbf{w}}$ of Index $1 \leq I \leq 10$					
Index	$\mathbf{w}$	Monomials of $f_{\mathbf{w}}$	d	b_2	K-E
1	(1,2,3,5)	$z_0^{10}, z_1^5, z_2^3z_1, z_3^2, \ldots (17)^*$	10	9	?
1	(1,3,5,7)	$z_0^{15}, z_1^5, z_2^3, z_3^2z_0, \ldots (19)$	15	9	?
1	(1,3,5,8)	$z_0^{16}, z_1^5z_0, z_2^3z_0, z_3^2, \ldots (20)$	16	10	?
1	(2,3,5,9)	$z_0^9, z_1^6, z_2^3z_1, z_3^2, \ldots (13)$	18	7	Y
1	(3,3,5,5)	$g_5(z_0, z_1), f_3(z_2, z_3)$	15	5	Y
1	(3,5,7,11)	$z_0^6z_2, z_1^5, z_2^2z_3, z_3^2z_0, \ldots (8)$	25	5	Y
1	(3,5,7,14)	$z_0^7z_2, z_1^5z_0, g_2(z_2^2, z_3), \ldots (9)$	28	6	Y
1	(3,5,11,18)	$g_2(z_0^6, z_3), z_1^5z_2, z_2^3z_0, \ldots (10)$	36	6	Y
1	(5,14,17,21)	$z_0^7z_3, z_1^4, z_2^3z_0, z_3^2z_1, z_0^5z_1z_2$	56	4	Y
1	(5,19,27,31)	$z_0^{10}z_3, z_1^4z_0, z_2^3, z_3^2z_1, z_0^7z_1z_2$	81	3	Y
1	(5,19,27,50)	$z_0^{20}, z_0^{10}z_3, z_3^2, z_1^5z_0, z_2^3z_1, z_0^7z_1^2z_3$	100	4	Y
1	(7,11,27,37)	$z_0^{10}z_1, z_1^4z_3, z_2^3, z_3^3z_0$	81	3	Y
1	(7,11,27,44)	$z_0^{11}z_1, z_2^3z_0, z_1^8, z_1^4z_3, z_3^2, z_0^4z_1^3z_2$	88	4	Y
1	(9,15,17,20)	$z_0^5z_1, z_1^4, z_2^3z_0, z_3^3$	60	3	Y
1	(9,15,23,23)	$z_0^6z_1, z_1^4z_0, z_2^3, z_2^2z_3, z_2z_3^2, z_3^3$	69	5	Y
1	(11,29,39,49)	$z_0^8z_2, z_1^4z_0, z_2^3, z_3^2z_1$	127	3	Y
1	(11,49,69,128)	$z_0^{17}z_2, z_1^5z_0, z_2^4, z_2^2z_3, z_3^2$	256	2	Y
1	(13,23,35,57)	$z_0^8z_1, z_1^4z_2, z_2^2z_3, z_3^2z_0$	127	3	Y
1	(13,35,81,128)	$z_0^{17}z_1, z_1^5z_2, z_2^3z_0, z_3^2$	256	2	Y
2	(2,3,4,5)	$z_0^6, z_1^4, z_2^3, z_3^2z_0, \ldots (10)$	12	5	?
2	(2,3,4,7)	$z_0^7, z_1^4z_0, z_2^3z_0, z_3^2, \ldots (11)$	14	6	?
2	(3,4,5,10)	$z_0^5z_2, z_1^5, z_2^4, z_3^2, \ldots (9)$	20	5	Y
2	(3,4,6,7)	$g_3(z_0^2, z_2), z_1^3z_2, z_3^2z_1, z_0z_3z_1^2, z_0^2z_1^3$	18	6	?
2	(3,4,10,15)	$z_0^{10}, z_1^5z_3, z_2^3, z_3^2, \ldots (10)$	30	7	Y
2	(3,7,8,13)	$z_0^7z_2, z_1^3z_2, z_2^2z_3, z_3^2z_0, z_0^5z_1, z_0^3z_1z_3, z_0^2z_1z_2^2$	29	5	?
2	(3,10,11,19)	$z_0^{10}z_3, z_1^3z_2, z_2^2z_3, z_3^2z_0, z_0^7z_1^2, z_0^4z_1z_3, z_0^3z_1z_2^3$	41	5	?
2	(5,13,19,22)	$z_0^7z_3, z_1^4z_0, z_2^3, z_3^2z_1, z_0^5z_1z_2$	57	3	Y
2	(5,13,19,35)	$z_0^{14}, z_0^7z_3, z_3^2, z_1^5z_0, z_2^3z_1, z_0^5z_1^2z_2$	70	3	Y
2	(6,9,10,13)	$z_0^6, z_1^4, z_2^3z_0, z_3^2z_2, z_0^3z_1^2$	36	4	Y
2	(7,8,19,25)	$z_0^7z_1, z_1^4z_3, z_2^3, z_3^2z_0, z_0^2z_1^3z_2$	57	3	Y
2	(7,8,19,32)	$z_0^8z_1, z_1^8, z_1^4z_3, z_3^2, z_2^3z_0, z_0z_2^3, z_0^3z_1^3z_2$	64	4	Y
2	(9,12,13,16)	$z_0^4z_1, z_1^4, z_2^3z_0, z_3^3$	48	3	Y
2	(9,12,19,19)	$z_0^5z_1, z_1^4z_0, z_2^3, z_2^2z_3, z_2z_3^3, z_3^3$	57	5	Y
2	(9,19,24,31)	$z_0^9, z_1^3z_2, z_2^3z_0, z_3^2z_1$	81	3	Y
2	(10,19,35,43)	$z_0^7z_2, z_1^5z_0, z_2^3, z_3^2z_1$	105	3	Y
2	(11,21,28,47)	$z_0^7z_2, z_1^5, z_2^3z_1, z_3^2z_0$	105	3	Y
2	(11,25,32,41)	$z_0^6z_3, z_1^3z_2, z_2^3z_0, z_3^2z_1$	107	3	Y
2	(11,25,34,43)	$z_0^{10}, z_1^4z_0, z_2^2z_3, z_3^2z_1$	111	3	Y
2	(11,43,61,113)	$z_0^{15}z_2, z_1^5z_0, z_2^3z_1, z_3^2$	226	2	Y
2	(13,18,45,61)	$z_0^9z_1, z_1^5z_2, z_2^3, z_3^2z_0$	135	3	Y

Table 1. (cont.) Sporadic Examples of $\mathcal{Z}_{\mathbf{w}}$ of Index $1 \leq I \leq 10$					
Index	$\mathbf{w}$	Monomials of $f_{\mathbf{w}}$	d	b_2	K-E
2	(13,20,29,47)	$z_0^6z_3, z_1^3z_3, z_2^3z_1, z_3^2z_0$	107	3	Y
2	(13,20,31,49)	$z_0^7z_1, z_1^4z_2, z_2^3z_3, z_3^2z_0$	111	3	Y
2	(13,31,71,113)	$z_0^{15}z_1, z_1^5z_2, z_2^3z_0, z_3^2$	226	2	Y
2	(14,17,29,41)	$z_0^5z_3, z_1^5z_0, z_2^2z_3, z_3^2z_1$	99	3	Y
3	(5,7,11,13)	$z_0^4z_3, z_1^4z_0, z_2^3, z_3^2z_1, z_0^3z_1z_2$	33	3	?
3	(5,7,11,20)	$z_0^8, z_1^5z_0, z_2^3z_1, z_3^2, z_0^4z_3, z_0^3z_1^2, z_2^2$	40	4	Y
3	(11,21,29,37)	$z_0^6z_2, z_1^4z_0, z_2^2z_3, z_3^2z_1$	95	3	Y
3	(11,37,53,98)	$z_0^{13}z_2, z_1^5z_0, z_2^3z_1, z_3^2$	196	2	Y
3	(13,17,27,41)	$z_0^6z_1, z_1^4z_2, z_2^2z_3, z_3^2z_0$	95	3	Y
3	(13,27,61,98)	$z_0^{13}z_1, z_1^5z_2, z_2^3z_0, z_3^2$	196	2	Y
3	(15,19,43,74)	$z_0^7z_2, z_1^7z_0, z_2^3z_1, z_3^2$	148	2	Y
4	(5,6,8,9)	$z_0^3z_3, z_1^4, z_2^3, z_3^2z_1, z_0^2z_1z_2$	24	3	?
4	(5,6,8,15)	$z_0^6, z_1^5, z_2^3z_1, z_3^2, z_0^3z_3, z_0^3z_3, z_0^2z_1^2z_2$	30	4	?
4	(9,11,12,17)	$z_0^5, z_1^3z_2, z_2^3z_0, z_3^2z_1$	45	3	?
4	(10,13,25,31)	$z_0^5z_2, z_1^5z_0, z_2^3, z_3^2z_1$	75	3	Y
4	(11,17,20,27)	$z_0^4z_3, z_1^3z_2, z_2^3z_0, z_3^2z_1$	71	3	?
4	(11,17,24,31)	$z_0^5z_2, z_1^4z_0, z_2^2z_3, z_3^2z_1$	79	3	Y
4	(11,31,45,83)	$z_0^{11}z_2, z_1^5z_0, z_2^3z_1, z_3^2$	166	2	Y
4	(13,14,19,29)	$z_0^4z_2, z_1^3z_3, z_2^3z_1, z_3^2z_0$	71	2	?
4	(13,14,23,33)	$z_0^5z_1, z_1^4z_2, z_2^2z_3, z_3^2z_0$	79	3	Y
4	(13,23,51,83)	$z_0^{11}z_1, z_1^5z_2, z_2^3z_0, z_3^2$	166	2	Y
5	(11,13,19,25)	$z_0^4z_2, z_1^4z_0, z_2^2z_3, z_3^2z_1$	63	3	?
5	(11,25,37,68)	$z_0^9z_2, z_1^5z_0, z_2^3z_1, z_3^2$	136	2	Y
5	(13,19,41,68)	$z_0^9z_1, z_1^5z_2, z_2^3z_0, z_3^2$	136	2	Y
6	(7,10,15,19)	$z_0^5z_1, z_1^3z_2, z_2^3, z_3^2z_0$	45	3	?
6	(11,19,29,53)	$z_0^7z_2, z_1^5z_0, z_2^4, z_3^2$	106	2	Y
6	(13,15,31,53)	$z_0^7z_1, z_1^5z_2, z_2^3z_0, z_3^2$	106	2	Y
7	(11,13,21,38)	$z_0^5z_2, z_1^5z_0, z_2^3z_1, z_3^2$	76	2	Y
8	(7,11,13,23)	$z_0^5z_1, z_1^3z_2, z_2^3z_0, z_3^2$	46	2	?
8	(7,18,27,37)	$z_0^9z_1, z_1^3z_2, z_2^3, z_3^2z_0$	81	3	?
9	(7,15,19,32)	$z_0^7z_1, z_1^3z_2, z_2^3z_0, z_3^2$	64	2	?
10	(7,19,25,41)	$z_0^9z_1, z_1^3z_2, z_2^3z_0, z_3^2$	82	2	?
10	(7,26,39,55)	$z_0^{13}z_1, z_1^3z_2, z_2^3, z_3^2z_0$	117	3	?

* (for lack of space only the total number of monomial terms in $f_{\mathbf{w}}$ is indicated)

The computer program indicates that there are neither series solutions nor sporadic solutions satisfying the hypothesis of Theorem 4.5 for $I > 10$. In fact, an easy argument shows that there are no such solutions for sufficiently large I. *

* The code for the C program used to generate the tables of the Theorem 4.5 are available at the following URL http://www.math.unm.edu/~galicki/papers/publications.html.

REMARKS 4.6: As already mentioned when $2I = w_0 + w_1$ or $2I < 3w_0$ the sufficient condition for the existence of the Kähler-Einstein metric fails. In all such cases techniques of [DK] say nothing about the corresponding $\mathcal{Z}_w$. Table 2 below gives the classical examples of $\mathcal{Z}_w$ that can be written as smooth hypersurfaces in weighted projective spaces and which by other methods admit K-E metrics [TY]. We only indicate a particular polynomial as the most general such hypersurface involves many such monomials which for reasons of space we do not include. We are not aware of any non-smooth orbifold examples for which the conditions of Theorem 4.5 fail, but are known to admit Kähler-Einstein metrics by another method.

Table 2. Classical del Pezzo surfaces as hypersurfaces in $\mathbb{CP}^3(\mathbf{w})$

Index	$\mathbf{w}$	$f_{\mathbf{w}}$	d	$\mathcal{Z}_{\mathbf{w}}$
1	(1,1,1,1)	$z_0^3+z_1^3+z_2^3+z_3^3$	3	$\mathbb{CP}(2)\#6\overline{\mathbb{CP}(2)}$
1	(1,1,1,2)	$z_0^4+z_1^4+z_2^4+z_3^2$	4	$\mathbb{CP}(2)\#7\overline{\mathbb{CP}(2)}$
1	(1,1,2,3)	$z_0^6+z_1^6+z_2^3+z_3^2$	6	$\mathbb{CP}(2)\#8\overline{\mathbb{CP}(2)}$
2	(1,1,1,1)	$z_0^2+z_1^2+z_2^2+z_3^2$	2	$\mathbb{CP}(1)\times\mathbb{CP}(1)$
3	(1,1,1,1)	$z_0+z_1+z_2+z_3$	1	$\mathbb{CP}(2)$

5. Proof of Theorem 4.5

The proof goes as follows: we begin by proving two lemmas which by Lemma 4.2 will reduce the problem to one in which the integers m_2 and m_1 of 4.1 are bounded. The computer programs then give a printout of solutions of 4.1 which give all log del Pezzo orbifolds $\mathcal{Z}_{\mathbf{w}}$ such that $(\mathcal{Z}_{\mathbf{w}}, D)$ might be klt for all $D \equiv -\frac{2+\epsilon}{3}K_{\mathcal{Z}_{\mathbf{w}}}$. In the case that m_0 is unbounded, these are the *series* solutions, and the case when m_0 is bounded, they are the *sporadic* solutions. Corollary 3.7 is sufficient, for most of the sporatic solutions, to check that $(\mathcal{Z}_{\mathbf{w}}, D)$ is klt for all $D \equiv -\frac{2+\epsilon}{3}K_{\mathcal{Z}_{\mathbf{w}}}$. However, for the series solutions the klt condition is much more tedious. Indeed, in several cases we are unable to give an answer. The results are collected in tables. In the last column of the tables we indicate whether the klt condition holds (Y) or is unknown (?). Since there exists a Kähler-Einstein orbifold metric whenever $(\mathcal{Z}_{\mathbf{w}}, D)$ is klt for all $D \equiv -\frac{2+\epsilon}{3}K_{\mathcal{Z}_{\mathbf{w}}}$, this column is labeled K-E. The second to the last column gives the second Betti number of $\mathcal{Z}_{\mathbf{w}}$ which is computed by the method described in section 6. Also, where space permits, we give the monomials that make up the defining weighted homogeneous polynomial. This is indicated in the tables for most of the sporadic solutions.

LEMMA 5.1: *Suppose $2I \geq 3w_0$. Then for any $\epsilon > 0$, there exists $D \in |-K_{\mathcal{Z}_{\mathbf{w}}}|$ such that $(\mathcal{Z}_{\mathbf{w}}, \frac{2+\epsilon}{3}D)$ is not klt.*

PROOF: Consider the section $z_0 \in H^0(\mathcal{Z}_{\mathbf{w}}, \mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(w_0))$ and let D be its zero divisor. Then by hypothesis $rD \in |\mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(-K_{\mathcal{Z}_{\mathbf{w}}})|$ for $r \geq \frac{3}{2}$. But then $\frac{2+\epsilon}{3}D$ can never be klt at a generic point $y \in D$, for $\epsilon > 0$, since it has multiplicity greater than 1. ∎

LEMMA 5.2: *Suppose $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}(w_0, w_1, w_2, w_3)$ is a hypersurface of index I with $2I = w_0 + w_1$. Then for any $\epsilon > 0$, there exists $D \in |-K_{\mathcal{Z}_{\mathbf{w}}}|$ such that $(\mathcal{Z}_{\mathbf{w}}, \frac{2+\epsilon}{3}D)$ is not klt.*

PROOF: First we notice in the proof of Lemma 4.2 that $2I = w_0 + w_1$ occurs in precisely one case and in this case we have $w_3 = w_1 + w_2 - I$. But then $\mathbf{w}$ must have the form

$$\mathbf{w} = (I - n, I + n, w, w + n)$$

for some $w \in \mathbb{Z}^+$ with $w \geq I + n$, and some non-negative integer $n < I$ and in this case the degree $d = 2w + n + I$. Now suppose that $z_0^{a_0} z_1^{a_1} z_2^{a_2} z_3^{a_3}$ is a monomial occurring with nonzero coefficient in the polynomial defining $\mathcal{Z}_{\mathbf{w}}$. We claim that if $a_0 = 0$ then $a_1 \neq 0$. Indeed, if $a_0 = a_1 = 0$ then

$$a_2 w + a_3 w = 2w + 2I + n.$$

But $w \geq I + n$ so the only possible solutions would require $a_2 + a_3 = 3$ and $w = 2I + n$ and one readily checks that these hypersurfaces are never quasi-smooth. Thus the divisor $D = \{z_0 = 0\} \cap \mathcal{Z}_{\mathbf{w}}$ has at least two components, E and F, where E is the line $z_0 = z_1 = 0$ and F is defined, inside the weighted projective plane $\{z_0 = 0\}$, by a polynomial $f(z_1, z_2, z_3)$. Moreover, $f(z_1, z_2, z_3) = z_2^2 + z_1^2(g(z_1, z_2, z_3))$. Note that the point $P = (0, 0, 0, 1) \in \mathcal{Z}_{\mathbf{w}}$ since $w + n$ does not divide the degree of $\mathcal{Z}_{\mathbf{w}}$. Thus if $\pi : \mathbb{C}^2 \longrightarrow \mathcal{Z}_{\mathbf{w}}$ is a local cover of the quotient singularity at $P = (0, 0, 0, 1)$ then $\pi^* D = \pi^* E + \pi^* F$ has multiplicity at least $1 + \mathrm{mult}_0(\pi^* F)$ at the origin. To compute the multiplicity of $\pi^* F$ at the origin, let $Y = Z(f(z_1, z_2, z_3))$. Then $Y \cap \mathcal{Z}_{\mathbf{w}} = F \cup G$, where G does not contain the point $(0, 0, 0, 1)$. Thus

$$\mathrm{mult}_0(\pi^* F) = \mathrm{mult}_0(\pi^* Z(f)) \geq 2.$$

Consequently, $(\mathcal{Z}_{\mathbf{w}}, \frac{2+\epsilon}{3}D)$ is never klt as it always has multiplicity > 2 at 0. But $\frac{I}{I-n}D \in |-K_{\mathcal{Z}_{\mathbf{w}}}|$ so this completes the proof of the lemma. ∎

The analysis of most of the sporadic examples of Table 1 is easily done with help of Corollary 3.7 which can restated for this purpose as:

COROLLARY 5.3: *Let $\mathbf{w} = (w_0, w_1, w_2, w_3)$ and $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}(\mathbf{w})$ be a quasi-smooth surface of degree $d = w_0 + w_1 + w_2 + w_3 - I$. Then $\mathcal{Z}_{\mathbf{w}}$ admits a Kähler-Einstein metric if $2Id < 3w_0 w_1$. If the line $(z_0 = z_1 = 0) \not\subset \mathcal{Z}_{\mathbf{w}}$ then $2Id < 3w_0 w_2$ is also sufficient. If the point $(0, 0, 0, 1) \notin \mathcal{Z}_{\mathbf{w}}$ then $2Id < 3w_0 w_3$ is also sufficient.*

We begin with the more complicated analysis of the infinite series examples. In what follows $\mathbb{P}$ will frequently denote the appropriate weighted projective space, with the weights $\mathbf{w}$ being understood; similarly $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}$ will denote a hypersurface in the weighted projective space, depending on an integer parameter k for all of the series examples.

• We consider now the hypersurface $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}(4, 2k+1, 2k+1, 4k)$ given by the zero set of the homogeneous polynomial of degree $d = 8k + 4$

$$f_{\mathbf{w}}(z_0, z_1, z_2, z_3) = z_0^{2k+1} + z_0 z_3^2 + g(z_1, z_2),$$

where g is of degree 4 and we assume that the polynomial g has 4 distinct roots on the projective line. Thus $-K_{\mathcal{Z}_{\mathbf{w}}} = \mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(2)$. The only singularities of $\mathcal{Z}_{\mathbf{w}}$ lie at the singular points of $\mathbb{P}(\mathbf{w})$. We can apply Corollary 3.7 to see that $(\mathcal{Z}_{\mathbf{w}}, D)$ is klt at all smooth points of $\mathbb{P}(\mathbf{w})$, for all $k > 0$, for all $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$ since

$$d \cdot \kappa = (8k+4)2 < 4(2k+1)(2k+1).$$

We now pass to the singular points of $\mathcal{Z}_{\mathbf{w}}$ which are slightly more involved as we need to consider a desingularization. We first treat the point $P = (0,0,0,1)$. We would like to apply the preceding discussion and use Shokurov's inversion of adjunction (Lemma 2.5) for which we need to verify that $\mathrm{mult}_0(\pi^* D) \leq 2$ for any effective $\mathbb{Q}$–divisor $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$: here $\pi : (\mathbb{C}^2, 0) \to (\mathcal{Z}_{\mathbf{w}}, P)$ is a local cover of the quotient singularity at P. We will consider the linear series $|z_1, z_2|$ on $\mathcal{Z}_{\mathbf{w}}$. This has only isolated base points on $\mathcal{Z}_{\mathbf{w}}$, including the point P in question, and consequently if E is a general member of this system then $D \cap E$ will be proper. Pulling back to $\mathbb{C}^2$ to compute the intersection multiplicity at 0 we find, since the ramification of π over P is of degree $4k$,

$$\mathrm{mult}_0(\pi^* D) \leq 4kD \cdot E = \frac{2(2k+1)(8k+4)4k}{4(2k+1)^2 4k} = 2,$$

and thus we may apply Lemma 2.5. Hence it is sufficient, in order to establish that $(\mathcal{Z}_{\mathbf{w}}, D)$ is log–canonical at P, to show that

$$p_*^{-1}(\pi^* D)|E$$

is a sum of points each with coefficient at most one.

Consider the hypersurface $z_0 = 0$ on $\mathcal{Z}_{\mathbf{w}}$. This is a union of four lines, L_1, L_2, L_3, L_4, one for each (projective) zero of the polynomial $g(z_1, z_2)$. Thus we have

5.4 $$L_1 + L_2 + L_3 + L_4 \in |\mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(4)|.$$

Note that if an effective divisor $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$ does not contain any of the lines L_i, then computing $(L_1 + L_2 + L_3 + L_4)D$ will immediately establish that $(\mathcal{Z}_{\mathbf{w}}, D)$ is klt at P. Thus the representatives of $-K_{\mathcal{Z}_{\mathbf{w}}}$ which are of concern are those containing some or all of the lines L_i. So suppose

5.5 $$D \equiv a_1 L_1 + a_2 L_2 + a_3 L_3 + a_4 L_4 + D',$$

where D' meets each of the four lines properly. We are interested in bounding the a_i which can be accomplished intersection theoretically. In particular we compute

5.6 $$L_i \cdot L_j = \frac{1}{4k} \qquad \text{for} \qquad i \neq j.$$

To see this, note that the four lines L_i are algebraically equivalent. Thus, using 5.4 above gives

5.7 $$\mathcal{Z}_{\mathbf{w}} \cdot \mathcal{O}_{\mathbb{P}}(4) \cdot \mathcal{O}_{\mathbb{P}}(2k+1) = 4(L_i \cdot \mathcal{O}_{\mathbb{P}}(2k+1)) \qquad \text{for all } i.$$

The left hand side of 5.7 is $\frac{4(8k+4)(2k+1)}{4(2k+1)^2 4k} = \frac{1}{k}$. On the other hand one can check, choosing the appropriate representative for $\mathcal{O}_{\mathbb{P}}(2k+1)$ that $\mathcal{O}_{\mathbb{P}}(2k+1) \cdot \mathcal{Z}_{\mathbf{w}} = L_j + C$, where C does not meet L_i; 5.6 follows immediately.

Intersecting 5.4 with $\mathcal{O}_{\mathbb{P}}(1)$ gives

5.8 $$\mathcal{O}_{\mathbb{P}}(1) \cdot L_i = \frac{1}{4(2k+1)k}, \qquad \forall\ i.$$

Using 5.6 and 5.8 we can compute L_i^2:

5.9 $$L_i^2 = L_i \cdot \mathcal{O}_{\mathbb{P}}(4) - \sum_{j \neq i} L_i \cdot L_j = \frac{1-6k}{(2k+1)4k}.$$

We have, by 5.8

5.10 $$D \cdot L_i = \frac{1}{2k(2k+1)}.$$

Using 5.5 we obtain from 5.10

5.11 $$a_i L_i^2 + \sum_{j \neq i} a_j L_j \cdot L_i + D' \cdot L_i = \frac{1}{2k(2k+1)}.$$

To compute the terms in 5.11 observe first that $\sum_{i=1}^4 a_i \leq 2$; this follows from the fact that $\mathrm{mult}_0(\pi^* D) \leq 2$. Next, note that

$$D' \cdot L_i \leq D \cdot D = \mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(2) \cdot \mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(2) = \frac{4(8k+4)}{4(2k+1)^2 4k} = \frac{1}{k(2k+1)}.$$

Finally, using 5.6 and 5.9, 5.11 becomes

$$\frac{a_i(1-6k)}{(2k+1)4k} + \frac{1}{4k} \sum_{j \neq i} a_j + \frac{1}{k(2k+1)} \geq \frac{1}{2k(2k+1)}.$$

Clearing denominators then yields

$$a_i \leq \frac{k+1}{2k}.$$

Thus if $k \geq 2$ we have $a_i \leq \frac{3}{4}$.

Next we bound $\mathrm{mult}_0(\pi^* D')$. Since $L_i \cap D'$ is proper for any i we compute, using 5.8

$$\mathrm{mult}_0(\pi^* D') \leq 4k L_i \cdot D \leq \frac{2}{2k+1}.$$

Returning to the inversion of adjunction set–up, we see that $p_*^{-1}(\pi^*(D))$ is a sum of four points (corresponding to the pull–backs of the four of the lines L_i) having total multiplicity at most $\frac{3}{4}$ and another divisor with total degree at most $\frac{2}{2k+1}$. A simple computation then shows that $\frac{11}{13}D$ is klt at $P = (0,0,0,1)$ for any $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$.

Next we turn to the points $P_i = (0, a_i, b_i, 0)$ on $\mathcal{Z}_{\mathbf{w}}$; there are exactly four of these by our assumption on the homogeneous polynomial $f(z_1, z_2)$ and the singularities have index $2k+1$. Since $\mathrm{BS}(\mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(w_0w_1w_2) \otimes \mathcal{I}_{P_i})$ is a finite set of points, we will apply Lemma 2.12 with $A = \mathcal{O}_{\mathbb{P}}(8k+4)$. Then we have

$$A \cdot D = \frac{(8k+4)(2)(8k+4)}{4(2k+1)(2k+1)4k} = \frac{2}{k}.$$

We can take $a = 4$ in Lemma 2.12 since the linear series $\mathcal{O}_{\mathbb{P}}(8k+4)$ allows for an isolated zero at P_i of multiplicity 4. Thus for $k \geq 2$ we see that $(\mathcal{Z}_{\mathbf{w}}, \frac{3}{4}D)$ is klt at P_i.

Finally, we deal with the points $Q_j = (a_j, 0, 0, b_j)$, where the quotient singularities are of index 4. We again apply Lemma 2.12, this time taking $A = \mathcal{O}_{\mathbb{P}}(2k+1)$. This linear series has sections with an isolated singularity of multiplicity 1 at any of the Q_j. We compute

$$A \cdot D = \frac{2(2k+1)(8k+4)}{4(2k+1)^2 4k} = \frac{1}{2k}$$

Thus for $k \geq 2$ we see that $(\mathcal{Z}_{\mathbf{w}}, \frac{3}{4}D)$ is klt at Q_j. Putting together all of our computations establishes that $(\mathcal{Z}_{\mathbf{w}}, \frac{3}{4}D)$ is klt for $k \geq 2$ and for any $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$.

• We consider now the hypersurface $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}(3, 3k+1, 6k+1, 9k+3)$ given by the zero set of the homogeneous polynomial of degree $d = 18k+6$

$$f_{\mathbf{w}}(z_0, z_1, z_2, z_3) = z_0^{6k+2} + z_1^3 z_3 + z_2^3 z_0 + z_3^2.$$

Here $-K_{\mathcal{Z}_{\mathbf{w}}} = \mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(2)$. The worst singularity in this case is at $x = (0,0,1,0)$, where the index is $6k+1$. Since $\mathrm{BS}(\mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(w_0w_1) \otimes \mathcal{I}_x)$ consists of finitely many points, Corollary 3.7 says that $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ is klt at x for $k \geq 1$ for any $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$.

Next we consider the singular line $z_0 = z_2 = 0$, where the index is $3k+1$. This intersects $\mathcal{Z}_{\mathbf{w}}$ in the finitely many points. Applying the third formula of Corollary 3.7 shows that $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ is klt at each of these points for $k \geq 1$.

Finally, there are index 3 singularities along the line $z_1 = z_2 = 0$ which again intersects $\mathcal{Z}_{\mathbf{w}}$ in finitely many points. Applying Corollary 3.7 to these shows that $(\mathcal{Z}_{\mathbf{w}}, \frac{7}{8}D)$ is klt at these points for $k \geq 1$. Putting everything together we see that $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ is klt for any $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$ provided $k \geq 1$.

• Next we consider the hypersurface $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}(6, 6k+5, 12k+8, 18k+15)$ given by the zero set of the homogeneous polynomial of degree $d = 36k+30$

$$f_{\mathbf{w}}(z_0, z_1, z_2, z_3) = z_0^{6k+5} + z_1^6 + z_1^3 z_3 + z_0 z_2^3 + z_3^2.$$

Here we have $\mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(-K_{\mathcal{Z}_{\mathbf{w}}}) = \mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(4)$. This variety contains fewer of the singular points of $\mathbb{P}$ than the previous example and asymptotically in k, the proportions of these weights are identical and thus this also must be α-klt for all k sufficiently large and for appropriate α. More specifically, we must deal with one point at infinity, $x = (0,0,1,0)$ which has index $12k+8$. Using $\mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(6(6k+5))$ in Corollary 3.7 we see that $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ is klt at x.

Next there is the line $z_0 = z_2 = 0$, where the index of $\mathbb{P}$ is $6k+5$. Applying Corollary 3.7 as above gives $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ klt for $k \geq 1$. Finally, there are is the line $z_1 = z_2 = 0$ of index 3. It follows once more from Corollary 3.7 that $(\mathcal{Z}_{\mathbf{w}}, \frac{7}{8}D)$ is klt at each of these points for all $k \geq 2$. There are singularities of index 2 along the line $z_1 = z_3 = 0$; $(\mathcal{Z}_{\mathbf{w}}, D)$ is klt at these points for all $k \geq 1$. Thus we find that for all $k \geq 2$, $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ is klt for any $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$.

• Next we consider the hypersurface $\mathcal{Z}_{\mathbf{w}} = \mathbb{P}(3, 3k+1, 3k+2, 3k+2)$ given by the zero set of the homogeneous polynomial of degree $d = 9k+6$

$$f_{\mathbf{w}}(z_0, z_1, z_2, z_3) = z_0^{3k+2} + z_0 z_1^3 + g(z_2, z_3),$$

where g is a homogeneous polynomial of degree three with distinct zeroes. The only singular points to be analyzed in this case are $(0,1,0,0)$ and $(0,0,a_i,b_i)$, where (a_i,b_i) are the zeroes of g. We first examine $P = (0,1,0,0)$, where $\mathbb{P}$ has a singularity of index $3k+1$. Let $\pi : (\mathbb{C}^2, 0) \to (\mathcal{Z}_{\mathbf{w}}, P)$ be a local cover and let $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$. Intersecting D with a general member of $\mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(3k+2)$ establishes that

$$\mathrm{mult}_0(\pi^* D) \leq 2$$

so that we can apply Shokurov's inversion of adjunction as before. As before we write $L_1 + L_2 + L_3$ for the zero scheme of z_0 on $\mathcal{Z}_{\mathbf{w}}$. We compute

$$\begin{aligned} L_i \cdot L_j &= \frac{1}{3k+1}, \qquad i \neq j, \\ \mathcal{O}_{\mathbf{P}}(1) \cdot L_i &= \frac{1}{(3k+1)(3k+2)}, \\ L_i^2 &= \frac{-1-6k}{(3k+1)(3k+2).} \end{aligned}$$

For $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$ effective we write $D = a_1 L_1 + a_2 L_2 + a_3 L_3 + D'$ as before, with D' meeting the three lines properly, and we find, expanding $D \cdot L_i$ and using the estimates $D \cdot D' \leq D \cdot D$, $a_1 + a_2 + a_3 \leq 2$

$$\frac{(6k+1)a_i}{(3k+1)(3k+2)} + \frac{a_i - 2}{3k+1} \leq \frac{2}{(3k+1)(3k+2)}.$$

Expanding yields $a_i \leq \frac{6k+6}{9k+3}$. Moreover, intersecting with $z_0 = 0$ shows that

$$\mathrm{mult}_0(\pi^* D') \leq \frac{2}{3(3k+2)}.$$

Thus, with notation as before, $p_*^{-1}\pi^* D$ is a sum of points with multiplicity at most $\frac{6k+6}{9k+3} + \frac{2}{3(3k+2)}$ and inversion of adjunction applies to show that $(\mathcal{Z}_{\mathbf{w}}, \frac{3}{4}D)$ is klt at P for all $k \geq 1$.

The other singular points of $\mathcal{Z}_{\mathbf{w}}$ like along the line $z_0 = z_1 = 0$ and these have index $3k+2$. Let $Q_i = (0, 0, a_i, b_i) \in \mathcal{Z}_{\mathbf{w}}$ be one of these three points. The analysis of D above for $P = (0, 1, 0, 0)$ applies at these points as well, unchanged though exactly one of the lines L_i passes through Q_i. In order to justify the use of Lemma 2.5 we need to show that $\mathrm{mult}_0(\pi^* D) \leq 2$ for a local cover of $\mathbb{P}$ at Q_i. This can be done with Lemma 2.12 using the linear series $\mathcal{O}_{\mathcal{Z}_{\mathbf{w}}}(3(3k+1))$ allowing for a singularity of multiplicity 3 at Q_i. Thus we conclude that $(\mathcal{Z}_{\mathbf{w}}, \frac{3}{4}D)$ is klt for all $k \geq 1$ and for all $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$.

• Next we consider the hypersurface $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}(6, 6k+3, 6k+5, 6k+5)$ given by the zero set of the homogeneous polynomial of degree $d = 18k+15$

$$f_{\mathbf{w}}(z_0, z_1, z_2, z_3) = z_0 z_1^3 + z_0^{2k+2} z_1 + g(z_2, z_3),$$

where again g is homogeneous of degree 3 and $\mathcal{Z}_{\mathbf{w}}$. This passes through two points at infinity $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ and also intersects the singular lines $z_0 = z_1 = 0$ and $z_2 = z_3 = 0$ in finitely many points. The analysis of $(0, 1, 0, 0)$ and $(0, 0, a_i, b_i)$ is identical to the prior case, yielding $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ is klt at these points for all $k \geq 1$. For the points along the line $z_2 = z_3 = 0$, other than $(0, 1, 0, 0)$,the index does not depend on k, and hence these are certainly klt for k sufficiently large; more specifically, $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ is also klt at these points for $k \geq 2$. Hence $(\mathcal{Z}_{\mathbf{w}}, \frac{5}{7}D)$ is klt for all $k \geq 2$ and for all $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$.

• Lastly we consider the hypersurface $\mathcal{Z}_{\mathbf{w}} \subset \mathbb{P}(3, 3k, 3k+1, 3k+1)$ given by the zero set of the homogeneous polynomial of degree $d = 9k+3$

$$f_{\mathbf{w}}(z_0, z_1, z_2, z_3) = z_0^{3k+1} + z_0 z_1^3 + g(z_2, z_3),$$

where g is homogeneous of degree 3. This contains only one singular point at infinity $(0, 1, 0, 0)$ as well as the points $(0, 0, a_i, b_i)$ and the points $(c_i, d_i, 0, 0)$. Routine computations exactly as above show that $(\mathcal{Z}_{\mathbf{w}}, \frac{3}{4}D)$ is klt for $k \geq 2$ and for all $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$.

Remark 5.12: Let us make a few comments for the series which we have not analyzed. For the two series of index 6, the simple intersection theoretic argument which establishes that $\mathrm{mult}_0(\pi^* D) \leq 2$ on a local cover of one of the singular points, where $D \equiv -K_{\mathcal{Z}_{\mathbf{w}}}$, fails. The desired bound may well still hold but more detailed analysis of the other points of intersection of D and the appropriate divisor of Corollary 3.7 would be necessary to establish this. In the other cases which we have not analyzed, the singularities of "bad" divisor $D = \{z_0 = 0\} \cap \mathcal{Z}_{\mathbf{w}}$ require a more subtle analysis because in these cases the remaining three weights are distinct and hence it is more difficult to compute the contribution of D to the tangent cone at the origin of the appropriate local cover; when two of the remaining three weights are the same, one quickly reduces to $\mathbb{P}(1,1,k)$ which is singular only at $(0,0,1)$ and it is easy to check which tangent directions D gives on the resolution. In principle, however, each of these cases could be decided with more involved computations.

6. The Topology of the Link L_f

Recall the well-known construction of Milnor [Mil] for isolated hypersurface singularities: There is a fibration of $(S^{2n+1}-L_f)\longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n\vee S^n\cdots\vee S^n$. The *Milnor number* μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $(w_0,\ldots,w_n)$ by the formula [MO]

6.1
$$\mu=\mu(L_f)=\prod_{i=0}^{n}(\frac{d}{w_i}-1).$$

The closure $\bar{F}$ of F is a manifold with boundary that is homotopy equivalent to F, and whose boundary is precisely the link L_f. Then the topology of L_f is determined by the *monodromy map* induced by the $S^1_{\mathbf{w}}$ action. Milnor and Orlik [MO] use these facts to give an algorithm for computing the Betti number $b_{n-1}(L_f)$ from the characteristic polynomial $\Delta(t)$ of the monodromy map. The procedure is this. Associate to any monic polynomial f with roots $\alpha_1,\ldots,\alpha_k\in\mathbb{C}^*$ its divisor

6.2
$$\mathrm{div}\ f=<\alpha_1>+\cdots+<\alpha_k>$$

as an element of the integral ring $\mathbb{Z}[\mathbb{C}^*]$ and let $\Lambda_n=\mathrm{div}(t^n-1)$. The 'rational weights' used in [MO] are just $\frac{d}{w_i}$, and are written in irreducible form, $\frac{d}{w_i}=\frac{u_i}{v_i}$. The divisor of the characteristic polynomial is then determined by

6.3.
$$\mathrm{div}\ \Delta(t)\ =\prod_i(\frac{\Lambda_{u_i}}{v_i}-1)\ =1+\sum a_j\Lambda_j,$$

where $a_j\in\mathbb{Z}$ and the second equality is obtained by using the relations $\Lambda_a\Lambda_b=\gcd(a,b)\Lambda_{lcm(a,b)}$. The second Betti number of the link is then given by

6.4
$$b_2(L_f)=1+\sum_j a_j.$$

Furthermore, the following proposition was proved in [BG3]:

Proposition 6.5: *Let L_f be the link of an isolated singularity defined by a weighted homogeneous polynomial f in four complex variables with weights $\mathbf{w}$. Suppose further that the weights $\mathbf{w}$ are well-formed, then Tor $(H_2(L_f,\mathbb{Z}))=0$.*

Now a well-known theorem of Smale [Sm] says that any simply connected compact 5-manifold which is spin, and whose second homology group is torsion free, is diffeomorphic to $S^5\#l(S^2\times S^3)$ for some non-negative integer l. Furthermore, it is known [BG2,Mor] that any simply connected Sasakian-Einstein manifold is spin. Combining this with the development above gives

THEOREM 6.6: *Let L_f be the link associated to a well-formed weighted homogeneous polynomial f in four complex variables. Suppose also that L_f is spin, in particular, if L_f admits an Sasakian-Einstein metric. Then L_f is diffeomorphic to $S^5\#l(S^2\times S^3)$, where $l=b_2(L_f)=1+\sum_j a_j$.*

7. The Moduli Problem for Sasakian-Einstein 5-Manifolds

In this section we discuss the moduli problem for Sasakian-Einstein manifolds. It is not our intention here to discuss the general moduli problem for Sasakian-Einstein structures, but rather to present some results regarding the 5 dimensional non-regular examples described in the previous sections together with the known regular Sasakian-Einstein 5-manifolds. In what follows by moduli space we shall mean certain sets of objects (sections of vector bundles) modulo the action of the group of diffeomorphisms that are diffeotopic to the identity. In the case of orbifolds diffeomorphism means diffeomorphism in the sense of orbifolds. Thus, in the case of complex moduli any two representatives are deformation equivalent.

We begin by briefly discussing the case complex structures on the del Pezzo surfaces obtained by blowing up $\mathbb{CP}^2$ at l distinct points for $0\leq l\leq 8$. We shall always assume that no two of these points lie on a line nor any three lie on a conic which we refer to as *in general position.* The complex structures on $\mathcal{Z}_l=\mathbb{CP}^2\#l\overline{\mathbb{CP}}^2$ are then determined by the complex coordinates of the l points modulo the action of the complex automorphism group $\mathfrak{G}(\mathcal{Z}_l)$. Since we can fix precisely four points in general position with the action of $GL(3,\mathbb{C})$, the moduli space of complex structures $\mathcal{M}_l^{\mathbb{C}}$ on $\mathcal{Z}_l$ for $l=0,\ldots,4$ is a single point, whereas for $l=5,6,7,8$ it is an open connected manifold of complex dimension $2(l-4)$.

It is known that when $l>2$, there exists [Siu,TY,Ti1] a unique [BM], up to complex automorphism, Kähler-Einstein structure associated to each complex structure. It follows that in this case ($l>2$) there is a 1-1 correspondence between the complex structures on $\mathcal{Z}_l$ and the homothety classes of positive Kähler-Einstein metrics modulo complex automorphisms. This gives the identification of moduli spaces

7.1 $$\mathcal{M}_l^{\mathbb{C}}\simeq\mathcal{M}_l^{KE},\quad \text{for } l>2$$

where $\mathcal{M}_l^{KE}$ denotes the moduli space of homothety classes of Kähler-Einstein metrics on $\mathcal{Z}_l$. Moreover, when $l=1,2$ the space $\mathcal{M}_l^{KE}$ is well-known to be empty, whereas, $\mathcal{M}_l^{\mathbb{C}}$ is a single point space. Furthermore, for $\mathbb{CP}^2$ and $\mathbb{CP}^1\times\mathbb{CP}^1$ both $\mathcal{M}_l^{\mathbb{C}}$ and $\mathcal{M}_l^{KE}$ are single point spaces. When $4<l<9$, it is possible that for two inequivalent complex structures $J_1,J_2\in\mathcal{M}_l^{\mathbb{C}}$ the Einstein metrics g_1, g_2 solving the corresponding Monge-Ampere equations coincide. However, it follows from a theorem of Pontecorvo [Pon] that up to complex conjugation this can happen only when the metric is anti-self-dual, that is, the anti-self-dual part W_+ of the Weyl conformal tensor must vanish. But it is well-known (cf. [Boy]) that a compact anti-self-dual Kähler surface has zero scalar curvature. So this cannot happen for del Pezzo surfaces. The precise result is given below in Proposition 7.13, and the argument works equally well for compact Kähler orbifolds.

In the more general case of log del Pezzo surfaces $\mathcal{Z}$ that is complex compact surfaces with positive first Chern class and at most quotient singularities much less is known. Here we are dealing with Kähler orbifolds and as discussed earlier the existence problem of finding Kähler-Einstein orbifold metrics is still open. However, the Bando-Mabuchi uniqueness theorem carries over to the orbifold case, so when a positive Kähler-Einstein orbifold metric exists it is unique up to homothety and complex automorphisms. Thus, generally, if $\mathcal{M}^{KE}$ denotes the moduli space of homothety classes of Kähler-Einstein metrics on $\mathcal{Z}$, and $\mathcal{M}^{\mathbb{C}}$ denotes the moduli space of complex structures on $\mathcal{Z}$, then there is an injective map

7.2 $$\mathcal{M}^{KE}\longrightarrow\mathcal{M}^{\mathbb{C}}.$$

Next we relate the moduli of Kähler-Einstein structures on $\mathcal{Z}$ to the moduli of Sasakian-Einstein structures on $\mathcal{S}$. First we give a result in general dimension from our previous work [BG1], and then specialize to dimension five.

PROPOSITION 7.3: *Two $2n+1$ dimensional rank one Sasakian structures $\mathcal{S}=(\xi,\eta,\Phi,g)$ and $\mathcal{S}'=(\xi',\eta',\Phi',g')$ are equivalent if and only if their corresponding spaces of leaves $(\mathcal{Z},\omega,J,h)$ and $(\mathcal{Z}',\omega',J',h')$ are equivalent as Kähler orbifolds. Furthermore, $\mathcal{S}=(\xi,\eta,\Phi,g)$ is Sasakian-Einstein if and only if $(\mathcal{Z},\omega,J,h)$ is Kähler-Einstein with scalar curvature $4n(n+1)$.*

REMARK 7.4: Any Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ has a canonically equivalent structure, namely the *conjugate* Sasakian structure $\bar{\mathcal{S}}=(-\xi,-\eta,-\Phi,g)$. This corresponds on the space of leaves to the complex conjugate Kähler structure, namely $(\mathcal{Z},-\omega,-J,h)$.

Notice that the correspondence in Proposition 7.4 is between rank one Sasakian-Einstein structures and Kähler-Einstein structures with a fixed scalar curvature, or equivalently homothety classes of Kähler-Einstein structures. Now there are two possible types of deformations of Sasakian-Einstein structures, those that deform the foliation and those that do not. The latter all lie in one of the families $\mathfrak{F}(\xi)$ discussed in section 1, and the former correspond to different base orbifolds $\mathcal{Z}$ (assuming both the original and deformed Sasakian-Einstein structures are rank one). We believe that Sasakian-Einstein structures lying in different families $\mathfrak{F}(\xi)$ correspond to distinct components of the moduli space of Sasakian-Einstein structures, but we do not prove this here.

Now let us fix some notation. Let $\mathcal{S}_l=S^5\#l(S^2\times S^3)$ denote by $\mathcal{M}_l^{SE}$ the moduli space of Sasakian-Einstein structures on $\mathcal{S}_l$, and let $\mathcal{M}_l^{reg,SE}$ and $\mathcal{M}_l^{1,SE}$ denote the moduli space of regular Sasakian-Einstein structures, and rank one Sasakian-Einstein structures on $\mathcal{S}_l$, respectively. Then we have natural inclusions

7.5 $$\mathcal{M}_l^{reg,SE}\subset\mathcal{M}_l^{1,SE}\subset\mathcal{M}_l^{SE}.$$

Combining our discussion above with [FK,BG1,Ti1-Ti3] we obtain for the regular case:

PROPOSITION 7.6: *The following hold:*

(i) $\mathcal{M}_l^{reg,SE}$ *is not empty if and only if* $l = 0, 1, 3, 4, 5, 6, 7, 8$.

(ii) $\mathcal{M}_l^{reg,SE}$ *is a single point when* $l = 0, 1, 3, 4$.

(iii) $\mathcal{M}_l^{reg,SE}$ *is a connected complex manifold of dimension* $2(l-4)$ *when* $l = 5, 6, 7, 8$. *Furthermore, up to conjugation there is precisely one Sasakian structure sharing the same Einstein metric* g.

To analyze the moduli problem in the non-regular case, we begin by describing the group of complex automorphisms $\mathfrak{G}_{\mathbf{w}}$ of the weighted projective 3-space $\mathbb{P}(\mathbf{w})$. We shall assume that $\mathbb{P}(\mathbf{w})$ is well-formed. Recall that $\mathbb{P}(\mathbf{w})$ can be defined as a scheme $\mathrm{Proj}(S(\mathbf{w}))$, where

$$S(\mathbf{w}) = \bigoplus_d S^d(\mathbf{w}) = \mathbb{C}[z_0, z_1, z_2, z_3].$$

The ring of polynomials $\mathbb{C}[z_0, z_1, z_2, z_3]$ is graded with grading defined by the weights $\mathbf{w} = (w_1, w_1, w_2, w_3)$. As a projective variety we can embed $\mathbb{P}(\mathbf{w}) \subset \mathbb{CP}^N$ and then the group $\mathfrak{G}_{\mathbf{w}}$ is a subgroup of $PGL(N, \mathbb{C})$. $\mathbb{P}(\mathbf{w})$ is a toric variety and we can describe $\mathfrak{G}_{\mathbf{w}}$ explicitly as follows: Let $\mathbf{w} = (w_0, w_1, w_2, w_3)$ be ordered as before. We consider the group $G(\mathbf{w})$ of automorphisms of the graded ring $S(\mathbf{w})$ defined on generators by

7.7
$$\varphi_{\mathbf{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} f_0^{(w_0)}(z_0, z_1, z_2, z_3) \\ f_1^{(w_1)}(z_0, z_1, z_2, z_3) \\ f_2^{(w_2)}(z_0, z_1, z_2, z_3) \\ f_3^{(w_3)}(z_0, z_1, z_2, z_3) \end{pmatrix},$$

where $f_i^{(w_i)}(z_0, z_1, z_2, z_3)$ is an arbitrary weighted homogeneous polynomial of degree w_i in (z_0, z_1, z_2, z_3). This is a finite dimensional Lie group and it is a subgroup of $GL(N, \mathbb{C})$. Projectivising, we get $\mathfrak{G}_{\mathbf{w}} = \mathbb{P}_{\mathbb{C}}(G(\mathbf{w}))$.

Note that when $\mathbf{w} = (1, 1, 1, 1)$ then $G(\mathbf{w}) = GL(4, \mathbb{C})$. Other than this case three weights are never the same if $\mathbb{P}(\mathbf{w})$ is well-formed. If two weights coincide then $G(\mathbf{w})$ contains $GL(2, \mathbb{C})$ as a subgroup. Finally, when all weights are distinct we can write

7.8
$$\varphi_{\mathbf{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} a_0 z_0 \\ a_1 z_1 + f_1^{(w_0)}(z_1) \\ a_2 z_2 + f_2^{(w_2)}(z_0, z_1) \\ a_3 z_3 + f_3^{(w_3)}(z_0, z_1, z_2) \end{pmatrix},$$

where $(a_0, a_1, a_2, a_3) \in (\mathbb{C}^*)^4$ and $f_i^{(w_i)}$, $i = 1, 2, 3$ are weighted homogeneous polynomials of degree w_i. The simplest situation occurs when $f_1 = f_2 = f_3$ are forced to vanish. Then $\mathfrak{G}_{\mathbf{w}} = (\mathbb{C}^*)^3$ is the smallest it can possibly be as $\mathbb{P}(\mathbf{w})$ is toric. This is, in fact, common to many examples of the log del Pezzo suraces of Table 1. More precisely, we have

LEMMA 7.9: *Let $w_0 < w_1 < w_2 < w_3$. If for each $i = 1, 2, 3$ the weight w_i is not a $\mathbb{Z}^+$-linear combination of smaller weights with non-negative integer coefficients than $\mathfrak{G}_{\mathbf{w}} = (\mathbb{C}^*)^3$.*

Let $S^d_{\mathbf{w}} \subset S(\mathbf{w})$ be the vector subspace spanned by all monomials in (z_0, z_1, z_2, z_3) of degree $d = |w| - I$, and let $\hat{S}^d(\mathbf{w}) \subset S^d(\mathbf{w})$ denote subset all quasi-smooth elements, i.e. those polynomials $f_{\mathbf{w}} \in S^d_{\mathbf{w}}$ such that conditions 3.4 hold. Then we define $m^d_{\mathbf{w}}$ to be the dimension of the subspace generated by $\hat{S}^d_{\mathbf{w}}$. Now the automorphism group $G(\mathbf{w})$ acts on $S^d_{\mathbf{w}}$ leaving the subset $\hat{S}^d(\mathbf{w})$ of quasi-smooth polynomials invariant. Thus, for each log del Pezzo surface that is a solution of Theorem 4.5 we define the moduli space

$$\mathcal{M}^d_{\mathbf{w}} = \hat{S}^d_{\mathbf{w}}/G(\mathbf{w}) = \mathbb{P}(\hat{S}^d_{\mathbf{w}})/\mathfrak{G}_{\mathbf{w}}, \tag{7.10}$$

with $n^d_{\mathbf{w}} = \dim(\mathcal{M}^d_{\mathbf{w}})$. Now there is an injective map

$$\mathcal{M}^d_{\mathbf{w}} \longrightarrow \mathcal{M}^{\mathbb{C}}(\mathcal{Z}_{\mathbf{w}}), \tag{7.11}$$

and by our results of sections 4 and 5, each element in $\mathcal{M}^d_{\mathbf{w}}$ correponds to a unique homothety class of Kähler-Einstein metrics modulo $\mathfrak{G}_{\mathbf{w}}$ and hence, to a unique Sasakian-Einstein structure on the corresponding 5-manifold $\mathcal{S}_l$ modulo the group $\mathfrak{G}_{\mathbf{w}}$ acting as CR automorphisms. The results with non-trivial moduli are collected in Table 3 below.

EXAMPLE 7.12: As an illustration, we shall calculate the moduli spaces of complex structures of the three classical del Pezzo surfaces of Table 2. We begin with the cubic in $\mathbb{P}(1,1,1,1)$. The subset $\hat{S}^3_{(1,1,1,1)} \subset \mathbb{C}^{20}$ is a dense open complex submanifold. In this case $G(\mathbf{w}) = GL(4, \mathbb{C})$ and the quotient $\mathcal{M}^3_{(1,1,1,1)}$ is a complex manifold of dimension 4. It is well-known that in this case we have the identification $\mathcal{Z}_f \simeq \mathcal{Z}_6$ so that $\mathcal{M}^3_{(1,1,1,1)}$ can be identified with $\mathcal{M}^{\mathbb{C}}_6$.

The second example is a degree 4 surface in $\mathbb{P}(1,1,1,2)$. Here $\hat{S}^4(1,1,1,2) \subset \mathbb{C}^{22}$ since the general weighted polynomial $f(\mathbf{z})$ of degree 4 can be written as the sum

$$f(\mathbf{z}) = g^{(4)}(z_0, z_1, z_2) + g^{(2)}(z_0, z_1, z_2)z_3 + \lambda z_3^2.$$

The group $G((1,1,1,2))$ is defined by

$$\varphi_{\mathbf{w}} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \mathbb{A} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \\ \alpha z_3 + \phi^{(2)}(z_0, z_1, z_2) \end{pmatrix}, \qquad \mathbb{A} \in GL(3, \mathbb{C}), \quad \alpha \in \mathbb{C}^*,$$

where $\phi^{(2)}(z_0, z_1, z_2)$ is an arbitrary homogeneous polynomial of degree 2 in (z_0, z_1, z_2). One can see that the action is free and, hence, $\mathcal{M}^4_{(1,1,1,2)}$ is a complex manifold of dimension $n^4_{(1,1,1,2)} = 6$. It is known that the smooth member $\mathcal{Z}_f \subset \mathbb{P}(1,1,1,2)$ is diffeomorphic to $\mathcal{Z}_7$ so that $\mathcal{M}^4_{(1,1,1,2)}$ can be identified with $\mathcal{M}^{\mathbb{C}}_7$.

The third example is a degree 6 surface in $\mathbb{P}(1,1,2,3)$. Here $\hat{S}^6(1,1,2,3) \subset \mathbb{C}^{23}$ as the general weighted polynomial $f(\mathbf{z})$ of degree 6 can be written as the sum

$$f(\mathbf{z}) = g^{(6)} + g^{(4)}z_2 + g^{(2)}z_2^2 + \lambda_1 z_2^3 + g^{(3)}z_3 + g^{(1)}z_2z_3 + \lambda_2 z_3^2,$$

where $g^{(i)} = g^{(p)}(z_0, z_1)$ is a homogeneous polynomial of degree $p = 1,2,3,4,6$. The group $G((1,1,2,3))$ in defined by

$$\varphi_{\mathbf{w}}\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \mathbb{A}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \\ \alpha_2 z_2 + \phi_2^{(2)}(z_0,z_1) \\ \alpha_3 z_3 + \phi_3^{(3)}(z_0,z_1) + \phi_3^{(1)}(z_0,z_1)z_2 \end{pmatrix}, \qquad \mathbb{A} \in GL(2,\mathbb{C}), \quad \alpha_2, \alpha_3 \in \mathbb{C}^*,$$

and $\phi_i^{(p)}$, $i = 1,2,3$ are homogeneous polynomials in (z_0, z_1) of degree p. One can see that $\dim(G(1,1,2,3)) = 15$ and $\mathcal{M}^6_{(1,1,2,3)}$ is a connected complex manifold of dimension $n^6_{(1,1,2,3)} = 8$. Again, the general smooth member $\mathcal{Z}_f \subset \mathbb{P}(1,1,2,3)$ is diffeomorphic to $\mathcal{Z}_8$ so that $\mathcal{M}^4_{(1,1,2,3)}$ can be identified with $\mathcal{M}^{\mathbb{C}}_8$.

One can carry out similar calculations for any of the log del Pezzo surfaces. In Table 1 below we tabulate all the examples of log del Pezzo surfaces of Theorem 4.5 which admit Kähler-Einstein metrics and for which $n^d_{\mathbf{w}} \geq 1$ and indicate the corresponding link L_f in S^7 :

Table 3. Log del Pezzo surfaces with $n^d_{\mathbf{w}} \geq 1$

Index	$\mathbf{w}$	d	$m^d_{\mathbf{w}}$	$n^d_{\mathbf{w}}$	$\mathcal{S}_l = L_f \subset S^7$
1	(2,2k+1,2k+1,4k+1)	8k+4	12	5	$\#7(S^2 \times S^3)$
1	(2,3,5,9)	18	13	5	$\#6(S^2 \times S^3)$
2	(3,4,10,15)	30	10	3	$\#6(S^2 \times S^3)$
1	(3,5,7,14)	28	9	4	$\#5(S^2 \times S^3)$
1	(3,5,11,18)	36	10	3	$\#5(S^2 \times S^3)$
2	(3,4,5,10)	20	9	3	$\#4(S^2 \times S^3)$
1	(3,5,7,11)	25	8	3	$\#4(S^2 \times S^3)$
1	(3,3,5,5)	15	10	2	$\#4(S^2 \times S^3)$
2	(7,8,19,32)	64	7	2	$\#3(S^2 \times S^3)$
3	(5,7,11,20)	40	7	2	$\#3(S^2 \times S^3)$
1	(5,14,17,21)	56	5	1	$\#3(S^2 \times S^3)$
1	(5,19,27,50)	100	6	1	$\#3(S^2 \times S^3)$
1	(7,11,27,44)	88	6	1	$\#3(S^2 \times S^3)$
2	(6,9,10,13)	36	5	1	$\#3(S^2 \times S^3)$
1	(5,19,27,31)	81	5	1	$\#2(S^2 \times S^3)$
2	(7,8,19,25)	57	5	1	$\#2(S^2 \times S^3)$

Next we turn to the equivalence problem for the Einstein metrics. As mentioned previously for the log del Pezzo surfaces with a Y in the last column of Table 1 and the tables of Theorem 4.5, there is a unique homothety class of Kähler-Einstein metrics corresponding to each point of $\mathcal{M}^d_{\mathbf{w}}$. But the question remains whether two inequivalent Kähler-Einstein structures can share the same Riemannian metric. This is answered by a Theorem of Pontecorvo [Pon] which can be restated for our purposes as:

PROPOSITION 7.13: *Let (J,J') be two distinct log del Pezzo structures on the same underlying orbifold $\mathcal{Z}$ that are both compatible with the same Einstein metric g. Then J and J' are complex conjugates, i.e. $J'=-J$.*

PROOF: By Proposition 3.1 of [Pon] if $J'\neq -J$ then the metric g is anti-self-dual, i.e. $W_+=0$. From [Boy] anti-self-dual Kähler metrics must have vanishing scalar curvature, and this cannot happen for log del Pezzo surfaces which are positive. ∎

Conjugate Kähler-Einstein structures correspond to the same point of the moduli space $\mathcal{M}^d_{\mathbf{w}}$, so each point of $\mathcal{M}^d_{\mathbf{w}}$ corresponds to a distinct diffeomorphism class of Kähler-Einstein metrics with scalar curvature $4n(n+1)$. There are similar results on the Sasakian level which are essentially due to Tanno [Tan] and Tachibana and Yu [TaYu]. We reformulate this result as follows:

PROPOSITION 7.14: *Let M be a $(4n+1)$-dimensional compact manifold. Let $\mathcal{S}=(\xi,\eta,\Phi,g)$ and $\mathcal{S}'=(\xi',\eta',\Phi',g)$ be two distinct Sasakian structures on M sharing the same Riemannian metric g. Suppose further that (M,g) is not a sphere with the standard round Sasakian metric. Then $\mathcal{S}'$ and $\mathcal{S}$ are conjugate Sasakian structures, i.e. $\mathcal{S}'=(-\xi,-\eta,-\Phi,g)$.*

PROOF: Since (M,g) is not a round sphere, a theorem of Tachibana and Yu [TaYu] says that $g(\xi,\xi')$ is a constant, say a. By Schwarz inequality $|a|\leq 1$ and since the Sasakian structures are distinct we must have $-1\leq a<1$. If $a=-1$ then $\mathcal{S}$ and $\mathcal{S}'$ are conjugate Sasakian structures. So assume $|a|<1$, then following Tanno [Tan] we define

$$\xi^{''}=\frac{\xi'-a\xi}{\sqrt{1-a^2}}.$$

Now we have

$$g(\xi,\xi^{''})=0,\qquad g(\xi^{''},\xi^{''})=1,$$

and it follows from the characterization of Sasakian structures in terms of the Riemannian curvature (cf. [BG2]) that $\xi^{''}$ defines a Sasakian structure $\mathcal{S}^{''}$ which is orthogonal to $\mathcal{S}$. But then this would define a 3-Sasakian structure [Tan] which cannot exist in dimension $4n+1$. Thus, $\mathcal{S}$ and $\mathcal{S}'$ are conjugate Sasakian structures. ∎

Now combining Theorems 1.23, 4.5, 6.6, and Proposition 7.14 with the information in Tables 1 and 3 proves Theorem A of the Introduction.

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