Sasakian Geometry

and

Einstein Metrics

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- theorems of Smale and Barden
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Credits: Many results described in these lectures are based on several papers written in collaboration with **Charles Boyer, János Kollár, and Michael Nakamaye**.

All references and these slides can be found at **http://www.math.unm.edu/~galicki**.

LECTURE 1

Canonical metrics in contact geometry

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• Contact and Sasakian Manifolds

Definition 1.1: *A $(2n+1)$-dimensional manifold M is a* **contact manifold** *if there exists a 1-form η, called a* **contact 1-form**, *on M such that*

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A **contact structure** *on M is an equivalence class of such 1-forms, where $\eta' \sim \eta$ if there is a nowhere vanishing function f on M such that $\eta' = f\eta$.*

Lemma 1.2: *On a contact manifold (M, η) there is a unique vector field ξ, called the* **Reeb vector field**, *satisfying the two conditions*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

Lemma 1.3: *Let (M, η) be a contact manifold. The cone $(\mathcal{C}(M), \omega) := (M \times \mathbb{R}_+, d(r^2\eta))$ is symplectic.*

Definition 1.4: *One-dimensional foliation $\mathcal{F}_\xi$ defined by the Reeb field ξ is called the* **characteristic foliation**. *The characteristic foliation $\mathcal{F}_\xi$ is said to be* **quasi-regular** *if there is a positive integer k such that each point has a foliated coordinate chart (U, x)*

so that each leaf of $\mathcal{F}_\xi$ passes through U at most k times. If $k = 1$ then the foliation is called **regular**. *Finally, we say that $\mathcal{F}_\xi$ is* **irregular** *if it is not quasi-regular.*

In the regular case we have the following famous theorem of **Boothby and Wang** [1958]:

Theorem 1.5: *Let (M, η) be a regular compact contact manifold. Then M is a total space of a principal circle bundle $\pi : M \to \mathcal{Z}$ over the space of leaves $\mathcal{Z} := M/\mathcal{F}_\xi$. Furthermore, $\mathcal{Z}$ is a compact symplectic manifold with symplectic form Ω, $[\Omega] \in H^2(\mathcal{Z}, \mathbb{Z})$, and η is a connection form on the bundle with curvature $d\eta = \pi^*\Omega$.*

$$\boxed{\begin{array}{ccc} (\mathcal{C}(M), \omega) & \hookleftarrow & (M, \eta) \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & (\mathcal{Z}, \Omega) \end{array}} \tag{1.1}$$

Remark 1.6: When (M, η) is not regular but quasi-regular the Boothby-Wang fibration theorem requires only minor modification: in general $\mathcal{Z}$ is not a **smooth manifold** but it is a **compact symplectic orbifold** and M is not a principle circle bundle but an **orbibundle** (or a V-bundle).

Question 1.7: Consider the orbibundle (quasi-regular) version of the diagram (1.1). **What are the most natural Riemannian metrics adapted to such geometric structure?**

It is believed and accepted that the most natural and interesting Riemannian metrics adapted to a symplectic or an almost complex structure are the **Kähler metrics**. Hence, one would like to introduce some metric structure on (1.1) such that π is an orbifold Riemannian submersion on a Kähler base. In particular, the Reeb vector field ξ must be a Killing field. But this is not sufficient. It turns out that the "right answer" to this question leads to Sasakian Geometry introduced by **Sasaki [1960]** under a cumbersome name of **normal contact metric structure**. We begin with one of the "older" definition of the Sasakian structure.

Definition 1.8: *A Riemannian manifold (M, g) is called a* **Sasakian manifold** *if there exists a Killing vector field ξ of unit length on M so that the tensor field Φ of type $(1, 1)$, defined by $\Phi(X) := \nabla_X \xi$, satisfies the condition*

$$(\nabla_X \Phi)(Y) \;=\; g(\xi, Y)X - g(X, Y)\xi$$

for any pair of vector fields X and Y on M. We refer to the quadruple $\mathcal{S} = (g, \eta, \xi, \Phi)$ as a **Sasakian structure** *on M, where η is the 1-form dual vector field ξ.*

It is easy to see that η is a contact form whose Reeb vector field is ξ. In particular, $\mathcal{S} = (g, \eta, \xi, \Phi)$ is a special type of **metric contact structure**.

It is far from obvious why Sasakian metrics should be considered as natural candidates. At first glance, the curvature restriction which is part of the definition appears rather artificial. That is until one realizes that it is **this condition that makes both the cone $\mathcal{C}(M)$ and the transverse space $\mathcal{Z}$ of the fundamental foliation Kählerian**.

• Structure Theorems

Theorem 1.9 *Let* (M, g, η, ξ, Φ) *be a compact quasi-regular Sasakian manifold of dimension* $2n + 1$*, and let* $\mathcal{Z}$ *denote the space of leaves of the characteristic foliation. Then* $\mathcal{Z}$ *is a Hodge orbifold with Kähler metric* h *and Kähler form* Ω *which defines an integral class* $[\Omega] \in H^2_{orb}(\mathcal{Z}, \mathbb{Z})$ *so that* $\pi : (M, g) \rightarrow (\mathcal{Z}, h)$ *is an orbifold Riemannian submersion. The fibers of* π *are totally geodesic submanifolds of* M *diffeomorphic to* S^1*.*

$$\boxed{\begin{array}{ccc} (\mathcal{C}(M), \omega, \bar{g}) & \hookleftarrow & (M, g, \xi, \eta, \Phi) \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & (\mathcal{Z}, \Omega, h, J) \end{array}}$$

Theorem 1.10 *Let* $(\mathcal{Z}, h)$ *be a Hodge orbifold with Kähler metric* h *and Kähler form* Ω*. Let* $\pi : M \rightarrow \mathcal{Z}$ *be the* S^1 *V-bundle whose first Chern class is* $[\Omega]$*, and let* η *be a connection 1-form in* M *whose curvature is* $2\pi^*\Omega$*. Then* M *with the metric* $\pi^*h + \eta \otimes \eta$ *is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the group of the bundle* S^1*, the total space* M *is a smooth manifold.*

Proposition 1.11: *Let* (M, ξ, η, Φ, g) *be a Sasakian manifold. Then the metric cone on* M *defined by* $(\mathcal{C}(M), \bar{g}) := (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ *is Kähler.*

Remark 1.12: Since **Erich Kähler's** seminal **1933** article

"...several dominant figures of the mathematical scene of the XXth century have, step after step along a 50 year period, transformed the subject into a major area of Mathematics that has influenced the evolution of the discipline much further than could have conceivably been anticipated by anyone..."

writes **J.-P. Bourguignon** in his tributary article *The Unabated Vitality of Kählerian Geometry*. Truly impressive names of "dominant figures" follow. Sasakian Geometry has not been as lucky. There always has been interesting work in the area but for unclear reasons it has never attracted people with the same broad vision, people who would set out to formulate and then work on fundamental problems. Yet, arguably in view of the above theorems, Sasakian manifolds are at least as interesting as Kähler ones. Perhaps more interesting as, while being smooth, they naturally include intricate geometry of compact Kähler orbifolds with cyclic quotient singularities.

• Orbifold Chern Classes

Let $(\mathcal{Z}, h, J, \omega_h)$ be a Kähler manifold. In a local complex chart the metric h is simply a Hermitian matrix $h_{i\bar{j}}$. It was Kähler who discovered that locally the Ricci tensor of a Kähler metric can be written as

$$R_{i\bar{j}} = -\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}\ln(\det(h_{i\bar{j}})).$$

Only much later first **Chern** and then **Calabi** realized the importance of this remarkable formula. If one defines the associated 2-form

$$\rho_\omega = -\frac{i}{2}R_{i\bar{j}}dz^i \wedge d\bar{z}^j$$

which is called the **Ricci form** one realizes that the cohomology class of ρ_ω **does not depend on the choice of the Kähler metric** within Kähler class. In fact, it is easy to see that Ricci form represents the cohomology class $2\pi c_1(\mathcal{Z})$.

Properly told, in a modern language, a story of Sasakian manifolds should start exactly here, with the first Chern class. In order to account for some important **subtleties related to orbifold singularities** some care needs to be exercised. There are three different but equivalent approaches.

[**Basic Cohomology**] First, one can use the foliation language and the notion of basic cohomology to define the so-called basic Chern classes $c_k(\mathcal{F}_\xi)$ of the fundamental foliation $\mathcal{F}_\xi$, in particular the basic first Chern class $c_1(\mathcal{F}_\xi) \in H^2_B(\mathcal{F}_\xi)$. This approach is perhaps the most geometric and the objects that define transverse invariants are basic forms on M.

[**Haefliger's Orbifold Cohomology**] Since $\mathcal{Z}$ as a compact complex orbifold one can use the orbifold cohomology groups $H^*_{orb}(\mathcal{Z},\mathbb{Z})$ of **Haefliger [1984]** and define $c_1^{orb}(\mathcal{Z})$ as an element of $H^2_{orb}(\mathcal{Z},\mathbb{Z})$.

[**Algebraic Geometry**] In algebraic geometry one of the fundamental objects is **the canonical line bundle** $K_{\mathcal{Z}}$, and its dual the **anti-canonical bundle** $K_{\mathcal{Z}}^{-1}$. Cohomology class corresponding to $K_{\mathcal{Z}}$, called **the canonical class** is often denoted also by $K_{\mathcal{Z}}$ and it is the negative of the first Chern class $c_1(\mathcal{Z}) := c_1(K_{\mathcal{Z}}^{-1})$. One could try to extend such definitions to the category of **complex orbifolds**. However, in general a complex orbifold is not the same as a **complex algebraic variety with quotient singularities**. In fact, very interesting complex orbifolds are **smooth** from the algebraic point of view. We explain with the standard example.

Example 1.13 **[Weighted Projective Spaces]** Let $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{Z}_+^n$ with $\gcd(w_0, \ldots, w_n) = 1$ and $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$. We define the $\mathbb{C}^*_{\mathbf{w}}$-action on $\mathbb{C}^{n+1}$:

$$\lambda \cdot (z_0, \ldots, z_n) := (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n),$$

and extend this action to $\mathbb{C}[z_0, \ldots, z_n]$

$$\lambda \cdot f(z_0, \ldots, z_n) := f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n).$$

$$\mathbb{P}^n_{\mathbb{C}}(\mathbf{w}) \equiv \mathbb{P}^n_{\mathbb{C}}(w_0, \ldots, w_n) := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*_{\mathbf{w}},$$

is called the **weighted projective space**. The eigenspace of the $\mathbb{C}^*_{\mathbf{w}}$-action on $\mathbb{C}[z_0, \ldots, z_n]$ with eigenvalue λ^d is

$$H^0(\mathbb{P}^n_{\mathbb{C}}(w_0, \ldots, w_n), \mathcal{O}(d)),$$

the space of **weighted homogeneous polynomials** of weight $\mathbf{w}$ and degree d.

Consider a triple of pairwise relatively prime positive integers (a, b, c). Its is easy to see that

$$\mathbb{P}^2_{\mathbb{C}}(ab, bc, ca) \simeq \mathbb{P}^2_{\mathbb{C}}(a, c, ca) \simeq \mathbb{P}^2_{\mathbb{C}}(a, 1, a) \simeq \mathbb{P}^2_{\mathbb{C}}(1, 1, 1),$$

where " $\simeq$ " indicates equivalence of algebraic varieties but **not** orbifold equivalence. Although all equivalent to the smooth complex projective plane these orbifolds (7 of them) are all different and they do have different orbifold first Chern class. On the other hand they are all Fano.

The next example is more interesting. Consider

$f(z_0, z_1, z_2) = z_0^c + z_1^a + z_2^b \in H^0(\mathbb{P}^2_{\mathbb{C}}(ab, bc, ca), \mathcal{O}(abc))$.

$f = 0$ defines a **weighted homogeneous hypersurface** in $X \subset \mathbb{P}^2_{\mathbb{C}}(ab, bc, ca)$. One can see that regardless the choice of (a, b, c) we have $X(a, b, c) \simeq \mathbb{P}^1_{\mathbb{C}}$. However, as we shall see later, **as orbifolds**, $X(a, b, c)$ are very different depending on the sign of

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1.$$

In particular, $X(a, b, c)$ is Fano, only when this sign is positive.

Both of the above examples are special because the orbifold structure consists of codimension 1 singularities. The space $\mathbb{P}^2_{\mathbb{C}}(ab, bc, ca)$ not **well-formed**. In order to study and understand such complex orbifolds we codify the data coming from the codimension 1 singularities by a single $\mathbb{Q}$-divisor, called the *branch divisor* of the orbifold, $\Delta := \sum(1 - \frac{1}{m_j})D_j$. Then we can think of a complex orbifold $\mathcal{Z}$ as a pair $(\mathcal{Z}, \Delta)$. With this in mind we define the orbifold first Chern class as $c_1^{orb}(\mathcal{Z}) \in H^2(\mathcal{Z}, \mathbb{Q})$

$$c_1^{orb}(\mathcal{Z}) \equiv c_1^{orb}(\mathcal{Z}, \Delta) = c_1(\mathcal{Z}) - \sum\big(1 - \frac{1}{m_j}\big)c_1([D_i]).$$

The orbifold canonical class $K_{\mathcal{Z}}^{orb} = -c_1^{orb}(\mathcal{Z})$ is the negative of the orbifold first Chern class.

Definition 1.14 *Let M be a compact quasi-regular Sasakian manifold with the transverse space $\mathcal{Z}$. We say that M is* **positive (negative) Sasakian** *when $c_1^{orb}(\mathcal{Z}) > 0$ $(c_1^{orb}(\mathcal{Z}) < 0)$. In the case $c_1^{orb}(\mathcal{Z})$ is trivial we say that M is* **null Sasakian**.

Definition-Proposition 1.15: *A Sasakian manifold (M, g, η, ξ, Φ) is* **Sasakian-Einstein***(SE) if the metric g is also Einstein. For any 2n+1-dimensional Sasakian manifold $\mathrm{Ric}(X, \xi) = 2n\eta(X)$ implying that any SE metric must have positive scalar curvature. Thus any complete SE manifold must have a finite fundamental group. Furthermore, the metric cone on M $(C(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ is Kähler Ricci-flat (Calabi-Yau=CY).*

Definition 1.16: *Let (M, g) be a Riemannian manifold of dimension m. We say that (M, g) is* **3-Sasakian** *if the metric cone $(\mathcal{C}(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ on M is hyperkähler.*

With these definition the following table lists some obvious interesting special families of Sasakian manifolds. We now explain why some of these families are interesting to a Riemannian geometer in search of special metrics.

Cone Geometry of $\mathcal{C}(M)$	M	Transverse Geometry of $\mathcal{F}_\xi$
Symplectic	Contact	Symplectic
Kähler	Sasakian	Kähler
Kähler	positive Sasakian	Fano, $c_1^{orb}(\mathcal{Z}) > 0$
Kähler	null Sasakian	Calabi-Yau, $c_1^{orb}(\mathcal{Z}) = 0$
Kähler	negative Sasakian	canonical, $c_1^{orb}(\mathcal{Z}) < 0$
Calabi-Yau	Sasakian-Einstein	Fano, Kähler-Einstein
Hyperkähler	3-Sasakian	$\mathbb{C}$-contact, Fano, Kähler-Einstein

• [**Positive Sasakian Geometry**] My own interest in Sasakian geometry started over a decade ago with the very last row of the table. That is because 3-Sasakian spaces are **automatically Einstein** and they are intimately related to **quaternion Kähler** and **twistor** geometries. However, 3-Sasakian manifolds are a very special case of positive Sasakian and SE manifolds. Positive Sasakian manifolds provide a foundation for studying and constructing SE metrics and positive Ricci curvature Sasakian metrics in any dimension. The main idea can be captured in the following

Theorem 1.17 *Let (M,g) be a Sasakian manifold such that $(\mathcal{Z},h)$ is Fano. Then $\mathcal{Z}$ admits an orbifold Kähler metric h' of positive Ricci curvature which can be lifted to a positive Ricci curvature Sasakian metric g' on M. If $\mathcal{Z}$ admits* **an orbifold Kähler-Einstein (KE)** *metric in the same Kähler class then this metric defines a unique SE metric on M.*

The first statement of the above theorem follows from an orbifold version of the famous **Yau's Theorem [1978]**. The second part is an orbibundle version of the old result of **Kobayashi [Kob63]** adapted to the Sasakian situation.

• **[Null and Negative Sasakian Geometry]** SE manifolds are necessarily of positive scalar curvature. However, there is an important class of special Sasakian metrics called η-Einstein (SηE) for which

$$\mathrm{Ric}(X,Y) = \lambda g(X,Y) + \nu\eta(X)\eta(Y).$$

Here (λ, ν) are constants and $\lambda + \nu = 2n = \dim(M) - 1$. The importance of (S$\eta$E) metrics follows from the following

Theorem 1.18 *Let (M, g, η, ξ, Φ) be a compact quasi-regular SηE manifold of dimension $2n+1$, and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation. Then $\mathcal{Z}$ is a Hodge orbifold with KE metric h with Einstein constant $\lambda + 2$.*

Now, in complete analogy with the Kähler case we have three possibilities, depending on the sign of the KE metric on the base:

$\lambda > -2; M$ is positive Sasakian and the metric can be easily deformed to an SE metric by the so-called **transverse homothety transformation**. Hence, the existence (non-existence) of such metrics is completely equivalent to the existence (non-existence) of an orbifold KE metric on $\mathcal{Z}$.

$\lambda = -2; M$ is null Sasakian and the existence of SηE metric rests assured on the orbifold version of the Calabi Conjecture. This was proved for foliations by **El Kacimi-Alaoui [1978]**. Such metrics are never Einstein but null Sasakian geometry is **intimately linked to the geometry of the CY orbifolds**.

$\lambda < -2; M$ is negative Sasakian and the existence of SηE metric rests assured on the orbifold version of the theorem proved independently by **Aubin** and **Yau** which states that any compact Kähler manifold $\mathcal{Z}$ with $c_1(\mathcal{Z}) < 0$ admits **a unique KE** with $\mathrm{Ric} = -g$. While SηE metric in the negative case is never Einstein, one can use it to obtain **a Lorentzian** metric on the same space which **is** SE.

• Existence and Obstructions

In the case $c_1(\mathcal{Z}) > 0$ the existence of an orbifold KE metric can be obstructed. This is true even for smooth manifolds where several obstructions are known:

- **Matsushima's obstruction**: the complex Lie algebra $\mathfrak{h}(\mathcal{Z})$ of holomorphic vector fields on $\mathcal{Z}$ must be reductive [**Matsushima 1957**], i.e., $\mathfrak{h}(\mathcal{Z}) = \mathfrak{h}(\mathcal{Z})) \oplus [\mathfrak{h}(\mathcal{Z}), \mathfrak{h}(\mathcal{Z})]$.
- **Futaki character**: **Futaki** [**1983**] introduced a functional on the Lie algebra of holomorphic vector fields $\mathfrak{h}(\mathcal{Z})$. On a KE manifold this functional must vanish.
- **generalized Futaki invariants**: In **1994 Ding and Tian** introduced a generalization of Futaki invariant. Remarkably, using this new invariant, they showed that there are log del Pezzo surfaces with no holomorphic vector fields which do not admit KE metrics. Later, Tian showed that, in fact, there are smooth Fano 3-folds with no holomorphic vector fields which admit no KE metric. This disproved a "folklore" conjecture of Calabi.
- **Generalized Futaki invariants and stability Tian** [**1997**]

- **Existence results via continuity method**: Continuity method was proposed by Calabi as an approach to the original Calabi conjecture. It has been used successfully by Yau to prove the conjecture and later by several people to establish existence of KE metrics on Fano manifolds and orbifolds.

More generally, we can list the following obstructions to various Sasakian structures:

- Sasakian manifolds are **orientable**
- Sasakian manifolds are **contact**
- compact irregular Sasakian manifolds exist **but** any irregular Sasakian structure can be approximated by a sequence of quasi-regular structures [**Rukimbira, 1994**]. As a consequence, **a compact contact manifold which does not admit a quasi-regular contact structure cannot carry a Sasakian structure**
- on a compact Sasakian manifold odd Betti numbers (up to the middle dimension) are even
- any compact simply-connected SηE manifold is spin (SE in particular)

- any compact positive Sasakian manifold has finite fundamental group (SE in particular)
- any compact 3-Sasakian manifold has vanishing odd Betti numbers (up to the middle dimension) **[Galicki-Salamon, 1996]**

• Classification of Sasakian 3-Manifolds

Three dimensional Sasakian geometry is well understood, culminating in the recent uniformization theorem due to **[Belgun 2000]**. Geiges **[Geiges 1997]** showed that a compact 3-manifold admits a Sasakian structure if and only if it is diffeomorphic to one of the following:

1. S^3/Γ with $\Gamma \subset \mathfrak{I}_0(S^3) = SO(4)$.
2. $\widetilde{SL}(2,\mathbb{R})/\Gamma$, where is universal cover of $SL(2,\mathbb{R})$ and $\Gamma \subset \mathfrak{I}_0(\widetilde{SL}(2,\mathbb{R}))$.
3. Nil^3/Γ with $\Gamma \subset \mathfrak{I}_0(\mathrm{Nil}^3)$.

Here Γ is a discrete subgroup of the connected component $\mathfrak{I}_0$ of the corresponding isometry group with respect to a 'natural metric', and Nil^3 denotes the 3 by 3 nilpotent real matrices, otherwise known as the Heisenberg group. These are three of the eight model geometries of **Thurston**, and correspond precisely to the compact **Seifert bundles** with non-zero Euler characteristic.

• Examples of SE Manifold

Example 1.19: [3-Sasakian Manifolds] Naively, one would expect that such spaces may be hard to find because of all the geometric structure they must have. But it turns out that it is precisely this extra structure that allows for some explicit constructions of families of 3-Sasakian manifolds.

- **"flat example"**

$$\begin{tikzcd}
 & \mathbb{H}^n \setminus \{0\} \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathbb{CP}^{2n-1} \arrow[dr] & & S^{4n-1} \arrow[ll] \arrow[dl] \\
 & \mathbb{PH}^{n-1} &
\end{tikzcd}$$

- **homogeneous examples**

$$\begin{tikzcd}
 & \mathcal{C}(G/L) \arrow[dl] \arrow[dr] \arrow[dd] & \\
G/L \cdot U(1) \arrow[dr] & & G/L \arrow[ll] \arrow[dl] \\
 & G/L \cdot Sp(1) &
\end{tikzcd}$$

- **toric examples** **[Boyer,Galicki, Mann, Rees'98]**

$$\begin{tikzcd}[column sep=small]
 & \mathbb{H}^n \setminus \{0\} \arrow[dl] \arrow[dr] & \\
\mathbb{CP}^{2n-1} \arrow[dr] & & S^{4n-1} \arrow[dl] \\
 & \mathbb{PH}^{n-1} \arrow[dd, "QUOTIENT"] & \\
T^k & & \\
 & \mathbb{R}_+ \times \mathcal{S}(\Omega) \arrow[dl] \arrow[dr] & \\
\mathcal{Z}(\Omega) \arrow[dr] & & \mathcal{S}(\Omega) \arrow[dl] \\
 & \mathcal{O}(\Omega) &
\end{tikzcd}$$

Here, Ω is an integral matrix which defines a homomorphism $f_\Omega : T^k \to U(n)$. In the special case when $\dim(\mathcal{S}(\Omega)) = 7$, there are choices of Ω for any $k \geq 1$ which make $\mathcal{S}(\Omega)$ smooth. Since $b_2(\mathcal{S}(\Omega)) = k$ we conclude that there exist Einstein manifolds with **arbitrarily large second Betti number**. These were first such examples.

- first **non-toric examples** **[Boyer,Galicki, Piccinni'02]** (in dimension 11,15) were obtained by non-Abelian reduction of the flat example, where instead of the torus $G = T^k$ one takes $G = Sp(1) \times T^k$.

- more recently non-toric examples in dimension 7 were obtained by **Grove, Wilking, and Ziller**. They use orbifold bundle construction with the examples of orbifold twistor space and self-dual Einstein metrics $\mathcal{Z}_k \to \mathcal{O}_k$ discovered by **Nigel Hitchin in 1996.** The self-dual Einstein metric on $\mathcal{O}_k$ is defined on $S^4 \setminus \mathbb{RP}^2$ and it has $\mathbb{Z}_k$ orbifold singularity along $\mathbb{RP}^2$. However, it turns out that the bundle $M_k \to \mathcal{Z}_k$ is actually smooth. In particular, one can compute integral cohomology ring of M_k. For odd k the 3-Sasakian manifold M_k is a rational homology 7-sphere with non-zero torsion depending on k. Hence, there exist infinitely many rational homology 7-spheres which have 3-Sasakian metrics.

Remark 1.20: A recent result of **Dearricott [2004]** links metrics of **positive sectional curvature to 3-Sasakian geometry**. Dearricott proves that when the quaternionic Kähler orbifold base is of positive sectional curvature then the 3-Sasakian metric can be deformed to a positive sectional curvature metric. Non-trivial examples exist only in dimension 7. In principle, a 3-Sasakian structure on some exotic 7-sphere could lead to **a construction of positive sectional curvature metric on a non-standard 7-sphere**.

Example 1.21: *Other Regular Sasakian-Einstein Manifolds:*

- **complex Hopf fibration**:

$$\begin{array}{ccc} \mathbb{C}^n \setminus \{0\} & & \\ & \searrow & \\ \downarrow & & S^{2n-1} \\ & \swarrow & \\ \mathbb{PC}^{n-1} & & \end{array}$$

- There are many **homogeneous examples**; all compact homogeneous Kähler-Einstein spaces are classified and they are of the form $\mathcal{Z} = G/P$. Hence, one can replace the complex projective space with $\mathcal{Z} = G/P$ and apply the usual Kobayashi construction.

- There are also many **inhomogeneous examples**. One can take any smooth compact Fano variety $\mathcal{Z}$ which admits a Kähler-Einstein metric. For example, in the case of complex surfaces it is know exactly which Fano (del Pezzo) surfaces admit KE metric. The complete classification was established by **Tian [1989]**.

Until 2001, examples of smooth irregular Sasakian-Einstein manifolds were rare (with the exception of inhomogeneous 3-Sasakian spaces mentioned above). In the second lecture we will describe a constructions of special Sasakian metric on Brieskorn manifolds.

LECTURE 2

Einstein Metrics on Brieskorn Manifolds

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•Sasakian Geometry of Links

Let $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{Z}_+^n$ with $\gcd(w_0, \ldots, w_n) = 1$, $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ and $\mathbb{C}^*_{\mathbf{w}}$ be the corresponding "weighted action" on $\mathbb{C}^{n+1}$. In **Example 1.13** we defined the corresponding weighted projective space $\mathbb{P}^n_{\mathbb{C}}(\mathbf{w})$ and the space $H^0(\mathbb{P}^n_{\mathbb{C}}(w_0, \ldots, w_n), \mathcal{O}(d))$, of weighted homogeneous polynomials of weight $\mathbf{w}$ and degree d.

Definition 2.1 *Let $f(\mathbf{z}) \in H^0(\mathbb{P}^n_{\mathbb{C}}(w_0, \ldots, w_n), \mathcal{O}(d))$. We define the following three spaces*

$$Y_f := \{\mathbf{z} \in \mathbb{C}^{n+1} \;\mid\; f(\mathbf{z}) = 0\} \subset \mathbb{C}^{n+1}.$$

$$\mathcal{Z}_f := (Y_f \setminus \{0\})/\mathbb{C}^*_{\mathbf{w}} \subset \mathbb{P}^n_{\mathbb{C}}(\mathbf{w}).$$

$$L_f := Y_f \cap S^{2n+1} \subset S^{2n+1}(1) \subset \mathbb{C}^{n+1}.$$

The space $\mathcal{Z}_f$ is called **weighted homogeneous hypersurface**, *Y_f is the* **affaine cone** *of $\mathcal{Z}_f$ and L_f is called the* **link**.

We are interested in the case $\{0\} \in \mathbb{C}^{n+1}$ is the only singularity of $f(\mathbf{z})$. In such case $\mathcal{Z}_f$ is called **quasi-smooth** and L_f is then a **link of isolated hypersurface singularity.**

The link L_f is endowed with a natural quasi-regular Sasakian structure inherited as a Sasakian submanifold of the sphere S^{2n+1} with its "weighted" Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which in the standard coordinates

$\{z_j = x_j + iy_j\}_{j=0}^n$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is determined by

2.1

$$\eta_{\mathbf{w}} = \frac{\sum_{i=0}^n (x_i dy_i - y_i dx_i)}{\sum_{i=0}^n w_i(x_i^2 + y_i^2)}, \qquad \xi_{\mathbf{w}} = \sum_{i=0}^n w_i(x_i\partial_{y_i} - y_i\partial_{x_i}),$$

and the standard Sasakian structure (ξ, η, Φ, g) on S^{2n+1}.

The quotient of S^{2n+1} by the "weighted S^1-action" generated by the vector field $\xi_{\mathbf{w}}$ is the weighted projective space $\mathbb{P}^n_{\mathbb{C}}(\mathbf{w})$ and we have a commutative diagram:

2.2

$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \downarrow \pi & & \downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are orbifold Riemannian submersions.

Remark 2.2: The well-known construction of Milnor for isolated hypersurface singularities shows that L_f is (n-2)-connected. Hence, the only non-vanishing integral homology groups are $H_{n-1}(L_f, \mathbb{Z})$ and $H_n(L_f, \mathbb{Z})$.

Proposition 2.3: *The link L_f is*

1. *negative Sasakian if and only if $d - \sum w_i > 0$,*
2. *null Sasakian if and only if $d - \sum w_i = 0$,*
3. *positive Sasakian if and only if $d - \sum w_i < 0$.*

At first glance Proposition 2.3 appears to be a reasonable generalization of the case of smooth hypersurface of degree d in $\mathbb{P}^n_{\mathbb{C}}$. To appreciate subtleties of codimension 1 singularities, let us consider the following example.

Example 2.4: **[Brieskorn 3-Manifolds $L(a,b,c)$]**. Let $f(z_0,z_1,z_2) = z_0^a + z_1^b + z_2^c$ and we denote $L_f = L(a,b,c)$ and $\mathcal{Z}_f = X(a,b,c)$. Note that the relevant sign in Proposition 2.3 is that of

$$abc - ab - bc - ca = abc(1 - 1/a - 1/b - 1/c).$$

Milnor [1975] studied the links $L(a,b,c)$ showing that their geometry indeed depends on this sign. He called the three types of links **hyperbolic** (negative Sasakian) **Euclidean** (null Sasakian), and **spherical** (positive Sasakian). Milnor shows that these 3 cases correspond to 3 Thurston geometries

1. $\widetilde{SL}(2,\mathbb{R})/\Gamma$,
2. Nil^3/Γ.
3. S^3/Γ.

Of course, there are only few spherical cases: $L(2,2,k) = S^3/\mathbb{Z}_k$, $L(2,3,5) = S^3/I^*$, $L(2,3,4) = S^3/T^*$, $L(2,3,3) = S^3/O^*$ and even fewer Euclidean links. What is important is that $\pi_1(L(a,b,c))$ is finite only for spherical links. In particular, Euclidean and hyperbolic links

cannot have a metric of positive Ricci curvature. Only spherical links obviously have such a metric. Yet, it is easy to check that when the exponents are pairwise relatively prime the transverse space $X(a,b,c) \simeq \mathbb{P}^1_{\mathbb{C}}$. Hence, there are infinitely many hyperbolic links such that their transverse geometry is a $\mathbb{P}^1_{\mathbb{C}}$ with orbifold singularities. Clearly, looking at $X(a,b,c)$ as a complex curve does not tell the whole story we want. But our notion of orbifold first Chern class consistently explains Sasakian geometry of these links.

•Milnor Fibration Theorem

There is a fibration of $(S^{2n+1} - L_f) \longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n \vee \cdots \vee S^n$. The **Milnor number** μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $\mathbf{w}$ by the formula $\mu = \mu(L_f) = \prod_{i=0}^{n}(\frac{d}{w_i} - 1)$. The closure $\bar{F}$ of F has the same homotopy type as F and is a compact manifold with boundary precisely the link L_f. So the reduced homology of F and $\bar{F}$ is only non-zero in dimension n and $H_n(F, \mathbb{Z}) \approx \mathbb{Z}^{\mu}$.

Using the Wang sequence of the Milnor fibration together with Alexander-Poincare duality gives the exact sequence

$$0 \to H_n(L_f, \mathbb{Z}) \to H_n(F, \mathbb{Z}) \xrightarrow{\mathbb{I}-h_*} H_n(F, \mathbb{Z}) \to H_{n-1}(L_f, \mathbb{Z}) \to 0$$

where h_* is the **monodromy** map (or characteristic map) induced by the $S^1_{\mathbf{w}}$ action. From this we see that $H_n(L_f, \mathbb{Z}) = \ker(\mathbb{I} - h_*)$ is a free Abelian group, and $H_{n-1}(L_f, \mathbb{Z}) = \mathrm{Coker}(\mathbb{I} - h_*)$ which in general has torsion, but whose free part equals $\ker(\mathbb{I} - h_*)$. So the topology of L_f is encoded in the monodromy map h_*. There is a well-known algorithm due to **Milnor and Orlik [1970]** for computing the free part of $H_{n-1}(L_f, \mathbb{Z})$ in terms of the characteristic polynomial $\Delta(t) = \det(t\mathbb{I} - h_*)$, namely the Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$.

There are certain cases when simple calculations yield a lot of information about homology of the link. Such is the case when the link is rational homology sphere.

Proposition 2.5: *The following hold:*

1. *L_f is a rational homology sphere iff $\Delta(1) \neq 0$.*
2. *L_f is a homology sphere iff $|\Delta(1)| = 1$.*
3. *If L_f i s a rational homology sphere, then the order of $H_{n-1}(L_f, \mathbb{Z})$ equals $|\Delta(1)|$.*

Remark 2.6 **Orlik [1978]** proposed an algorithmic way of computing torsion of any L_f. He made a conjecture that his algorithm always produces correct answer. The conjecture was later proved by **Randell [1980]** for special links (such as Brieskorn-Pham links discussed next). As far as I know this conjecture is still open.

Example 2.7:

- $\mathbf{w} = (1,1,1,1), \quad f = z_0 + z_1 + z_2 + z_3, \quad d = 1.$
$$\mathcal{Z}_f = \mathbb{CP}(2), \qquad L_f = S^5.$$
- $\mathbf{w} = (1,1,1,1), \quad f = z_0^2 + z_1^2 + z_2^2 + z_3^2, \quad d = 2.$
$$\mathcal{Z}_f = \mathbb{CP}(1) \times \mathbb{CP}(1), \qquad L_f = S^2 \times S^3.$$
- $\mathbf{w} = (1,1,1,1), \quad f = z_0^3 + z_1^3 + z_2^3 + z_3^3, \quad d = 3.$
$$\mathcal{Z}_f = \mathbb{CP}(2)\#6\overline{\mathbb{CP}(2)} \quad , \qquad L_f = 6\#(S^2 \times S^3).$$
- $\mathbf{w} = (1,1,1,2), \quad f = z_0^4 + z_1^4 + z_2^4 + z_3^2, \quad d = 4.$
$$\mathcal{Z}_f = \mathbb{CP}(2)\#7\overline{\mathbb{CP}(2)} \quad , \qquad L_f = 7\#(S^2 \times S^3).$$
- $\mathbf{w} = (1,1,2,3), \quad f = z_0^6 + z_1^6 + z_2^3 + z_3^2, \quad d = 6.$
$$\mathcal{Z}_f = \mathbb{CP}(2)\#8\overline{\mathbb{CP}(2)} \quad , \qquad L_f = 8\#(S^2 \times S^3).$$

- $\mathbf{w} = (1,1,1,k)$, $f = z_0^{k+1} + z_1^{k+1} + z_2^{k+1} + z_0 z_3$, $d = k+1$.

$$L_f = k\#(S^2 \times S^3).$$

- $\mathbf{w} = (1,2,3,5)$, $f = z_0^{10} + z_1^5 + z_2^3 z_1 + z_3^2$, $d = 10$

$$L_f = 9\#(S^2 \times S^3).$$

- $\mathbf{w} = (11,29,39,49)$, $f = z_0^8 z_2 + z_1^4 z_0 + z_2^3 + z_3^2 z_1$, $d = 127$

$$L_f = 2\#(S^2 \times S^3).$$

- $\mathbf{w} = (13,35,81,128)$, $f = z_0^{17} z_1 + z_1^5 z_2 + z_2^3 z_0 + z_3^2$, $d = 256$

$$L_f = S^2 \times S^3.$$

- $\mathbf{w} = (17,34,75,125,175)$, $f =$ homework, $d = 425$

$$|H_3(L_f, \mathbb{Z})| = 17^{12}.$$

- $\mathbf{w} = (127,2266,3651,6043,8435)$, $f =$ homework, $d = 20521$

$$|H_3(L_f, \mathbb{Z})| = 20521.$$

- $\mathbf{w} = (127,2392,3399,6043,8561)$, $f =$ homework, $d = 20521$

$$|H_3(L_f, \mathbb{Z})| = 20521.$$

• Brieskorn-Pham Links

In his famous work, in **1966 Brieskorn** considered special links often called **Brieskorn-Pham links** (or BP for short) $L(\mathbf{a})$ defined by

$$\sum_{i=0}^{n} |z_i^2| = 1, \qquad f_{\mathbf{a}}(\mathbf{z}) = z_0^{a_0} + \cdots + z_n^{a_n} = 0.$$

To the vector $\mathbf{a} = (a_0, \cdots, a_n) \in \mathbb{Z}_+^{n+1}$ one associates a graph $G(\mathbf{a})$ whose $n+1$ vertices are labeled by $a_0, \cdots, a_n$. Two vertices a_i and a_j are connected iff $\gcd(a_i, a_j) > 1$. Let C_{ev} denote the connected component of $G(\mathbf{a})$ determined by the even integers. Note that all even vertices belong to C_{ev}, but C_{ev} may contain odd vertices as well. Then we have the so-called **Brieskorn Graph Theorem**

Theorem 2.8: *The link $L(\mathbf{a})$ (with $n \geq 3$) is homeomorphic to the (2n-1)-sphere if and only if either of the following conditions hold*

1. *$G(\mathbf{a})$ contains at least two isolated points,*
2. *$G(\mathbf{a})$ contains one isolated point and C_{ev} has an odd number of vertices and for any distinct $a_i, a_j \in C_{ev}$, $\gcd(a_i, a_j) = 2$.*

•Homotopy Groups of Spheres

Recall that by seminal work of **Milnor, Kervaire and Milnor, and Smale**, for each $k \geq 5$, **differentiable homotopy spheres of dimension k form an Abelian group** Θ_k, where the group operation is **connected sum**. Θ_k has a subgroup bP_{k+1} consisting of those homotopy k-spheres which bound parallelizable manifolds V_{k+1}. Kervaire and Milnor proved that $bP_{2m+1} = 0$ for $m \geq 1$, $bP_{4m+2} = 0$, or $\mathbb{Z}_2$ and is $\mathbb{Z}_2$ if $4m + 2 \neq 2^i - 2$ for any $i \geq 3$. The most interesting groups are bP_{4m} for $m \geq 2$. These are cyclic of order

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1} - 1) \text{ numerator } \left(\frac{4B_m}{m}\right),$$

where B_m is the m-th Bernoulli number. Thus, for example $|bP_8| = 28$, $|bP_{12}| = 992$, $|bP_{16}| = 8128$ and $|bP_{20}| = 130,816$. In the first two cases these include all exotic spheres. The correspondence is given by

$$KM : \Sigma \mapsto \frac{1}{8}\tau(V_{4m}(\Sigma)) \mathrm{mod} |bP_{4m}|,$$

where $V_{4m}(\Sigma)$ is any parallelizable manifold bounding Σ and τ is its **signature**.

In the case of BP links, by the Milnor Fibration Theorem we can take V to be the Milnor fiber which is diffeomorphic to $V_{\mathbf{a}}^{2n} = \{\mathbf{z} \in \mathbb{C}^{n+1} \mid f_{\mathbf{a}}(z_0, \cdots, z_n) = 1\}$. Brieskorn shows that the signature of $V_{\mathbf{a}}^{2n}$ can be written combinatorially as

$$\#\{\mathbf{x} \in \mathbb{Z}^{2m+1} \mid 0 < x_i < a_i \text{ and } 0 < \sum_{j=0}^{2m} \frac{x_i}{a_i} < 1 \mathrm{mod} 2\}$$

$$-\#\{\mathbf{x} \in \mathbb{Z}^{2m+1} \mid 0 < x_i < a_i \text{ and } 1 < \sum_{j=0}^{2m} \frac{x_i}{a_i} < 2 \mathrm{mod} 2\},$$

where $n = 2m$.

Example 2.9: Let us consider the Brieskorn-Pham link $L(6k-1, 3, 2, 2, 2)$. By Brieskorn Graph Theorem this is a homotopy 7-sphere. One can easily compute the signature using the above formula to find out that $\tau(L(6k-1, 3, 2, 2, 2)) = 8k$. Hence $L(5, 3, 2, 2, 2) = \Sigma_1^7$ is an exotic 7-sphere and it is called Milnor generator (all others can be obtained from it by taking connected sums).

Remark 2.10: Note that the links $L(6k-1, 3, 2, 2, 2)$ are all positive Sasakian manifolds. Similarly, all links in Example 2.7 are positive, in particular $\#_k(S^2 \times S^3)$ has a positive Sasakian structure for all $k \geq 0$. Hence, they all admit Sasakian metric of positive Ricci curvature. In addition, we can consider the links $L(6k-1,3,2,\ldots,2)$ (of dimension $3\bmod(4)$) and $L(p,2,2,\ldots,2)$ (of dimension $1\bmod(4)$).

Theorem 2.11: *All homotopy spheres that are boundaries of parallalizable manifolds admit Sasakian metric of positive Ricci curvature.*

Theorem 2.12: *For any $k\geq 0$, $\#_k(S^2\times S^3)$ admits a Sasakian metric of positive Ricci curvature.*

Theorem 2.11 was first established using surgery theory techniques by **Wraith [1998]**. Our proof which uses techniques described in these lectures **[Boyer, Galicki, Nakamaye'2003]** appear later in Topology.

Theorem 2.12 is a special case of a well-known result of **Sha and Yang [1991]** who showed that positive Ricci curvature metrics can exist on manifolds with **arbitrarily large second Betti number**. By **Gromov's [1980]** famous theorem that cannot happen for **metrics with positive sectional curvature**.

Question 2.13: **What about Einstein condition?** *Suppose $\mathcal{Z}_f$ is Fano. Could one perhaps prove existence* **(continuity method?)** *of a KE metric on $\mathcal{Z}_f$? Every time this can successfully be done we automatically get an SE metric on the link L_f.*

• KE Metrics on Fano Orbifolds

Let $(\mathcal{Z}, J, g, \omega_g)$ be a compact Käher manifold (orbifold). We would like to know if one can always find a Kähler-Einstein metric h such that $[\omega_h] = [\omega_g]$. Recall that on a Kähler-Einstein manifold

$$\rho_g = \lambda\omega_g,$$

where the metric can be normalized so that $\lambda = \pm 1$ or 0. Since $2\pi c_1(\mathcal{Z}) = \lambda[\omega_g]$ we have
• When $c_1(\mathcal{Z}) > 0$ we must have $\lambda = +1$.
• When $c_1(\mathcal{Z}) < 0$ we must have $\lambda = -1$.
• When $c_1(\mathcal{Z}) = 0$ we must have $\lambda = 0$.
Suppose there exists an Einstein metric h with Kähler from in the same cohomology class. We have globally defined functions $\phi, f \in C^\infty(\mathcal{Z}, \mathbb{R})$ such that

$$\rho_g - \lambda\omega_g = i\partial\bar{\partial} f, \qquad \omega_h - \omega_g = i\partial\bar{\partial}\phi.$$

After fixing appropriate normalization we get, in local coordinates, the following **Monge-Ampère equation**

$$\frac{\det(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j})}{\det(g_{i\bar{j}})} = e^{f-\lambda\phi}.$$

The case of $\lambda = -1$ is actually the simplest and was solved by **Aubin** and independently by **Yau**. The case

$\lambda = 0$ follows from the Yau's solution to the original Calabi Problem. We are interested in $\lambda = +1$ case.

One tries to solve the Monge-Ampère equation

$$\frac{\det(g_{i\bar{j}} + \partial_i \bar{\partial}_{\bar{j}} \phi_t)}{\det(g_{i\bar{j}})} = e^{-t\phi_t + tf + C_t}, \quad g_{i\bar{j}} + \partial_i \bar{\partial}_{\bar{j}} \phi_t > 0$$

We start with ϕ_0 and $C_0 = 0$ and we try to reach $t = 1$, where the metric will be Kähler-Einstein. The so called **continuity method** sets out to show that the interval where solutions exist is both open and closed. **Openness follows from the Implicit Function Theorem** but there are well-known **obstructions to closedness**. It turns out that the critical step is a 0th order estimate. That is, as the values of t for which the Monge-Ampère equation is solvable approach a critical value $t_0 \in [0, 1]$, a subsequence of the ϕ_t converges to a function ϕ_{t_0} which is the sum of a C^∞ and of a plurisubharmonic function. **Tian** showed that we only need to prove

$$\int_{\mathcal{Z}} e^{-\gamma t \phi_{t_0}} \omega_0^n < +\infty$$

for some $\gamma \in (\frac{n}{n+1}, 1)$, where ω_0 is the Kähler form of $g_0 = g$.

The existence problem for positive KE metrics has been studied by many people: **Yau, Tian, Siu, Nadel**, and most recently by **Demailly and Kollár** who work in the orbifold category.

Theorem 2.14 **[Boyer,Galicki,Kollár' 2003]** *Let* $\mathcal{Z}(\mathbf{a})$ *be the transverse space of the BP link* $L(\mathbf{a})$*. Let* $C_i = \mathrm{lcm}(a_0, \ldots, \hat{a}_i, \ldots, a_n)$*,* $b_i = \gcd(C_i, a_i)$*. Then* $\mathcal{Z}(\mathbf{a})$ *is Fano and it has a Kähler-Einstein metric if*

1. $1 < \sum_{i=0}^{n} \frac{1}{a_i}$,
2. $\sum_{i=0}^{n} \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_i\{\frac{1}{a_i}\}$, *and*
3. $\sum_{i=0}^{n} \frac{1}{a_i} < 1 + \frac{n-1}{n-1} \min_{i,j}\{\frac{1}{b_i b_j}\}$.

In this case the link $L(\mathbf{a})$ *admits a SE metric with one-dimensional isometry group.*

Remark 2.15 More generally, we consider weighted homogeneous perturbations

$$Y(\mathbf{a}, p) := (\sum_{i=0}^{n} z_i^{a_i} + p(z_0, \ldots, z_n) = 0) \subset \mathbb{C}^{n+1},$$

where weighted $\mathrm{degree}(p) = d = \mathrm{lcm}(a_0, \ldots, a_n)$. The genericity condition we need, which is always satisfied by $p \equiv 0$ is: **The intersections of $Y(\mathbf{a}, p)$ with any number of hyperplanes $(z_i = 0)$ are all smooth outside the origin.** Theorem 2.14 holds true for any $Y(\mathbf{a}, p)/\mathbb{C}^*_{\mathbf{w}}$ and $L(\mathbf{a}, p)$.

•Einstein Metric on Spheres

We would like to demonstrate the power of Theorem 2.14 in the case $L(\mathbf{a}, p)$ is restricted to be a homotopy sphere. First, **little history**:

• **[Gauss, Riemann]** Any standard sphere S^n, $n > 1$, admits a metric of constant positive sectional curvature. These canonical metrics are $SO(n+1)$-homogeneous and Einstein, i.e., the Ricci curvature tensor is a constant positive multiple of the metric.

•**[1974]** The spheres S^{4m+3}, $m > 1$ are known to have another $Sp(m+1)$-homogeneous Einstein metric discovered by **Jensen**. The metric is obtained from the "quaternionic Hopf fibration" $S^3 \to S^{4m+3} \to \mathbb{HP}^m$. Since both base and fiber are Einstein spaces with positive Einstein constant we obtain two Einstein metrics in the canonical variation. The second metric is also called "squashed sphere" metric in some physics literature.

•**[1978]** In addition, S^{15} has a third $\mathrm{Spin}(9)$-invariant homogeneous Einstein metric discovered by **Bourguignon and Karcher** in 1978. The existence of such metric has to do with the fact that S^{15}, in addition to fibering over $\mathbb{HP}^2$, also fibers over S^8 with fiber S^7. Thus 15-sphere admits 3 different homogeneous Einstein metrics.

•**[1982] Ziller** proved that these are the only homogeneous Einstein metrics on spheres.

•**[1998] Böhm** obtained infinite sequences of non-isometric Einstein metrics, of positive scalar curvature, on S^5, S^6, S^7, S^8, and S^9. Böhm's metrics are of cohomogeneity one and they are not only the first inhomogeneous Einstein metrics on spheres but also the first non-canonical Einstein metrics on even-dimensional spheres.

•**[2003] Boyer, Galicki, Kollár** *Einstein metric on spheres*, **Annals of Mathematics**; to appear.

Theorem 2.14A: *On S^5 there are at least* **68** *inequivalent families of SE metrics. Some of these families admit non-trivial continuous SE deformations. The biggest constructed family has* **real dimension 10**.

Remarks:

• Our metrics differ from Böhm's. His have rather large isometry group while all ours (and this comment applies to all dimensions) are, by construction, of "maximal" cohomogeneity, i.e., they have **one-dimensional isometry group**.

• Böhm gets infinitely many non-isometric Einstein metrics on S^5 but he does not get any **continuous families** as is typically the case with our metrics.

• We note that the numbers **"68"** or **"10"** in Theorem 2.14A have no special significance: more families and higher-dimensional moduli can most likely be obtained using the same basic techniques. Nonetheless, in each odd dimension, they are to illustrate the abundance of metrics we get: both the number of families as well as the dimension of the moduli of the largest family increases **doubly exponentially** in each next odd dimension.

Theorem 2.14B: *Let Σ^7_i, be a homotopy 7-sphere corresponding to the element $i \in bP_8 \simeq \mathbb{Z}_{28} \simeq \Theta_7$ in the Kervaire-Milnor group. Σ^7_i admits at least n_i inequivalent families of SE metrics, where $(n_1, \ldots, n_{28})$ =(376, 336, 260, 294, 231, 284, 322, 402, 317, 309, 252, 304, 258, 390,409, 352, 226, 260, 243, 309, 292, 452, 307, 298, 230, 307, 264, 353), giving a total of* **8610** *cases. In each oriented diffeomorphism class some of the families depend on a moduli. In particular, the standard 7-sphere Σ^7_{28} admits an* **82-dimensional** *family of inequivalent SE metrics.*

Theorem 2.14C: *In dimensions $(4n + 1)$ both the standard and the so-called Kervaire spheres admit many families of inequivalent SE metrics for each $n \geq 2$.*

Remarks:

• In dimension 9, for example, **the Kervaire sphere is exotic**, i.e., $bP_{10} = \mathbb{Z}_2$. A computer search yields more than $3 \cdot 10^6$ inequivalent families of SE metrics on both spheres.

• In dimension 13 the Kervaire sphere is diffeomorphic to the standard sphere as $bP_{14} = 0$. In sharp contrast with the fact that **the only Einstein metric known on** S^{13} **is the standard one** we find

Theorem 2.14D: *The standard sphere S^{13} admits over 10^9* **distinct families** *of SE metrics one of them depending on* **21,300,113,901,610 parameters***.*

Conjecture: *In any odd dimension, all homotopy spheres which bound parallelizable manifolds admit families of SE metrics.*

Theorem 2.14E [BGKT'2003]: *The conjecture is true in dimensions 11 and 15. More precisely, each homotopy sphere (992 possible oriented diffeomorphism types) in dimension 11 admits at least one SE metric and each homotopy 15-sphere which bounds a parallelizable manifold (8128 possible oriented diffeomorphism types) admits at least one SE metric.*

Remark:

• Theorem 2.14E required some considerable computing time. For example, using the same machines and the same codes would require about **3000 years** to extend the result to the case of 19-spheres (bP_{20} has 130,816 elements).

LECTURE 3

Sasakian Geometry of Barden Manifolds

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•Theorems of Smale and Barden

In **1965 Barden** proved the following remarkable theorem, extending an earlier result of **Smale [1962]**

Theorem 3.1: *The class of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism. Furthermore, any such M is diffeomorphic to one of the spaces*

$$M_{j;k_1,\ldots,k_s} = X_j \# M_{k_1} \# \cdots \# M_{k_s},$$

where $-1 \leq j \leq \infty$, $s \geq 0$, $1 < k_1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$. A complete set of invariants is provided by $H_2(M, \mathbb{Z})$ and an additional diffeomorphism invariant $i(M) = j$ which depends only on the second Stiefel-Whitney class $w^2(M)$.

In these lectures we will refer to a simply connected, closed, oriented, smooth, 5-manifold as a **Barden manifold**.

BUILDING BLOCKS

- $X_{-1} = SU(3)/SO(3)$; $\quad H_2(X_{-1}, \mathbb{Z}) = \mathbb{Z}_2$,
- X_∞ =non-trivial S^3 bundle over S^2; $\quad H_2(X_\infty, \mathbb{Z}) = \mathbb{Z}$,
- $X_j, \quad j \in \mathbb{N}, \quad H_2(X_j, \mathbb{Z}) = \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$
- $X_0 = S^5$,
- $M_\infty = S^2 \times S^3 \qquad H_2(M_\infty, \mathbb{Z}) = \mathbb{Z}$,
- $M_\alpha, \quad \alpha \in \mathbb{N}; \quad H_2(M_\alpha, \mathbb{Z}) = \mathbb{Z}_\alpha \oplus \mathbb{Z}_\alpha$

When M is spin $i(M) = j = 0$ as is $w^2(M) = 0$ and Barden's result is the extension of the well-known theorem of **Smale [1962]** for spin 5-manifolds.

The following theorem follows from an old result of **J. Gray [1959]**:
Theorem 3.2 *A Barden manifold M admits an almost contact structure (reduction of the structure group to $U(2) \times \mathbf{1}$) if and only if $j = 0$ or $j = \infty$.*

In **1991 Geiges** showed that there are no other obstructions to contactness. That is every almost contact Barden manifold admits a contact structure. Geiges also shows that

Theorem 3.3 *A Barden manifold* M *admits a* **regular contact structure** *if and only if* $H_2(M,\mathbb{Z})$ *has no torsion.*

Hence, the only Barden manifolds with regular contact structure are either $\#kM_\infty$ (spin) or $X_\infty\#(k-1)M_\infty$ (non-spin). Both can be constructed explicitly as circle bundles over $\mathcal{Z} = \mathbb{P}^2_{\mathbb{C}}\#k\overline{\mathbb{P}^2_{\mathbb{C}}}$.

•Some Questions and Problems

Question 3.4 *Let* M *Barden manifold with* $j = 0,\infty$. *When does* M *admit a* **quasi-regular contact structure**? *Can* $H_2(M,\mathbb{Z})$ *obstruct existence of such a structure?*

We saw that most 3-manifolds, even though being contact, do not support quasi-regular contact structures. The ones that do are Seifert fibered and they also admit Sasakian structures. The above question can be rephrased: **Which Barden manifolds are Seifert fibrations over compact symplectic orbifolds?**

Questions 3.5 *Let* M *be a Barden manifold with* $j = 0,\infty$. *When does* M *admit a Sasakian structure? Can one describe all Sasakian structures on* **a given (on any?)** M?

Question 3.6 *Let M Barden manifold. When does M admit a* **metric of positive Ricci curvature***?*

The expected answer is **ALL** but remarkably little is known about the subject. One positive result was mentioned earlier is due to **Sha** and **Yang [1991]** from which it follows that such metrics exist on any $\#_k M_\infty$. In addition, both X_{-1} and X_∞ carry such metrics (even Einstein metrics). The relevant question in the context of Sasakian geometry is

Questions 3.7 *Let M be a Barden manifold with $j = 0, \infty$. When does M admit a positive Sasakian structure? More generally, given* **some (any)** *M, can one describe all positive Sasakian structures on M?*

Full answer to the above question would amount to a complete classification of **cyclic log del Pezzo surfaces**. Unlike smooth log del Pezzo surfaces the latter occur with remarkable abundance and are very poorly understood. However, we will see that a complete answer can be given in some special cases.

Question 3.8 *Let M be a Barden manifold. When does M admit an* **Einstein metric (SE)***?*

There are several constructions that lead to such Einstein metrics in dimension 5. We shall briefly review all of them:

• [**Regular Sasakian-Einstein Manifolds**] These are circle bundle over del Pezzo surfaces with KE metrics and therefore we have a complete classification via

Theorem 3.9: [**Tian'1990**] *The following del Pezzo surfaces admit Kähler-Einstein metrics: $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}, \ \ 3 \leq n \leq 8$. Furthermore, the moduli space of KE structures in each case is completely understood.*

Theorem 3.10: [**Friedrich-Kath'1990**] *Let $\mathcal{S}_l = S^5 \# l M_\infty$.*

1. *For each $l = 0, 1, 3, 4$, there is precisely one regular SE structure on $\mathcal{S}_l$.*
2. *For each $5 \leq l \leq 8$ there is a $2(l-4)$ complex parameter family of inequivalent regular SE structures on $\mathcal{S}_l$.*
3. *For $l = 2$ or $l \geq 9$ there are no regular SE structures on $\mathcal{S}_l$.*

• [**Homogeneous Einstein Metrics**] The symmetric metrics on $X_0 = S^5$, $X_{-1} = SU(3)/SO(3)$ are Einstein. Furthermore, $M_\infty = S^2 \times S^3$ admits infinitely many homogeneous Einstein metrics discovered by **Wang and Ziller [1990]**.

• [**Böhm Metrics**] **Böhm** has explicitly constructed cohomogeneity one Einstein metrics on $X_0 = S^5$ and on $M_\infty = S^2 \times S^3$.

• [**Cohomogeneity One Bundle Metrics on M_∞ and X_∞**] Several groups of string theory physicists have recently obtained explicit examples of cohomogeneity one Einstein metrics on X_∞ and SE metrics on M_∞ (**Lu-Page-Pope, Hashimoto-Sakaguchi-Yasui, Gauntlett-Martelli-Sparks-Waldram, Gibbons-Hartnoll-Yasui [all in 2004]**) .

Question 3.11 *Let M be a Barden manifold. How big is the moduli of the* **Einstein (SE)** *metrics on M? Can the moduli space be infinite dimensional?*

Problem/Question 3.12 *Classify all null Sasakian Barden manifolds. Can one describe all null Sasakian structure on a given M?*

The second question would involve understanding all **orbifold K3-surfaces with cyclic quotient singularities**.

$$
\begin{array}{l}
\{\textbf{Barden Manifolds}\} \\
\qquad\quad \cup \\
\{\textbf{Contact Manifolds}\} \\
\qquad\quad \cup \\
\{\textbf{Sasakian Manifolds}\} \\
\qquad\quad \cup \\
\{\textbf{Positive Sasakian Manifolds}\} \\
\qquad\quad \cup \\
\{\textbf{Sasakian – Einstein Manifolds}\}
\end{array}
$$

•Barden Manifolds as Links and Seifert Bundles

Definition [**Orlik-Weigreich'75, Kollár'04**] *Let $\mathcal{Z}$ be a normal complex space. A* **Seifert $\mathbb{C}^*$-bundle over** $\mathcal{Z}$ *is a normal complex space Y together with a morphism $f : Y \to \mathcal{Z}$ and a $\mathbb{C}^*$-action on Y such that*

1. *f is Stein (that is, the preimage of an open Stein subset of $\mathcal{Z}$ is Stein).*
2. *For every $x \in \mathcal{Z}$, the $\mathbb{C}^*$-action on the reduced fiber $Y_x := \mathrm{red} f^{-1}(x)$ is $\mathbb{C}^*$-equivariantly biholomorphic to the natural $\mathbb{C}^*$-action on $\mathbb{C}^*/\mathbb{Z}_m$, for some $m = m(x)$, where $\mathbb{Z}_m \subset \mathbb{C}^*$ denotes the mth roots of unity.*

The number $m(x)$ is called the multiplicity of the Seifert fiber over x. We assume that generically it is equal to 1.

Every Seifert $\mathbb{C}^*$-bundle over $\mathcal{Z}$ contains a real hypersurface $M \subset Y$ with an S^1-action. We call $f : M \to \mathcal{Z}$ the **Seifert S^1-bundle over** $\mathcal{Z}$.

•Vanishing Torsion Case

In this case we only have to worry about $\#kM_\infty$ (spin) or $X_\infty\#(k-1)M_\infty$ (non-spin). The latter are never SE so we leave this case out for the moment. We saw that $\#kM_\infty$ all occur as positive Sasakian links. Using various link models of Barden manifolds we were able to show that

Theorem 3.13 **[Boyer-Galicki-Nakamaye]** *Barden manifolds $\#kM_\infty$ admit families of quasi-regular (not regular) SE structure for $k = 0, \ldots, 9$.*

With Boyer we conjectured that SE metrics should exist on any $\#kM_\infty$. Using Seifert bundle approach in a recent paper Kollár has proved this conjecture showing:

Theorem 3.14 **[Kollar'04]** *Barden manifolds $\#kM_\infty$ admit families of $(2k-2)$-dimensional quasi-regular SE structure for each $k > 6$.*

The only remaining question now is that concerning the moduli. To be more precise, on may ask

Problem 3.15 *Describe the moduli of SE structures on any Barden manifolds $\#kM_\infty$? Start with S^5, $S^2 \times S^3$?*

•Pure Torsion Case (R.H.S.)

Using Theorem 2.14 and Orlik's algorithm we get the following list of rational homology 5-spheres which admit SE metrics

$L(\mathbf{a})$	Torsion
$L(3,3,3,k)$, $\gcd(k,3)=1, k>5$	$\mathbb{Z}_k \oplus \mathbb{Z}_k$
$L(2,4,4,k)$, $\gcd(k,2)=1$, $k>10$	$\mathbb{Z}_k \oplus \mathbb{Z}_k$
$L(2,3,6,k)$, $\gcd(k,6)=1$, $k>12$	$\mathbb{Z}_k \oplus \mathbb{Z}_k$

The three series above satisfy $\sum_{i=0}^{2} \frac{1}{a_i} = 0$. In the case when $\sum_{i=0}^{2} \frac{1}{a_i} < 0$ one can easily see that there are 16 more rational homology 5-spheres which satisfy inequalities of Theorem 2.14. An example of such a link is $L(3,4,4,4)$ whose 2-torsion equals $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Hence, $L(3,4,4,4)$ is diffeomorphic to $M_3 \# M_3 \# M_3$. In particular, we get the following

Theorem 3.16 **[Boyer-Galicki'04]** *The Barden manifold M_k admits SE structure for each $k>5$ prime to 3 and for each $k>10$ prime to 2.*

Using Seifert bundle approach Kollár was able to show that very few Barden manifolds admit positive Sasakian structures. Actually **2-torsion group obstructs both Sasakian and positive Sasakian geometry**.

Theorem 3.17 **[Kollár'04]** *Let M be a positive Barden manifold. Then the torsion group of $H_2(M,\mathbb{Z})$ is one of the following:*

1. $(\mathbb{Z}_m)^2$ *for any* $m \in \mathbb{Z}_+$,
2. $(\mathbb{Z}_n)^4, \quad n = 3, 4, 5,$
3. $(\mathbb{Z}_3)^6,$
4. $(\mathbb{Z}_3)^8,$
5. $(\mathbb{Z}_2)^{2n}$, *for any* $m \in \mathbb{Z}_+$.

Theorem 3.18 [**Kollár'04**] *Let M be a Barden manifold which is a rational homology sphere. Suppose M admits a positive Sasakian structure and the torsion group of $H_2(M, \mathbb{Z})$ contains an element of order at least 12. Then*

1. $M = M_m$ *for some m not divisible by 30.*
2. *The number of Seifert bundle structures (characteristic foliations) on M varies between 1 and 4 depending on n modulo 30.*
3. *Each Seifert bundle structure gives raise to a 2-dimensional family of SE metrics parameterized by the moduli space of genus 1 curves.*

Theorem 3.18 is a special case of a more general statement. It turns out that most contact of rational homology 5-spheres do not admit any Sasakian structure. For example,

Theorem 3.19 [**Kollár'04**] *Let M be a Barden manifold which is a rational homology sphere. If M admits a Sasakian structure, and the order of 2-torsion is p^{2k} for p prime and k odd then*

$$M = \#kM_p.$$

Hence, for instance, for any $p > 1$ the R.H.S. $M_p \# M_{p^2}$ admits no Sasakian structure. It is possible that the rational homology 5-spheres that do not admit Sasakian structure are simply **examples of contact 5-manifolds with no quasi-regular Reeb vector field.**

•Mixed Case

Note that the links in the previous table have companions with non-trivial second Betti number and by Theorem 2.14 they too admit Sasakian-Einstein metrics. We list the relevant information in the table below:

$L(\mathbf{a})$	$b_2(L(\mathbf{a}))$
$L(3,3,3,3n)$, $n > 2$	6
$L(2,4,4,2n)$, $\gcd(n,2) = 1$, $n > 5$	3
$L(2,4,4,4n)$, $n > 2$	7
$L(2,3,6,2n)$, $\gcd(n,3) = 1$, $n > 12$	2
$L(2,3,6,3n)$, $\gcd(n,2) = 1$, $n > 12$	4
$L(2,3,6,6n)$, $n > 4$	8

Theorem 3.20 [Boyer-Galicki'04] *The Barden manifold $M_n \# k M_\infty$ admits SE structure for each $k = 2,3,4,6,7,8$ and each $n > 12$.*

Again, with his Seifert bundle approach was able to prove a much stronger result:

Theorem 3.21 [**Kollár'04**] *Let M be a Barden manifold. Suppose M admits a positive Sasakian structure and $H_2(M,\mathbb{Z}) = \mathbb{Z}^n \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ with $m \geq 12$.* Then

1. $n \leq 8$.
2. There are 93 cases for each $m \geq 12$.
3. Each Seifert bundle structure yields at least a 2-dimensional family of SE metrics.

Theorem 3.22 [**Kollár'04**] *Let M be a Barden manifold. Suppose M admits a positive Sasakian structure and $H_2(M,\mathbb{Z}) = \mathbb{Z}^n \oplus (\mathbb{Z}_5)^4$. Then*

1. *$M = M_5 \# M_5$.*
2. *The Seifert bundle structure (characteristic foliation) is unique.*
3. *M admits a 4-dimensional family of SE metrics naturally parameterized by the moduli of genus 2 curves.*

Theorem 3.23 [**Kollár'04**] *Let M be a Barden manifold. Suppose M admits a null Sasakian structure. Then $M = \# k M_\infty$ with $1 < k \leq 21$.*

Most of the cases already occur on the famous list of **Reid** of the weighted homogeneous hypersurfaces $X_{f,d} \subset$

$\mathbb{PC}(w_0, w_1, w_2, w_3)$. However, $k = 2, 17$ are missing. It is probably too hard to classify all Seifert bundle structures. But one might be able to obtain complete description for some cases.