Contact Structures Hypersurface Singularities and Einstein Metrics

KRZYSZTOF GALICKI

Max-Planck-Institut für Mathematik, Bonn

and

University of New Mexico, Albuquerque

Srni, January 2004

CONTENTS

REFERENCES

1. C. P. Boyer, K. Galicki, *Einstein metrics on rational homology spheres*, preprint 2003.
2. C. P. Boyer, K. Galicki, J. Kollár, *Einstein metric on spheres*, preprint 2003.
3. C. P. Boyer, K. Galicki, J. Kollár, E. Thomas, *Einstein metrics on exotic spheres in dimension 7,11, and 15*, preprint 2003.
4. C. P. Boyer, K. Galicki, M. Nakamaye, *On Sasakian-Einstein Geometry of 5-Manifolds*, **Mathematische Annalen 325** (2003), 485-524.
5. C. P. Boyer, K. Galicki, M. Nakamaye, *Sasakian Geometry, Homotopy Spheres, and Positive Ricci Curvature,* **Topology 42** (2003), 981-1002.
6. C. P. Boyer, K. Galicki, *New Einstein Manifolds in Dimension Five,* **Journal of Differential Geometry 57** (2001), 443-467.
7. J.-P. Demailly, J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, **Ann. Scient. Ec. Norm. Sup. Paris 34** (2001), 525-556.
8. C. P. Boyer, K. Galicki, *Sasakian Manifolds*, **Oxford University Press** (hopefully in 2005).

More, including these slides, at
http://www.math.unm.edu/~galicki.

0. Introduction

Some Problems and Questions

We want to ask questions regarding the existence of positive Ricci curvature metrics (including the case constant positive Ricci curvature, i.e., Einstein with positive Einstein constant) on contact M^{2n+1}. Let us start with

• **dim(M)=3**: Well-know result of Martinet says that very orientable 3-manifold admits a contact structure. Another celebrated theorem of Hamilton shows that proving existence of a positive Ricci curvature on every simply connected 3-manifold is equivalent to proving **Poincaré Conjecture**. On the other hand Every Einstein 3-manifold is of constant curvature.

• **dim(M)=5**: In 1965 Barden proved the following remarkable theorem

THEOREM: *The class of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism. Furthermore, any such M is diffeomorphic to one of the spaces $M_{j;k_1,\ldots,k_s} = X_j \# M_{k_1} \# \cdots \# M_{k_s}$, where $-1 \leq j \leq \infty$, $s \geq 0$, $1 < k_1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$. A complete set of invariants is provided by $H_2(M, \mathbb{Z})$ and an additional diffeomorphism invariant $i(M) = j$ which depends only on the second Stiefel-Whitney class $w^2(M)$.*

In these lectures we will refer to a simply connected, closed, oriented, smooth, 5-manifold as a **Barden manifold**.

BUILDING BLOCKS

$X_{-1} = SU(3)/SO(3); \quad H_2(X_{-1}, \mathbb{Z}) = \mathbb{Z}_2,$

$X_\infty =$ non – trivial S^3 bundle over $S^2; \quad H_2(X_\infty, \mathbb{Z}) = \mathbb{Z},$

$X_j, \quad j \in \mathbb{N}, \quad H_2(X_j, \mathbb{Z}) = \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$

$X_0 = S^5,$

$M_\infty = S^2 \times S^3 \qquad H_2(M_\infty, \mathbb{Z}) = \mathbb{Z},$

$M_\alpha, \quad \alpha \in \mathbb{N}; \quad H_2(M_\alpha, \mathbb{Z}) = \mathbb{Z}_\alpha \oplus \mathbb{Z}_\alpha$

When M is spin $i(M) = j = 0$ as is $w^2(M) = 0$ and Barden's result is the extension of the well-known theorem of Smale for spin 5-manifolds. By an old theorem of John Gray M admits an almost contact structure when $j = 0, \infty$ and by another result of Geiges M is in such a case necessarily contact. We have the following inclusions:

$$\begin{array}{lcl}
\{\textbf{Barden Manifolds}\} & & \\
\quad\quad \cup & & \\
\{\textbf{Contact Manifolds}\} & \supset & \{\textbf{Sasakian Manifolds}\} \\
\quad\quad \cup & & \\
\{\textbf{Spin Manifolds}\} & & \\
\quad\quad \cup & & \\
\{\textbf{Positive Sasakian Manifolds}\} & & \\
\quad\quad \cup & & \\
\{\textbf{Sasakian – Einstein Manifolds}\} & &
\end{array}$$

A. Which **Barden Manifolds** admit metrics of positive Ricci curvature?

The expected answer is **ALL** but remarkably little is known about the subject. One positive result was proved by Sha and Yang which, in dimension 5, reads

THEOREM: $\#n(S^2 \times S^3)$ *admits a positive Ricci curvature metric for all* $n \in \mathbb{N}$.

In addition, the symmetric metrics on S^5, X_{-1} and a bundle metric on X_∞ are also of positive Ricci curvature.

B. Which **Barden Manifolds** admit Einstein metrics of positive Ricci curvature?

There are several constructions that lead to such Einstein metrics in dimension 5.

[**Kobayashi Bundle Construction**] An old construction of Kobayashi asserts that certain unique simply connected circle bundle over any del Pezzo surface with a Kähler-Einstein metric admits an Einstein metric of positive scalar curvature. Existence results of Siu, Tian-Yau, and Tian settle the problem as we have

THEOREM: *The following del Pezzo surfaces admit Kähler-Einstein metrics:* $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^2\#n\overline{\mathbb{CP}^2}, \quad 3 \leq n \leq 8.$ *Furthermore, the moduli space of K-E structures in each case is completely understood.*

There are two del Pezzo surfaces which do not admit any K-E metrics due to theorem of Matsushima: the existence is obstructed by holomorphic vector fields. These are blow-ups of $\mathbb{CP}^2$ at one or two points.

As and immediate consequence, for 5-manifolds, we have the following (Friedrich-Kath, Boyer-Galicki):

THEOREM: *Let* $\mathcal{S}_l = S^5 \# l(S^2 \times S^3)$.

1) *For each* $l = 0, 1, 3, 4$, *there is precisely one regular Sasakian-Einstein structure on* $\mathcal{S}_l$.
2) *For each* $5 \leq l \leq 8$ *there is a* $2(l-4)$ *complex parameter family of inequivalent regular Sasakian-Einstein structures on* $\mathcal{S}_l$.
3) *For* $l = 2$ *or* $l \geq 9$ *there are no regular Sasakian-Einstein structures on* $\mathcal{S}_l$.

[**Homogeneous Einstein Metrics**] The symmetric metrics on $X_0 = S^5$, $X_{-1} = SU(3)/SO(3)$ are Einstein. Furthermore, $M_\infty = S^2 \times S^3$ admits infinitely many homogeneous Einstein metrics (Wang and Ziller).

[**Böhm Metrics**] Böhm has explicitly constructed cohomogeneity one Einstein metrics on S^5 and $S^2 \times S^3$.

• **dim(M)=2n+1** In higher dimensions there are many constructions of positive Ricci curvature metrics as well as Einstein metrics. We will be more selective in our choice of problems and will restrict our interest to the case when M is homeomorphic (but not necessarily diffeomorphic) to the sphere S^{2n+1}.

Questions A1: Do any exotic 7-spheres admit metrics of positive Ricci curvature?

This question was answered by Wraith who showed that

Theorem: *All spheres which bound parallelizable manifolds admit metrics of positive Ricci curvature.*

By a result of Hitchin it is known that some exotic spheres (first in dimension 9) cannot admit a metric of positive scalar curvature. In particular such spheres will not admit metrics of positive Ricci curvature. It is still not know if any of the exotic spheres admit metrics of positive sectional curvature.

Any standard sphere S^n admits a metric of constant sectional curvature. These canonical metrics are homogeneous and Einstein. The spheres S^{4m+3}, $m > 1$ are known to have another $Sp(m+1)$-homogeneous Einstein metric discovered by Jensen in 1974. In addition, S^{15} has a third Spin(9)-invariant homogeneous Einstein metric discovered by Bourguignon and Karcher. In 1982 Ziller proved that these are the only homogeneous Einstein metrics on spheres. No other Einstein metrics on spheres were known until 1998 when Böhm constructed infinite sequences of non-isometric Einstein metrics, of positive scalar curvature, on S^5, S^6, S^7, S^8, and S^9. Böhm's metrics are of cohomogeneity one and they are not only the first inhomogeneous Einstein metrics on spheres but also the first non-canonical Einstein metrics on even-dimensional spheres. Even with Böhm's result Einstein metrics on spheres appeared to be rare.

Questions B1: Are there other inhomogeneous Einstein metrics on standard spheres?

Questions B2: Do some exotic spheres admit Einstein metrics of positive scalar curvature?

Some Answers and Theorems

THEOREM 1: [Boyer,–] *For every integer $k > 2$ that is either relatively prime to 3 or 2, there exist cohomogeneity 4 Sasakian-Einstein metrics, depending on two real parameters, on a simply connected rational homology 5-sphere M^5_k with $w_2(M^5_k) = 0$, and $H_2(M^5_k, \mathbb{Z})$ having order k^2. Furthermore, in each odd dimensions there are infinitely many rational homology spheres admitting families of Sasakian-Einstein metrics with one dimensional isometry group.*

THEOREM 2: [Boyer,–,Nakamaye] *For every integer $l \geq 1$, $l\#(S^2 \times S^3)$ admits a Sasakian metric with positive Ricci curvature.*

THEOREM 3: [Boyer,–,Nakamaye] *All spheres which bound parallelizable manifolds admit Sasakian metrics of positive Ricci curvature.*

THEOREM 4: [Boyer,–,Nakamaye] *$\mathcal{S}_l = l\#(S^2 \times S^3)$ admit infinite families of quasi-regular Sasakian-Einstein metrics for $1 < l \leq 9$. $\mathcal{S}_1$ admits at least 14 quasi-regular inequivalent Sasakian-Einstein metrics.*

THEOREM 5: [Boyer,–,Kollár] *On S^5 we obtain at least 68 inequivalent families of Sasakian-Einstein metrics. Some of these admit non-trivial continuous Sasakian-Einstein metrics deformations. The biggest constructed family has has (real) dimension 10.*

THEOREM 6: [Boyer,–,Kollár] *All 28 oriented diffeomorphism classes on S^7 admit inequivalent families of Sasakian-Einstein metrics structures, some of them (in each diffeomorphism class) depending on moduli.*

In each case, the number of families ranges from 231 to 452. Moreover, there are fairly large moduli. For example, the standard 7-sphere admits an 82-dimensional family of Sasakian-Einstein metrics. All these metrics have one-dimensional isometry group.

THEOREM 7: [Boyer,–,Kollár] *The* $(4n+1)$*-dimensional standard and Kervaire spheres both admit many families of inequivalent Sasakian-Einstein metrics for each* $n \geq 2$*.*

A partial computer search yielded more than $3 \cdot 10^6$ cases for S^9 and more than 10^9 cases for S^{13}, including a 21300113901610-dimensional family. The only Einstein metric on S^{13} known thus far was the standard one!

CONJECTURE: [Boyer,–,Kollár] *All odd-dimensional homotopy spheres which bound parallelizable manifolds admit Sasakian-Einstein metrics.*

THEOREM 8: [Boyer,–,Kollár, Thomas] *The conjecture is true in dimension 11 and 15. More precisely, each homotopy sphere (992 possible diffeomorphism types) in dimension 11 admits at least one Sasakian-Einstein metric and each homotopy 15-sphere which bound a parallelizable manifold (8128 possible diffeomorphism types) admits at least one Sasakian-Einstein metric.*

THEOREM: [Kollár] *For every* $l \geq 6$*, there are infinitely many* $(2l-2)$*-dimensional families of Einstein metrics on* $\mathcal{S}_l = l\#(S^2 \times S^3)$*.*

1. Contact and Sasakian Manifolds

Contact transformations arose in the theory of Analytical Mechanics developed in the 19th century by Hamilton, Jacobi, Lagrange, and Legendre. But its first systematic treatment was given by Sophus Lie. Consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x^1, \cdots, x^n; y^1, \cdots, y^n; z)$, and a 1-form η given by

$$\eta = dz - \sum_i y^i dx^i. \tag{1.1}$$

It is easy to see that η satisfies $\eta \wedge (d\eta)^n \neq 0$. A 1-form on $\mathbb{R}^{2n+1}$ that satisfies this equation is called a **contact form**. Locally we have the following

THEOREM 1.2 [Darboux] *Let η be a 1-form on $\mathbb{R}^{2n+1}$ that satisfies $\eta \wedge (d\eta)^n \neq 0$. Then there is an open set $U \subset \mathbb{R}^{2n+1}$ and local coordinates $(x^1, \cdots, x^n; y^1, \cdots, y^n; z)$ such that η has the form (1.1) in U.*

DEFINITION 1.3: *A $(2n+1)$-dimensional manifold M is a* **contact manifold** *if there exists a 1-form η, called a* **contact 1-form**, *on M such that*

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A **contact structure** *on M is an equivalence class of such 1-forms, where $\eta' \sim \eta$ if there is a nowhere vanishing function f on M such that $\eta' = f\eta$.*

Lemma 1.4: *On a contact manifold (M, η) there is a unique vector field ξ, called the Reeb vector field, satisfying the two conditions*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

Definition 1.5: *An almost contact structure on a differentiable manifolds M is a triple (ξ, η, Φ), where Φ is a tensor field of type $(1,1)$ (i.e. an endomorphism of TM), ξ is a vector field, and η is a 1-form which satisfy*

$$\eta(\xi) = 1 \quad \text{and} \quad \Phi \circ \Phi = -\mathbb{I} + \xi \otimes \eta,$$

where $\mathbb{I}$ is the identity endomorphism on TM. A smooth manifold with such a structure is called an **almost contact manifold**.

Let (M, η) be a contact manifold with a contact 1-form η and consider $\mathcal{D} = \ker\ \eta \subset TM$. The subbundle $\mathcal{D}$ is maximally non-integrable and it is called the **contact distribution**. The pair $(\mathcal{D}, \omega)$, where ω is the restriction of $d\eta$ to $\mathcal{D}$ gives $\mathcal{D}$ the structure of a symplectic vector bundle. We denote by $\mathcal{J}(\mathcal{D})$ the space of all almost complex structures J on $\mathcal{D}$ that are compatible with ω, that is the subspace of smooth sections J of the endomorphism bundle $\mathrm{End}(\mathcal{D})$ that satisfy

$$(1.6)\quad J^2 = -\mathbb{I}, \quad d\eta(JX, JY) = d\eta(X, Y), \quad d\eta(X, JX) > 0$$

for any smooth sections X, Y of $\mathcal{D}$. Notice that each $J \in \mathcal{J}(\mathcal{D})$ defines a Riemannian metric $g_{\mathcal{D}}$ on $\mathcal{D}$ by setting

$$(1.7)\qquad g_{\mathcal{D}}(X, Y) = d\eta(X, JY).$$

One easily checks that $g_{\mathcal{D}}$ satisfies the compatibility condition $g_{\mathcal{D}}(JX,JY) = g_{\mathcal{D}}(X,Y)$. Furthermore, the map $J \mapsto g_{\mathcal{D}}$ is one-to-one, and the space $\mathcal{J}(\mathcal{D})$ is contractible. A choice of J gives M an almost CR structure.

Moreover, by extending J to all of TM one obtains an almost contact structure. There are some choices of conventions to make here. We define the section Φ of $\mathrm{End}(TM)$ by $\Phi = J$ on $\mathcal{D}$ and $\Phi\xi = 0$, where ξ is the Reeb vector field associated to η. We can also extend the transverse metric $g_{\mathcal{D}}$ to a metric g on all of M by

$$1.8 \quad g(X,Y) = g_{\mathcal{D}} + \eta(X)\otimes\eta(Y) = d\eta(X,\Phi Y) + \eta(X)\otimes\eta(Y)$$

for all vector fields X, Y on M. One easily sees that g satisfies the compatibility condition $g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y)$.

Definition 1.9: *A contact manifold M with a contact form η, a vector field ξ, a section Φ of* $\mathrm{End}(TM)$, *and a Riemannian metric g which satisfy the conditions*

$$\eta(\xi) = 1, \qquad \Phi^2 = -\mathbb{I} + \xi\otimes\eta,$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

is known as a metric contact structure on M.

Definition-Proposition 1.10: *A Riemannian manifold (M,g) is called a* **Sasakian manifold** *if any one, hence all, of the following equivalent conditions hold:*

(i) There exists a Killing vector field ξ of unit length on M so that the tensor field Φ of type $(1,1)$, *defined by*

$\Phi(X) = \nabla_X \xi$, *satisfies the condition*

$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi$$

for any pair of vector fields X and Y on M.

(ii) There exists a Killing vector field ξ of unit length on M so that the Riemann curvature satisfies the condition

$$R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi,$$

for any pair of vector fields X and Y on M.

(iii) The metric cone on M $(C(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ is Kähler.

We refer to the quadruple $\mathcal{S} = (g, \eta, \xi, \Phi)$ as a **Sasakian structure** on M, where η is the 1-form dual vector field ξ. It is easy to see that η is a contact form whose Reeb vector field is ξ. In particular $\mathcal{S} = (g, \eta, \xi, \Phi)$ is a special type of *metric contact structure.*

The vector field ξ is nowhere vanishing, so there is a 1-dimensional foliation $\mathcal{F}_\xi$ associated with every Sasakian structure, called the *characteristic foliation.* We will denote the space of leaves of this foliation by $\mathcal{Z}$. Each leaf of $\mathcal{F}_\xi$ has a holonomy group associated to it. The dimension of the closure of the leaves is called the *rank* of $\mathcal{S}$. We shall be interested in the case $\mathrm{rk}(\mathcal{S}) = 1$.

DEFINITION 1.11: *When $rk(\mathcal{S}) = 1$ we say that Sasakian structure $\mathcal{S}$ is* **quasi-regular**. *If $\mathcal{F}_\xi$ defines a principal S^1 bundle, we say that $\mathcal{S}$ is* **regular**.

When $\mathcal{S}$ is compact and quasi-regular then then $\mathcal{Z}$ has a structure of a Riemannian orbifold (or a V-manifold). $\mathcal{Z}$ is a smooth manifold in the regular case.

DEFINITION-PROPOSITION 1.12: *A Sasakian space* (M, g) *is* **Sasakian-Einstein** *if the metric* g *is also Einstein. For any 2n+1-dimensional Sasakian manifold* $\mathrm{Ric}(X, \xi) = 2n\eta(X)$ *implying that any Sasakian-Einstein metric must have positive scalar curvature. Thus any complete Sasakian-Einstein manifold must have a finite fundamental group. Furthermore the metric cone on* M $(C(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ *is Kähler Ricc-flat (Calabi-Yau).*

The following are the two fundamental structure theorems for Sasakian and Sasakian-Einstein manifolds:

THEOREM 1.13: [Boyer,–] *Let* (M, g) *be a compact quasi-regular Sasakian manifold of dimension* $2n+1$, *and let* $\mathcal{Z}$ *denote the space of leaves of the characteristic foliation. Then*

(i) The leaf space $\mathcal{Z}$ *is a Hodge orbifold with Kähler metric* h *and Kähler form* ω *which defines an integral class* $[\omega]$ *in* $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$ *so that* $\pi : (M, g) \longrightarrow (\mathcal{Z}, h)$ *is an orbifold Riemannian submersion. The fibers of* π *are totally geodesic submanifolds of* M *diffeomorphic to* S^1.

(ii) (M, g) *is Sasakian-Einstein iff* $(\mathcal{Z}, h)$ *is Kähler-Einstein with scalar curvature* $4n(n+1)$.

THEOREM 1.14: [Boyer,–] *Let* $(\mathcal{Z}, h)$ *be a Hodge orbifold. Let* $\pi : M \longrightarrow \mathcal{Z}$ *be the* S^1 *V-bundle whose first Chern class is* $[\omega]$, *and let* η *be a connection 1-form in* M *whose curvature is* $2\pi^*\omega$, *then* M *with the metric* $\pi^* h + \eta \otimes \eta$ *is a Sasakian orbifold. Furthermore, if all the local uniformizing groups* *inject into the group of the bundle S^1, the total space M is a smooth Sasakian manifold.*

DEFINITION 1.15: *A Sasakian manifold (M, g) is called a* **positive** *if the transverse geometry $\mathcal{Z}$ is* **Fano**.

Positivity, or the Fano condition just as in the smooth case can be expressed in terms of orbifold Chern classes of the $\mathcal{Z}$ (first Chern class $c_1(\mathcal{Z})$), or in terms of basic Chern classes of the characteristic foliation $\mathcal{F}_\xi$ (first Chern class $c_1(\mathcal{F}_\xi)$).

THEOREM 1.16: [Boyer,–, Nakamaye] *Any positive Sasakian manifold (M, g) admits a Sasakian metric g' of positive Ricci curvature.*

The following is an orbifold version of the famous Kobayashi bundle construction of Einstein metrics on bundles over positive Kähler-Einstein manifolds.

THEOREM 1.17: *Let $(\mathcal{Z}, h)$ be a compact Fano orbifold with $\pi_1^{orb}(\mathcal{Z}) = 0$ and Kähler-Einstein metric h. Let $\pi : M \longrightarrow \mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$. Suppose further that the local uniformizing groups of $\mathcal{Z}$ inject into S^1. Then with the metric $g = \pi^* h + \eta \otimes \eta$, M is a compact simply connected Sasakian-Einstein manifold.*

A very special class of Sasakian-Einstein spaces is naturally related to several quaternionic geometries.

DEFINITION 1.18: *Let (M, g) be a Riemannian manifold of dimension m. We say that (M, g) is 3-Sasakian if the metric cone $(\mathcal{C}(M), \bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$ on M is hyperkähler.*

1.19

$$\begin{tikzcd} \mathcal{C}(M) & M \arrow[l, hook'] \arrow[d, "\pi"] \\ & \mathcal{Z} \end{tikzcd}$$

$\mathcal{C}(M)$: CONE GEOMETRY	M	$M/\mathcal{F}_\xi$: TRANSVERSE GEOMETRY
SYMPLECTIC	**CONTACT**	**SYMPLECTIC**
KÄHLER	**SASAKIAN**	**KÄHLER**
KÄHLER	**POSITIVE SASAKIAN**	**FANO**
CALABI-YAU	**SASAKIAN EINSTEIN**	**FANO KÄHLER-EINSTEIN**
HYPER-KÄHELER	**3-SASAKIAN**	**FANO, $\mathbb{C}$-CONTACT KÄHLER-EINSTEIN**

HYPERKAHLER
GEOMETRY

$$
\begin{tikzcd}
 & & \mathbb{R}_+ \times M \arrow[dl] \arrow[dr] \arrow[dd] & & \\
\text{TWISTOR GEOMETRY} & \mathcal{Z} \arrow[dr] & & M \arrow[ll] \arrow[dl] & \text{3-SASAKIAN GEOMETRY} \\
 & & \mathcal{O} & &
\end{tikzcd}
$$

QUATERNIONIC KAHLER
GEOMETRY

EXAMPLES OF S-E MANIFOLDS

EXAMPLE 1.20: *3-Sasakian Manifolds:*

(a) "flat example"

$$
\begin{tikzcd}
 & \mathbb{H}^n \setminus \{0\} \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathbb{CP}^{2n-1} \arrow[dr] & & S^{4n-1} \arrow[ll] \arrow[dl] \\
 & \mathbb{PH}^{n-1} &
\end{tikzcd}
$$

(b) homogeneous examples

$$
\begin{tikzcd}
 & \mathcal{C}(G/L) \arrow[dl] \arrow[dr] \arrow[dd] & \\
G/L \cdot U(1) \arrow[dr] & & G/L \arrow[ll] \arrow[dl] \\
 & G/L \cdot Sp(1) &
\end{tikzcd}
$$

(c) toric examples [Boyer,–, Mann, Rees'98]

$$\begin{tikzcd} & \mathbb{H}^n \setminus \{0\} \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathbb{CP}^{2n-1} \arrow[dr] & & S^{4n-1} \arrow[ll] \arrow[dl] \\ & \mathbb{PH}^{n-1} & \end{tikzcd}$$

$$T^k \qquad \Big\downarrow \textit{QUOTIENT}$$

$$\begin{tikzcd} & \mathbb{R}_+ \times \mathcal{S}(\Omega) \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathcal{Z}(\Omega) \arrow[dr] & & \mathcal{S}(\Omega) \arrow[ll] \arrow[dl] \\ & \mathcal{O}(\Omega) & \end{tikzcd}$$

Here, Ω is an integral matrix which defines a homomorphism $f_\Omega : T^k \longrightarrow U(n)$. In the case $\dim(\mathcal{S}(\Omega)) = 7$, there are choices of Ω for any $k \geq 1$ which make $\mathcal{S}(\Omega)$ smooth. Since $b_2(\mathcal{S}(\Omega)) = k$ we conclude that there exist Einstein manifolds with arbitrarily large second Betti number. These were first such examples.

(d) first non-toric examples [Boyer,–, Piccinni'02] (in dimension 11,15) were obtained by non-Abelian reduction of the flat example, where instead of the torus $G = T^k$ one takes $G = Sp(1) \times T^k$.

(e) more recently non-toric examples in dimension 7 were obtained by Grove, Wilking, and Ziller. They use orbifold bundle construction with the examples of orbifold twistor space and self-dual Einstein metrics $\mathcal{Z}_k \longrightarrow \mathcal{O}_k$ discovered by Nigel Hitchin in 1996. the self-dual Einstein metric on $\mathcal{O}_k$ is defined on $S^4 \setminus \mathbb{RP}^2$ and it has $\mathbb{Z}_k$ orbifold singularity along $\mathbb{RP}^2$. However, it turns out that the bundle $M_k \longrightarrow \mathcal{Z}_k$ is actually smooth. In particular, one can compute integral cohomology ring of M_k. For odd k the 3-Sasakian manifold M_k is a rational homology 7-sphere with non-zero torsion depending on k. Hence, there exist infinitely many rational homology 7-spheres which have 3-Sasakian metrics.

Example 1.21: *Other Sasakian-Einstein Manifolds:*

(a) complex Hopf fibration:

$$\begin{tikzcd} \mathbb{C}^n \setminus \{0\} \arrow[dd] \arrow[dr] & \\ & S^{2n-1} \arrow[dl] \\ \mathbb{PC}^{n-1} & \end{tikzcd}$$

(b) There are many homogeneous examples; all compact homogeneous Kähler-Einstein spaces are classified and they are of the form $\mathcal{Z} = G/P$. Hence, one can replace the complex projective space with $\mathcal{Z} = G/P$ and apply the usual Kobayashi construction.

(c) There are also many inhomogeneous examples. One can take any smooth compact Fano variety $\mathcal{Z}$ which admits a Kähler-Einstein metric. For example, in the case of complex surfaces it is know exactly which Fano (del Pezzo) surfaces admit a K-E metric. These were already mentioned in the introduction.

Until recently, examples of smooth irregular Sasakian-Einstein manifolds were rare (with the exception of inhomogeneous 3-Sasakian spaces mentioned above). On the other there are plentiful examples of compact irregular Sasakian, or even positive Sasakian manifolds. In these lectures we will concentrate on one particular example.

2. Links of isolated hypersurface singularities

[The basic reference on links is Milnor's beautiful book: *Singular Points of Complex Hypersurfaces.*]

Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action given by $(z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n)$, where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w} \in (\mathbb{Z}^+)^{n+1}$, and we shall assume that $\gcd(w_0, \ldots, w_n) = 1$. Let f be a weighted homogeneous polynomial, that is $f \in \mathbb{C}[z_0, \ldots, z_n]$ and satisfies

$$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n),$$

where $d \in \mathbb{Z}^+$ is the *degree* of f. We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity or to further simplify matters that it is the only singularity.

We are interested in the *link* L_f defined by

2.1
$$L_f = \{f = 0\} \cap S^{2n+1},$$

where

2.2
$$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \,|\, \sum_{j=0}^{n} |z_j|^2 = 1\}$$

is the unit sphere in $\mathbb{C}^{n+1}$. L_f is endowed with a natural quasi-regular Sasakian structure inherited as a Sasakian submanifold of the sphere S^{2n+1} with its "weighted" Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which in the standard coordinates $\{z_j = x_j + iy_j\}_{j=0}^n$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is determined by

2.3
$$\eta_{\mathbf{w}} = \frac{\sum_{i=0}^n (x_i dy_i - y_i dx_i)}{\sum_{i=0}^n w_i(x_i^2 + y_i^2)}, \qquad \xi_{\mathbf{w}} = \sum_{i=0}^n w_i(x_i\partial_{y_i} - y_i\partial_{x_i}),$$

and the standard Sasakian structure (ξ, η, Φ, g) on S^{2n+1}.

The quotient of S^{2n+1} by the "weighted S^1-action" generated by the vector field $\xi_{\mathbf{w}}$ is the weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$, and we have a commutative diagram:

2.4
$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are orbifold Riemannian submersions. L_f is the total space of the principal S^1 V-bundle over the orbifold $\mathcal{Z}_f$ whose first Chern class is $\frac{c_1(\mathcal{Z})}{I} \in H^2(\mathcal{Z}_f, \mathbb{Q})$, and $\eta_{\mathbf{w}}$ is a connection in this V-bundle whose curvature is $d\eta$.

PROPOSITION 2.5: *The orbifold $\mathcal{Z}_f$ is Fano if and only if $d - \sum w_i < 0$.*

Now, recall the well-known construction of Milnor for isolated hypersurface singularities: There is a fibration of $(S^{2n+1} - L_f) \longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n \vee S^n \cdots \vee S^n$. The *Milnor number* μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $(w_0, \ldots, w_n)$ by the formula

2.6
$$\mu = \mu(L_f) = \prod_{i=0}^{n} (\frac{d}{w_i} - 1).$$

The closure $\bar{F}$ of F has the same homotopy type as F and is a compact manifold with boundary precisely the link L_f. So the reduced homology of F and $\bar{F}$ is only non-zero in dimension n and $H_n(F, \mathbb{Z}) \approx \mathbb{Z}^\mu$. Using the Wang sequence of the Milnor fibration together with Alexander-Poincare duality gives the exact sequence

2.7
$$0 \to H_n(L_f, \mathbb{Z}) \to H_n(F, \mathbb{Z}) \xrightarrow{\mathbb{I} - h_*} H_n(F, \mathbb{Z}) \to H_{n-1}(L_f, \mathbb{Z}) \to 0$$

where h_* is the *monodromy* map (or characteristic map) induced by the $S^1_{\mathbf{w}}$ action. From this we see that $H_n(L_f, \mathbb{Z}) = \ker(\mathbb{I} - h_*)$ is a free Abelian group, and $H_{n-1}(L_f, \mathbb{Z}) = \mathrm{Coker}(\mathbb{I} - h_*)$ which in general has torsion, but whose free part equals $\ker(\mathbb{I} - h_*)$. So the topology of L_f is encoded in the monodromy map h_*. There is a well-known algorithm due to Milnor and Orlik for computing the free part of $H_{n-1}(L_f, \mathbb{Z})$ in terms of the characteristic polynomial $\Delta(t) = \det(t\mathbb{I} - h_*)$, namely the Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$. First we mention an important immediate consequence of the exact sequence (2.7) which is due to Milnor:

Proposition 2.8: *The following hold:*

(i) L_f *is a rational homology sphere if and only if* $\Delta(1) \neq 0$.

(ii) L_f *is a homology sphere if and only if* $|\Delta(1)| = 1$.

(iii) If L_f *is a rational homology sphere, then the order of* $H_{n-1}(L_f, \mathbb{Z})$ *equals* $|\Delta(1)|$.

Example 2.9:

(a) $\mathbf{w} = (1,1,1,1), \quad f = z_0 + z_1 + z_2 + z_3, \quad d = 1.$

$$\mathcal{Z}_f = \mathbb{CP}(2), \qquad L_f = S^5.$$

(b) $\mathbf{w} = (1,1,1,1), \quad f = z_0^2 + z_1^2 + z_2^2 + z_3^2, \quad d = 2.$

$$\mathcal{Z}_f = \mathbb{CP}(1) \times \mathbb{CP}(1), \qquad L_f = S^2 \times S^3.$$

(c) $\mathbf{w} = (1,1,1,1), \quad f = z_0^3 + z_1^3 + z_2^3 + z_3^3, \quad d = 3.$

$$\mathcal{Z}_f = \mathbb{CP}(2)\#6\overline{\mathbb{CP}(2)} \quad , \qquad L_f = 6\#(S^2 \times S^3).$$

(d) $\mathbf{w} = (1,1,1,2), \quad f = z_0^4 + z_1^4 + z_2^4 + z_3^2, \quad d = 4.$

$$\mathcal{Z}_f = \mathbb{CP}(2)\#7\overline{\mathbb{CP}(2)} \quad , \qquad L_f = 7\#(S^2 \times S^3).$$

(e) $\mathbf{w} = (1,1,2,3), \quad f = z_0^6 + z_1^6 + z_2^3 + z_3^2, \quad d = 6.$

$$\mathcal{Z}_f = \mathbb{CP}(2)\#8\overline{\mathbb{CP}(2)} \quad , \qquad L_f = 8\#(S^2 \times S^3).$$

(f) $\mathbf{w} = (1,1,1,k)$, $f = z_0^{k+1} + z_1^{k+1} + z_2^{k+1} + z_0 z_3$, $d = k+1$.

$$L_f = k\#(S^2 \times S^3).$$

(g) $\mathbf{w} = (1,2,3,5)$, $f = z_0^{10} + z_1^5 + z_2^3 z_1 + z_3^2$, $d = 10$

$$L_f = 9\#(S^2 \times S^3).$$

(h) $\mathbf{w} = (11,29,39,49)$, $f = z_0^8 z_2 + z_1^4 z_0 + z_2^3 + z_3^2 z_1$, $d = 127$

$$L_f = 2\#(S^2 \times S^3).$$

(i) $\mathbf{w} = (13,35,81,128)$, $f = z_0^{17} z_1 + z_1^5 z_2 + z_2^3 z_0 + z_3^2$, $d = 256$

$$L_f = S^2 \times S^3.$$

(j) $\mathbf{w} = (17,34,75,125,175)$, $f =$ homework, $d = 425$

$$|H_3(L_f, \mathbb{Z})| = 17^{12}.$$

(k) $\mathbf{w} = (127,2266,3651,6043,8435)$, $f =$ homework, $d = 20521$

$$|H_3(L_f, \mathbb{Z})| = 20521.$$

(l) $\mathbf{w} = (127,2392,3399,6043,8561)$, $f =$ homework, $d = 20521$

$$|H_3(L_f, \mathbb{Z})| = 20521.$$

(m) $\mathbf{w} = (6, 2(6k-1), 3(6k-1), 3(6k-1), 3(6k-1))$,
$f = z_0^{6k-1} + z_1^3 + z_2^2 + z_3^2 + z_4^2$,
$d = 6(6k-1)$

$$Z_f \simeq \mathbb{P}(2,1,1,1), \qquad L_f \simeq S^7.$$

But the smooth structure depends on k.

(n) $\mathbf{w} = (2, 2p+1, 2p+1, 2p+1, 2p+1, 2p+1)$,
$f = z_0^{2p+1} + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2$,
$d = 2(2p+1)$

$$Z_f \simeq \mathbb{P}(1,1,1,1,1), \qquad L_f \simeq S^9.$$

But the smooth structure depends on p.

In some of the above examples the homogeneous polynomial f contains no "mixed" monomial terms of the form $z_i^{c_i} z_j^{c_j}$. Such an f is called of Brieskorn-Pham type (abbreviated BP type). In his famous work, in 1966 Brieskorn considered links $L(\mathbf{a})$ defined by

2.10 $$\sum_{i=0}^{n} |z_i^2| = 1, \qquad f_{\mathbf{a}}(\mathbf{z}) = z_0^{a_1} + \cdots + z_n^{a_n} = 0.$$

To the vector $\mathbf{a} = (a_0, \cdots, a_n) \in \mathbb{Z}_+^{n+1}$ one associates a graph $G(\mathbf{a})$ whose $n+1$ vertices are labeled by $a_0, \cdots, a_n$. Two vertices a_i and a_j are connected if and only if $\gcd(a_i, a_j) > 1$. Let C_{ev} denote the connected component of $G(\mathbf{a})$ determined by the even integers. Note that all even vertices belong to C_{ev}, but C_{ev} may contain odd vertices as well. Then we have the so-called **Brieskorn Graph Theorem**

THEOREM 2.11: *The following hold:*

(i) *The link $L(\mathbf{a})$ is a rational homology sphere if and only if either $G(\mathbf{a})$ contains at least one isolated point, or C_{ev} has an odd number of vertices and for any distinct $a_i, a_j \in C_{ev},\ \gcd(a_i, a_j) = 2$.*

(ii) *The link $L(\mathbf{a})$ is an integral homology sphere if and only if either $G(\mathbf{a})$ contains at least two isolated points, or $G(\mathbf{a})$ contains one isolated point and C_{ev} has an odd number of vertices and for any distinct $a_i, a_j \in C_{ev}$, $\gcd(a_i, a_j) = 2$.*

Recall that by seminal work of Milnor, Kervaire and Milnor and Smale, for each $n \geq 5$, differentiable homotopy spheres of dimension n form an Abelian group Θ_n, where the group operation is connected sum. Θ_n has a subgroup bP_{n+1} consisting of those homotopy n-spheres which bound parallelizable manifolds V_{n+1}. Kervaire and Milnor proved that $bP_{2m+1} = 0$ for $m \geq 1$, $bP_{4m+2} = 0$, or $\mathbb{Z}_2$ and is $\mathbb{Z}_2$ if $4m + 2 \neq 2^i - 2$ for any $i \geq 3$. The most interesting groups are bP_{4m} for $m \geq 2$. These are cyclic of order

$$2.12 \qquad |bP_{4m}| = 2^{2m-2}(2^{2m-1} - 1) \text{ numerator } \left(\frac{4B_m}{m}\right),$$

where B_m is the m-th Bernoulli number. Thus, for example $|bP_8| = 28, |bP_{12}| = 992, |bP_{16}| = 8128$ and $|bP_{20}| = 130,816$. In the first two cases these include all exotic spheres. The correspondence is given by

2.13 $$KM : \Sigma \mapsto \frac{1}{8}\tau(V_{4m}(\Sigma))\mathrm{mod}|bP_{4m}|,$$

where $V_{4m}(\Sigma)$ is any parallelizable manifold bounding Σ and τ is its signature. Let Σ_i denote the exotic sphere with $KM(\Sigma_i) = i$. Now, Brieskorn Graph theorem tells us for which $\mathbf{a}$ the BP link $L(\mathbf{a})$ is a homotopy sphere. By (2.13) we need to be able to compute the signature to determine the diffeomorphism types of various links. We restrict our interest just to the case when $m = 2k+1$.

In this case, the diffeomorphism type of a homotopy sphere $L(\mathbf{a}) \in bP_{2m-2}$ is determined by the signature $\tau(M)$ of a parallelizable manifold M whose boundary is $\Sigma_{\mathbf{a}}^{2m-3}$. By the Milnor Fibration Theorem we can take M to be the Milnor fiber $M_{\mathbf{a}}^{2m-2}$ which, for links of isolated singularities coming from weighted homogeneous polynomials is diffeomorphic to the hypersurface $\{\mathbf{z} \in \mathbb{C}^m \mid f_{\mathbf{a}}(z_0, \cdots, z_{m-1}) = 1\}$.

Brieskorn shows that the signature of $M_{\mathbf{a}}^{2m-2}$ can be written combinatorially as

2.14 $$\tau(M_{\mathbf{a}}^{4k}) =$$

$$\#\{\mathbf{x} \in \mathbb{Z}^{2k+1} \mid 0 < x_i < a_i \text{ and } 0 < \sum_{j=0}^{2k} \frac{x_i}{a_i} < 1\mathrm{mod}2\}$$

$$-\#\{\mathbf{x} \in \mathbb{Z}^{2k+1} \mid 0 < x_i < a_i \text{ and } 1 < \sum_{j=0}^{2k} \frac{x_i}{a_i} < 2\mathrm{mod}2\},$$

where $m = 2k+1$.

Using a formula of Eisenstein, Zagier has rewritten this formula as:

2.15 $$\tau(M^{4k}_{\mathbf{a}}) =$$

$$\frac{(-1)^k}{N}\sum_{j=0}^{N-1}\cot\frac{\pi(2j+1)}{2N}\cot\frac{\pi(2j+1)}{2a_0}\cdots\cot\frac{\pi(2j+1)}{2a_{2k}},$$

where N is any common multiple of the a_i's. Both formulas are quite well suited to computer use. We wrote a C code which we call `sig.c`. For any m-tuple with $m = 2k+1 = 5, 7, 9$ `sig.c` computes the signature $\tau(\mathbf{a}) := \tau(M^{4k}_{\mathbf{a}})$ and the diffeomorphism type of the link using either of the above formulas.

EXAMPLE 2.16: Let us consider the Breiskorn-Pham link $L(5,3,2,2,2)$. By Brieskorn Graph Theorem this is a homotopy 7-sphere. One can easily compute the signature using (2.14) to find out that $\tau(L(5,3,2,2,2)) = 8$. Hence $L(5,3,2,2,2) = \Sigma^7_1$ is an exotic 7-sphere and it is called Milnor generator (all others can be obtained from it by taking connected sums). Similarly one does not need a computer to find that the signature of any $L(6k-1,3,2,2,2)$ to see that these so-called Brieskorn spheres realize all 28 diffeomorphism types of 7-spheres.

QUESTION 2.17: *Suppose that $\mathcal{Z}_f$ is Fano. Could one perhaps prove existence of an Kähler-Einstein metric on $\mathcal{Z}_f$? Every time this can successfully be done we automatically get a Sasakian-Einstein metric on the link L_f.*

3. Kähler-Einstein Metrics on Fano Orbifolds

Recall that on a Kähler manifold Ricci curvature 2-form ρ_ω of any Kähler metric represents the cohomology class $2\pi c_1(M)$. The well-known Calabi Conjecture is the question whether or not the converse is also true. To be more specific we begin with a couple of definitions

DEFINITION 3.1 *Let (M, J, g, ω_g) be a compact Kähler manifold. The Kähler cone of M*

$$K(M) = \left\{ \begin{array}{c} [\omega] \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R}) \ \ | \, [\omega] = [\omega_h] \\ \text{for some Kaehler metric } h \end{array} \right\}$$

is the set of all possible Kähler classes on M.

It is easy to show that $K(M)$ is a convex open set in $H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$.

DEFINITION 3.2 *Let (M, J, g, ω_g) be a compact Kähler manifold and $K(M)$ it Kähler cone. For any fixed Kähler class $[\omega] \in K(M)$ we define*

$$\mathcal{K}_{[\omega]} = \{h \in \Gamma(\odot^2(TM)) \ \ | \, h \text{ his Kaehler and } [\omega] = [\omega_h]\}$$

to be the space of all Kähler metrics in a given cohomology class.

Global $i\partial\bar{\partial}$-lemma provides for a very simple description of the space of Kähler metrics $\mathcal{K}_{[\omega]}$. Suppose we have a Kähler metric g in a with Kähler class $[\omega_g] = [\omega] \in K(M)$. If $h \in \mathcal{K}_{[\omega]}$ is another Kähler metric then, up to a constant, there exists a global function $\phi \in C^\infty(M, \mathbb{R})$ such that

$\omega_h - \omega_g = i\partial\bar{\partial}\phi$. We could fix constant by requiring, for example, that $\int_M \phi\omega_g^n = 0$. Hence, we have

COROLLARY 3.3 *Let (M, J, g, ω_g) be a compact Kähler manifold with $[\omega_g] = [\omega] \in K(M)$. Then, relative to the metric g the space of all Kähler metric $\mathcal{K}_{[\omega]}$ can be described as*

$$\mathcal{K}_{[\omega]} = \{\phi \in C^\infty(M, \mathbb{R}) \mid \omega_h = \omega_g + i\partial\bar{\partial}\phi > 0, \int_M \phi\omega_g^n = 0\},$$

where the 2-form $\omega_h > 0$ means that $\omega_h(X, JY)$ is a Hermitian metric on M.

We have the following theorem

THEOREM 3.4 *Let (M, J, g, ω_g) be a compact Kähler manifold, $[\omega_g] = \omega \in K(M)$ the corresponding Kähler class and ρ_g the Ricci form. Consider any $(1,1)$-form Ω on M such that $[\Omega] = 2\pi c_1(M)$. Then there exists a unique Kähler metric $h \in \mathcal{K}_{[\omega]}$ such that $\Omega = \rho_h$.*

The above statement is the celebrated Calabi Conjecture which was posed by Eugene Calabi in 1954. The conjecture in its full generality was eventually proved by Yau in 1976. In the Fano case when $c_1(M) > 0$, i.e., when the first Chern class can be represented by a positive-definite real, closed $(1,1)$-form ρ' on M, the conjecture implies that the Kähler form of M can be represented by a metric of positive Ricci curvature.

Let us reformulate the problem using the global $i\partial\bar{\partial}$-lemma. We start with a given Kähler metric g on M in Kähler class $[\omega_g] = [\omega]$. Since ρ_g also represents $2\pi c_1(M)$ there exists a globally defined function $f \in C^\infty(M, \mathbb{R})$ such that

$$\rho_g - \Omega = i\partial\bar{\partial} f.$$

Appropriately, f may be called *a discrepancy potential function* for the Calabi problem and we could fix the constant by asking that $\int_M (e^f - 1)\omega_g = 0$.

Now, supposed the desired solution of the problem is a metric $h \in \mathcal{K}_{[\omega]}$. We know that the Kähler form of h can be written as

$$\omega_h = \omega_g + i\partial\bar{\partial}\phi,$$

for some smooth function $\phi \in C^\infty(M, \mathbb{R})$. We normalize ϕ as in previous corollary. Combining these two equations we see that

$$\rho_h - \rho_g = i\partial\bar{\partial} f.$$

If we define a smooth function $F \in C^\infty(M, \mathbb{R})$ relating the volume forms of the two metrics $\omega_h^n = e^F \omega_g^n$ then the left-hand side of the above equation takes the following form

$$i\partial\bar{\partial} F = \rho_h - \rho_g = i\partial\bar{\partial} f,$$

and, hence, simply $i\partial\bar{\partial}(F - f) = 0$. Hence, $F = f + c$. But since we normalized $\int_M (e^f - 1)\omega_g = 0$ we must have $c = 0$. Hence, $F = f$, or $\omega_h^n = e^f \omega_g^n$. We can now give two more equivalent formulations of the Calabi Problem.

THEOREM 3.5 *Let (M, J, g, ω_g) be a compact Kähler manifold, $[\omega_g] = \omega \in K(M)$ the corresponding Kähler class and ρ_g the Ricci form. Consider any $(1,1)$-form on M such that $[\Omega] = 2\pi c_1(M)$. Let $\rho_g - \Omega = i\partial\bar{\partial} f$, with $\int_M (e^f - 1)\omega_g = 0$.*

(i) There exists a unique Kähler metric $h \in \mathcal{K}_{[\omega]}$ whose volume form ω_h^n equals to $e^f \omega_g^n$.

(ii) Let $(\mathcal{U}, z_1, \ldots, z_n)$ be a local complex chart on M with respect to which the metric $g = (g_{i\bar{j}})$. Then, up to a constant, there exists a unique smooth function ϕ in $\mathcal{K}_{[\omega]}$, which satisfies the following equation

$$\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right)}{\det(g_{i\bar{j}})} = e^f.$$

The equation in (ii) is called the Monge-Ampére equation. Part (i) gives a very simple geometric characterization of the Calabi-Yau theorem. On a compact Kähler manifold one can always find a metric with arbitrarily prescribed volume form. The uniqueness part of this theorem was already proved by Calabi. This part involves only Maximum Principle. The existence proof uses continuity methods and it involves several difficult a priori estimates. These were found by Yau in 1978. We have

Corollary 3.6 *Let (M, J, g, ω_g) be a compact Kähler manifold with $c_1(M) = 0$. Then M admits a unique Kähler Ricci-flat metric.*

It is "folklore" that Calabi-Yau Conjecture is also true for compact orbifold. In the context of Sasakian geometry and fundamental foliations the transverse space $\mathcal{Z}$ is typically a compact Kähler orbifold. In the context of foliations a **transverse Yau theorem** was proved by El Kacimi-Alaoui in 1990.

Let (ξ, η, Φ, g) be a Sasakian structure on M, and consider the contact subbundle $\mathcal{D} = \ker\ \eta$. There is an orthogonal splitting of the tangent bundle as

$$TM = \mathcal{D} \oplus L_\xi, \tag{3.7}$$

where L_ξ is the trivial line bundle generated by the Reeb vector field ξ. The contact subbundle $\mathcal{D}$ is just the normal bundle to the characteristic foliation $\mathcal{F}_\xi$ generated by ξ. It is naturally endowed with both a complex structure $J = \Phi|\mathcal{D}$ and a symplectic structure $d\eta$. Hence, $(\mathcal{D}, J, d\eta)$ gives M a *transverse Kähler* structure with Kähler form $d\eta$ and metric $g_{\mathcal{D}}$ defined by

$$g_{\mathcal{D}}(X, Y) = d\eta(X, JY) \tag{3.8}$$

which is related to the Sasakian metric g by

$$g = g_{\mathcal{D}} \oplus \eta \otimes \eta.$$

Recall that a smooth p-form α on M is called *basic* if

$$\xi \rfloor \alpha = 0, \qquad \mathcal{L}_\xi \alpha = 0, \tag{3.9}$$

and we let Λ^p_B denote the sheaf of germs of basic p-forms on M, and by Ω^p_B the set of global sections of Λ^p_B on M. The sheaf Λ^p_B is a module under the ring, Λ^0_B, of germs of smooth basic functions on M. We let $C^\infty_B(M) = \Omega^0_B$ denote global sections of Λ^0_B, i.e. the ring of smooth basic functions on M. Since exterior differentiation preserves basic forms we get a de Rham complex

$$\cdots \longrightarrow \Omega^p_B \xrightarrow{\ d\ } \Omega^{p+1}_B \longrightarrow \cdots \tag{3.10}$$

whose cohomology $H^*_B(\mathcal{F}_\xi)$ is called the *basic cohomology* of $(M,\mathcal{F}_\xi)$. The basic cohomology ring $H^*_B(\mathcal{F}_\xi)$ is an invariant of the foliation $\mathcal{F}_\xi$ and hence, of the Sasakian structure on M. It is related to the ordinary de Rham cohomology $H^*(M,\mathbb{R})$ by the long exact sequence

3.11
$$\to H^p_B(\mathcal{F}_\xi)\longrightarrow H^p(M,\mathbb{R})\xrightarrow{j_p} H^{p-1}_B(\mathcal{F}_\xi)\xrightarrow{\delta} H^{p+1}_B(\mathcal{F}_\xi)\to$$

where δ is the connecting homomorphism given by $\delta[\alpha]_B = [d\eta\wedge\alpha]_B = [d\eta]_B\cup[\alpha]_B$, and j_p is the composition of the map induced by $\xi\rfloor$ with the well known isomorphism $H^r(M,\mathbb{R})\approx H^r(M,\mathbb{R})^{S^1}$ where $H^r(M,\mathbb{R})^{S^1}$ is the S^1-invariant cohomology defined from the S^1-invariant r-forms $\Omega^r(M)^{S^1}$. Here we denote cohomology classes in $H^p_B(\mathcal{F}_\xi)$ by $[\cdot]_B$ in order to distinguish them from the ordinary cohomology classes. We also note that $d\eta$ is basic even though η is not.

Next we exploit the fact that the transverse geometry is Kähler. Let $\mathcal{D}_\mathbb{C}$ denote the complexification of $\mathcal{D}$, and decompose it into its eigenspaces with respect to J, that is, $\mathcal{D}_\mathbb{C} = \mathcal{D}^{1,0}\oplus\mathcal{D}^{0,1}$. Similarly, we get a splitting of the complexification of the sheaf Λ^1_B of basic one forms on M, namely

$$\Lambda^1_B\otimes\mathbb{C} = \Lambda^{1,0}_B\oplus\Lambda^{0,1}_B.$$

We let $\Lambda^{p,q}_B$ denote the sheaf of germs of basic forms of type (p,q), and as in the usual case there is a splitting

3.12
$$\Lambda^r_B\otimes\mathbb{C} = \bigoplus_{p+q=r}\Lambda^{p,q}_B,$$

as well as the *basic Dolbeault complex*

$$0 \longrightarrow \Lambda_B^{p,0} \xrightarrow{\bar\partial} \Lambda_B^{p,1} \xrightarrow{\bar\partial} \cdots \longrightarrow \Lambda_B^{p,n} \longrightarrow 0, \tag{3.13}$$

together with *basic Dolbeault cohomology groups* $H_B^{p,q}(\mathcal{F}_\xi)$. Most of the usual results about Kähler geometry carry over to transverse Kähler geometry.

Now the contact subbundle $\mathcal{D}$ is a complex vector bundle and thus has a first Chern class $c_1(\mathcal{D}) \in H^2(M,\mathbb{Z})$. Consider the long exact sequence (3.11) together with the natural map $H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{R})$ whose kernel is the torsion part of $H^2(M,\mathbb{Z})$. We have

$$\begin{array}{ccccccccc} & & & & H^2(M,\mathbb{Z}) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathbb{R} & \xrightarrow{\delta} & H_B^2(\mathcal{F}_\xi) & \xrightarrow{\iota_*} & H^2(M,\mathbb{R}) & \longrightarrow & \cdots \end{array} \tag{3.14}$$

As in (3.11) the map δ is given by $\delta(c) = c[d\eta]$ where $c \in \mathbb{R}$. Now on a Sasakian manifold the vector bundle $\mathcal{D}^{1,0}$ is holomorphic with respect to the CR-structure, so we can compute the free part of $c_1(\mathcal{D}) = c_1(\mathcal{D}^{1,0})$ from the transverse Kähler geometry in the usual way. That is $c_1(\mathcal{D})$ can be represented by a basic real closed $(1,1)$-form ρ_B. The class $c_1^B = [\rho_B] \in H_B^2(\mathcal{F}_\xi)$ is independent of the transverse metric and basic connection used to compute it, and depends only on the foliated manifold $(M,\mathcal{F}_\xi)$ with its CR-structure. It is described by El Kacimi-Alaoui and called the *basic first Chern class of* $\mathcal{D}$ there. Alternatively, we can think of c_1^B

as the negative of the first Chern class of the "transverse canonical bundle" $K = (\Lambda_B^{1,0})^n$ of M.

THEOREM 3.15: [El Kacimi-Alaoui] *If c_1^B is represented by a real basic* $(1,1)$ *form ρ^T, then it is the Ricci curvature form of a unique transverse Kähler form ω^T in the same basic cohomology class as $d\eta$.*

In the language of positive Sasakian manifolds this theorem can be used to show

THEOREM 3.16: [Boyer,–,Nakamaye] *Any positive Sasakian manifold admits a metric of positive Ricci curvature.*

One of the consequences of the Yau's theorem is that a compact Kähler manifold with $c_1(M) = 0$ must admit a Ricci-flat, hence, Einstein metric. More generally, we can consider existence of Kähler-Einstein metrics with arbitrary Einstein constant λ. By scaling we can assume that $\lambda = 0, \pm 1$. Specifically, let (M, J, g, ω_g) be a compact Käher manifold. We would like to know if one can always find a Kähler-Einstein metric $h \in \mathcal{K}_{[\omega_g]}$. Recall that on a Kähler-Einstein manifold $\rho_g = \lambda\omega_g$. This implies that $2\pi c_1(M) = \lambda[\omega_g]$. Now, if $c_1(M) > 0$ we must have $\lambda = +1$ because $[\omega_g]$ is the Kähler class. Similarly, when $c_1(M) < 0$ the only allowable sign of a Kähler-Einstein metric on M is $\lambda = -1$. Clearly, when $c_1(M) = 0$ we must have $\lambda = 0$ as $[\omega_g] \neq 0$. As we have already pointed out the $\lambda = 0$ case follows from Yau's solution to the Calabi conjecture. For the reminder of this lecture we shall assume that $\lambda = \pm 1$.

Let (M, J, g, ω_g) be a Kähler manifold and $[\omega_g] = [\omega] \in K(M)$ the Kähler class. Let us reformulate the existence problem using the global $i\partial\bar{\partial}$-lemma. Suppose there exists an Einstein metric $h \in \mathcal{K}_{[\omega]}$. Starting with the original Kähler metric g on M we have a globally defined function $f \in C^\infty(M, \mathbb{R})$ such that

$$\rho_g - \lambda\omega_g = i\partial\bar{\partial}f.$$

As before we will call f *a discrepancy potential function.* We also fix the constant by asking that $\int_M (e^f - 1)\omega_g = 0$. Let $h \in \mathcal{K}_{[\omega]}$ be an Einstein metric for which $\rho_h = \lambda\omega_h$. Using global $i\partial\bar{\partial}$-lemma once again we have a globally defined function $\phi \in C^\infty(M, \mathbb{R})$ such that $\omega_h - \omega_g = i\partial\bar{\partial}\phi$. We shall fix the constant in ϕ later. Using these two equations we easily get

$$\rho_g - \rho_h = i\partial\bar{\partial}(f - \lambda\phi).$$

Defining F so that $\omega_h^n = e^F\omega_g^n$ we can write this equation as

$$i\partial\bar{\partial}F = i\partial\bar{\partial}(f - \lambda\phi).$$

This implies that $F = f - \lambda\phi + c$. We have already fixed the constant in f so c depends only on the choice of ϕ. We can make $c = 0$ by choosing ϕ such that $\int_M (e^{f-\lambda\phi} - 1)\omega_g^n = 0$. Hence, we have the following

PROPOSITION 3.17 *Let (M, J) be a compact Kähler manifold with $\lambda c_1(M) > 0$, where $\lambda = \pm 1$. Let $[\omega] \in K(M)$ be a Kähler class and g, h two Kähler metrics in $\mathcal{K}_{[\omega]}$ with Ricci forms ρ_g, ρ_h. Let $f, \phi \in C^\infty(M, \mathbb{R})$ be defined by $\rho_g - \lambda\omega_g = i\partial\bar{\partial}f$, $\omega_h - \omega_g = i\partial\bar{\partial}\phi$. Fix the relative constant of $f - \lambda\phi$ by setting $\int_M (e^{f-\lambda\phi} - 1)\omega_g = 0$. Then the* *metric h is Einstein with Einstein constant λ if and only if ϕ satisfies the following Monge-Ampére equation*

$$\omega_h^n = e^{f-\lambda\phi}\omega_g^n,$$

in a local complex chart $(\mathcal{U}, z_1, \ldots, z_n)$ written as

$$\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\right)}{\det(g_{i\bar{j}})} = e^{f-\lambda\phi}.$$

Note that by setting $\lambda = 0$ we get the Monge-Ampére equation for the original Calabi problem. The character of the Monge-Ampére equation above very much depends on the choice of λ. The case of $\lambda = -1$ is actually the simplest as the necessary a priori C^0-estimates can be derived using the Maximum Principle. This was done by Aubin and independently by Yau. We have

THEOREM 3.18 *Let (M, J, g, ω_g) be a compact Kähler manifold with $c_1(M) < 0$. Then there exists a unique Kähler metric $h \in \mathcal{K}_{[\omega_g]}$ such that $\rho_h = -\omega_h$.*

When $k = +1$ the problem is much harder. It has been known for quite some time that there are actually non-trivial obstructions to the existence. Let $\mathfrak{h}(M)$ be the complex Lie algebra of all holomorphic vector fields on M. Matsushima proved that one a compact Kähler-Einstein manifold with $c_1(M) > 0$ $\mathfrak{h}(M)$ must be reductive, i.e., $\mathfrak{h}(M) = Z(\mathfrak{h}(M) \oplus [\mathfrak{h}(M), \mathfrak{h}(M)]$.

Here one tries to solve the Monge-Ampére equation

$$\frac{\det(g_{i\bar{j}} + \partial_i\bar{\partial}_{\bar{j}}\phi_t)}{\det(g_{i\bar{j}})} = e^{-t\phi_t+f}, \quad g_{i\bar{j}} + \partial_i\bar{\partial}_{\bar{j}}\phi_t > 0$$

for $t \in [0,1]$. Yau's Theorem tells us that this has a solution for $t=0$, and we try to solve this for $t=1$, where the metric will be Kähler-Einstein The so called "continuity method" sets out to show that the interval where solutions exist is both open and closed. Openness follows from the Implicit Function Theorem, but there are well known obstructions to closedness. This problem has been studied by many people; Yau, Tian, Siu, Nadel, and most recently by Demailly and Kollár who work in the orbifold category. Closedness is equivalent to the uniform boundedness of the integrals

$$\int_{\mathcal{Z}} e^{-\gamma t \phi_t} \omega_0^n$$

for any $\gamma \in (\frac{n}{n+1}, 1)$, where ω_0 is the Kähler form of h_0. This means that the *multiplier ideal sheaf* $\mathcal{J}(\gamma\phi) = \mathcal{O}_{\mathcal{Z}}$ for all $\gamma \in (\frac{n}{n+1}, 1)$.

Here is an example of the estimates used to obtained many (but not all) theorems mentioned in the introduction

THEOREM 3.19 [Boyer,–,Kollár] *Let $\mathcal{Z}(\mathbf{a})$ be the transverse space of the BP link $L(\mathbf{a})$. Let $C_i = \mathrm{lcm}(a_1, \ldots, \hat{a}_i, \ldots, a_n)$, $b_i = \gcd(C_i, a_i)$. Then $\mathcal{Z}(\mathbf{a})$ is Fano and it has a Kähler-Einstein metric if*

(i) $1 < \sum_{i=1}^m \frac{1}{a_i}$,

(ii) $\sum_{i=1}^m \frac{1}{a_i} < 1 + \frac{m-1}{m-2} \min_i\{\frac{1}{a_i}\}$, *and*

(iii) $\sum_{i=1}^m \frac{1}{a_i} < 1 + \frac{m-1}{m-2} \min_{i,j}\{\frac{1}{b_i b_j}\}$.

In this case the link $L(\mathbf{a})$ admits a Sasakian-Einstein metric with one-dimensional isometry group.