Nuclear Physics B
Volume 289 (1987), 573-588.

New Matter Couplings in N=2 Supergravity

Krzysztof Galicki

Abstract: We explicitly construct a whole new class of nonlinear matter couplings in N=2 supergravity theory in four dimensions. These are the first examples of couplings described by non-symmetric and non-homogeneous σ-model manifolds that include the recently found Pedersen self-dual metric with constant negative scalar curvature. We give 4n-dimensional and multi-center generalizations of this σ-model coupling and discuss their relation to the hyperkähler geometry of Taub-NUT and multi-Taub-NUT metrics.

1. Introduction

It is well known that local $N = 2$ supersymmetry in $d = 4$ dimensions allows for scalar couplings under certain restrictions on the geometry of the underlying σ-model manifold. The scalar fields must parametrize a quaternionic Kähler $Sp(n)\cdot Sp(1)$ holonomy) manifold of the constant negative scalar curvature [1]. Witten and Bagger explicitly constructed the most general form of such a coupling in terms of the corresponding metric, its Levi-Civita connection and its Riemann curvature tensor. They also gave a list of all examples of quaternionic Kähler manifolds known in the mathematical literature. In particular, all of them are homogeneous spaces. Most known examples, except for those found by Alekseevskii [2], are also locally symmetric. Until recently, no other examples were constructed.

The first two nonlinear matter couplings in $N = 2$ supergravity theory were derived by Breitenlohner and Sohnius [3]. These are σ-models with scalar matter fields parametrizing non-compact symmetric homogeneous spaces: the quaternionic projective ball $\mathbb{HH}^n = Sp(n,1)/Sp(n)\times Sp(1)$ and the Wolf space $X^n = U(n,2)/U(n)\times U(2)$. Later, de Wit et al. gave a very general form of $N = 2$ supergravity coupled to an arbitrary number of Yang-Mills and scalar multiplets [4]. They pointed out that both of the above couplings can be also obtained in their formalism. But, as we noticed in [5], it seems that Witten and Bagger's lagrangian is more general, since it gives the form of all interactions in terms of geometrical properties of an arbitrary quaternionic Kähler manifold with negative scalar curvature. De Wit's lagrangian can describe only those scalar couplings which correspond to σ-model quaternionic Kähler manifolds of a specific kind: Namely, those that

During the preparation of this work both authors was supported by NSF grant PHY-8507627.

can be obtained as quaternionic quotients by quaternionic isometries of quaternionic projective spaces $\mathbb{HP}^n = Sp(n+1)/Sp(n) \times Sp(1)$ and their non-compact or semi-Riemannian analogues (such as for instance $\mathbb{HH}^n$). It is true that all the Wolf spaces are indeed quaternionic quotients of quaternionic projective space. For example, in our previous work (see [6]) we have shown how to obtain $Y^n = O(n+4)/O(n) \times O(4)$ as an $SU(2)$ quotient of $\mathbb{HP}^n$. (The generalization to the non-compact case of $O(n+4)/O(n) \times O(4)$ σ-model is straightforward.) But it is unlikely that all manifolds with $Sp(n) \cdot Sp(1)$ holonomy are quaternionic quotients of quaternionic projective or hyperbolic spaces. In particular, we do not know if such is the case with the five examples of quaternionic Kähler manifolds which are cosets of exceptional Lie groups. However, the formalism of [4], although not the most general in that sense, allows for a search of new matter couplings and the method is geometrically equivalent to our quaternionic quotient when applied to quaternionic projective or hyperbolic spaces. It is also of particular interest for mathematicians since very few examples of non-symmetric manifolds of this kind are known.

In this paper we want to present new matter couplings. Some of them correspond to quaternionic Kähler manifolds that (to our knowledge) were not known in mathematical literature before. All of the examples discussed are non-symmetric quaternionic Kähler manifolds of constant negative scalar curvature, so that the kinetic term for the scalar fields in the lagrangian has the correct sign (no ghosts). In dimension 4 (the real dimension of the σ-model manifold), one of our models involves the recently constructed self-dual, $U(2)$-invariant Pedersen metric with negative cosmological constant [7]. We also present its $4n$-dimensional generalization with isometry group $U(n-1) \times Sp(1)$, constructed by Hitchin [8]. They are in some sense quaternionic analogues of $4n$-dimensional hyperkähler Taub-NUT metrics which, as we show, they give in the limit ofscalar curvature going to zero. Next, we derive new σ-model couplings with an arbitrary number of parameters that are quaternionic multi-Taub-NUT generalizations of the Pedersen metric. Finally, we discuss some other new interesting examples.

As in our previous paper [5], where we were considering σ-models couplings on compact orbifolds, we shall use de Wit's formalism throughout. We find it very useful in presenting our results. The method which we use, and which we have described in [6, 9] has a very elegant geometrical interpretation in terms of quaternionic quotients. This type of geometrical approach is particularly important as far as our understanding of global properties of a σ-model manifold is concerned. It is practically impossible to describe the global properties of the manifold from the form of the couplings in the lagrangian only. These give us the metric and, consequently, the curvatures only locally. Usually we do not know if this metric is complete or non-singular. The invariance under local (or global) $N = 2$ supersymmetry transformations is also not enough to determine if a coupling given by some lagrangian corresponds to a smooth quaternionic Kähler (or hyperkähler) riemannian manifold or not. Thus, we also need some geometrical description of the method in which we derive our new metrics, a description which would provide a tool to investigate their global topological properties. Just as hyperkähler quotients in the case of $N = 2$ global supersymmetry [10], our quaternionic quotient description provides us with such a tool. In particular, we were able to show that new couplings presented in [5] correspond to quaternionic riemannian orbifolds with two disjoint singular sets. In general, the singularity structure can even be worse than in the orbifold case. Then, although we formally deal with an $N = 2$ locally supersymmetric action, it is not clear if such couplings always make sense. We would like to address this particular problem in a future work.

Our paper is organized as follows. In Section 2 we briefly review the notation of de Wit et al. [4] and some results of [5]. Using this formalism, in Section 3 we derive a whole class of new σ-model couplings. In Section 4 we investigate some properties of these new metrics. In particular we show that they are smooth and that some of them give the various multi-Taub-NUT hyperkähler metrics in the decoupling limit (the global supersymmetry limit). Finally, in Section 5 we summarize our results.

2. $N = 2$ supergravity-matter couplings

We only very briefly review the formalism of de Wit et al. [4] for $N = 2$ supergravity coupled to an arbitrary number of scalar and vector multiplets together with some of the results presented in our previous work [5]. We refer the interested reader to the original papers.

In the conformal tensor calculus one can couple the Weyl multiplet

$$\{\ e^a_\mu,\ \psi^i_\mu,\ b_\mu,\ A_\mu,\ V^i_{\mu j},\ T^{ij}_{ab},\ \chi^i,\ D\ \} \tag{2.1}$$

to $(m+1)$ Yang-Mills multiplets

$$\{\ X^I,\ \Omega^I_i,\ W^I_\mu,\ Y^I_{ij}\ \} \tag{2.2}$$

and $2r$ hypermultiplets

$$\{\ A^\alpha_i,\ \xi^\alpha\}, \tag{2.3}$$

where

$\alpha = 0, ..., 2r-1$	— "matter" representation index of some Lie group G
$i, j = 1, 2$	— $SU(2)$ index
$I, J = 0, ..., m$	— G-group index in the adjoint representation
$\mu; a, b$	— spacetime indices (curved and flat).

In the Weyl multiplet e^a_μ is the vierbein, ψ^i_μ is the gravitino $SU(2)$ doublet, b_μ is the gauge field for dilatations, A_μ and $V^i_{\mu j}$ are the gauge fields for chiral $U(1)$ and $SU(2)$ respectively, T^{ij}_{ab} is a real $SU(2)$ antisymmetric tensor, χ^i is a spinor doublet, and D is a real scalar. In the vector multiplet X^I is a complex scalar, Ω^I_i is a real spinor doublet and Y^I_{ij} is a real $SU(2)$ triplet. In the hypermultiplet, ξ^α is a spinor and A^α_i are $2r$ complex scalar fields subject to the following reality condition

$$\overline{A^\alpha_i} = \epsilon^{ij}\rho_{\alpha\beta}A^\beta_j \stackrel{\text{def}}{=} A^i_\alpha, \tag{2.4}$$

where ϵ^{ij} is $SU(2)$ invariant antisymmetric tensor and $\rho_{\alpha\beta}$ is skew symmetric matrix of the form

$$\rho = \begin{pmatrix} \mathbb{O} & \mathbb{I}_r \\ -\mathbb{I}_r & \mathbb{O} \end{pmatrix} \tag{2.5}$$

It follows from (2.4) that A^i_α are global coordinates on r-dimensional quaternionic vector space $\mathbb{H}^r$. We introduce a flat pseudo-riemannian metric on $\mathbb{H}^r$ with signature $(p/2, q/2)$ given by a diagonal matrix $\mathbf{d}$

$$\mathbf{d} = \begin{pmatrix} -\mathbb{I}_{p/2} & & & \\ & \mathbb{I}_{q/2} & & \\ & & -\mathbb{I}_{p/2} & \\ & & & \mathbb{I}_{q/2} \end{pmatrix} \tag{2.6}$$

where $p + q = 2r$. Then the scalar product

$$(A_i,\ A^i) \stackrel{\text{def}}{=} d_\alpha{}^\beta A^\alpha_i A^i_\beta \tag{2.7}$$

is $SU(2)$ and $Sp(p/2, q/2)$ invariant. Next,we require our action to be gauge invariant with respect to the following gauge transformations

$$\delta_\lambda A^\alpha_i = g\lambda^I T_I{}^\alpha{}_\beta A^\beta_i, \tag{2.8}$$

where $\mathbb{T}_I$ are antihermitian generators of some Lie group G. Gauge invariance of the constructed lagrangian (see (4.13) in [4]) restricts G to be some subgroup of $Sp(p/2, q/2)$. The authors of [4] start with an $N = 2$ superconformally and gauge invariant action with all three multiplets introduced above. Next, they show that the gauge fields for dilatation, chiral $SU(2) \times U(1)$ and S-supersymmetry decouple algebraically and thus the resulting action is that of $N = 2$ Poincarè supergravity coupled to matter.

The gauge fixing condition for dilatation and the algebraic equation for the D-field lead to the constraint

$$(A_i,\ A^i) = -2/\kappa^2, \tag{2.9}$$

where κ is Newton's constant. Taking into account the $SU(2)$ chiral gauge invariance, after solving the auxiliary field equation for $V^i_{\mu j}$, we see that the scalar fields A^α_i must parametrize a quaternionic homogeneous space

$$\mathbb{HP}^{(r-1)}(q/2, p/2) \stackrel{\text{def}}{=} Sp(q/2, p/2)/Sp(q/2, p/2 - 1) \times Sp(1). \tag{2.10}$$

From now on we shall be interested only in the case when $p = 2$. Then (2.10) is just a quaternionic hyperbolic space $\mathbb{HH}^{r-1} = Sp(r-1, 1)/Sp(r-1) \times Sp(1)$. It is a non-compact quaternionic Kähler manifold of constant negative scalar curvature and a definite metric. As pointed out in [4] and extensively discussed in or previous work [5], if we do not have non-propagating auxiliary gauge fields then the scalar fields are restricted to parametrize $\mathbb{HH}^{r-1}$. However, if we introduce auxiliary gauge degrees of freedom W^I_μ (without corresponding kinetic term in the action) then our σ-model manifold is just a quaternionic quotient of $\mathbb{HH}^{r-1}$ by some subgroup K of the isometry group $Sp(r-1,1)$. All these isometries preserve the quaternionic structure of $\mathbb{HH}^{r-1}$. As shown in [9] this is sufficient for the consistent quaternionic reduction.

The Y^I_{ij} fields equations, where I is now the index of the quotient group K of non-propagating gauge fields, yield the following constraints on $\mathbb{A} = (A^\alpha_i)$

$$\mu(\mathbb{A}) \stackrel{\text{def}}{=} d_\alpha{}^\beta A^k_\beta \epsilon_{ki} T_I{}^\alpha{}_\gamma A^\gamma_j = 0, \quad I = 1, ..., \text{dim} K \tag{2.11}$$

which we recognize as a quaternionic moment map for the quotient. The quotient manifold

$$M = \mathbb{HH}^{r-1} /\!/\!/ K \stackrel{\text{def}}{=} \{\mathbb{A} \subset \mathbb{HH}^{r-1} : \quad \mu(\mathbb{A}) = 0\}/K \tag{2.12}$$

is again a smooth quaternionic Kähler manifold provided that the global action of $K \subset Sp(r-1,1)$ is free and in the case of non-compact isometries also proper. (Otherwise M does not have the differentiable structure of a riemannian manifold.)

Let us write the bosonic part of the whole lagrangian. For simplicity, we omit the fermionic part as well as all other couplings except for the σ-model scalar fields to the determinant of the vierbein (the gravitational field). This is enough to describe our new couplings uniquely and all other couplings can be obtained without difficulty. Thus the relevant part of the action is

$$e^{-1}\mathcal{L}_\sigma = -d_\alpha{}^\beta \hat{\partial}_\mu A^\alpha_i \hat{\partial}^\mu A^i_\beta + \frac{1}{4\kappa^2} V^i_{\mu j} V^{\mu j}{}_i \tag{2.13}$$

where

$$V^i_{\mu j} = \kappa^2 d_\alpha{}^\beta A^i_\beta \overleftrightarrow{\partial}_\mu A^\alpha_j \tag{2.14}$$

and

$$\hat{\partial}_\mu A^\alpha_i \stackrel{\text{def}}{=} \partial_\mu A^\alpha_i - g W^I_\mu T_I{}^\alpha{}_\beta A^\beta_i . \tag{2.15}$$

The W^I_μ gauge fields are auxiliary and it can be implicitly written in terms of scalars A^α_i:

$$g W^I_\mu d_\alpha{}^\beta [T_{(I,\beta}{}^\sigma A^i_\sigma T_{J)}{}^\alpha{}_\tau A^\tau_i] = d_\alpha{}^\beta [A^i_\beta \overleftrightarrow{\partial}_\mu T_I{}^\alpha{}_\sigma A^\sigma_i]. \tag{2.16}$$

Then (2.13) simplifies after substituting (2.14-16)

$$e^{-1}\mathcal{L}_\sigma = -(\partial_\mu A_i,\, \partial^\mu A^i) + \frac{1}{4}\kappa^2 (A_i,\, \overleftrightarrow{\partial}_\mu A^j)(A_j,\, \overleftrightarrow{\partial}^\mu A^i) + g^2 W^I_\mu W^{\mu J} (\mathbb{T}_J A_i,\, \mathbb{T}_I A^i). \tag{2.17}$$

Here the generators $\mathbb{T}$ for a Lie algebra of the subgroup K only. Eq. (2.17) is a general σ-model coupling with scalar fields parametrizing a non-compact quaternionic Kähler manifold $M \stackrel{\text{def}}{=} \mathbb{HH}^{r-1}//K$ (assuming that the action of K on $\mathbb{HH}^{r-1}$ has required properties of being free and proper). Its quaternionic dimension is $(r-1-\text{dim}K)$. In the next section we shall be interested only in one parameter subgroups of the isometry group of $\mathbb{HH}^{r-1}$. Thus $\text{dim}K = 1$, $K \subset Sp(r-1,1)$, and $\text{dim}M = 4(r-2)$.

3. New couplings from the quaternionic reduction

We would like to investigate the following infinitesimal action of a one-parameter non-compact group of isometries of $\mathbb{HH}^{r-1}$ [8]

$$\delta_s(A)^\alpha_i = gsT^\alpha{}_\beta A^\beta_i, \tag{3.1}$$

where

$$\mathbb{T} = \left(\begin{array}{c|c} \begin{array}{cc|c} \begin{array}{cc} 0 & \lambda \\ \lambda & 0 \end{array} & \mathbb{O} \\ \hline \mathbb{O} & i\mathbb{I}_{r-2} \end{array} & \mathbb{O} \\ \hline \mathbb{O} & \begin{array}{c|c} \begin{array}{cc} 0 & \lambda \\ \lambda & 0 \end{array} & \mathbb{O} \\ \hline \mathbb{O} & -i\mathbb{I}_{r-2} \end{array} \end{array} \right). \tag{2.23}$$

Notice that $\mathbb{T}$ in (2.23) is antihermitian with respect to our metric. It is easy to see that $T^\alpha{}_\beta$ generates a non-compact group isomorphic to $\mathbb{R} \simeq SO(1,1) \subset Sp(r-1,1)$. Before we proceed with our investigation of the σ-model coupling which is given by this particular action we want to introduce a new notation. It will explicitly exhibit the quaternionic structure of the theory and at the same time it will dramatically simplify many calculations. First notice that because of the reality condition (2.4) one can write A^α_i as a $2 \times 2r$ matrix of the following form

$$\mathbb{A} = \begin{pmatrix} \Phi_- & \Phi_+ \\ -\Phi^*_+ & \Phi^*_- \end{pmatrix}, \tag{3.2}$$

where Φ_-, Φ_+ are r-dimensional complex vectors. Now, let us introduce an r-dimensional quaternionic vector

$$\mathbf{u} \stackrel{\text{def}}{=} \Phi_+ + j\Phi^*_-, \tag{3.3}$$

or

$$u^\alpha = A_2^\alpha + jA_2^{\alpha+r}, \qquad \alpha = 0, ..., r-1, \tag{3.4}$$

where j is another quaternionic unit. The metric (2.6) can now be written in the form

$$\mathbf{d} = \begin{pmatrix} -\mathbb{I}_{p/2} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_{q/2} \end{pmatrix} \tag{3.5}$$

and scalar product associated with it we shall denote by $(\mathbf{u},\ \mathbf{u})$. Now it is easy to rewrite everything in terms of our quaternionic fields. The constraint (2.9) becomes

$$(A_i,\ A^i) = 2(\mathbf{u},\ \mathbf{u}) = -2/\kappa^2 \tag{3.6}$$

and (2.11)

$$\mu(\mathbf{u}) = (\mathbf{u}, \mathbb{T}\mathbf{u}) = 0. \tag{3.7}$$

Notice that $\overline{(\mathbf{u}, \mathbb{T}\mathbf{u})} = -(\mathbf{u}, \mathbb{T}\mathbf{u})$ since $\mathbb{T}$ is antihermitian with respect to the above metric. Thus (3.7) are three real constraints on $\mathbf{u}$ (or equivalently $\mathbb{A}$). Here the matrix $\mathbb{T}$ is an antihermitian operator in $\mathbb{H}^r$ and is given by

$$\mathbb{T} = \left(\begin{array}{cc|c} 0 & \lambda & \\ & & \mathbb{O} \\ \lambda & 0 & \\ \hline & & \\ \multicolumn{2}{c|}{\mathbb{O}} & i\mathbb{I}_{r-2} \\ & & \end{array}\right) \tag{3.8}$$

Furthermore, we write the lagrangian (2.17) in the following form

$$e^{-1}\mathcal{L}_\sigma = -2(\partial_\mu \mathbf{u},\ \partial^\mu \mathbf{u}) + 2\kappa^2(\mathbf{u}, \partial_\mu \mathbf{u})(\partial^\mu \mathbf{u}, \mathbf{u}) + 2g^2 W_\mu W^\mu (\mathbb{T}\mathbf{u}, \mathbb{T}\mathbf{u}), \tag{3.9}$$

where the auxiliary field W_μ is given by

$$W_\mu = g^{-1}(\mathbf{u}, \partial_\mu \mathbb{T}\mathbf{u})(\mathbb{T}\mathbf{u}, \mathbb{T}\mathbf{u})^{-1}. \tag{3.10}$$

The chiral $SU(2)$ gauge transformation is very simple and it is realized by quaternionic multiplication by unit quaternions from the right

$$\mathbf{u} \sim \mathbf{u}\nu, \quad \overline{\nu}\nu = 1 \tag{3.11}$$

Substituting (3.10) into (3.9) we get

$$e^{-1}\mathcal{L}_\sigma = -2(\partial_\mu \mathbf{u},\ \partial^\mu \mathbf{u}) + 2\kappa^2(\mathbf{u}, \partial_\mu \mathbf{u})(\partial^\mu \mathbf{u}, \mathbf{u}) + 2(\mathbf{u}, \partial_\mu \mathbb{T}\mathbf{u})^2(\mathbb{T}\mathbf{u}, \mathbb{T}\mathbf{u})^{-1} \tag{3.12}$$

where the quaternionic scalar fields are subject to (3.7) and (3.6). Also, we consider only equivalence classes, the equivalence relation given by the $\mathbb{R}$ transformation

$$\mathbf{u} \sim e^{s\mathbb{T}}\mathbf{u} \tag{3.13}$$

Notice that

$$e^{s\mathbb{T}} = \left(\begin{array}{cc|c} \cosh \lambda s & \sinh \lambda s & \\ & & \mathbb{O} \\ \sinh \lambda s & \cosh \lambda s & \\ \hline & & \\ \multicolumn{2}{c|}{\mathbb{O}} & e^{is}\mathbb{I}_{r-2} \end{array}\right) \tag{3.14}$$

is a group element parametrized by $s \in \mathbb{R}$. (The sinh and cosh denote the hyperbolic functions). We now show that the above action is free everywhere on the ball (not only on the submanifold given by (3.7)). But we know that (3.7) is an invariant submanifold for this action. Thus, the group acts invariantly and freely also on (3.7). Let us assume that the action in question is not free. Let us take some $\mathbf{u}$ on the quaternionic ball that is fixed by some non-trivial element of this action. Let u^o, u^1 be the first and the second component of the vector $\mathbf{u}$. Due to (3.6) $u^o \neq 0$. There must exist a nontrivial $s \neq 0$ such that

$$\begin{pmatrix} \cosh \lambda s & \sinh \lambda s \\ \sinh \lambda s & \cosh \lambda s \end{pmatrix} \begin{pmatrix} u^o \\ u^1 \end{pmatrix} = \begin{pmatrix} u^o \\ u^1 \end{pmatrix}, \nu \tag{3.15}$$

where ν is some unit quaternion. Solving this two-dimensional system of equations we conclude that ν must be equal to one and that $2\cosh \lambda s = 2$, thus the only element of $\mathbb{R}$ that fixes our point in the ball parametrized by $\mathbf{u}$ is $s = 0$ – the trivial one. Consequently the action must be free on the ball. Notice that the submatrix $e^{is}\mathbb{I}_{r-2}$ could be replaced by an arbitrary matrix and this new $\mathbb{R}$ action would still be free.

Moreover, one can check that this action is proper in the following sense: An action of a non-compact isometry group on the riemannian manifold M

$$g: \quad x \longrightarrow \; g(x) \in M, \quad x \in M, \; g \in G$$

is a 'proper' action when the map $G \times M \longrightarrow \; M \times M$ given by

$$(g, x) \longrightarrow \; (g(x), x)$$

has the property that the inverse images of compact sets are compact. Then, provided that the above action is free, the quotient space M/G is again a smooth riemannian manifold [11]. (It is trivial to see that M must be non-compact, otherwise the action of G is never proper).

Thus we know that our manifold is a smooth riemannian $4(r-2)$-dimensional with complete quaternionic Kähler metric. It is non-compact and of constant negative scalar curvature. The isometry group is easily seen to be $U(r-2) \times Sp(1)$. For $r = 2$ the metric (as guaranteed by he quotient construction) is a self dual gravitational instanton with negative cosmological constant (the Weyl tensor is self-dual) and is the recently found Pedersen metric [7]. In higher dimensions we obtain the generalization found by Hitchin

[8]. We now show that the metrics just constructed are related to hyperkähler Taub-NUT metrics in the sense that they produce them in the scalar curvature going to zero limit (or global supersymmetry limit). Since Taub-NUT metrics are not locally symmetric, our metrics are not either by the argument of [6]. They are also non-homogeneous, because all non-compact homogeneous examples were classified by Alekseevskii [2] and ours are not non-symmetric Alekeevskii's spaces.

Let us introduce a global non-homogeneous coordinate system on $\mathbb{HH}^{r-1}$. Since $u^o \neq 0$ we define

$$w^\alpha = \kappa^{-1} u^\alpha (u^o)^{-1}, \qquad \alpha = 1, ..., r-1. \tag{3.16}$$

The constraint (3.6) can be rewritten as

$$\overline{u}^o u^o [-1 + \kappa^2 (\mathbf{w}, \mathbf{w})] = -1/\kappa^2, \tag{3.17}$$

where now $(\mathbf{w}, \mathbf{w}) \stackrel{\text{def}}{=} \sum_{\alpha=1}^{r-1} \overline{w}^\alpha w^\alpha$ We observe that our projective coordinates are bound to a ball $0 \leq (\mathbf{w}, \mathbf{w}) < 1/\kappa^2$. Also the constraint (3.17) is solved provided

$$\overline{u}^o u^o = \frac{1}{\kappa^2} [1 - \kappa^2 (\mathbf{w}, \mathbf{w})]^{-1} \tag{3.18}$$

$\mathbf{w}$, as defined above, is also invariant under multiplication of $\mathbf{u}$ from the right. Thus we can fix this $Sp(1)$ action by choosing u^o to be real and given by

$$u^o = \sqrt{\frac{1}{\kappa^2} [1 - \kappa^2 (\mathbf{w}, \mathbf{w})]^{-1}}. \tag{3.19}$$

Now, let us see how the $\mathbb{R}$ action can be expressed in the new coordinate system. We have

$$\varphi_s(w^1) = (\cosh(\lambda s) w^1 + \kappa^{-1} \sinh \lambda s)(\kappa \sinh(\lambda s) w^1 + \cosh \lambda s)^{-1}$$

$$\varphi_s(\mathbf{w}) = e^{is} \mathbf{w} (\kappa \sinh(\lambda s) w^1 + \cosh \lambda s)^{-1}, \quad \alpha = 2, ..., r. \tag{3.20}$$

Next, we want to rewrite our moment map constraint (3.7). We get the following

$$\sum_{\alpha=2}^{r} \overline{w}^\alpha i w^\alpha = \frac{\lambda}{\kappa} (w^1 - \overline{w}^1) \tag{3.21}$$

The above constraint is clearly invariant with respect to the transformations of w by (3.20).

Finally, we can write our scalar part of the $N = 2$ supergravity lagrangian in terms of the w fields:

$$e^{-1} \mathcal{L}_\sigma = -\frac{2(\partial_\mu \mathbf{w}, \partial^\mu \mathbf{w})}{(1 - \kappa^2 (\mathbf{w}, \mathbf{w}))} + \frac{2\kappa^2 (\partial_\mu \mathbf{w}, \mathbf{w})(\mathbf{w}, \partial^\mu \mathbf{w})}{(1 - \kappa^2 (\mathbf{w}, \mathbf{w}))^2} +$$

$$+ \frac{2\{-\lambda\kappa \partial_\mu w^1 + \kappa^2 \sum_\alpha \overline{w}^\alpha i \partial_\mu w^\alpha + \frac{1/2\kappa^4 (\sum_\alpha \overline{w}^\alpha i w^\alpha)(\sum_\alpha \partial_\mu \overline{w}^\alpha w^\alpha + \sum_\alpha \overline{w}^\alpha \partial_\mu w^\alpha)}{(1 - \kappa^2 (\mathbf{w}, \mathbf{w}))}\}^2}{\kappa^4 (\lambda^2 \kappa^{-2} - \lambda^2 \overline{w}^1 w^1 + \sum_\alpha \overline{w}^\alpha w^\alpha)(1 - \kappa^2 (\mathbf{w}, \mathbf{w}))} \tag{3.22}$$

where all sums are over $\alpha = 2, ..., r-1$. The above lagrangian is gauge invariant under the $\mathbb{R}$ transformation (3.20). In fact, it is easy to fix the gauge and solve the constraint (3.21) which is linear in w^1 field. One can check that there is always a gauge transformation which makes w^1 purely imaginary, *i.e.*, ,

$$\overline{w}^1 + w^1 = 0, \tag{3.23}$$

which means that in this gauge

$$\frac{\kappa}{2\lambda} \sum_{\alpha=2}^{r-1} \overline{w}^\alpha i w^\alpha = w^1. \tag{3.24}$$

We now can insert (3.24) into the lagrangian (3.22) to express it in terms of $(r-2)$ quaternionic scalar fields w^α. The w^α's are still bound to be in the following region inside the ball $\sum_{\alpha=2}^{r-1} \overline{w}^\alpha w^\alpha < 1/\kappa^2$

$$\sum_{\alpha=2}^{r-1} \overline{w}^\alpha w^\alpha + \frac{\kappa^2}{4\lambda^2} (\sum_{\alpha=2}^{r-1} \overline{w}^\alpha i w^\alpha)(\overline{\sum_{\alpha=2}^{r-1} \overline{w}^\alpha i w^\alpha}) < 1/\kappa^2. \tag{3.25}$$

The boundary of this region is given by some quartic equation and is topologically equivalent to S^{4r-9}. In the case when $r = 3$, the situation is even simpler because then (3.25) reads

$$\sum_{\alpha=2}^{r-1} \overline{w}^\alpha w^\alpha + \frac{\kappa^2}{4\lambda^2} (\sum_{\alpha=2}^{r-1} \overline{w}^\alpha w^\alpha)^2 < 1/\kappa^2 \tag{3.26}$$

which is equivalent to

$$\sum_{\alpha=2}^{r} \overline{w}^\alpha w^\alpha < \frac{2\lambda^2}{\kappa^2} (\sqrt{1+\lambda^{-2}} - 1). \tag{3.27}$$

Thus w^α's parametrize a 4-ball with radius $\frac{2\lambda^2}{\kappa^2}(\sqrt{1+\lambda^{-2}} - 1)$. In higher dimensions the situation is slightly more complicated and given by the eq.(3.25). Now we have the full geometry: A global and local description of our σ-model manifold with quaternionic Kähler metric on it and with the resulting interactions in the lagrangian. The full supersymmetric action can easily be recovered with the help of [4]. We want to show the analogy of this model with that of hyperkähler Taub-NUT.

4. The global supersymmetry limit: other examples

Let us rewrite our group transformations (3.20) in its infinitesimal form

$$\delta_s w^1 = \frac{\lambda}{\kappa} s - \lambda \kappa s (w^1)^2$$

$$\delta_s w^\alpha = is - \kappa \lambda s w^1 w^\alpha, \quad \alpha = 2, ..., r-1 \tag{4.1}$$

Let us also assume that $\lambda = \Lambda\kappa$. We shall examine the decoupling limit of Newton's constant going to zero. This is also the global supersymmetry limit and, consequently, the resulting metric should be hyperkähler at least locally. The transformation rules in the limit become

$$\delta_s w^1 = \Lambda s$$
$$\delta_s w^\alpha = is, \quad \alpha = 2, ...r-1 \tag{4.2}$$

and the constraint equations read

$$\sum_{\alpha=2}^{r-1} \overline{w}^\alpha i w^\alpha = \Lambda(w^1 - \overline{w}^1) \tag{4.3}$$

Finally, the lagrangian (3.22) has a simple form

$$\lim_{\kappa\to 0} \mathcal{L}_\sigma(\overline{\mathbf{w}}, \mathbf{w}) = \mathcal{L}_{\text{Taub-NUT}} = -2\sum_{\alpha=1}^{r-1} \partial_\mu \overline{w}^\alpha \partial^\mu w^\alpha + \frac{2(-\Lambda\partial_\mu w^1 + \sum_{\alpha=2}^{r-1} \overline{w}^\alpha i \partial_\mu w^\alpha)^2}{(\sum_{\alpha=2}^{r-1} \overline{w}^\alpha w^\alpha + \Lambda^2)}. \tag{4.4}$$

The lagrangian (4.4) is invariant under local gauge transformations given by (4.2). Now we can fix the gauge so that

$$w^1 + \overline{w}^1 = 0 \tag{4.5}$$

and then solving (4.3) which is linear in w^1 we obtain

$$\frac{1}{2}\mathcal{L}_{\text{Taub-NUT}}(\overline{\mathbf{w}}, \mathbf{w}) = -\frac{1}{4\Lambda^2}\partial_\mu \overline{(\sum_\alpha \overline{w}^\alpha i w^\alpha)} \partial^\mu (\sum_\alpha \overline{w}^\alpha i w^\alpha) +$$
$$-\sum_\alpha \partial_\mu \overline{w}^\alpha \partial^\mu w^\alpha + \frac{(\sum_\alpha \overline{w}^\alpha i \overleftrightarrow{\partial}_\mu w^\alpha)^2}{4(\sum_\alpha \overline{w}^\alpha w^\alpha + \Lambda^2)}. \tag{4.6}$$

But this is exactly the formulation of the Taub-NUT hyperkähler σ-models given by hyperkähler quotient of the quaternionic vector spaces. The scalar curvature going to zero limit takes us from the Pedersen metric and its $4n$-dimensional generalization to the Taub-NUT metrics. This limit makes sense even globally since in both cases the topologies are the same, namely $\mathbb{R}^{4n}$.

Since it is known how to construct multi-center Taub-NUT metrics via hyperkähler quotients [12] it is rather straightforward to generalize our quaternionic Taub-NUT metric to a multi-center case. We shall demonstrate this for the two-center metric. We introduce the following abelian two-parameter action on $\mathbb{H}^3$ given in terms of infinitesimal generators

$$\mathbb{T}_\lambda = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad \mathbb{T}_p = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & ip & 0 \\ 0 & 0 & 0 & ip \end{pmatrix} \tag{4.7}$$

The first is the generator of $SO(1,1) \subset Sp(1,3)$ and the second one generates a circle action with $U(1) \subset Sp(1) \times Sp(3) \subset Sp(1,3)$. Thus our two parameter group action is given by the following matrix multiplication of the homogeneous vector u^α from the left.

$$\varphi_{s,r}(\mathbf{u})^\alpha = \begin{pmatrix} \cosh\lambda s & \sinh\lambda s & 0 & 0 \\ \sinh\lambda s & \cosh\lambda s & 0 & 0 \\ 0 & 0 & e^{is} & 0 \\ 0 & 0 & 0 & e^{is} \end{pmatrix} \begin{pmatrix} e^{2\pi ir} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2\pi ipr} & 0 \\ 0 & 0 & 0 & e^{2\pi ipr} \end{pmatrix} \begin{pmatrix} u^o \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} \tag{4.8}$$

where $s, \lambda \in \mathbb{R}$ p is an integer and $r \in [0,1)$ if p is even or $r \in [0,1/2)$ if p is odd. As before, the action generated by $\mathbb{T}_\lambda$ is free. The one-parameter subgroup generated by the $U(1)$ generator also acts freely but only on constraints given by the moment maps

$$\mathcal{Z} \stackrel{\text{def}}{=} \{\mathbf{u} \in \mathbb{HH}^3 \quad : \mu_\lambda(\mathbf{u}) = (\mathbf{u}, \mathbb{T}_\lambda \mathbf{u}) = 0, \mu_p(\mathbf{u}) = (\mathbf{u}, \mathbb{T}_p \mathbf{u}) = 0\} \tag{4.9}$$

Also, any one parameter subgroup of this $\mathbb{R} \times U(1)$ acts freely on the constraints. Thus $\mathbb{HH}^3 /\!/\!/ bbr \times U(1)$ is a smooth self-dual 4-manifold with negative cosmological constant. In the scalar curvature going to zero limit ($\lambda \to 0$ and $1/p \to 0$) we obtain a two-center Taub-NUT metric given by the familiar hyperKähler quotient of $\mathbb{H}^3$. The limiting procedure involves taking $p = \Lambda_2 \kappa$ where κ is our coupling constant. Thus taking a smooth limit of $\kappa \to 0$ takes us away from integer values of p. For irrational values of p our $U(1)$ action becomes non-compact. We do not know if our metric would make sense globally for arbitrary real $p > 1$ but we think this is the case. Generalization of the above example to multi-center metrics involves taking more $U(1)$ generators and is rather straightforward.

Let us come back for a moment to the metric discussed in Section 3 and point out that the $\mathbb{R}$ action introduced there could be modified in the $U(1)$ sector by introducing arbitrary weights. This does not introduce fixed points but breaks the isometry group of a quotient metric to a subgroup of $Sp(1) \times U(r-2)$. We believe that this, in general, would correspond to certain deformation of $4 < 4n$-dimensional analogues of the Pedersen metric.

Finally we would like to present our last example which is to some extent a quaternionic analog of the Eguchi-Hanson metric [13]. Let us examine the following $U(1)$ action on the quaternionic hyperbolic space $\mathbb{HR}^r$

$$\varphi_t(\mathbf{u}, u_0) = (e^{2\pi ipt}\mathbf{u},\ e^{2\pi iqt}u_0) \tag{4.10}$$

where $t \in [0,1)$ if $(p+q)$ is odd and where $t \in [0,1/2)$ if $(p+q)$ is even, p, q are relatively prime positive integers. It is easy to see that this action is a quaternionic isometry and that it is locally free. In the case of quaternionic projective space this action was also locally free but never free and would lead to the compact quaternionic orbifolds constructed in

[9]. But now the situation is a bit different: Let us examine it carefully. Again, since $u^o \neq 0$ we introduce the Fubini-Study projective coordinates

$$w_\alpha = u_\alpha u_o^{-1}, \quad \text{for} \quad \alpha = 1, ..., r. \tag{4.11}$$

In these coordinates the action (4.10) becomes

$$\varphi_t(\mathbf{w}) = e^{2\pi i p t}\mathbf{w}e^{-2\pi i q t}, \tag{4.12}$$

and the invariant submanifold of the zero moment map is given as

$$\mathcal{Z} \cong \{\mathbf{w} \in \mathbb{HH}^r : \quad -iq + p\sum_\alpha \overline{w}_\alpha i w_\alpha = 0\} \tag{4.13}$$

We then write

$$\mathbf{w} = \mathbf{w}_+ + j\overline{\mathbf{w}}_- \tag{4.14}$$

where $\mathbf{w}_+, \mathbf{w}_- \in \mathbb{C}^n \cup \{||\mathbf{w}_+||^2 + ||\mathbf{w}_-||^2 < 1\}$ and observe that

$$\varphi_t(\mathbf{w}_+, \mathbf{w}_-) = (e^{2\pi i(p-q)t}\mathbf{w}_+, \; e^{2\pi i(p+q)t}\mathbf{w}_-)$$

$$\mathcal{Z} = \{(\mathbf{w}_+, \mathbf{w}_-) \in \mathbb{HH}^r : \quad ||\mathbf{w}_-||^2 - ||\mathbf{w}_+||^2 = -q/p, \quad \overline{\mathbf{w}}_- \cdot \mathbf{w}_+ = 0\}. \tag{4.15}$$

It is not difficult now to see that the action has non-trivial isotropy in $\mathcal{Z}$ exactly at the points where $\mathbf{w}_- = 0$. This gives us a singular set Σ_0 described explicitly as

$$\begin{aligned} \Sigma_0 &= \{(\mathbf{w}_+, \mathbf{w}_-) \in \mathcal{Z} : \quad \mathbf{w}_- = 0\}/S^1 \\ &= \{(\mathbf{w}_+, 0) \in \mathcal{Z} : \quad ||\mathbf{w}_-||^2 = q/p\}/S^1 \cong \mathbb{P}^{n-1}_{\mathbb{C}}. \end{aligned}$$

One easily checks that the isotropy group $\Gamma_0 = \{t \in S^1 : \quad \varphi_t(x) = x\}$, for points x corresponding to Σ_0, is

$$\Gamma_0 \cong \begin{cases} \mathbb{Z}_{p-q}, & \text{if } (p+q) \text{ is odd;} \\ \mathbb{Z}_{\frac{p-q}{2}}, & \text{if } (p+q) \text{ is even.} \end{cases} \tag{4.16}$$

Thus for

$$p - q = \begin{cases} 1, & \text{if } (p+q) \text{ is odd;} \\ 2, & \text{if } (p+q) \text{ is even.} \end{cases} \tag{4.17}$$

our $U(1)$ action is free. For any other choice of p, q we obtain yet another example of a quaternionic orbifold with one singular set given by Σ_0. But in the case of (4.17) our quotient construction gives a smooth quaternionic Kähler metric. It is easy to see what is the boundary at infinity of the 4-manifold given in this construction. It is the lens space $L_{p-1} \stackrel{\text{def}}{=} S^3/\mathbb{Z}_{p-1}$ for $(2p-2)$ even (p and $p-2$ relatively prime) and L_{2p-1} for even p. It follows that we can obtain a self-dual 4-manifold with any L_n as its boundary in infinity. This is remarkably similar to the hyperkähler multi-Eguchi-Hanson self-dual instanton. The boundary at infinity has the same topology. But the manifolds are different. It is not quite clear how these metrics correspond to the multi-Eguchi-Hanson metrics in the scalar curvature going to zero limit. The naive limit produces a singular object: a cone on the lens space. This is the isolated point singularity discussed by Hitchin; it can be resolved, giving rise to the multi-Eguchi-Hanson instantons [14]. We would like to address the issue of global properties of these metrics in our future work.

5. Conclusions

Using the formalism of $N = 2$ supergravity theory coupled to a matter system together with the methods of quaternionic quotient we constructed a whole new class of very interesting σ-model couplings admitting $N = 2$ local supersymmetry. Our results are interesting from the point of view of $N = 2$ supergravity because of the variety of new matter couplings we proposed. It would be interesting to see some quantum properties of the various models. On the other hand our results are also very significant from the point of view of the differential geometry of quaternionic manifolds with $Sp(n)\cdot Sp(1)$ holonomy to which all our couplings correspond. All the manifolds we discussed are non-symmetric and non-homogeneous quaternionic Kähler manifolds. We investigated connections between our metrics and some non-compact hyperhähler metrics which they give in the scalar curvature going to zero limit. It would be an interesting question to ask whether all known non-compact examples of hyperhähler metrics are related in this way to quaternionic metrics. As we showed this is indeed the case with the Taub-NUT and multi-Taub-NUT hyperhähler metrics for which we constructed their quaternionic analogues. Our last example seems to be a quaternionic analogue of multi-Eguchi-Hanson metrics. Recently, the hyperhähler quotient led to the construction of new self-dual gravitational instantons conjectured by Hitchin with S^3/Γ as their boundary at infinity [15]. (Γ is a discrete subgroup of $SU(2)$). In particular, it would be very interesting to see if also those hyperhähler metrics have their quaternionic analogues. We expect to address some of these questions in our future work.

Acknowledgements

I would like to thank Professor Hitchin for pointing out to me the action which leads to the Pedersen metric in the quotient construction. I would like to thank also Professor Roček for many suggestions and very useful discussions on hyperkähler quotients. I thank Professor LeBrun for helpful discussions.

References

[1] J. Bagger and E. Witten, *Matter couplings in $N = 2$ supergravity*, Nucl. Phys. B222 (1983), 1.

[2] D. V. Alekseevskii, *Classification of quaternionic spaces with transitive solvable group of motions*, Math. USSR-Izv. 9 (1975), 297.

[3] P. Breitenlohner and M. Sohnius, *Matter couplings and non-linear σ-models in $N = 2$ supergravity*, Nucl. Phys. B187 (1981), 409.

[4] B. de Wit, P. G. Lauwers, and A. Van Proyen, *Lagrangians of $N = 2$ supergravity-matter systems*, Nucl.Phys. B255 (1985), 569.

[5] K. Galicki, *Quaternionic Kähler and hyperkähler nonlinear σ-models*, Nucl. Phys. B271 (1986), 402.

[6] K. Galicki, *A Generalization of the Momentum Mapping Construction for Quaternionic Kähler Manifolds*, Commun. Math. Phys. 108 (1987), 117.

[7] H. Pedersen, *Einstein Metrics, Spinning Top Motions and Monopoles*, Math. Ann. 274 (1986), 35.

[8] N. J. Hitchin, not published.

[9] K. Galicki and B. H. Lawson, Jr., *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. 282 (1988), 1.

[10] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyperkähler metrics and supersymmetry*, Commun. Math. Phys. 108 (1987), 535.

[11] V. Arnold, *Mathematical Methods in Classical Mechanics*, Springer-Verlag, New Yrok 1978.

[12] M. Roček, *Supersymmetry and nonlinear σ-models*, Physics D15 (1985), 75.

[13] G. W. Gibson and S. Hawking, *Gravitational multi-instantons*, Phys. Lett. B78 (1978), 430.

[14] N. J. Hitchin, *Polygons and gravitons*, Math. Proc. Camb. Phil. Soc., 85 (1979), 455.

[15] P. B. Kronheimer, *Instantons gravitationneles et singularités de Klein*, Comptes rendus, 303, série I, (1986), 53.

Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840

January 1987

and

Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
Wrocław, Poland