

Duality and Yang-Mills Fields on Quaternionic Kähler Manifolds

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ABSTRACT: The concept of a self-dual connection on a four dimensional Riemannian manifold is generalized to the $4n$ -dimensional case of any quaternionic Kähler manifold. The generalized self-dual connections are minima of a modified Yang-Mills functional. It is shown that our definitions give a correct framework for a mapping theory of quaternionic Kähler manifolds. The mapping theory is closely related to the construction of Yang-Mills fields on such manifolds. We also discuss some monopole-like equations.

1. Introduction

A quaternionic Kähler manifold is a Riemannian manifold whose holonomy group can be reduced to a subgroup of $Sp(n) \cdot Sp(1)$, $n > 1$ [1, 2]. By definition, such manifold has dimension $4n$. As demonstrated by Salamon [2, 3] it can be also viewed as a higher dimensional analogy of the anti-self-dual Einstein 4-manifold. The bundle of 2-forms on a quaternionic Kähler manifold M has the following irreducible decomposition as representation of $Sp(n) \cdot Sp(1)$

$$\Lambda^2 T^* M = S^2 \mathbb{H} \oplus S^2 \mathbb{E} \oplus \left(S^2 \mathbb{H} \oplus S^2 \mathbb{E} \right)^\perp, \quad (1.1)$$

where \mathbb{H} and \mathbb{E} are vector bundles associated to the standard representations of $Sp(n)$ and $Sp(1)$ respectively. This decomposition resembles the decomposition of $\Lambda^2 T^* M$ into the direct sum of self-dual and anti-self-dual 2-forms when M is four dimensional. Just as in the four dimensional case we are able to interpret the decomposition (1.1) in terms of the Hodge $*$ -operator.

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If the curvature of a connection ∇ is in either the $S^2\mathbb{H}$ or the $S^2\mathbb{E}$ part of (1.1) then ∇ is a minimum of the Yang-Mills functional and if the curvature is in the orthogonal complement of $S^2\mathbb{H} \oplus S^2\mathbb{E}$ then ∇ is most likely a saddle point. We have found that the Yang-Mills functional can be modified so that whenever the curvature of ∇ is in one and only one component of (1.1) the connection is its minimum.

We demonstrate that our definitions are compatible with the description of Yang-Mills fields on 4-manifolds and that it gives a correct framework for mapping theory of quaternionic Kähler manifolds. On the other hand, when the energy functional is interpreted as a classical Lagrangian, our quaternionic mapping theory yields many new examples of quantum field theories with $SU(2)$ (or $SO(3)$) gauge symmetry and composite gauge fields: 4-dimensional sigma models. We show that some fundamental properties of the well-known 4-dimensional σ -models on the quaternionic projective spaces are shared by such models on arbitrary quaternionic Kähler manifolds. Finally, we demonstrate that our formalism provides a global picture for the generalized monopole equation of Pedersen and Poon [4].

2. Duality

Let M be a $4n$ -dimensional Riemannian manifold whose holonomy group is contained in $Sp(n) \cdot Sp(1) \subset SO(4n)$. Then the cotangent bundle of M can be identified with

$$T^*M = \mathbb{E} \otimes \mathbb{H},$$

where \mathbb{E} and \mathbb{H} are the standard representations of $Sp(n)$ and $Sp(1)$ respectively. Then $S^2\mathbb{H}$ is a real rank 3 subbundle of $\text{End}TM$. Locally, at each $x \in M$, $S^2\mathbb{H}$ has a basis $\{I, J, K\}$ satisfying

$$I^2 = J^2 = -\mathbb{I}, \quad IJ = -JI = K. \quad (2.1)$$

The metric g on M is compatible with the bundle $S^2\mathbb{H}$ in the sense that for each $A \in S^2\mathbb{H}_x$, g is Hermitian with respect to A , *i.e.*, $g(AX, AY) = g(X, Y)$ for all $X, Y \in T_xM$. One can use the metric to define an isomorphism

$$\text{End}TM \cong T^*M \otimes T^*M$$

under which $S^2\mathbb{H}$ is isometrically embedded in $\Lambda^2 T^*M$. Explicitly, any element $A \in S^2\mathbb{H}_x$ is mapped into ω_A by

$$\omega_A(X, Y) = g(AX, Y), \quad X, Y \in T_xM.$$

Let $\{\omega_1, \omega_2, \omega_3\}$ be a local orthogonal frame of $S^2\mathbb{H} \subset \Lambda^2 T^*M$. For convenience of further computations let us normalize $\{\omega_1, \omega_2, \omega_3\}$ to have length $2n$ and then define

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3. \quad (2.2)$$

This Ω is a globally defined, non-degenerate 4-form on M and it is parallel. It is usually called the fundamental 4-form or the quaternionic structure on M as its parallelism

determines reduction of the structure group on M . The condition $\nabla\Omega = 0$ can be used to define quaternionic Kähler geometry in dimension bigger than 4. In dimension 4 we shall say that M is quaternionic Kähler if it is self-dual and Einstein. The parallelism of Ω immediately implies that $d\Omega = 0$. Recently, Swann [5] showed that the converse is also true provided $\dim M \geq 12$.

Pointwisely, Ω can be described as follows. At any point $x \in M$, $T_x^*M = \mathbb{E}_x \otimes \mathbb{H}_x$, where \mathbb{E}_x is the $2n$ -dimensional complex representation of $Sp(n)$ and \mathbb{H}_x is the 2-dimensional complex representation of $Sp(1)$. Let ω_E and ω_H be the symplectic forms on \mathbb{E}_x and \mathbb{H}_x respectively and j_E and j the quaternionic structures. Then the metric g on T_x^*M can be expressed as

$$g = \omega_E \otimes \omega_H. \quad (2.3)$$

Let $\{e^j, j_E e^j : j = 1, \dots, n\}$ be a symplectic basis on \mathbb{E}_x and $\{h, jh\}$ a symplectic basis on \mathbb{H}_x . We define

$$\begin{aligned} \omega_0^j &\doteq \frac{1}{\sqrt{2}}(e^j \otimes h + j_E e^j \otimes jh) \\ \omega_1^j &\doteq \frac{i}{\sqrt{2}}(e^j \otimes h - j_E e^j \otimes jh) \\ \omega_2^j &\doteq \frac{1}{\sqrt{2}}(j_E e^j \otimes h - e^j \otimes jh) \\ \omega_3^j &\doteq \frac{i}{\sqrt{2}}(j_E e^j \otimes h + e^j \otimes jh). \end{aligned} \quad (2.4)$$

Now $\{\omega_0^j, \omega_1^j, \omega_2^j, \omega_3^j, j = 1, \dots, n\}$ forms an orthonormal basis on T_x^*M . Let

$$\begin{aligned} \omega_1 &\doteq \sum_{j=1}^n (\omega_0^j \wedge \omega_1^j + \omega_2^j \wedge \omega_3^j) \\ \omega_2 &\doteq \sum_{j=1}^n (\omega_0^j \wedge \omega_2^j - \omega_1^j \wedge \omega_3^j) \\ \omega_3 &\doteq \sum_{j=1}^n (\omega_0^j \wedge \omega_3^j + \omega_1^j \wedge \omega_2^j). \end{aligned} \quad (2.5)$$

Then $\{\omega_1, \omega_2, \omega_3\}$ forms an orthogonal basis on $S^2\mathbb{H}_x$. We shall choose Ω as in (2.2). The orthogonal basis for $S^2\mathbb{E}_x$ can be written as

$$\left\{ \begin{aligned} \Sigma_0^{ij} &\doteq (\omega_0^i \wedge \omega_0^j + \omega_1^i \wedge \omega_1^j) + (\omega_2^i \wedge \omega_2^j + \omega_3^i \wedge \omega_3^j), \quad 1 \leq i < j \leq n, \\ \Sigma_1^{ij} &\doteq (\omega_0^i \wedge \omega_1^j - \omega_2^i \wedge \omega_3^j) + (\omega_0^j \wedge \omega_1^i - \omega_2^j \wedge \omega_3^i), \quad 1 \leq i \leq j \leq n, \\ \Sigma_2^{ij} &\doteq (\omega_0^i \wedge \omega_2^j + \omega_1^i \wedge \omega_3^j) + (\omega_0^j \wedge \omega_2^i + \omega_1^j \wedge \omega_3^i), \quad 1 \leq i \leq j \leq n, \\ \Sigma_3^{ij} &\doteq (\omega_0^i \wedge \omega_3^j - \omega_1^i \wedge \omega_2^j) + (\omega_0^j \wedge \omega_3^i - \omega_1^j \wedge \omega_2^i), \quad 1 \leq i \leq j \leq n. \end{aligned} \right. \quad (2.6)$$

Σ_0^{ij} give $n(n-1)/2$ basis elements and Σ_A^{ij} , $A = 1, 2, 3$, give $n(n+1)/2$ basis elements respectively. One can easily check that

$$\text{vol}(M) = \frac{1}{(2n+1)!} \Omega^n \quad (2.7)$$

$$\text{vol}(M) = \frac{1}{12n(2n+1)} \Omega \wedge * \Omega, \quad (2.8)$$

where $\text{vol}(M)$ is the volume form of M and “*” is the Hodge *-operator. As a consequence we have

$$* \Omega = \frac{6}{(2n-1)!} \Omega^{n-1}. \quad (2.9)$$

Note that all these equations are valid even when n is equal to 1.

DEFINITION 2.1: A 2-form ω on M is c -self-dual if

$$* \omega = c \omega \wedge \Omega^{n-1}. \quad (2.10)$$

When $n = 1$ then $c^2 = 1$, because $*^2 = 1$, and the above equation is reduced to the conformally invariant self-dual or anti-self-dual equations on a 4-dimensional oriented Riemannian manifold. Notice that the above definition depends on the choice of both the fundamental 4-form Ω and the constant c . In dimension higher than 4, as we shall now see, there are three different constants c that give non-trivial solutions to (2.10). Similar equations were studied in [6].

THEOREM 2.2: Let ω be a non-zero c -self-dual 2-form. Then $c = c_i$, $i = 1, 2, 3$, where

$$c_1 = \frac{6n}{(2n+1)!}, \quad c_2 = \frac{-1}{(2n-1)!}, \quad c_3 = \frac{3}{(2n-1)!}. \quad (2.11)$$

Moreover, when $c = c_1$ then $\omega \in S^2\mathbb{H}$, when $c = c_2$ then $\omega \in S^2\mathbb{E}$, and when $c = c_3$ then ω is in the orthogonal complement of $S^2\mathbb{H} \oplus S^2\mathbb{E}$ in $\Lambda^2 T^*M$.

PROOF: As the basis for $S^2\mathbb{H}$ is given in (2.5) and the basis for $S^2\mathbb{E}$ in (2.6) the proof is an easy exercise in linear algebra. Therefore we only spell out the constraints on the coefficients of the 2-form ω . Using the orthonormal basis $\{\omega_0^j, \omega_1^j, \omega_2^j, \omega_3^j : j = 1, \dots, n\}$ any 2-form ω can be written as

$$\omega = \sum_{i,j,\alpha,\beta} \omega_{(\alpha)(\beta)}^{(i)(j)} \omega_\alpha^i \wedge \omega_\beta^j.$$

Then $*\omega = c_1 \omega \wedge \Omega^{n-1}$ if and only if

$$\begin{cases} \omega_{(0)(1)}^{(i)(i)} = \omega_{(2)(3)}^{(i)(i)} = \omega_{(0)(1)}^{(j)(j)} = \omega_{(2)(3)}^{(j)(j)} \\ \omega_{(0)(2)}^{(i)(i)} = -\omega_{(1)(3)}^{(i)(i)} = \omega_{(0)(2)}^{(j)(j)} = -\omega_{(1)(3)}^{(j)(j)} \\ \omega_{(0)(3)}^{(i)(i)} = \omega_{(1)(2)}^{(i)(i)} = \omega_{(0)(3)}^{(j)(j)} = \omega_{(1)(2)}^{(j)(j)} \end{cases} \quad (2.12)$$

for all i, j

$$\omega_{(0)(0)}^{(i)(j)} = \omega_{(1)(1)}^{(i)(j)} = \omega_{(0)(2)}^{(i)(j)} = \omega_{(3)(3)}^{(i)(j)} = 0 \quad \forall i, j \quad (2.13)$$

and

$$\omega_{(\alpha)(\beta)}^{(i)(j)} = 0 \quad \forall i \neq j \quad \forall \alpha, \beta. \quad (2.14)$$

Similarly, $*\omega = c_2\omega \wedge \Omega^{n-1}$ if and only if

$$\begin{cases} \omega_{(0)(1)}^{(i)(j)} = -\omega_{(2)(3)}^{(i)(j)}, \omega_{(0)(2)}^{(i)(j)} = \omega_{(1)(3)}^{(i)(j)}, \omega_{(0)(3)}^{(i)(j)} = -\omega_{(1)(2)}^{(i)(j)} & \forall i, j \\ \omega_{(\alpha)(\alpha)}^{(i)(j)} = \omega_{(\beta)(\beta)}^{(i)(j)}, & \forall i, j, \alpha, \beta \\ \omega_{(\alpha)(\beta)}^{(i)(j)} = \omega_{(\beta)(\alpha)}^{(i)(j)}, & \forall i, j, \alpha, \beta, \alpha \neq \beta. \end{cases} \quad (2.15)$$

Finally, $*\omega = c_3\omega \wedge \Omega^{n-1}$ if and only if

$$\begin{cases} \sum_{i=1}^n \omega_{(0)(1)}^{(i)(i)} = \sum_{i=1}^n \omega_{(0)(2)}^{(i)(i)} = \sum_{i=1}^n \omega_{(0)(3)}^{(i)(i)} = 0 \\ \sum_{\alpha=0}^3 \omega_{(\alpha)(\alpha)}^{(i)(j)} = 0 \quad \forall i, j \\ \omega_{(0)(1)}^{(i)(j)} + \omega_{(0)(1)}^{(j)(i)} = \omega_{(2)(3)}^{(i)(j)} + \omega_{(2)(3)}^{(j)(i)} & \forall i, j \\ \omega_{(0)(2)}^{(i)(j)} + \omega_{(0)(2)}^{(j)(i)} = -\left(\omega_{(1)(3)}^{(i)(j)} + \omega_{(1)(3)}^{(j)(i)}\right) & \forall i, j \\ \omega_{(0)(3)}^{(i)(j)} + \omega_{(0)(3)}^{(j)(i)} = \omega_{(1)(2)}^{(i)(j)} + \omega_{(1)(2)}^{(j)(i)} & \forall i, j. \quad \blacksquare \end{cases} \quad (2.16)$$

DEFINITION 2.3: Let P be a principal bundle on M with connection ∇ . This connection is c -self-dual if its curvature 2-form is c -self-dual.

DEFINITION 2.4: For any real constant c , a generalized ‘‘Yang-Mills’’ functional on the space of connections on P is defined by

$$YM_c(\nabla) \doteq \frac{1}{2} \int_M [\|F\|^2 + c^2 \|F \wedge \Omega^{n-1}\|^2] \text{vol}(M), \quad (2.17)$$

where F is the curvature of the connection.

$YM_c(\nabla)$ has the following Euler-Lagrange equations

$$d * F + c^2 \left(d * (F \wedge \Omega^{n-1}) \right) \wedge \Omega^{n-1} = 0. \quad (2.18)$$

Notice that

$$\begin{aligned}
0 &\leq \| *F - cF \wedge \Omega^{n-1} \|^2 \\
&= \| *F \|^2 - 2\langle *F, cF \wedge \Omega^{n-1} \rangle + c^2 \| F \wedge \Omega^{n-1} \|^2 \\
&= \| F \|^2 - 2c(\text{tr}F \wedge F) \wedge \Omega^{n-1} + c^2 \| F \wedge \Omega^{n-1} \|^2 \\
&= \| F \|^2 - 16c\pi^2 p_1(P) \wedge \Omega^{n-1} + c^2 \| F \wedge \Omega^{n-1} \|^2
\end{aligned}$$

or

$$c(8\pi^2)p_1(P) \wedge \Omega^{n-1} \leq \frac{1}{2}[\| F \|^2 + c^2 \| F \wedge \Omega^{n-1} \|^2],$$

where $p_1(P)$ is the first Pontrjagin class of the bundle P on M . Hence, after integrating over M , we get

$$8\pi^2 c \int_M p_1(P) \wedge \Omega^{n-1} \text{vol}(M) \leq YM_c(\nabla). \quad (2.19)$$

The equality holds if and only if

$$*F = cF \wedge \Omega^{n-1},$$

i.e., if F is c -self-dual. In such case we shall call the connection ∇ itself a c -self-dual-connection. As $p_1(P)$ is a topological invariant of the bundle P , we define

$$Q(P) \doteq 8\pi^2 \int_M p_1(P) \wedge \Omega^{n-1} \text{vol}(M) \quad (2.20)$$

and call it a topological charge of the bundle P . We have just demonstrated that

PROPOSITION 2.5: *Any c -self-dual connection is a minimum of the Yang-Mills energy functional $YM_c(\nabla)$.*

The following result is due to [7].

PROPOSITION 2.6: *Any c -self-dual connection is an extremum of the Yang-Mills energy functional $YM(\nabla)$. Moreover, c_1 - and c_2 -self-dual connections are minimizing.*

PROOF: Suppose ∇ is a c -self-dual connection. Then

$$d * F = cd * (F \wedge \Omega^{n-1}) = 0$$

as $dF = d\Omega = 0$. Hence, $d * F = 0$ or ∇ is a Yang-Mills connection.

Let us write $F(\nabla) \in \Lambda^2 T^*M$ as

$$F(\nabla) = F_1 + F_2 + F_3,$$

where $F_1 \in S^2\mathbb{H}$, $F_2 \in S^2\mathbb{E}$, and $F_3 \in (S^2\mathbb{H} \oplus S^2\mathbb{E})^\perp$. Then

$$YM(\nabla) = \frac{1}{2} \int_M \left(\| F_1 \|^2 + \| F_2 \|^2 + \| F_3 \|^2 \right) \text{vol}(M)$$

because (1.1) is an orthogonal decomposition with respect to the usual norm $\|\cdot\|$ on $\Lambda^2 T^*M$. Notice that the topological charge of P can be written in terms of the components of $F(\nabla)$

$$Q(P) = \int_M \text{tr}(F \wedge F) \wedge \Omega^{n-1} \text{vol}(M) = \int_M \left(\frac{1}{c_1} \|F_1\|^2 + \frac{1}{c_2} \|F_2\|^2 + \frac{1}{c_3} \|F_3\|^2 \right) \text{vol}(M).$$

Hence, we can write $YM(\nabla)$ as

$$\begin{aligned} 2YM(\nabla) &= c_1 Q(P) + \int_M \left(\left(1 - \frac{c_1}{c_2}\right) \|F_2\|^2 + \left(1 - \frac{c_1}{c_3}\right) \|F_3\|^2 \right) \text{vol}(M) = \\ &= c_1 Q(P) + \int_M \left(\left(1 + \frac{3}{2n+1}\right) \|F_2\|^2 + \left(1 - \frac{1}{2n+1}\right) \|F_3\|^2 \right) \text{vol}(M) \end{aligned} \quad (2.21)$$

or

$$\begin{aligned} 2YM(\nabla) &= c_2 Q(P) + \int_M \left(\left(1 - \frac{c_2}{c_1}\right) \|F_1\|^2 + \left(1 - \frac{c_2}{c_3}\right) \|F_3\|^2 \right) \text{vol}(M) = \\ &= c_2 Q(P) + \int_M \left(\left(1 + \frac{2n+1}{3}\right) \|F_1\|^2 + \frac{4}{3} \|F_3\|^2 \right) \text{vol}(M) \end{aligned} \quad (2.22)$$

or

$$\begin{aligned} 2YM(\nabla) &= c_3 Q(P) + \int_M \left(\left(1 - \frac{c_3}{c_1}\right) \|F_1\|^2 + \left(1 - \frac{c_3}{c_2}\right) \|F_2\|^2 \right) \text{vol}(M) = \\ &= c_3 Q(P) + \int_M \left((-2n) \|F_1\|^2 + 4 \|F_2\|^2 \right) \text{vol}(M) \end{aligned} \quad (2.23)$$

It follows now from (2.21), (2.22) and Theorem 2.2 that c_1 - and c_2 -self-dual connections are minima of $YM(\nabla)$. \blacksquare

We do not know of any examples of c_3 -self-dual connections but (2.23) seems to indicate that, if they exist, they will be unstable.

3. Quaternionic maps and sigma models

In this chapter we introduce a new concept of quaternionic maps. We shall do it in such a way that it generalizes the theory of holomorphic mappings between Kähler manifolds. On the other hand we shall see that it is also very natural in studying instantons on 4-manifolds and 4-dimensional σ -models with composite $SU(2)$ (or $SO(3)$) gauge fields and Yang-Mills fields on quaternionic Kähler manifolds.

It is well-known that, if one defines a quaternionic Kähler submanifold to be a submanifold with a quaternionic structure given by restrictions, then it is automatically a totally geodesic submanifold [4]. We shall therefore not insist that the whole quaternionic structure be preserved by such mappings. Instead we adopt a weaker definition.

DEFINITION 3.1: Let M, N be quaternionic Kähler manifolds. A map f from M to N is called quaternionic if $f^*S^2\mathbb{H}_N \subset S^2\mathbb{H}_M$.

The following theorem is in an obvious analogy to the well-known result stating that holomorphic maps between Kähler manifolds are energy minimizing.

THEOREM 3.2: On the space of differentiable mappings between two compact oriented quaternionic Kähler manifolds M and N define the following functional

$$E(f) \doteq \frac{1}{2} \sum_{i=1}^3 \int_M \left(\|f^*\omega_i\|^2 + c^2 \|f^*\omega_i \wedge \Omega^{m-1}\|^2 \right) \text{vol}(M), \quad (3.1)$$

where $c = c_1 = \frac{6m}{(2m+1)!}$, $4m = \dim M$, and

$$Q(f) \doteq \int_M f^*\Omega_N \wedge \Omega_M^{m-1}. \quad (3.2)$$

Then $cQ(f) \leq E(f)$ and equality holds if and only if the map f is quaternionic.

PROOF: Let Ω_M, Ω_N be the fundamental 4-forms on M and N respectively. Once they are fixed $Q(f)$ is a homotopy invariant. As usual, we shall call it the degree or the topological charge of f .

Let $\{\omega_1, \omega_2, \omega_3\}$ be a local orthogonal frame on $S^2\mathbb{H}_N$ such that

$$\Omega_N = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.$$

We have to show that $E(f)$ is well-defined. If $\omega_i = \sum_j \phi_{ij} \mu_j$ is an $SO(3)$ -rotation of the frame field on $S^2\mathbb{H}_N$ then pointwisely

$$f^*\omega_i = \sum_{j=1}^3 (\phi_{ij}) f^*\mu_j.$$

Furthermore,

$$\begin{aligned} \sum_{i=1}^3 \|f^*\omega_i\|^2 &= \sum_{i=1}^3 f^*\omega_i \wedge *f^*\omega_i = \sum_{i=1}^3 \sum_{j,k} (\phi_{ij} f^*\mu_j) \wedge *(\phi_{ik} f^*\mu_k) = \\ &= \sum_{i=1}^3 \sum_{j,k} (\phi_{ij} \phi_{ik}) (f^*\mu_j \wedge *f^*\mu_k) = \sum_{i=1}^3 \sum_{j,k} (\phi_{ji}^{-1} \phi_{ik}) (f^*\mu_j \wedge *f^*\mu_k) = \end{aligned}$$

$$= \sum_{j,k} \delta_{jk} f^* \mu_j \wedge * f^* \mu_k = \sum_{j=1}^3 f^* \mu_j \wedge * f^* \mu_j = \sum_{j=1}^3 \|f^* \mu_j\|^2.$$

Similarly,

$$\begin{aligned} \sum_{i=1}^3 \|f^* \omega_i \wedge \Omega^{m-1}\|^2 &= \sum_{i=1}^3 \left(f^* \omega_i \wedge \Omega^{m-1} \right) \wedge * \left(f^* \omega_i \wedge \Omega^{m-1} \right) = \\ &= \sum_{i=1}^3 \sum_{j,k} \left(\phi_{ij} \phi_{ik} \right) \left(f^* \mu_j \wedge \Omega^{m-1} \right) \wedge * \left(f^* \mu_k \wedge \Omega^{m-1} \right) = \\ &= \sum_{j=1}^3 \left(f^* \mu_j \wedge \Omega^{m-1} \right) \wedge * \left(f^* \mu_j \wedge \Omega^{m-1} \right) = \sum_{j=1}^3 \|f^* \mu_j \wedge \Omega^{m-1}\|^2. \end{aligned}$$

Hence, $E(f)$ is independent of the choice of any normalized frame on $S^2\mathbb{H}_N$ and therefore well defined. Now the inequality $cQ(f) \leq E(f)$ follows from

$$0 \leq \| * f^* \omega_i - c f^* \omega_i \wedge \Omega^{m-1} \|^2$$

which can be written as

$$c \left\langle * f^* \omega_i, f^* \omega_i \wedge \Omega^{m-1} \right\rangle \leq \frac{1}{2} \left(\|f^* \omega_i\|^2 + c^2 \|f^* \omega_i \wedge \Omega^{m-1}\|^2 \right). \quad (3.3)$$

Since

$$\left\langle * f^* \omega_i, f^* \omega_i \wedge \Omega^{m-1} \right\rangle = f^* \omega_i \wedge f^* \omega_i \wedge \Omega^{m-1}$$

and

$$f^* \Omega_N = \sum_{i=1}^3 f^* \omega_i \wedge f^* \omega_i,$$

the inequality $cQ(f) \leq E(f)$ is simply obtained by summation of (3.3) over i and integration over M .

Finally, when $c = \frac{6m}{(2m+1)!}$, the assertion that $cQ(f) = E(f)$ is equivalent to the requirement that

$$* f^* \omega = c f^* \omega \wedge \Omega_M^{m-1}$$

holds for all $\omega \in S^2\mathbb{H}_N$, or that $f^* \omega \in S^2\mathbb{H}_M$ by Theorem 2.2., *i.e.*, f is quaternionic. \blacksquare

EXAMPLE 3.3: If $\dim M = 4$, $S^2\mathbb{H}_M \cong \Lambda_+^2$. As the Hodge $*$ -operator is conformally invariant, any orientation preserving conformal automorphism is a quaternionic map in our sense.

In [8] Atiyah gave a geometric construction for all basic $SU(2)$ -instantons, *i.e.*, anti-self-dual Yang-Mills fields on the Euclidean 4-sphere with topological charge -1 , as follows: The Euclidean 4-sphere is viewed as the quaternionic projective line $\mathbb{H}\mathbb{P}^1$. The tautological bundle is the bundle \mathbb{H} with charge -1 . The natural connection ∇ of \mathbb{H} is anti-self-dual. Let f be an orientation preserving conformal automorphism which is not an isometry. Then $f^*\nabla$, the pull-back connection on $f^*\mathbb{H}$, is a new anti-self-dual connection.

EXAMPLE 3.4: The above example can be easily generalized as follows: The quaternionic projective space $\mathbb{H}\mathbb{P}^n$ has a tautological bundle \mathbb{H} . By definition, any element of $GL^+(n+1, \mathbb{H})$ is an orientation preserving quaternionic linear map. In other words, if $f \in GL^+(n+1, \mathbb{H})$ is considered as an automorphism of $\mathbb{H}\mathbb{P}^n$, then $f^*\mathbb{H}$ is isomorphic to \mathbb{H} . It follows that $f^*S^2\mathbb{H} \equiv S^2\mathbb{H}$ and hence f is a quaternionic map. As the natural connection ∇ on \mathbb{H} is c_1 -self-dual, so is $f^*\nabla$. Besides, as long as f is not an isometry, $f^*\nabla$ is not gauge equivalent to ∇ . We do not know if these are all c_1 -self-dual connections on $\mathbb{H}\mathbb{P}^n$.

EXAMPLE 3.5: Another well-known example of a mapping which in our language is quaternionic is a general $SU(2)$ -instanton over 4-sphere with the topological charge k [8, 9]. The $S^2\mathbb{H}$ bundle on the quaternionic projective space $\mathbb{H}\mathbb{P}^k$ has a canonical $Sp(1)$ -connection and all instantons over S^4 are induced by an appropriate choice of $f : S^4 \rightarrow \mathbb{H}\mathbb{P}^k$. In fact f can be described explicitly as follows: If $\mathbf{u} \in \mathbb{H}\mathbb{P}^k$ is a local (Fubini-Study) quaternionic coordinate on the quaternionic projective space and $x \in S^4$ is a local quaternionic coordinate on the 4-sphere identified with the quaternionic projective line $\mathbb{H}\mathbb{P}^1$ then

$$\mathbf{u}(x) = \left[\vec{\lambda} \cdot (\mathbb{B} - x\mathcal{I}) \right]^\dagger \quad (3.4)$$

where $\vec{\lambda} = (\lambda_1, \dots, \lambda_k)$ is a quaternionic row vector, \mathbf{u} is a quaternionic column vector, \mathbb{B} is a symmetric quaternionic $n \times n$ matrix, \dagger denotes quaternionic conjugation and transposition, and $(\vec{\lambda}, \mathbb{B})$ are subject to the following two conditions

$$\begin{cases} \text{Im}(\mathbb{B}^\dagger \mathbb{B} + \vec{\lambda}^\dagger \vec{\lambda}) = 0 \\ \left(\forall x \in \mathbb{H}\mathbb{P}^1 \quad (\mathbb{B} - x\mathcal{I})\vec{\xi} = 0, \quad \vec{\lambda} \cdot \vec{\xi} = 0 \quad \text{where } \vec{\xi} \in \mathbb{H}^k \right) \Rightarrow \vec{\xi} = 0. \end{cases} \quad (3.5)$$

In the same way k -instantons over the complex projective plane can be generated by quaternionic maps from $bbc\mathbb{P}^2 \rightarrow \mathbb{H}\mathbb{P}^{2k}$ [10, 11].

The energy functional (3.1) may also be interpreted as an $SO(3)$ locally gauge invariant lagrangian of an interesting class of nonlinear field theories called σ -models. In particular, if $\dim M = 4$, one can think of M as a physical, possibly curved, space-time and $f(x)$, $x \in M$, becomes an N -valued classical field with the action functional given by $E(f)$. $E(f)$ is manifestly invariant with respect to the global coordinate transformation on M (diffeomorphisms of M) as well as it is gauge invariant under the following gauge transformations

$$\left(f^*\omega_i \right)_x \rightarrow \sum_j \Phi_{ij}(x) \left(f^*\omega_j \right)_x, \quad (3.6)$$

where $\Phi_{ij}(x)$ is a local $SO(3)$ transformation and $(f^*\omega_i)$ is the curvature 2-form of a gauge field A_j on M defined as follows

$$d(f^*\omega_i) = \sum_{j,k} \epsilon_{ijk} A_j \wedge f^*\omega_k. \quad (3.7)$$

The gauge potential 1-form on A_j transforms in the usual way

$$\delta(\epsilon_{ijk} A_k) = -d\Phi_{ij}(x). \quad (3.8)$$

$A_j(f)$ depends on the choice of $f(x)$, *i.e.*, it is a composite gauge field. If $N = \mathbb{H}\mathbb{P}^n$ and $\mathbf{u} \in \mathbb{H}\mathbb{P}^n$ as before then

$$A(\mathbf{u}) = -\frac{1}{2} \frac{\mathbf{u}^\dagger \cdot d\mathbf{u} - d\mathbf{u}^\dagger \cdot \mathbf{u}}{1 + \mathbf{u}^\dagger \cdot \mathbf{u}} = iA_1 + jA_2 + kA_3.$$

This particular example was introduced and extensively studied by Gürsey and Tze [12]. Here we see that many interesting global and local properties of $\mathbb{H}\mathbb{P}^n$ -model are common for a large class of field theoretical models based on $E(f)$. All of them have duality equations built in and all possess global topological invariants.

4. Generalized Bogomolny equations

In this section we discuss some special solutions of the c -self-duality equations. If $M = \mathbb{R}^4 \ni (x_0, x_1, x_2, x_3)$ and P is a principal bundle over M then one can study x_0 -invariant solutions to the usual self-dual equations. They are called time invariant instantons or monopoles. In our case, let $M = \mathbb{R}^{4n} \simeq \mathbb{R}^4 \otimes \mathbb{R}^n \ni \{x_\alpha^i\}_{\alpha=0,\dots,3}^{i=1,\dots,n}$, P be a principal bundle over M , and let $YM_c(\nabla)$ be our Yang-Mills functional. In an obvious analogy to the 4-dimensional case we can study x_0^i invariant c -self-dual connections on M or “ c -monopoles” on $\mathbb{R}^3 \otimes \mathbb{R}^n$. Let us start with the following observation.

PROPOSITION 4.1: *Let $M = \mathbb{R}^4 \otimes \mathbb{R}^n$ be the $4n$ -dimensional Euclidean flat space with global linear coordinates x_α^i , $\alpha = 0, 1, 2, 3$; $i = 1, \dots, n$. For any (x_1, \dots, x_n) in \mathbb{R}^n we define*

$$p: \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \otimes \mathbb{R}^n$$

by

$$(x_0, x_1, x_2, x_3) \longrightarrow x_\alpha^i = x_\alpha x^i. \quad (4.1)$$

Suppose P is a principal bundle over M with connection ∇ and curvature F . Then $p^*\nabla$ is an anti-self-dual connection on p^*P if

$$*F = -\frac{1}{(2n-1)!} F \wedge \Omega^{n-1}, \quad (4.2)$$

i.e., F is c_2 -self-dual.

PROOF: In the x_α^i -coordinates dx_α^i is exactly the 1-form ω_α^i of (2.4). Now a 2-form ω satisfies the equation

$$*F = -\frac{1}{(2n-1)!} F \wedge \Omega^{n-1}$$

if and only if

$$F = -\frac{1}{6} * (F \wedge *\Omega). \quad (4.3)$$

Using Theorem 2.2 we get the following solutions

$$\begin{cases} F_{(0)(1)}^{(i)(j)} = -F_{(2)(3)}^{(i)(j)}, F_{(0)(2)}^{(i)(j)} = F_{(1)(3)}^{(i)(j)}, F_{(0)(3)}^{(i)(j)} = -F_{(1)(2)}^{(i)(j)} \quad \forall i, j \\ F_{(\alpha)(\alpha)}^{(i)(j)} = F_{(\beta)(\beta)}^{(i)(j)}, \quad \forall i, j, \alpha, \beta \\ F_{(\alpha)(\beta)}^{(i)(j)} = F_{(\alpha)(\beta)}^{(j)(i)}, \quad \forall i, j, \alpha, \beta, \alpha \neq \beta. \end{cases} \quad (4.4)$$

Let us denote the components of p^*F by $F_{\alpha\beta}$. As a consequence of the chain rule we get

$$F_{\alpha\beta} = \sum_{i,j} x^i x^j F_{(\alpha)(\beta)}^{(i)(j)} \quad (4.5)$$

and therefore

$$\begin{aligned} F_{01} &= \sum_{i,j} x^i x^j F_{(0)(1)}^{(i)(j)} = -F_{23} \\ F_{02} &= \sum_{i,j} x^i x^j F_{(0)(2)}^{(i)(j)} = F_{13} \\ F_{03} &= \sum_{i,j} x^i x^j F_{(0)(3)}^{(i)(j)} = -F_{12}. \end{aligned} \quad (4.6)$$

In other words, $p^*\nabla$ is an anti-self-dual connection. ■

Recently, Pedersen and Poon used twistorial approach to find a generalization of the Bogomolny equations [5]. They introduced Yang-Mills-Higgs equations on $\mathbb{R}^3 \otimes \mathbb{R}^n$. If one considers monopoles on \mathbb{R}^3 as time invariant instantons on \mathbb{R}^4 the following simple geometric description of generalized monopoles comes with no surprise.

PROPOSITION 4.2: *Let x_μ^i , $\mu = 0, 1, 2, 3$; $i = 1, \dots, n$ be a global linear coordinate on $\mathbb{R}^4 \otimes \mathbb{R}^n$ and let*

$$p: \quad \mathbb{R}^4 \otimes \mathbb{R}^n \longrightarrow \mathbb{R}^3 \otimes \mathbb{R}^n$$

be a projection

$$(x_0^i, x_1^i, x_2^i, x_3^i) \longrightarrow (x_1^i, x_2^i, x_3^i).$$

If (∇, Φ^i) is a generalized monopole then

$$\nabla' \doteq p^* \nabla + \sum_i \Phi^i dx_0^i \quad (4.7)$$

is a c_2 -self-dual connection. Conversely, any c_2 -self-dual connection that is x_0^i -invariant determines a solution of the generalized monopole equation.

PROOF: The curvature F' of the connection ∇' is given by

$$F' = p^* F + \sum_j (\nabla \Phi^j) \wedge dx_0^j + \frac{1}{2} \sum_{i < j} [\Phi^i, \Phi^j] dx_0^i \wedge dx_0^j, \quad (4.8)$$

where F is the curvature 2-form of ∇ . Now, using equations (4.4), we get

$$\begin{cases} \nabla_{(i)} \Phi^j = F_{(i)(j)}, \nabla_{(j)} \Phi^i = -F_{(i)(j)}, \nabla_{(j)} \Phi^i = F_{(i)(j)} \quad \forall i, j \\ F_{(\alpha)(\alpha)} = \frac{1}{2} [\Phi^i, \Phi^j], \quad \forall i, j; \quad \alpha = 1, 2, 3 \\ \nabla_{(\alpha)} \Phi^j = \nabla_{(j)} \Phi^i \quad \forall i, j; \quad \alpha = 1, 2, 3 \end{cases} \quad (4.9)$$

which can be written as

$$\begin{cases} F_{(\alpha)(\beta)} = \sum_{\gamma} \epsilon_{\alpha\beta\gamma} \nabla_{(\gamma)} \Phi^j + \frac{1}{2} \delta_{\alpha\beta} [\Phi^i, \Phi^j] \quad \forall i, j; \quad \forall \alpha, \beta = 1, 2, 3 \\ \nabla_{(\alpha)} \Phi^j = \nabla_{(j)} \Phi^i \quad \forall i, j; \quad \alpha = 1, 2, 3. \end{cases} \quad (4.10)$$

The converse is obvious. ■

We can also obtain “monopole” analogues of c -self duality equations in the c_1 and c_3 cases. The first one is not interesting, however, because it yields n decoupled self-dual Bogomolny equations. In the second case we can explicitly write down the set of equations

$$\begin{cases} F_{(\alpha)(\beta)} + F_{(j)(i)} = \sum_{\gamma} \epsilon_{\alpha\beta\gamma} \left(\nabla_{(\gamma)} \Phi^j + \nabla_{(j)} \Phi^i \right), \quad \forall i, j; \quad \forall \alpha, \beta \\ \sum_{i=1}^n \nabla_{(\alpha)} \Phi^i = 0 \quad \forall \alpha, \quad [\Phi^i, \Phi^j] = - \sum_{\alpha=1}^3 F_{(\alpha)(\alpha)} \quad \forall i, j. \end{cases} \quad (4.11)$$

For $n = 1$ these are just the usual Bogomolny equations with the reversed orientation. We do not know any non-trivial solutions of (4.11) for $n > 1$ at the moment. Finally, let us remark that we could introduce additional invariance and reduce the c -self-duality equation to $2n$ dimensions, assuming that the c -self-dual equations on $\mathbb{R}^4 \otimes \mathbb{R}^n$ be both x_0^i and x_1^i -invariant. Then we obtain an analogue of the well-known vortex equation of the two-dimensional Yang-Mills-Higgs theory. Again c_2 case is the most natural generalization and we shall address this problem in a future work.

References

- [1] S. ISHIHARA, *Quaternion Kählerian manifolds*, J. Differential Geometry 9, 483(1974).
- [2] S. SALAMON, *Quaternionic Kähler manifolds*, Invent. Math. 67, 143-171(1982).
- [3] S. SALAMON, *Differential geometry of quaternionic manifolds*, Ann. Scient. Éc. Norm. Sup. 19, 31-55(1986).
- [4] H. PEDERSEN AND Y. S. POON, *Hyper-Kähler Metrics and a Generalization of the Bogomolny Equations*, Commun. Math. Phys. 117, 569-580(1989).
- [5] A. SWANN, *On the Fundamental 4-Form of a Quaternion-Hermitian Manifold*, Oxford University preprint (1990).
- [6] E. CORRIGAN, P. GODDARD AND A. KENT, *Some comments on the ADHM construction in 4k dimensions*, Commun. Math. Phys. 100, 1-13(1985).
- [7] M. M. CAPRIA AND S. SALAMON, *Yang-Mills fields on quaternionic spaces*, Nonlinearity 1, 517-530(1988).
- [8] M. F. ATIYAH, *Geometry of Yang-Mills Fields*, Lezioni Fermiane, Pisa 1979.
- [9] M. F. ATIYAH, V. G. DRINFELD, N. J. HITCHIN, AND YU. I. MANIN, *Construction of instantons*, Phys. Lett., 65A, 185-187(1978).
- [10] N. BUCHDAL, *Instantons on $\mathbb{C}P^2$* , J. Differential Geometry 24, 19-52(1986).
- [11] S. K. DONALDSON, *Vector bundles on flag manifolds and the Ward correspondence*, Geometry Today, Progress in Math., No. 60, Birkhauser, Boston, 1985.
- [12] F. GÜRSEY AND C. H. TZE, *Complex and quaternionic analyticity and gauge theories I*, Annals of Physics 128, 29-130(1980).

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