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Hypercomplex Structures from 3-Sasakian Structures

Charles P. Boyer Krzysztof Galicki Benjamin M. Mann

Abstract. This paper describes certain hypercomplex manifolds as circle V-bundles over 3-Sasakian orbifolds. Our techniques involve both 3-Sasakian and hypercomplex reduction. In general dimension most of the quotients exist only as hypercomplex orbifolds; however, we construct a large family of compact simply connected smooth 8-manifolds whose second integral homology group is free with arbitrary rank. We also construct hypercomplex manifolds in any dimension 4n whose second Betti number is either 1 or 2.

Introduction

A hypercomplex structure on a smooth manifold M^{4n} is a G-structure where $G = GL(n,\mathbb{H})$ that admits a necessarily unique torsion free connection, the Obata connection [Bon,Ob]. In particular, every such M has three complex structures $I, J,$ and K which satisfy the relations of the algebra of imaginary quaternions. Compact hypercomplex manifolds M of dimension 4 ('quaternionic curves') have been classified [Boy]. Such M are either hyperkähler, in which case they are either tori T^4 or $K3$ surfaces, or M is a quaternionic Hopf surface [Ka]. Thus, both the geometry and topology of compact hypercomplex 4-manifolds are quite restrictive. Every such manifold is locally conformally hyperkähler [Boy], and has Kodaira dimension $-\infty$ or 0. Topologically the second Betti number is either 0 or 22, and every compact simply connected hypercomplex 4-manifold is hyperkähler, in fact a $K3$ surface. By contrast for compact hypercomplex manifolds of dimension 8 ('quaternionic surfaces') and higher the situation is dramatically different. Very recently several new constructions involving compact hypercomplex manifolds that are not hyperkähler have made this difference apparent.

In the case of compact hyperkähler geometry all known examples of irreducible hyperkähler manifolds are deformation equivalent [Huy] to one of two possibilities, the Hilbert scheme of points on a $K3$ surface, or a generalized Kummer variety. Both of these examples are due to Beauville [Bea], and both are simply connected. Outside the world of hyperkähler geometry the simplest examples of hypercomplex manifolds are the ones that are locally conformally hyperkähler. All of these are generalized Hopf manifolds of Vaisman [Vai], and they admit a natural one-dimensional foliation which when the leaves are compact has a compact 3-Sasakian orbifold as its space of leaves [OrPi]. In particular, all such homogeneous examples have been classified [OrPi] by using the classification of homogeneous 3-Sasakian manifolds in [BGM2]. The connection to 3-Sasakian geometry also gives large families of examples of inhomogeneous hypercomplex manifolds constructed as flat bundles over 3-Sasakian manifolds $\mathcal{S}$. This is done by taking $\mathcal{S}$ to be any of the

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3-Sasakian manifolds recently constructed in [BGM3, BGMR] or a quotient thereof, and by choosing a homomorphism $\rho : \pi_1(\mathcal{S}) \longrightarrow S^1$. However, these spaces are never simply connected.

The case when M is not locally conformally hyperkähler is perhaps the most intriguing. Very little is known in general about such spaces. A class of homogeneous hypercomplex manifolds that are not locally conformally hyperkähler was studied by physicists interested in supersymmetric σ-models. In this regard, Spindel et. al. [SSTP] classified compact Lie groups which admit hypercomplex structures. Using different methods, Joyce [Joy1] later recovered this [SSTP] classification and developed a theory of homogeneous hypercomplex manifolds. He also introduced a construction he called hypercomplex quotient [Joy2] which is a variant of the hyperkähler reduction of [HKLR]. The authors [BGM4] used this reduction to study hypercomplex structures on the complex Stiefel manifold $V^{\mathbb{C}}_{2,n}$ of 2-frames in $\mathbb{C}^n$. The homogeneous hypercomplex structure on this complex Stiefel manifold was first discussed by Joyce in [Joy1] and later as a submanifold of a quaternionic vector space by Battaglia [Bat]. However, we studied families of hypercomplex structures on $V^{\mathbb{C}}_{2,n}$ most of which are inhomogeneous, and investigated the equivalence problem. The automorphism group of the hypercomplex structures given in [BGM2, BGM4] all contain an n-torus T^n with an additional $SU(2)$ rotating the hypercomplex structures. The quotient construction can be generalized to give n^2-parameter family of structures. This was observed by Pedersen and Poon [PP] who also proved that this family is complete. They developed a deformation theory of hypercomplex structures allowing them to compute the dimension of the moduli space in a neighborhood of the homogeneous hypercomplex structure [PP]. In dimension 8, the Stiefel manifold $V^{\mathbb{C}}_{2,3}$ is just $SU(3)$ and the hypercomplex geometry of this group manifold has been seen in several different contexts. Until the work of Joyce [Joy1] this was the only simply connected example that is not hyperkähler.

Joyce introduced yet another method of generating simply connected examples. In dimension 8 the construction can be described as follows: Let us consider any compact self-dual 4-manifold N and take the associated quaternionic line bundle of N which is known to be hypercomplex [S]. This bundle can be compactified and then twisted with another $U(1)$-bundle (an instanton) to produce new simply connected hypercomplex 8-manifolds. This construction has recently received more systematic study by Pedersen, Poon and Swann [PPS]. They proved that any hypercomplex manifold with a certain type of a free $U(2)$ action, up to $\mathbb{Z}/2$ cover, must arise this way. When $N = \mathbb{CP}^2$, twisting with the primitive instanton bundle over $\mathbb{CP}^2$, gives the homogeneous hypercomplex geometry of $SU(3)/\mathbb{Z}_2$. The authors give a detailed discussion of the $N = 2\mathbb{CP}^2$ case showing that there are twists giving simply connected 8-manifolds. We are not aware of any study of the hypercomplex structures in the $k > 2$ case, although, in principle, one could use the explicit description of the self-dual geometry of $k\mathbb{CP}^2$ due to LeBrun [LeB] and carry it out.

This paper presents a different approach and a bundle construction which employs the relation to 3-Sasakian geometry and 3-Sasakian reduction. We believe that our method has several important advantages. First, it appears that our method is a natural extension of the locally conformally hyperkähler geometry which is just a special case for us. Then we can utilize all the recent results on compact 3-Sasakian 7-manifolds [BGMR, BGM3, Bi]. Finally, our construction is very explicit and gives some insight into the world of compact simply connected hypercomplex 8-manifolds.

In the first half of this paper we describe hypercomplex structures on circle V-bundles $H(\mathcal{S})$ over 3-Sasakian orbifolds $\mathcal{S}$, and study some of their properties. We show that these hypercomplex structures define a unique complex structure which is never Kähler. In fact, $H(\mathcal{S})$ cannot admit any symplectic structure. Moreover, they are (in general, singular) holomorphic fibration whose fibers are elliptic curves over a base space that is a $\mathbb{Q}$-Fano projective variety, namely the twistor space of $\mathcal{S}$ described in [BG]. As a complex analytic manifold $H(\mathcal{S})$ has vanishing geometric genus, and the algebraic dimension $\mathfrak{a}(H(\mathcal{S}))$ is either $2n-1$ or $2n$; if $\pi_1(H(\mathcal{S}))$ is finite, $\mathfrak{a}(H(\mathcal{S}))$ must be $2n-1$. When the bundle $H(\mathcal{S})$ is flat then $H_1(H(\mathcal{S}),\mathbb{Q}) = \mathbb{Q}$ and $H(\mathcal{S})$ is locally conformally hyperkähler.

The second half of the paper contains our main results, where we construct hypercomplex toral quotients. This is done in arbitrary dimension in section 3 by applying Joyce's hypercomplex reduction [Joy2] to the hypercomplex Stiefel manifolds $\mathcal{N}(\mathbf{p})$ obtained previously [BGM2]. We explicitly construct hypercomplex orbifolds as toral quotients in every allowable dimension. However, two problems arise, the first theoretical, and the second technical. First, if we demand that the orbifolds be smooth manifolds then there will be bounds on both the second Betti number for a large enough fixed dimension and on the dimension for a large enough fixed second Betti number. This was described in the 3-Sasakian context in [BGM3]. Second, the techniques [BGMR] that we have developed for understanding the topology of such quotients are limited so far to the case when dim $H(\mathcal{S}) = 8$. Indeed, in this regard we show in section 4 that our smooth hypercomplex toral quotients of dimension 8 are simply connected and have no 2-torsion. In fact, so far this is the only method we have for distinguishing the toral quotients $H(\mathcal{S})$ from the trivial bundle over $\mathcal{S}$. In particular, we prove

THEOREM A: *Consider the following matrix*

$$\Omega = \begin{pmatrix} p_1 & p_2 & \dots & p_k & p_{k+1} & p_{k+2} \\ 1 & 0 & \dots & a_2 & b_2 & c_2 \\ \vdots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & a_k & b_k & c_k \end{pmatrix}.$$

Let all $k \times k$ minor determinants of Ω be non-zero and assume that $\gcd(a_i, b_i, c_i) = 1$ *for all $i = 2, \cdots, k$. Then, for each such Ω there exists a compact hypercomplex 8-manifold $H(\Omega)$ with $\pi_1(H(\Omega)) = 0$ and $\pi_2(H(\Omega)) = H_2(H(\Omega)) = \mathbb{Z}^{k-1}$. Furthermore, each $H(\Omega)$ has at least a real one parameter family of inequivalent hypercomplex structures on it.*

It is not difficult to see that for each k the conditions of this theorem can be satisfied. For $k = 1$ Theorem A reproduces the hypercomplex structures on the Stiefel manifold $\mathcal{N}(p_1, p_2, p_3) \simeq SU(3)$ discussed in [BGM2,BGM4]. These can be obtained as deformations [PP] of the homogeneous hypercomplex structure on $SU(3)$. But for $k > 1$ there is no homogeneous hypercomplex structure, and for large enough k the manifold $H(\Omega)$ cannot be homotopy equivalent to any homogeneous space. Each $H(\Omega)$ has a T^3 of hypercomplex symmetries and there also is an $SU(2)$ which rotates the two sphere of complex structures. Topologically, for a given $k > 1$, the bundles $H(\mathcal{S})$ have a lot in common with twisted compact associated bundles over $k\mathbb{CP}^2$. The precise relation between the two different constructions remains unclear. We only point out that our bundles $H(\mathcal{S})$ do not appear to have any free $U(2)$-action of the type considered in [PPS] which is a characteristic property of any twisted associated bundle over a compact quaternionic manifold. Our $U(2)$ action described in section 2 is clearly of the same type but it is only locally free, and for $k > 1$ it is never free regardless the choice of parameters $(\mathbf{p}, \Omega_1)$. One could, however, view our construction as an orbifold version of the construction of [Joy1, PPS] with the base space $\mathcal{O}$ being a compact self-dual Einstein orbifold constructed in [GN].

As a corollary of Theorem A, we have

COROLLARY B: *Let $H(\Omega)$ be as in Theorem A, and assume that the* 3*-Sasakian quotient $\mathcal{S}$ is smooth. Then for each positive integer l there are hypercomplex 8-manifolds $H(\Omega, l)$ with $\pi_1(H(\Omega, l)) \simeq \mathbb{Z}_l$ and $b_2(H(\Omega, l)) = k - 1$. Furthermore, each $H(\Omega, l)$ has at least a real one parameter family of hypercomplex structures.*

As an immediate consequence of [BGMR, BGM3, Bi] and Theorem 4.1 below we get

COROLLARY C: *There exist hypercomplex generalized Hopf manifolds $H(\Omega)$ of dimensions* $8, 12$ *and* 16 *having any second Betti number $b_2 \leq 4$. There exist hypercomplex generalized Hopf manifolds of the form $H(\Omega) = \mathcal{S}(\Omega) \times S^1$ of dimension 8 with $\pi_1(H(\Omega)) = \mathbb{Z}$, and $H_2(H(\Omega)) = \mathbb{Z}^k$ for any integer k. Furthermore, each of these manifolds has a real one parameter family of inequivalent hypercomplex structures.*

Our quotient construction also gives hypercomplex toral quotients of the same dimensions. However, as mentioned above we have been successful in analyzing the topology only in dimension 8. We certainly believe that these are all simply connected with possible second Betti numbers $1, 2$ and 3. Our construction also gives hypercomplex manifolds of dimension $4n$ with second Betti number either 1 or 2. We also believe that these are simply connected with second Betti number 1.

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§1. Hypercomplex Circle Bundles over 3-Sasakian Orbifolds

We are interested in hypercomplex manifolds that are circle V-bundles over 3-Sasakian orbifolds.

DEFINITION 1.1: *An orbifold $\mathcal{S}$ is said to have an almost contact* 3*-structure if there are vector fields ξ^a, one forms η^a, and* $(1,1)$ *tensor fields Φ^a, for $a = 1, 2, 3$ that are invariant* *under the action of all the local uniformizing groups of the orbifold and that satisfy the conditions*

1.3
$$\begin{aligned} \eta^a(\xi^b) &= \delta^{ab}, \\ \Phi^a\xi^b &= -\epsilon^{abc}\xi^c, \\ \Phi^a\circ\Phi^b - \xi^a\otimes\eta^b &= -\epsilon^{abc}\Phi^c - \delta^{ab}\mathbb{I}. \end{aligned}$$

If the almost contact 3-structure is a 3-Sasakian structure with respect to a metric g which is invariant under all the local uniformizing groups, then $\mathcal{S}$ is called a 3-*Sasakian orbifold.*

REMARK 1.2: The condition of invariance of tensor fields under the local uniformizing groups adopted here has quite strong consequences. It implies for example that each stratum in the singular locus has a 3-Sasakian structure. This can be contrasted to the case of both the twistor and quaternionic Kähler orbifolds [BG,GL]. Here the invariance condition is imposed on a certain bundle and the singular locus need not carry the original structure.

We are particularly interested in the case when the total space of a circle V-bundle over a 3-Sasakian orbifold is a smooth manifold. Let $\mathcal{S}$ be a 3-Sasakian orbifold with local uniformizing systems $\{U,\Gamma,\phi\}$. A circle V-bundle $H(\mathcal{S})$ over $\mathcal{S}$ is described [Ba1,Ba2] by locally trivial bundles $U\times S^1$ over the local uniformizing neighborhoods together with a map $\gamma\mapsto h_U$ defined by

1.3
$$h_U(\gamma)(x,u) = (\gamma^{-1}x, \eta_U(\gamma)(x)\cdot u),$$

where $\gamma\mapsto\eta_U(\gamma)(x)$ is a group homomorphism from Γ into the group of the bundle S^1.

LEMMA 1.4: *Let $H(\mathcal{S})$ be a circle V-bundle over an orbifold $\mathcal{S}$. The total space of $H(\mathcal{S})$ is a smooth manifold if and only if the homomorphism η_U is a monomorphism everywhere on $\mathcal{S}$.*

PROOF: The orbifold charts on $H(\mathcal{S})$ are given by open sets $\tilde{V}$ of the quotients $(U\times S^1)/\sim$ with respect to the action 1.3. If η_U is injective this action is free, so $\tilde{V}$ is homeomorphic to an open set of $\mathbb{R}^{n+1}$, where $n=\dim\ \mathcal{S}$. Conversely, let x be in the singular locus of the orbifold $\mathcal{S}$. Then x is a fixed point of the local uniformizing group Γ. If the map $\gamma\mapsto\eta_U(\gamma)$ has a non-trivial kernel K then K is a subgroup of Γ which fixes (x,u). Hence, in a neighborhood of x the quotient $(U\times S^1)/\sim$ is locally homeomorphic to $\mathbb{R}^{n+1}/K$. ∎

Henceforth we shall refer to a V-bundle on an orbifold simply as a bundle unless we wish to emphasize the orbifold aspects. Let $H(\mathcal{S})$ be a circle bundle over a 3-Sasakian orbifold $\mathcal{S}$. At this stage we do not assume that the total space $H(\mathcal{S})$ is a smooth manifold but only an orbifold of a certain type. There is natural almost hypercomplex structure on $H(\mathcal{S})$ defined as follows: Let $\pi: H(\mathcal{S})\to\mathcal{S}$ denote the natural projection and let $\hat{g}$ be a Riemannian metric on $H(\mathcal{S})$ that is invariant under the local uniformizing groups and such that $\pi : (H(\mathcal{S}), \hat{g}) \to (\mathcal{S}, g)$ is a Riemannian submersion. Let $\mathcal{V}_1$ denote the vertical subbundle of the tangent bundle $TH(\mathcal{S})$ to $H(\mathcal{S})$ and let Ξ be a nowhere vanishing section of $\mathcal{V}_1$ that generates the S^1 action on $H(\mathcal{S})$. The almost contact 3-structure on $\mathcal{S}$ allows us to define an almost hypercomplex structure on $H(\mathcal{S})$. This is done as follows. The metric $\hat{g}$ on $H(\mathcal{S})$ splits the tangent bundle $TH(\mathcal{S})$ as $TH(\mathcal{S}) \simeq \hat{\mathcal{H}} \oplus \mathcal{V}_1$ and π_* induces an isometry between the horizontal vector space $\hat{\mathcal{H}}_p$ at a point $p \in H(\mathcal{S})$ and the tangent space $T_{\pi(p)}\mathcal{S}$. For any vector field X on $\mathcal{S}$, we denote by $\hat{X}$ its horizontal lift to $H(\mathcal{S})$, that is, $\hat{X}$ is the unique *basic* vector field that is π-related to X. In particular, the three vector fields $\hat{\xi}^a$ generate a subbundle $\hat{\mathcal{V}}_3$ of $\hat{\mathcal{H}}$ that is isometric at every point to the bundle $\mathcal{V}_3$ on $\mathcal{S}$. Let $\tilde{\mathcal{H}}$ denote the orthogonal complement to $\hat{\mathcal{V}}_3$ in $\hat{\mathcal{H}}$, so that we have the further splitting $TH(\mathcal{S}) \simeq \hat{\mathcal{H}} \oplus \mathcal{V}_1 \simeq \tilde{\mathcal{H}} \oplus \hat{\mathcal{V}}_3 \oplus \mathcal{V}_1$. Since the Φ^a 's are sections of End $\mathcal{H} \oplus$ End $\mathcal{V}_3$ on $\mathcal{S}$ they lift to sections $\hat{\Phi}^a$ of End $\tilde{\mathcal{H}} \oplus$ End $\hat{\mathcal{V}}_3$ on $H(\mathcal{S})$ defined on basic vector fields by $\hat{\Phi}^a\hat{X} = \widehat{\Phi^a X}$ and extended to arbitrary sections of End $\tilde{\mathcal{H}} \oplus$ End $\hat{\mathcal{V}}_3$ by linearity. We further assume that $\hat{\xi}^a$ and $\hat{\Phi}^a$ are invariant under the local uniformizing groups of $H(\mathcal{S})$. Hence, we can define endomorphisms $\mathcal{I}^a$ on $TH(\mathcal{S})$ by

1.5
$$\begin{aligned} \mathcal{I}^a X &= -\hat{\Phi}^a X + \pi^*\eta^a(X)\Xi, \\ \mathcal{I}^a \Xi &= -\hat{\xi}^a, \end{aligned}$$

where X is any horizontal vector field on $H(\mathcal{S})$ (i.e., any section of $\hat{\mathcal{H}}$). If this almost hypercomplex structure is integrable, we call this a *compatible* (with the 3-Sasakian structure on $\mathcal{S}$) *hypercomplex structure* on $H(\mathcal{S})$. Every hypercomplex structure on $H(\mathcal{S})$ considered in this paper will be of this type.

Now the metric $\hat{g}$ is not necessarily hyperhermitian; however, it is easy to see that by rescaling along the fibers of π by a factor of $\hat{g}(\Xi, \Xi)^{-1}$ yields a hyperhermitian metric which we denote by h. We shall refer to the pair $(H(\mathcal{S}), \Xi)$ as a *framed circle bundle* on $\mathcal{S}$. Now, given a framed circle bundle $(H(\mathcal{S}), \Xi)$, the horizontal subbundle $\hat{\mathcal{H}}$ defines a $\mathfrak{u}(1)$ connection on $H(\mathcal{S})$ if and only if $[\Xi, X]$ is a section of $\hat{\mathcal{H}}$ whenever X is, and it is direct to prove

PROPOSITION 1.6: *Let $(H(\mathcal{S}), \Xi)$ be a framed circle bundle and let h be an hyperhermitian metric on $H(\mathcal{S})$. Then the following are equivalent:*

(i) Ξ is a Killing vector field with respect to h.

(ii) $\hat{\mathcal{H}}$ defines a $\mathfrak{u}(1)$ connection on $H(\mathcal{S})$.

(iii) The two conditions hold:

(a) $[\Xi, \hat{\xi}^a] = 0$ for all $a = 1, 2, 3$.

(b) $[\Xi, X]$ is a section of $\tilde{\mathcal{H}}$ whenever X is.

Next we examine the curvature of $\mathfrak{u}(1)$ connections of a framed circle bundle $(H(\mathcal{S}), \Xi)$

over a 3-Sasakian orbifold $\mathcal{S}$. Let $\hat{\eta}^0$ denote the 1-form on $H(\mathcal{S})$ that is dual to the vector field Ξ with respect to the hyperhermitian metric h, i.e., $\hat{\eta}^0(X) = h(\Xi, X)$ for any vector field X on $(H(\mathcal{S}), \Xi)$. In the case that $\hat{\mathcal{H}}$ defines a connection on $(H(\mathcal{S}), \Xi)$, one sees that $\hat{\eta}^0$ is the connection 1-form of $\hat{\mathcal{H}}$. The curvature of this connection is the 2-form defined by

1.7 $$\omega(X,Y) = d\hat{\eta}^0(X,Y),$$

where X, Y are any vector fields on $H(\mathcal{S})$. Now on a 3-Sasakian orbifold $\mathcal{S}$ the vector fields ξ^a form a basis for the Lie algebra $\mathfrak{su}(2)$. However, in general, the horizontal lifts $\hat{\xi}^a$ to $(H(\mathcal{S}), \Xi)$ do not. Certain components of the curvature 2-form ω are obstructions to lifting the Lie algebra. A computation shows that

1.8 $$[\hat{\xi}^a, \hat{\xi}^b] = 2\epsilon^{abc}\hat{\xi}^c - \omega(\hat{\xi}^a, \hat{\xi}^b)\Xi.$$

The following proposition is an immediate consequence of equation 1.8.

PROPOSITION 1.9: *Let $(H(\mathcal{S}), \Xi)$ be a framed circle bundle over a 3-Sasakian orbifold $\mathcal{S}$ and suppose that the horizontal subbundle $\hat{\mathcal{H}}$ defines a $\mathfrak{u}(1)$ connection on $H(\mathcal{S})$. Then the following conditions are equivalent:*

(i) The horizontal lifts $\hat{\xi}^a$ generate the Lie algebra $\mathfrak{su}(2)$.

(ii) The subbundle $\hat{\mathcal{V}}_3$ of $TH(\mathcal{S})$ is integrable.

(iii) The components $\omega(\hat{\xi}^a, \hat{\xi}^b)$ of the curvature of the $\mathfrak{u}(1)$ connection on $H(\mathcal{S})$ vanish.

More generally, if $(H(\mathcal{S}), \Xi)$ is any framed circle bundle over a 3-Sasakian orbifold and the horizontal subbundle $\hat{\mathcal{H}}$ defines a $\mathfrak{u}(1)$ connection on $H(\mathcal{S})$ then equation 1.8 together with condition (iiia) of Proposition 1.6 imply that the rank four subbundle $\mathcal{V}_4 = \hat{\mathcal{V}}_3 \oplus \mathcal{V}_1$ is integrable and thus defines a four-dimensional foliation $\mathcal{F}_4$ on $H(\mathcal{S})$. In the case that any (and hence all) of the conditions of Proposition 1.9 are satisfied the splitting $\mathcal{V}_4 = \hat{\mathcal{V}}_3 \oplus \mathcal{V}_1$ as bundles gives also a splitting as foliations, namely, $\mathcal{F}_4 = \mathcal{F}_1 \oplus \mathcal{F}_3$. Let us assume for simplicity that $\mathcal{S}$ is complete, hence, both $\mathcal{S}$ and $H(\mathcal{S})$ are compact. Then we have the identifications

$$H(\mathcal{S})/\mathcal{F}_1 = \mathcal{S} \quad \text{and} \quad H(\mathcal{S})/\mathcal{F}_4 = \mathcal{O},$$

where $\mathcal{O}$ is the quaternionic Kähler orbifold associated to $\mathcal{S}$ and an orbifold commutative diagram

1.10 $$\begin{array}{ccc} H(\mathcal{S}) & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ H(\mathcal{S})/\mathcal{F}_3 & \longrightarrow & \mathcal{O}. \end{array}$$

The leaves of the two vertical maps are of the form S^3/Γ, where Γ is a finite subgroup of $SU(2)$. So the leaves of the foliation $\mathcal{F}_4$ are products $S^1 \times S^3/\Gamma$. In the case that the generic leaf has $\Gamma =$ identity or $\mathbb{Z}_2$, $H(\mathcal{S})$ is a principal orbifold bundle over $\mathcal{O}$ with group $SU(2)$ or $SO(3)$, respectively. The first vertical map is an S^3/Γ orbifold bundle over $H(\mathcal{S})/\mathcal{F}_3$ and one can show that the horizontal subbundle $\tilde{\mathcal{H}} \oplus \mathcal{V}_1$ defines an $\mathfrak{su}(2)$ on $H(\mathcal{S})$. However, a similar result does not necessarily hold for the orbifold bundle $H(\mathcal{S}) \to \mathcal{O}$. We have

PROPOSITION 1.11: *Assume the hypothesis of Proposition 1.9 together with any (hence all) of its conditions. Suppose further that the generic leaf of the foliation $H(\mathcal{S})/\mathcal{F}_4$ is a Lie group G of the form $U(2)$ or $S^1 \times SO(3)$. Then the following are equivalent:*

(i) The horizontal subbundle $\tilde{\mathcal{H}}$ defines a $\mathfrak{u}(2)$ connection in the orbifold bundle $H(\mathcal{S}) \to \mathcal{O}$.

(ii) The components $\omega(\hat{\xi}^a, \tilde{X})$ of the curvature of the $\mathfrak{u}(1)$ connection in $H(\mathcal{S})$ vanish for any section $\tilde{X}$ of $\tilde{\mathcal{H}}$.

PROOF: Let $\tilde{X}$ be a section of $\tilde{\mathcal{H}}$ that is basic with respect to the orbifold bundle $H(\mathcal{S}) \to \mathcal{O}$. Then, using the fact (see [BGM2]) that $d\hat{\eta}^b(\hat{\xi}^a, \tilde{X}) = d\eta^b(\xi^b, X) = 0$, one obtains the equation

$$[\hat{\xi}^a, \tilde{X}] = [\xi^a, X]^{\hat{*}} - \omega(\hat{\xi}^a, \tilde{X})\Xi, \tag{1.12}$$

where the superscript $\hat{*}$ denotes the horizontal lift to $\hat{\mathcal{H}}$. The result follows from this equation. ∎

THEOREM 1.13: *Let $(H(\mathcal{S}), \Xi)$ be a framed circle bundle over a 3-Sasakian orbifold $\mathcal{S}$. Then the almost hypercomplex structure defined by Equation 1.5 is integrable if and only if the horizontal subbundle $\hat{\mathcal{H}}$ on $(H(\mathcal{S}), \Xi)$ defines a $\mathfrak{u}(1)$ connection and the following three conditions hold:*

(i) $\omega(\hat{\xi}^a, \hat{\xi}^b) = 0$ for any $a, b = 1, 2, 3$.

(ii) $\omega(\hat{\xi}^a, \tilde{X}) = 0$ for all $a = 1, 2, 3$ and any section $\tilde{X}$ of $\tilde{\mathcal{H}}$.

(iii) $\omega(\hat{\Phi}^a \tilde{X}, \hat{\Phi}^a \tilde{Y}) = \omega(\tilde{X}, \tilde{Y})$ for all $a = 1, 2, 3$ and for any sections $\tilde{X}, \tilde{Y}$ of $\tilde{\mathcal{H}}$, where ω denotes the curvature 2-form of the connection 1-form $\hat{\eta}^0$.

REMARK 1.14: Propositions 1.9 and 1.11 show that conditions (i) and (ii) are equivalent to the fact that the 1-form $\hat{\eta}^0$ defines the $\mathfrak{u}(1)$ part of a $\mathfrak{u}(2)$ connection in the orbifold bundle $H(\mathcal{S})$ over the quaternionic Kähler orbifold $\mathcal{O}$. Condition (iii) of the definition means that the curvature of this connection is of type $(1,1)$ with respect to each almost complex structure defined by $a = 1, 2, 3$.

PROOF OF THEOREM 1.13: Assuming that the horizontal subbundle $\hat{\mathcal{H}}$ defines a $\mathfrak{u}(1)$ connection on $H(\mathcal{S})$, one computes the Nijenhuis tensor of the almost complex structures $\mathcal{I}^a$. Integrability follows then from the vanishing of this tensor by the Newlander-Nirenberg theorem. Straightforward but tedious computations give for the non-identically zero components:

1.15
$$
\begin{aligned}
N^a(\tilde{X},\tilde{Y}) &= (\omega(\tilde{X},\tilde{Y}) - \omega(\hat{\Phi}^a\tilde{X},\hat{\Phi}^a\tilde{Y}))\Xi + (\omega(\hat{\Phi}^a\tilde{X},\tilde{Y}) + \omega(\tilde{X},\hat{\Phi}^a\tilde{Y}))\hat{\xi}^a;\\
N^a(\hat{\xi}^b,\hat{\xi}^c) &= (\omega(\hat{\xi}^b,\hat{\xi}^c) - \epsilon^{abc}\epsilon^{ace}\omega(\hat{\xi}^d,\hat{\xi}^e))\Xi - (\epsilon^{abe}\omega(\hat{\xi}^e,\hat{\xi}^c) + \omega(\hat{\xi}^b,\hat{\xi}^e))\hat{\xi}^a;\\
N^a(X,\hat{\xi}^b) &= \omega(\tilde{X},\hat{\xi}^a)\Xi + [\omega(\hat{\Phi}^a\tilde{X},\hat{\xi}^b) - \epsilon^{abc}(\omega(\hat{\Phi}^a\tilde{X},\hat{\xi}^c) + \omega(\tilde{X},\hat{\xi}^c))]\hat{\xi}^c;\\
N^a(\hat{\xi}^b,\Xi) &= \omega(\hat{\xi}^b,\hat{\xi}^a)\hat{\xi}^a + \epsilon^{abc}\omega(\hat{\xi}^c,\hat{\xi}^a)\Xi,
\end{aligned}
$$

and the result follows. ∎

§2. Some Topological and complex Analytical Properties of $H(\mathcal{S})$

In this section we investigate some general properties of our circle bundles $H(\mathcal{S})$.

LEMMA 2.1: *Let $H(\mathcal{S})$ be a compatible hypercomplex circle bundle over the 3-Sasakian orbifold $\mathcal{S}$. Then*

(i) The fibers of $H(\mathcal{S})$ are totally geodesic.

(ii) The leaves of the foliation $\mathcal{F}_4$ are totally geodesic.

(iii) The vector fields $\hat{\xi}^1,\hat{\xi}^2,\hat{\xi}^3,\Xi$ are everywhere linearly independent Killing fields which span the Lie algebra $\mathfrak{u}(2)$. In particular, the Lie algebra $\mathfrak{i}(H)$ of infinitesimal isometries contains $\mathfrak{u}(2)$.

(iv) The vector field $Z^a = \Xi + i\hat{\xi}^a$ is nowhere vanishing and holomorphic with respect to the complex structure $\mathcal{I}^a$.

(v) If the 3-Sasakian orbifold $\mathcal{S}$ is complete, the complex structures in the two-sphere of complex structures on $H(\mathcal{S})$ are all equivalent. Hence, the hypercomplex structure on $H(\mathcal{S})$ defines a unique complex structure.

PROOF: The proof of (i) and (ii) are standard computations involving the O'Neill tensors A and T (see [Bes]) and is left to the reader. ∎

To prove (iii) we first note that compatibility and Proposition 1.6 imply Ξ is a Killing vector field. Furthermore, Proposition 1.9 implies that the $\hat{\xi}^a$ generate the Lie algebra $\mathfrak{su}(2)$. The linear independence is clear. We need to show that the vector fields $\hat{\xi}^a$ are Killing fields for all $a = 1,2,3$. Now for any vector fields X,Y on $H(\mathcal{S})$ we have

2.2
$$(\mathcal{L}_{\hat{\xi}^a})h(X,Y) = \hat{\xi}^a h(X,Y) - h([\hat{\xi}^a,X],Y) - h(X,[\hat{\xi}^a,Y]).$$

Let X and Y be sections of $\hat{\mathcal{H}}$ which since $\mathcal{L}_{\hat{\xi}^a}h$ is a tensor field we can take to be basic, that is, the horizontal lifts of vector fields $\check{X}$ and $\check{Y}$ on $\mathcal{S}$. Then $h(X,Y) = g_{\mathcal{S}}(\check{X},\check{Y})$ and

$\xi^a h(X,Y) = \xi^a g_{\mathcal{S}}(\check{X},\check{Y})$, so the right hand side of 2.2 vanishes since ξ^a are Killing vector fields on $(\mathcal{S}, g_{\mathcal{S}})$. Now suppose that X is vertical which we can take to be Ξ. Then again the right hand side of equation 2.2 vanishes by Proposition 1.6 and Theorem 1.13. If Y is also vertical we can take $Y = \Xi$. In this case the first term vanishes since $h(\Xi,\Xi) = 1$ and the last two terms vanish by (iiia) of Proposition 1.6. This proves that $\hat{\xi}^a$ are Killing vector fields.

(iv): The vector field Z^a is clearly nowhere vanishing and is easily seen to be type $(1,0)$ with respect to the complex structure $\mathcal{I}^a$. A type $(1,0)$ vector field is holomorphic if and only if its real part preserves the complex structure, that is, $\mathcal{L}_\Xi \mathcal{I}^a = 0$. We have

2.3 $$\mathcal{L}_\Xi \mathcal{I}^a = -\mathcal{L}_\Xi \hat{\Phi}^a + \Xi \otimes \mathcal{L}_\Xi \hat{\eta}^a - [\Xi, \hat{\xi}^a] \otimes \hat{\eta}^0 - \hat{\xi}^a \otimes \mathcal{L}_\Xi \hat{\eta}^0.$$

The last three terms vanish by Proposition 1.6. To see that the first term vanishes we first notice that $\hat{\Phi}^a X$ is the $\hat{\mathcal{H}}$ component of $\nabla_X \hat{\xi}^a$ while the $\mathcal{V}_1$ component vanishes by compatibility. Hence, we can write

2.4 $$\mathcal{L}_\Xi \hat{\Phi}^a = \mathcal{L}_\Xi (\nabla \hat{\xi}^a) = (\mathcal{L}_\Xi \nabla)\hat{\xi}^a + \nabla[\Xi, \hat{\xi}^a].$$

The last term on the right vanishes by Proposition 1.6, while the first term vanishes since Ξ, being a Killing vector field, is an infinitesimal affine transformation. This proves (iv).

To prove (v) we notice that, since $\mathcal{S}$ is complete, Theorem A of [BGM2] implies that $\mathcal{S}$ is compact and, hence, $H(\mathcal{S})$ is also compact. So the vector field Z^a generates a global $\mathcal{I}^a$-holomorphic transformation on $H(\mathcal{S})$. Thus it suffices to show that

2.5 $$\mathcal{L}_{Z^a} \mathcal{I}^b = 2i\epsilon^{abc} \mathcal{I}^c.$$

To verify this identity we first notice that, since ξ^a are Killing vector fields on $\mathcal{S}$, they are infinitesimal affine transformations on $\mathcal{S}$. Now since $\mathcal{S}$ is 3-Sasakian, $\mathcal{L}_{\xi^a} \Phi^b = 2\epsilon^{abc}\Phi^c$ and this equation lifts to a similar equation with hats on $H(\mathcal{S})$. This fact, together with equation 2.4 above, implies that $\mathcal{L}_{Z^a} \hat{\Phi}^b = 2i\epsilon^{abc}\hat{\Phi}^c$. A similar argument shows that the remaining terms in the expression for $\mathcal{I}^a$ transform in the same way, verifying 2.5. This proves (v) and hence the lemma. ∎

A well-known residue theorem of Bott [Bot] implies that a compact complex manifold with a nowhere vanishing holomorphic vector field has zero Chern numbers. Thus we have

THEOREM 2.6: *Let $\mathcal{S}$ be a complete 3-Sasakian orbifold and H a smooth manifold, then the Chern numbers of $H(\mathcal{S})$ all vanish. Thus, if $\{F_k\}$ is any multiplicative sequence of Chern classes, then the F-genus of $H(\mathcal{S})$ vanishes. In particular, the Todd genus, the $\hat{A}$ genus, and Hirzebruch signature of $H(\mathcal{S})$ all vanish.*

REMARKS 2.7: Since, by (iv) of Lemma 2.1, all the complex structures in the hypercomplex structure are equivalent, we do not distinguish between them. So there is no ambiguity by not specifying the complex structure in Theorem 2.6. Theorem 2.6 also implies that the Euler number vanishes, but this is true for any principal bundle.

From now on we shall assume that $\mathcal{S}$ is complete, so that both $\mathcal{S}$ and $H(\mathcal{S})$ are compact. There are two important foliations on $H(\mathcal{S})$ that are related to the hypercomplex structure. First, we have the foliation $\mathcal{F}_4$ discussed previously. The leaves of this foliation have the form $S^1 \times S^3/\Gamma$, where Γ is a finite subgroup of $SU(2)$ and the space of leaves is the quaternionic Kähler orbifold $\mathcal{O}$. We have

Lemma 2.8: *Each leaf L of the foliation $\mathcal{F}_4$ on $H(\mathcal{S})$ is an elliptic Hopf surface of the form $S^1 \times S^3/\Gamma$ with respect to the hypercomplex structure $\mathcal{I}^a$ restricted to L.*

Proof: The almost hypercomplex structure $\mathcal{I}^a$ on $H(\mathcal{S})$ restricts to an almost hypercomplex structure on each leaf given by

$$\mathcal{I}^a_l = -\hat{\Phi}^a_l + \Xi \otimes \hat{\eta}^a - \hat{\xi}^a \otimes \hat{\eta}^0,$$

where $\hat{\Phi}^a_l$ denotes the tensor field $\hat{\Phi}^a$ restricted to the leaf L. This almost complex structure is clearly integrable. Thus, L is one of the hypercomplex Hopf surfaces discussed in [Boy]. Now since the $\hat{\xi}^a$'s are the horizontal lifts of the 3-Sasakian vector fields ξ^a on $\mathcal{S}$, the projection $\pi : H(\mathcal{S}) \to \mathcal{S}$ restricted to L maps surjectively onto a leaf S^3/Γ of the foliation $\mathcal{F}$ on $\mathcal{S}$. Moreover, this map is just projection onto the second factor in $L = S^1 \times S^3/\Gamma$. Furthermore, the vector field $\hat{\xi}^a$ is a generator of a circle subgroup $U(1)$ in $SU(2)$ and this subgroup acts locally freely on S^3/Γ. It is easy to see that this action preserves the complex structure given by $\mathcal{I}^a_l$. Hence, we have an elliptic foliation

2.9
$$\begin{array}{ccc} E & \longrightarrow & S^1 \times S^3/\Gamma \\ & & \Big\downarrow \\ & & \mathbb{P}^1/\Gamma. \end{array}$$

∎

The second foliation on $H(\mathcal{S})$ is the holomorphic foliation $\mathcal{F}_2$ generated by $Z^a = \Xi + i\hat{\xi}^a$. Without loss of generality we take $a = 1$. We want to identify the space of leaves $H(\mathcal{S})/\mathcal{F}_2$ with the twistor space $\mathcal{Z}$ of the 3-Sasakian orbifold $\mathcal{S}$ with its induced complex structure [BG]. As spaces this follows from the commutative diagram

2.10
$$\begin{array}{ccccc} & & H(\mathcal{S}) & & \\ & \overset{S^1}{\swarrow} & & \overset{E}{\searrow} & \\ \mathcal{S} & & \xrightarrow{S^1} & & \mathcal{Z}. \end{array}$$

We need to identify the complex structure on $H(\mathcal{S})/\mathcal{F}_2 \simeq \mathcal{Z}$ induced by the holomorphic foliation $\mathcal{F}_2$ with the standard complex structure on the twistor space $\mathcal{Z}$. It suffices to identify the almost complex structures since both structures are integrable. The almost complex structure on $\mathcal{Z}$ induced by the foliation $\mathcal{F}_2$ is the projection of the almost complex structure $\mathcal{I}^1$ restricted to the horizontal subbundle $\hat{\mathcal{H}} \oplus \hat{\mathcal{V}}_3^+$, where $\hat{\mathcal{V}}_3^+$ is the two-dimensional subbundle of $\hat{\mathcal{V}}_3$ generated by the vector fields $\hat{\xi}^2$ and $\hat{\xi}^3$. This restricted almost complex structure is just $-\hat{\Phi}^1$. Under the natural projection, $\pi : H(\mathcal{S}) \to \mathcal{S}$, this tensor field projects to the tensor field $-\Phi^1$ on $\mathcal{S}$. But from [BG] $-\Phi$ defines a CR-structure on the horizontal space $\ker \eta^1$ and this induces the complex structure on $\mathcal{Z}$. Hence, the two complex structures coincide. Then the results of [BG] imply

Theorem 2.11: *Let $H(\mathcal{S})$ be a compatible circle bundle over a 3-Sasakian orbifold $\mathcal{S}$. Then $H(\mathcal{S})$ is holomorphically foliated by elliptic curves and the space of leaves $\mathcal{Z}$ is a $\mathbb{Q}$-factorial Fano contact variety with a Kähler-Einstein metric of positive scalar curvature.*

Topologically the circle bundles on $\mathcal{S}$ are classified by $H^2(\mathcal{S},\mathbb{Z})$. For a given bundle $H(\mathcal{S})$ we can change the framing of the circle by $\Xi \mapsto \lambda\Xi$ for $\lambda \in \mathbb{R}^+$. This does not alter the bundle, but it does change the hypercomplex structure. In fact, the proof of Proposition 4.6 of [BGM2] gives

Proposition 2.12: *Each circle bundle $H(\mathcal{S})$ over $\mathcal{S}$ has a real one-parameter family of inequivalent hypercomplex structures.*

Remark 2.13: The inequivalent hypercomplex structures described in Proposition 2.12 determine inequivalent complex structures on $H(\mathcal{S})$.

Next we describe some topological information about circle bundle $H(\mathcal{S})$ over 3-Sasakian orbifolds $\mathcal{S}$. When $\mathcal{S}$ is a smooth manifold it is well-known that circle bundles on $\mathcal{S}$ are classified by elements of $H^2(\mathcal{S},\mathbb{Z})$. In the case that $\mathcal{S}$ and $H(\mathcal{S})$ are orbifolds, the theory is due to Haefliger and Salem [Hae,HS]. Let us briefly describe the orbifold cohomology and homotopy groups [Hae]. Let X be an orbifold of dimension n, and let P denote the bundle of orthonormal frames on X. It is a smooth manifold on which the orthogonal group $O(n)$ acts locally freely with the quotient X. Let $EO(n) \longrightarrow BO(n)$ denote the universal $O(n)$ bundle. Consider the diagonal action of $O(n)$ on $EO(n) \times P$ and denote the quotient by BX. Now there is a natural projection of BX onto X with generic fiber the contractible space $EO(n)$, and Haefliger defines the orbifold cohomology, homology, and homotopy groups by

$$H^i_{orb}(X,\mathbb{Z}) = H^i(BX,\mathbb{Z}), \quad H_i^{orb}(X,\mathbb{Z}) = H_i(BX,\mathbb{Z}), \quad \pi_i^{orb}(X) = \pi_i(BX). \tag{2.14}$$

This definition of π_1^{orb} is equivalent to Thurston's better known definition, and when X is a smooth manifold these orbifold groups coincide with the usual groups. Now with this in hand for the orbifold category, the circle V-bundles over $\mathcal{S}$ are classified [HS] by $H^2_{orb}(\mathcal{S},\mathbb{Z})$. Of course, rationally there is no difference since $H^i_{orb}(\mathcal{S},\mathbb{Z})\otimes\mathbb{Q} \simeq H^i(\mathcal{S},\mathbb{Z})\otimes\mathbb{Q}$. For example, the rational Gysin sequence

$$\cdots \longrightarrow H^p(\mathcal{S},\mathbb{Q}) \longrightarrow H^{p+2}(\mathcal{S},\mathbb{Q}) \longrightarrow H^{p+2}(H(\mathcal{S}),\mathbb{Q}) \longrightarrow H^{p+1}(\mathcal{S},\mathbb{Q}) \longrightarrow \cdots$$

applies. Combining this with known results of 3-Sasakian manifolds [BGM1,GS] one finds

Proposition 2.15: *Let $H(\mathcal{S})$ be a circle bundle over a 3-Sasakian orbifold $\mathcal{S}$ of dimension $4n-1$. Then the following relations for the Betti numbers of $H(\mathcal{S})$ hold:*

(i) $b_{2n}(H(\mathcal{S}))=0$.

(ii) $b_{2n-2}(H(\mathcal{S}))\leq b_{2n-1}(H(\mathcal{S}))=b_{2n-2}(\mathcal{S})$.

(iii) $b_{2p+1}(H(\mathcal{S}))\leq b_{2p}(\mathcal{S})$ *for* $1\leq p\leq n-1$.

(iv) $b_{2p}(H(\mathcal{S}))\leq b_{2p}(\mathcal{S})$ *for* $1\leq p\leq n-1$.

(v) $b_1(H(\mathcal{S}))=0,1$.

(vi) $b_2(H(\mathcal{S}))=\begin{cases} b_2(\mathcal{S})-1 & \text{if } b_1(H(\mathcal{S}))=0;\\ b_2(\mathcal{S}) & \text{if } b_1(H(\mathcal{S}))=1.\end{cases}$

PROOF: Left to the reader. ∎

We have the following immediate corollaries:

COROLLARY 2.16: *Let $\mathcal{S}$ be a 3-Sasakian orbifold of dimension 7, and $H(\mathcal{S})$ a smooth circle bundle over $\mathcal{S}$. If $b_1(H(\mathcal{S}))=0$, then we have*

$$b_p(H(\mathcal{S}))=\begin{cases} 1 & \text{if } p=0,8;\\ 0 & \text{if } p=1,4,7;\\ k-1 & \text{if } p=2,6;\\ k & \text{if } p=3,5,\end{cases}$$

whereas if $b_1(H(\mathcal{S}))=1$, then

$$b_p(H(\mathcal{S}))=\begin{cases} 1 & \text{if } p=0,1,7,8;\\ 0 & \text{if } p=4;\\ k & \text{if } p=2,3,5,6.\end{cases}$$

Thus, there are two types of hypercomplex circle bundles $H(\mathcal{S})$ in dimension 8, and the rational homology of each type is completely determined by the second Betti number. In the first type above the bundle cannot be flat and $H(\mathcal{S})$ cannot be locally conformally hyperkähler, whereas the second type has the rational homology of the trivial bundle which includes the class of flat bundles. Our next corollary gives important complex analytic information.

COROLLARY 2.17: *$H(\mathcal{S})$ has no symplectic structure. In particular, $H(\mathcal{S})$ admits no Kähler metric.*

Let $\mathbb{C}(X)$ denote the meromorphic function field of a complex space X, and let $\mathfrak{a}(X)$ denote the algebraic dimension of X. Consider the (singular) elliptic fibration $\pi: H(\mathcal{S})\longrightarrow\mathcal{Z}$. Since $\mathcal{Z}$ is a projective algebraic variety of dimension $2n-1$ [BG], $\mathfrak{a}(\mathcal{Z})=2n-1$ and π induces a monomorphism $\pi^*:\mathbb{C}(\mathcal{Z})\longrightarrow\mathbb{C}(H(\mathcal{S}))$. We have

PROPOSITION 2.18: *If* $\pi_1(H(\mathcal{S}))$ *is finite, then* $\mathfrak{a}(H(\mathcal{S})) = 2n-1$ *and* $\pi^* : \mathbb{C}(\mathcal{Z}) \longrightarrow \mathbb{C}(H(\mathcal{S}))$ *is an isomorphism.*

PROOF: We know that $2n-1 \leq \mathfrak{a}(H(\mathcal{S})) \leq 2n$, and we must show that $H(\mathcal{S})$ cannot be Moishezon. Since π_1 is finite, $H(\mathcal{S})$ has a finite simply connected unbranched cover with the same algebraic dimension. So it suffices to prove the result when $H(\mathcal{S})$ is simply connected. Suppose $H(\mathcal{S})$ were Moishezon, then by a theorem of Moishezon [M] there is a finite sequence of blow-ups making $H(\mathcal{S})$ projective algebraic, i.e., there is a dominant map $\phi : X \longrightarrow H(\mathcal{S})$, where X is a projective algebraic variety. But since $H_1(H(\mathcal{S}), \mathbb{Z}) = 0$ every elliptic curve $E_z = \pi^{-1}(z)$ is homologous to 0 in $H(\mathcal{S})$. But then $\phi^* E_z$ is an elliptic curve of X that is homologous to 0 in X. Thus, $\phi^* E_z \cdot H = 0$, where H is the restriction to X of a hyperplane divisor. But this is impossible in a projective algebraic variety. ∎

Let $p_g(H(\mathcal{S})) = h^{2n,0} = h^{0,2n}$ denote geometric genus and $h^{p,q}$ the Hodge numbers of $H(\mathcal{S})$. Then

PROPOSITION 2.19: $p_g(H(\mathcal{S})) = 0$.

PROOF: Let ω be a holomorphic section of the canonical line bundle K on $H(\mathcal{S})$. Then $d\omega = \partial\omega + \bar{\partial}\omega = 0$ so ω is also d-closed. But by Corollary 2.15 $H^{2n}(H(\mathcal{S}), \mathbb{R}) = 0$, so ω must be exact. Thus, by Stokes Theorem

$$\int \omega \wedge \bar{\omega} = 0,$$

and this implies that $\omega = 0$. ∎

We now turn our attention to the subclass of the class of bundles $H(\mathcal{S})$, namely flat bundles. Using results of [OrPi,V] we have

PROPOSITION 2.20: *Suppose* $H(\mathcal{S})$ *is smooth and the orbifold bundle* $\pi : H(\mathcal{S}) \to \mathcal{S}$ *is flat. Then* $H(\mathcal{S})$ *is locally conformally hyperkähler. In particular,* $H^1(H(\mathcal{S}), \mathbb{Q}) = \mathbb{Q}$.

Thus locally conformally hyperkähler geometry appears as a special category in our construction. The diagram 2.10 is well-known both in the context of generalized Hopf manifolds [Vai] as well as locally conformally hyperkähler spaces [OrPi]. Another interesting case is when $H(\mathcal{S})$ is hypercomplex homogeneous, that is the group of hypercomplex symmetries of $H(\mathcal{S})$ acts transitively. One can then use arguments similar to the ones employed in [BGM2] to show that

PROPOSITION 2.21: *Suppose* $H(\mathcal{S})$ *is hypercomplex homogeneous. Then all the leaves of the* $U(2)$ *action have to be diffeomorphic. In particular both* $\mathcal{S}$ *and* $\mathcal{O}$ *are smooth and homogeneous.*

Using the above result one can easily classify all the hypercomplex homogeneous circle bundles. We get a result first announced in [BGM5]:

Corollary 2.22: *Suppose $H(\mathcal{S})$ is hypercomplex homogeneous. Then $H(\mathcal{S})$ is one of the following:*

(i) $H(\mathcal{S}) = \mathbb{V}^{\mathbb{C}}_{n,2}$.

(ii) $H(\mathcal{S}) = \mathbb{V}^{\mathbb{C}}_{n,2}/\mathbb{Z}_k$ *with* $2 \leq k \in \mathbb{Z}^+$.

(iii) $H(\mathcal{S})$ *is locally conformally hyperkähler and then it is one of the spaces:* $(G/H) \times S^1$ *with* G/H *equal to* $S^{4n-1}, \mathbb{RP}^{n-1}, SU(m)/S(U(m-2) \times U(1)), m \geq 1, SO(k)/SO(k-4) \times Sp(1), k \geq 7, G_2/Sp(1), F_4/Sp(3), E_6/SU(6), E_7/Spin(12), E_8/E_7$*; or the unique non-trivial principal* S^1*-bundle over* $\mathbb{RP}^{4n-1}$*. All these bundles are flat.*

In all of the above cases there is a real one parameter family of hypercomplex structures.

The homogeneous hypercomplex structures of $\mathbb{V}^{\mathbb{C}}_{n,2}$ were studied by many authors [Joy2, Bat, BGM2, PP, PPS] in more than one context. Ornea and Piccini [OrPi] show that all locally conformally hyperkähler homogeneous spaces are precisely those given in (iii) above. Results of Borel and Remmert [BR], and Tits [Ti] say that any compact homogeneous complex manifold X is the total space of a bundle (a Tits bundle) with parallelizable fibers over a generalized flag manifold G/P, where G is a complex Lie group acting transitively and holomorphically on X, and P is a parabolic subgroup. This clearly applies to hypercomplex geometry as well, although for a general homogeneous hypercomplex manifold the associated complex structures may not all be equivalent, so there may be many Tits bundles associated to the same homogeneous hypercomplex manifold. However, for the circle bundles $H(\mathcal{S})$, the complex structures are all equivalent, and there is a unique Tits bundle whose parallelizable fibers are elliptic curves and whose generalized flag base space is a twistor space. In fact, Theorem 2.11 can be thought of as an analogue of the Tits bundle construction in the inhomogeneous case. The simplest examples of inhomogeneous hypercomplex structures on compact manifolds that are not hyperkähler are the inhomogeneous hypercomplex Hopf surfaces. In [BGM2] and [BGM4], the authors described a n-parameter family of inhomogeneous hypercomplex structures on the Stiefel manifolds $\mathbb{V}^{\mathbb{C}}_{n,2}$. It was recently shown by Pedersen and Poon [PP] that in a neighborhood of a homogeneous hypercomplex structure on $\mathbb{V}^{\mathbb{C}}_{n,2}$, the family of hypercomplex structures given in [BGM2,BGM4] is precisely the subfamily of T^3-equivariant deformations, and that the full versal deformation space is n^2-dimensional. In the following chapter we use hypercomplex reduction to construct many more new examples of inhomogeneous hypercomplex structures. They are somewhat different as there is no underlying homogeneous structure.

§3. Hypercomplex and 3-Sasakian Toral Quotients

Let $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{H}^n$ be the quaternionic coordinates on the n-dimensional quaternionic vector space $\mathbb{H}^n$ equipped with the flat metric. Consider the unit sphere

$$S^{4n-1} = \{\mathbf{u} \in \mathbb{H}^n \ \mid \ \sum_{\alpha=1}^{n} \overline{u}_\alpha u_\alpha = 1\}$$

with its canonical metric g_{can} obtained from the flat metric by the inclusion $S^{4n-1} \subset \mathbb{H}^n$. This sphere has two natural 3-Sasakian structures determined by whether one treats $\mathbb{H}^n$ as a right or left quaternionic vector space. We choose the left module structure on $\mathbb{H}^n$ and this chooses the right 3-Sasakian vector fields ξ_r^a. The subgroup of the isometry group $O(n)$ of the (S^{4n-1}, g_{can}) that normalizes this structure is $Sp(n)\cdot Sp(1) = \big(Sp(n)\times Sp(1)\big)/\mathbb{Z}_2$, where the $Sp(1)$ is the group generated by the 3-Sasakian vector fields ξ_r^a. The group $Sp(n)\cdot Sp(1)$ acts on the sphere as:

$$Sp(n)\times Sp(1)\times S^{4n-1}\to S^{4n-1}$$

3.1
$$\big((B,\sigma);\mathbf{u}\big)\longrightarrow B\mathbf{u}\sigma^{-1},$$

where $B\in Sp(n)$ is the quaternionic $n\times n$ matrix of the defining representation of $Sp(n)$, and $\sigma\in Sp(1)$ is a unit quaternion. The diagonal $\mathbb{Z}_2$ subgroup defined by $A=-\mathbb{I}, \sigma=-1$ acts trivially, and the factor group $Sp(n)\cdot Sp(1)$ acts effectively on the S^{4n-1}. The group $Sp(n)$ is precisely the subgroup of the isometry group which commutes with the 3-Sasakian $Sp(1)$ action, i.e., the group of automorphisms preserving the 3-Sasakian structure. Associated to any subgroup $G\subset Sp(n)$, there is a 3-Sasakian moment map [BGM1] $\mu_G : S^{4n-1}\longrightarrow \mathfrak{g}^*\otimes\mathbb{R}^3$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of G.

In this paper we shall consider a maximal torus $T^n\subset Sp(n)$ and its subgroups acting on S^{4n-1}. The maximal torus that we choose is that given in terms of its action on $\mathbb{H}^n$ by $u_\alpha\mapsto\tau_\alpha u_\alpha$. Then every quaternionic representation of a k-torus T^k on $\mathbb{H}^n$ can be described by a diagonal matrix of the form

3.2
$$\begin{pmatrix} \prod_{i=1}^k \tau_i^{a_1^i} & \dots & 0\\ \vdots & \ddots & \vdots \\ 0 & \dots & \prod_{i=1}^k \tau_i^{a_n^i}\end{pmatrix},$$

where $(\tau_1,..,\tau_k)\in S^1\times\cdots\times S^1 = T^k$ are the complex coordinates on T^k, and $a_j^i\in\mathbb{Z}$. In turn this representation defines a $k\times n$ integral *weight* matrix

3.3
$$\Omega=\begin{pmatrix} a_1^1 & a_2^1 & \dots & a_k^1 & \dots & a_n^1\\ a_1^2 & a_2^2 & \dots & a_k^2 & \dots & a_n^2\\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots\\ a_1^k & a_2^k & \dots & a_k^k & \dots & a_n^k\end{pmatrix}.$$

Let $\mathfrak{t}_k$ denote the Lie algebra of the k-torus T^k for any k. Then invariantly Ω defines an element of $\hom_{\mathbb{Z}}(\mathfrak{t}_n,\mathfrak{t}_k)\simeq\mathfrak{t}_k\otimes t_n^*$. This $\mathbb{Z}$-module parameterizes the quaternionic representations of T^k on $\mathbb{H}^n$. Such a representation gives rise to a moment map $\mu_\Omega : S^{4n-1}\to\mathfrak{t}_k^*\otimes\mathbb{R}^3$, and for "good" representations the quotient $\mu_\Omega^{-1}(0)/T^k$ is well-behaved, that is, at worst an orbifold. We shall give the precise conditions below. In terms of the quaternionic coordinates of $\mathbb{H}^n$ the moment map is given by the simple quadratic expression [BGMR]$\mu_\Omega = \sum_j \mu^j_\Omega e_j$, where

3.4 $$\mu^j_\Omega(\mathbf{u}) = \sum_\alpha \bar{u}_\alpha i a^j_\alpha u_\alpha,$$

and $\{e_j\}_{j=1}^k$ denotes the standard basis for $\mathbb{R}^k \simeq \mathfrak{t}^*_k$.

Two weight matrices $\Omega, \Omega' \in \mathcal{M}_{k\times n}(\mathbb{Z})$ are equivalent if there are $A \in GL(k,\mathbb{Z})$ and $w \in \mathcal{W}(Sp(n))$ such that $\Omega' = A\Omega w$. Equivalent weight matrices give rise to isomorphic 3-Sasakian quotients $\mu^{-1}_{\Omega'}(0)/T^k(\Omega') \simeq \mu^{-1}_\Omega(0)/T^k(\Omega)$. Actually, if $\Omega' = A\Omega$ for $A \in GL(k,\mathbb{Z})$ the quotients coincide; hence, $\mathcal{S}(\Omega)$ depends only on the equivalence class $[\Omega]$.

The conditions on Ω that guarantee "nice" behavior of the quotient are given by

THEOREM 3.5: [BGMR1,BGM3] *If all the k by k minor determinants $\Delta_{\alpha_1\ldots\alpha_k}$ of Ω are non-vanishing then the quotient $\mathcal{S}(\Omega) = \mu^{-1}_\Omega(0)/T^k(\Omega)$ is a 3-Sasakian orbifold, where the sequence $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$ label the columns of Ω. Let g denote the k^{th} determinantal divisor of Ω. Then $\mathcal{S}(\Omega)$ is a smooth manifold if and only if, in addition*

$$\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\alpha_k}) = g$$

for all sequences $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n$.

An Ω for which all k by k minor determinants are non-vanishing is called *non-degenerate*, and if the gcd condition of Theorem 3.5 also holds Ω is called *admissible.* From now on we shall assume that Ω is non-degenerate so that the $T^k(\Omega)$ action on $\mu^{-1}_\Omega(0)$ is locally free, and the quotient $\mathcal{S}(\Omega)$ is an orbifold. In [BGM3] a mod 2 reduction to smoothness was obtained. Namely, if $k > 1$ then $\dim \mathcal{S} = 7, 11, 15$, and if $k > 4$ then $\dim \mathcal{S} = 7$. Moreover, the following was proved in [BGMR] for the case $\dim \mathcal{S} = 7$, and by Bielawski [Bi] for general dimension:

THEOREM 3.6: *The second Betti number b_2 of the 3-Sasakian quotients $\mathcal{S}(\Omega)$ of Theorem 3.5 is k.*

The circle V-bundles $H(\mathcal{S})$ over $\mathcal{S}(\Omega)$ are classified by $H^2_{orb}(\mathcal{S}(\Omega),\mathbb{Z})$. We shall show in Theorem 4.1 below that when $\dim\, H(\mathcal{S}) = 8$ this group is $\mathbb{Z}^k$. For now our aim is to construct hypercomplex structures on the total space of these bundles. Choose a subgroup $T^{k-1} \subset T^k$. This gives an exact sequence (as $\mathbb{Z}$-modules)

3.7 $$0 \longrightarrow \mathfrak{t}_{k-1} \longrightarrow \mathfrak{t}_k \longrightarrow \mathfrak{t}_1 \longrightarrow 0$$

and tensoring with the free $\mathbb{Z}$-module $\mathfrak{t}^*_n$ gives the exact sequence

3.8 $$0 \longrightarrow \mathfrak{t}_{k-1} \otimes \mathfrak{t}^*_n \longrightarrow \mathfrak{t}_k \otimes \mathfrak{t}^*_n \longrightarrow \mathfrak{t}^*_n \longrightarrow 0.$$

A $k-1$ by n submatrix Ω_1 of Ω corresponds an element of $\mathfrak{t}_{k-1}\otimes\mathfrak{t}_n^*$ and a splitting of 3.8 corresponds to writing

$$\Omega=\begin{pmatrix}\mathbf{p}\\ \Omega_1\end{pmatrix}, \tag{3.9}$$

where $\mathbf{p}$ is an integral point of $\mathfrak{t}_n^*$. Now consider the action of the subtorus $T^{k-1}_{\Omega_1}$ on $\mu_\Omega^{-1}(0)$. It is locally free and the quotient $\mu_\Omega^{-1}(0)/T^{k-1}_{\Omega_1}$ is a circle V-bundle over $\mathcal{S}(\Omega)$. Now $\mathfrak{t}_k\otimes\mathfrak{t}_n^*$ is also a $GL(k,\mathbb{Z})$-module and the subgroup $K\subset GL(k,\mathbb{Z})$ that stabilizes $\mathfrak{t}_{k-1}\otimes\mathfrak{t}_n^*$ as $\mathbb{Z}$-submodules is the subgroup of $GL(k,\mathbb{Z})$ consisting of elements of the form

$$\begin{pmatrix}\pm1 & q_2 & \cdots & q_k\\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

with $q_i\in\mathbb{Z}$ and $B\in GL(k-1,\mathbb{Z})$. So the circle V-bundles over $\mathcal{S}(\Omega)$ are parameterized by the coset space $GL(k,\mathbb{Z})/K\simeq\mathbb{Z}^k$. If $\Omega=(\mathbf{p},\Omega_1)^t$ and $\Omega'=(\mathbf{p}',\Omega_1')^t$ differ by an element of the subgroup K, then $H(\mathbf{p},\Omega_1)$ and $H(\mathbf{p}',\Omega_1')$ are isomorphic circle bundles. We believe in this case that the hypercomplex structures that we construct are inequivalent but we shall not prove this here.

We prove the existence of hypercomplex structures on these V-bundles by applying the hypercomplex quotient construction of Joyce [Joy2] to the zero set of the circle moment map $\mu_{\mathbf{p}}:S^{4n-1}\longrightarrow\mathbb{R}^3$, where $\mathbf{p}$ is the integer point of $\mathfrak{t}_n^*$ of 3.9. If all the components p_i of $\mathbf{p}$ are non-zero then $\mu_{\mathbf{p}}^{-1}(0)$ is diffeomorphic [BGM1] to the Stiefel manifold $\mathbb{V}^{\mathbb{C}}_{n,2}$ of complex 2-planes in $\mathbb{C}^n$ and has a natural hypercomplex structure [BGM2] labeled by $\mathbf{p}$. In the case that some components $\mathbf{p}$ vanish, $\mu_{\mathbf{p}}^{-1}(0)$ is a singular stratified space. Nevertheless, we show that it has a natural hypercomplex structure to which the reduction procedure can be applied. We denote by $\mathcal{N}(\mathbf{p})$ the Stiefel manifold (as well as the singular stratified version) with the hypercomplex structure defined by $\mathbf{p}$, and the corresponding circle action will be denoted by $S^1(\mathbf{p})$. By a transformation in the Weyl group $\mathcal{W}(Sp(n))$ if necessary, we can always take this point to lie in the positive Weyl chamber C^n. Furthermore, by a transformation of $GL(k,\mathbb{Z})$ we can take $\mathbf{p}$ to be the first row of Ω.

The $(k-1)$-torus $T^{k-1}(\Omega_1)$ is a subgroup of the group Aut $\mathcal{N}(\mathbf{p})$ of hypercomplex automorphisms of $\mathcal{N}(\mathbf{p})$. Let $\nu_{\Omega_1}:\mathcal{N}(\mathbf{p})\longrightarrow\mathfrak{t}_{k-1}\otimes\mathbb{R}^3$ denote the restriction to $\mathcal{N}(\mathbf{p})$ of the projection of μ_Ω onto the last $k-1$ coordinates of $\mathfrak{t}_k$. Now $T^{k-1}(\Omega_1)$ also acts locally freely on $\nu_{\Omega_1}^{-1}(0)=\mu_\Omega^{-1}(0)$, so the quotient space $\nu_{\Omega_1}^{-1}(0)/T^{k-1}(\Omega_1)$ is also an orbifold. We shall denote this orbifold by $H(\mathbf{p},\Omega_1)$. The circle group $S^1(\mathbf{p})$ acts locally freely on $H(\mathbf{p},\Omega_1)$ and the quotient is just the orbifold $\mathcal{S}(\Omega)$.

THEOREM 3.10: *The orbifold $H(\mathbf{p},\Omega_1)$ is a circle V-bundle over the 3-Sasakian orbifold $\mathcal{S}(\Omega)$ and has a naturally induced hypercomplex structure which is compatible with the 3-Sasakian structure on $\mathcal{S}(\Omega)$. Furthermore, if the gcd condition of Theorem 3.5 is satisfied $H(\mathbf{p},\Omega_1)$ is a hypercomplex manifold.*

PROOF: Recall Joyce's [Joy2] hypercomplex reduction procedure. If a Lie group G acts locally freely on a hypercomplex manifold (M, I^a) preserving the hypercomplex structure then we look for a moment map $\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of G satisfying the two conditions

(i) $I^1 d\mu^1 = I^2 d\mu^2 = I^3 d\mu^3$.

(ii) For every $\xi \in \mathfrak{g}$, $I^a d\mu^a_\xi(\Xi) \neq 0$ for all $a = 1, 2, 3$, where Ξ is the vector field on M corresponding to $\xi \in \mathfrak{g}$.

We apply Joyce's reduction to the moment map $\nu_{\Omega_1} : \mathcal{N}(\mathbf{p}) \longrightarrow \mathfrak{t}_{k-1} \otimes \mathbb{R}^3$. First we consider the non-singular case when all components of $\mathbf{p}$ are non-vanishing. Notice that for each $a = 1, 2, 3$, $d\nu^a$ is a section of $\mathfrak{t}_{k-1} \otimes T^*\mathcal{N}(\mathbf{p})$ which is the restriction of a quadratic 1-form written in the flat quaternionic coordinates on $\mathbb{H}^n$. Now by viewing $\mathcal{N}(\mathbf{p})$ as the total space of a $U(2)$ principal V-bundle over a quaternionic Kähler orbifold [BGM2], one sees that a certain choice of $U(2)$-connection gives an exact sequence of vector bundles on $\mathcal{N}(\mathbf{p})$

3.11 $$0 \longrightarrow \mathcal{V}^*(\mathbf{p}) \longrightarrow T^*\mathcal{N}(\mathbf{p}) \longrightarrow Q^*(\mathbf{p}) \longrightarrow 0,$$

where $\mathcal{V}^*(\mathbf{p})$ is spanned by the connection 1-form. From Theorem 1.14 of [BGM2] the hypercomplex structure $\mathcal{I}^a(\mathbf{p})$ coincides on $Q^*(\mathbf{p})$ with the restriction of the flat hypercomplex structure I^a_+ on $\mathbb{H}^n$ associated with right quaternionic multiplication. Let ν_j denote the jth component of ν_{Ω_1} with respect to the basis of t_{k-1} determined by the $k-1$ rows of Ω_1 and let $\Xi_j(\Omega)$ be the corresponding vector field on $\mathcal{N}(\mathbf{p})$, where $j = 2, \cdots, k$. Let $\eta^0_j(\Omega)$ be the 1-form dual to the vector field $\Xi_j(\Omega)$ with respect to the restriction g of the flat metric in $\mathbb{H}^n$ to $\mathcal{N}(\mathbf{p}) \cap \nu^{-1}_{\Omega_1}(0)$. Then a straightforward computation shows that for each $a = 1, 2, 3$ we have
$$\mathcal{I}^a(\mathbf{p}) d\nu^a_j = -\eta^0_j(\Omega).$$
This immediately shows that condition (i) above is satisfied.

To check the transversality condition we notice that

$$\mathcal{I}^a(\mathbf{p}) d\nu^a_j(\Xi_j(\Omega)) = -g(\Xi_j(\Omega), \Xi_j(\Omega))$$

for each $a = 1, 2, 3$. Now the vector fields $\Xi_j(\Omega)$ can vanish on $\mathcal{N}(\mathbf{p})$, so the transversality condition (ii) does not hold on all of $\mathcal{N}(\mathbf{p})$. However, since T^{k-1} acts locally freely on $\nu^{-1}_{\Omega_1}(0)$[BGMR], the norms $g(\Xi_j(\Omega), \Xi_j(\Omega))$are non-vanishing there, so condition (ii) does hold on the zero set of the moment map. Thus, by continuity, condition (ii) holds on a tubular neighborhood W_Ω of $\nu^{-1}_{\Omega_1}(0)$ since no k by k minor determinant of Ω vanishes. Applying Joyce's theorem to the moment map ν_{Ω_1} on W_Ω we see that the quotient $\nu^{-1}_{\Omega_1}(0)/T^{k-1}$ is a hypercomplex orbifold and it will be a hypercomplex manifold when the gcd condition of Theorem 3.5 is satisfied.

Now consider the singular case when some components of $\mathbf{p}$ vanish. Since Ω is non-degenerate, there are at most $k-1$ components of $\mathbf{p}$ that vanish. Let l denote the number of non-vanishing components of $\mathbf{p}$. Then we must have $l \geq n-k+1$. From the moment map equations the singular locus of $\mathcal{N}(\mathbf{p})$ consists precisely of the subset, where the l quaternionic coordinates corresponding to the non-vanishing components of $\mathbf{p}$ vanish. Let $\mathcal{N}(\mathbf{p})_0$ denote the smooth locus of $\mathcal{N}(\mathbf{p})$. This has a hypercomplex structure precisely as in the non-singular case. Now the zero set $\nu_{\Omega_1}^{-1}(0)$ lies entirely in $\mathcal{N}(\mathbf{p})_0$, since on $\nu_{\Omega_1}^{-1}(0) = \mu_{\Omega}^{-1}(0)$ at most $n-k-1$ quaternionic coordinates can vanish [BGMR], and $l > n-k$. Since $\mathcal{N}(\mathbf{p})_0$ is a smooth manifold which is dense and open in $\mathcal{N}(\mathbf{p})$, there is a tubular neighborhood W of $\nu_{\Omega_1}^{-1}(0)$ lying entirely in $\mathcal{N}(\mathbf{p})_0$, and the remainder of the proof goes through as before. ∎

Next we consider conditions under which the hypercomplex orbifold $H(\mathbf{p},\Omega_1)$ is a smooth manifold. We assume as before that Ω is reduced, non-degenerate, and that its first row is $\mathbf{p} = (p_1, \cdots, p_n)$. The argument here follows closely the proof of Lemma 2.15 of [BGMR]. We note that on $\mu_{\Omega}^{-1}(0) \subset S^{4n-1}$ there are at most $n-k-1$ quaternionic coordinates that can simultaneously vanish. Thus, at least $k+1$ quaternionic coordinates are non-vanishing. So the conditions that determine a fixed point of the T^{k-1} action on $\mu_{\Omega}^{-1}(0)$ are

3.12 $$\prod_{i=2}^{k} \tau_i^{a^i_\beta} = 1 \quad \text{for each } \beta = \alpha_j,\ j = 1, \cdots k+1.$$

Here the α_j label the $k+1$ non-vanishing quaternionic coordinates. So these conditions must hold for every sequence $\alpha = (\alpha_1, \cdots, \alpha_{k+1})$ of length $k+1$ with the lexicographic ordering $1 \leq \alpha_1 < \cdots < \alpha_{k+1} \leq n$. Let M_α denote the $k-1$ by $k+1$ submatrix of Ω_1 obtained by choosing the $k+1$ columns corresponding to $\alpha = (\alpha_1, \cdots, \alpha_{k+1})$, and let $\Delta_a(\alpha)$ denote the $k-1$ by $k-1$ minor determinants of M_α for some given order $a = 1, \cdots, \frac{k(k+1)}{2}$. Then as in [BGMR] the conditions 3.12 are equivalent to the conditions that for each such sequence α we have

$$\tau_i^{\Delta_a(\alpha)} = 1.$$

We have thus arrived at

THEOREM 3.13: *The hypercomplex orbifold $H(\mathbf{p},\Omega_1)$ is a smooth manifold if and only if for each ordered sequence α as above the condition*

$$\gcd(\Delta_1(\alpha), \cdots, \Delta_{\frac{k(k+1)}{2}}(\alpha)) = 1$$

holds.

We concentrate on the two cases. The case that $n = k+2$ so $\dim\ H(\mathbf{p},\Omega_1) = 8$, and the case $k = 2$ for any dimension (0 mod 4). An analysis similar to that given in [BGM3] will show that there are smoothness bounds on both the dimension if $k > 2$, and on k if the dimension is greater than 8. Of course, if $\mathcal{S}(\Omega)$ is smooth then any circle bundle over it will be smooth. So it follows from the results of [BGMR] and [BGM3] that

Corollary 3.14: *There are infinite families of smooth hypercomplex toral quotients* $H(\mathbf{p},\Omega_1)$ *in dimension* 8 *for all* k. *Moreover, there are infinite families of smooth hypercomplex toral quotients* $H(\mathbf{p},\Omega_1)$ *for* $k = 2,3,4$ *and* $\dim\ H(\mathbf{p},\Omega_1) = 12$ *or* 16.

The conditions in Theorem 3.12 are easy when $k = 2$, and we have

Corollary 3.15: *Let* $k = 2$ *and write* Ω *as*

$$\Omega = \begin{pmatrix} p_1 & \cdots & p_n \\ q_1 & \cdots & q_n \end{pmatrix},$$

then the $4(n-k)$ *dimensional hypercomplex orbifold* $H(\mathbf{p},\mathbf{q})$ *is smooth if and only if* $\gcd(q_i,q_j,q_l) = 1$ *for all triples of components of* $\mathbf{q}$.

Remark 3.16: Note that we can relax the requirement that the entries of $\mathbf{p}$ are integral. If we take $\mathbf{p}\in\mathbb{R}^n$ then the orbifold $H(\mathbf{p},\Omega_1)$ exists as Joyce's hypercomplex quotient of $\mathcal{N}(\mathbf{p})$ by the T^{k-1} torus with the action defined by Ω_1 as long as the matrix Ω is non-degenerate. Just as in the case of the Stiefel manifold $\mathcal{N}(\mathbf{p})$, for general $\mathbf{p}$, we loose the bundle structure $H(\mathbf{p},\Omega_1)\rightarrow\mathcal{S}(\Omega)$. The foliation given by the circle action associated to the weight vector $\mathbf{p}$ may have non-compact leaves and the quotient will not be an orbifold. However, if all the components of $\mathbf{p}$ are rational, it follows from Proposition 2.12 for $\lambda\in\mathbb{R}^+$ that $H(\mathbf{p},\Omega_1)$ and $H(\lambda\mathbf{p},\Omega_1)$ describe inequivalent hypercomplex structures on the same manifold. Thus, for each circle bundle $H(\mathbf{p},\Omega_1)$ there is at least a 1 parameter family of inequivalent hypercomplex structures. When $\mathbf{p}$ and $\mathbf{q}$ are not proportional and both define a non-singular $k\times n$ matrix Ω, it is not clear whether in general the topology of $H(\mathbf{p},\Omega_1)$ depends on $\mathbf{p}$, or whether $H(\mathbf{p},\Omega_1)$ is diffeomorphic to $H(\mathbf{q},\Omega_1)$.

By construction, all our spaces $H(\mathbf{p},\Omega_1)$ have a locally free $U(2)$ action such that the space of leaves $\mathcal{O}(\Omega)$ is a quaternionic Kähler orbifold of positive scalar curvature. When $H(\mathbf{p},\Omega_1)$ is 8-dimensional $\mathcal{O}(\Omega)$ is a self-dual Einstein orbifold of positive scalar curvature. Using the orbifold version of the Gysin sequence one can show that $b_2(\mathcal{O}(\Omega)) = k$. All the 4-dimensional orbifolds $\mathcal{O}(\Omega)$ are examples of T^2-symmetric self-dual Einstein metrics. The $U(2)$ action on $H(\mathbf{p},\Omega_1)$ is only locally free and, in fact, never free if $k>1$. It has the characteristic property investigated by Pedersen, Poon, and Swann in the context of quaternionic geometry [PPS]. Consider the central $U(1)\subset U(2)$ and let X be the corresponding vector field for this action. Then: (1) X preserves the hypercomplex structure (i.e., the central $U(1)$ is a hypercomplex symmetry), (2) the triple $\{IX,JX,KX\}$ generates the action of $SU(2)\subset U(2)$, and (3) this $SU(2)$ rotates the hypercomplex structures $\{I,J,K\}$. One one can easily generalize Theorem 2.1 of [PPS] to show the following orbifold analogue:

Proposition 3.17: *Let* M *be a compact hypercomplex manifold with a locally free* $U(2)$ *action satisfying conditions (1-3). Then the space of leaves* $\mathcal{O}\simeq M/U(2)$ *is a compact orbifold which admits a quaternionic structure. Furthermore, if there exists a hyperhermitian metric* g *on* M *such that the* $U(2)$ *action is isometric and* $SU(2)$ *defines a 3-Sasakian structure on the orbifold* $\mathcal{S} = M/U(1)$ *then* $\mathcal{O}$ *is a quaternionic Kähler orbifold. In this case* M *is our circle bundle* $H(\mathcal{S})$.

When $\mathcal{O}$ is smooth one can invert this construction and show that (up to double cover) any such M can be reconstructed from a self-dual manifold $\mathcal{O}$ [PPS]. One takes the compact quaternionic associated bundle $\mathcal{V}(\mathcal{O})$ of $\mathcal{O}$ and twists it by an instanton bundle $\mathcal{P}$. It is clear that there also exists an appropriate orbifold analogue of this twisting construction. The construction can be used to obtain inhomogeneous hypercomplex structures on $\mathcal{V}(k\mathbb{CP}^2)$ twisted by some instanton bundle $\mathcal{P}$. When $k = 1$, one only gets $SU(3)/\mathbb{Z}_2$. When $k = 2$ there are many different twisted bundles $\mathcal{V}_{\mathcal{P}}(2\mathbb{CP}^2)$, some of them simply connected [PPS]. In principle, one could use this construction to obtain hypercomplex structures on $\mathcal{V}_{\mathcal{P}}(k\mathbb{CP}^2)$.

The bundles $\mathcal{V}_{\mathcal{P}}(k\mathbb{CP}^2)$ share a lot of topological and geometric properties of our 8-dimensional spaces $H(\mathbf{p}, \Omega_1)$. They have the same Betti numbers and the same symmetry group. However, our $H(\mathbf{p}, \Omega_1)$ do not seem to have any free $U(2)$-action and the orbifold character of the construction is crucial here. A complete study of the relevant spectral sequences should detect any differences. In the next section we describe the relevant spectral sequence for our 8-manifolds $H(\mathbf{p}, \Omega_1)$ but only up to total homological dimension 2. As will be seen, even this is quite intricate.

§4. The Topology of the 8-dimensional Toral Quotients

The purpose of this section is to describe some of the topology of both $\mathcal{S}(\Omega)$ and $H(\mathbf{p}, \Omega_1)$ in the case that $n = k + 2$, that is, when dim $\mathcal{S}(\Omega) = 7$ and dim $H(\mathbf{p}, \Omega_1) = 8$. Our methods are based on those of [BGMR] and so only work for these dimensions. However, in contrast to [BGMR], where only rational information was obtained, here we work with $\mathbb{Z}$ coefficients. Explicitly, we prove:

THEOREM 4.1: *Let Ω be non-degenerate, $\mathcal{S}(\Omega)$ and $H(\mathbf{p}, \Omega_1)$ be the 3-Sasakian and hypercomplex quotients defined in section 3 of dimensions 7 and 8, respectively. Further assume that $H(\mathbf{p}, \Omega_1)$ is a manifold. Then*

(i) $\pi_1(H(\mathbf{p}, \Omega_1)) = 0$.

(ii) $\pi_2(H(\mathbf{p}, \Omega_1)) = H_2(H(\mathbf{p}, \Omega_1), \mathbb{Z}) \simeq \mathbb{Z}^{k-1}$.

(iii) $\pi_1(\mathcal{S}(\Omega)) = \pi_1^{orb}(\mathcal{S}(\Omega)) = 0$.

(iv) $\pi_2^{orb}(\mathcal{S}(\Omega)) \simeq H_2^{orb}(\mathcal{S}(\Omega), \mathbb{Z}) \simeq H^2_{orb}(\mathcal{S}(\Omega), \mathbb{Z}) \simeq \mathbb{Z}^k$.

PROOF: Since Ω is non-degenerate, the zero set $\mu_\Omega^{-1}(0)$ is a compact smooth submanifold [BGMR] of S^{4n-1} of dimension $4n - 3k - 1 = k + 7$. Following the notation of [BGMR] we denote this zero set by $N(\Omega)$. Moreover, the right action of $Sp(1)$ on S^{4n-1} descends to an action on $N(\Omega)$, since $\mathfrak{t}_k^* \otimes \mathbb{R}^3$ is just k copies of the adjoint representation of $Sp(1)$, and $0 \in \mathfrak{t}_k^* \otimes \mathbb{R}^3$ is invariant. Furthermore, this induced action of $Sp(1)$ is free, since it is free on S^{4n-1}. Let $B(\Omega)$ denote the quotient manifold. So we have a principal $Sp(1)$-bundle

4.2 $$Sp(1) \longrightarrow N(\Omega) \longrightarrow B(\Omega).$$

Now $N(\Omega)$ also has an action of a $(k+2)$-torus T^{k+2} and in [BGMR] it was shown that the quotient space $Q(\Omega)$ of $N(\Omega)$ by $T^{k+2} \times Sp(1)$ is a closed polygon with $k+2$ vertices. The action of T^{k+2} factors through the principal $Sp(1)$ bundle $N(\Omega)$ to give a smooth surjective map

4.3 $$\pi : B(\Omega) \longrightarrow Q(\Omega)$$

whose generic fiber is T^{k+2}. We now analyze the structure of this map. Recall from [BGMR] (Lemma 3.11) that $Q(\Omega)$ consists of 3 strata $Q_i(\Omega)$ with $i = 1, 2, 3$. We have $Q(\Omega) = Q_0(\Omega) \sqcup Q_1(\Omega) \sqcup Q_2(\Omega)$, where $Q(\Omega)$ is homeomorphic to the closed disc $\bar{D}^2$. The regular values of π consists of the generic stratum $Q_2(\Omega)$ which is homeomorphic to the open disc D^2. The singular values of π consists of $Q_1(\Omega) \sqcup Q_0(\Omega)$ and is homeomorphic to the boundary $\partial\bar{D}^2 \simeq S^1$. $Q_1(\Omega)$ is homeomorphic to the disjoint union of $k+2$ copies of the open unit interval and $Q_0(\Omega)$ is a set of $k+2$ points. For $k = 1, 2, 3$ we define $B_i(\Omega) = \pi^{-1}(Q_i(\Omega))$. We then have

LEMMA 4.4: *There is a stratification*

4.4 $$B(\Omega) = B_0(\Omega) \sqcup B_1(\Omega) \sqcup B_2(\Omega)$$

and the fibers of $B_i(\Omega)$ *are tori* T^{k+i}.

Next,

LEMMA 4.5: $\pi_1(B(\Omega))$ *is Abelian.*

PROOF: Let γ be a loop in $B(\Omega)$ representing a class of $\pi_1(B(\Omega))$. We can take γ to be smooth. The inverse image of the singular values of π is $B_1(\Omega) \sqcup B_0(\Omega)$, and this has codimension 2 in $B(\Omega)$. Thus, by general position, γ can be homotoped to a loop in the generic stratum $B_2(\Omega)$. But since the base is contractible, it can be homotoped to a fibre T^{k+2} which has Abelian fundamental group. ∎

We can actually prove much more:

LEMMA 4.6: $H_1(B(\Omega)) = H_2(B(\Omega)) = 0$.

As the proof of this lemma is very technical and quite tedious, we shall return to it at the end of the section. We have a corollary of these two lemmas which is of interest in its own right.

COROLLARY 4.7: *Both* $N(\Omega)$ *and* $B(\Omega)$ *are 2-connected.*

PROOF: First consider $B(\Omega)$. By Lemma 4.5 $\pi_1(B(\Omega))$ is Abelian, so $\pi_1(B(\Omega)) = H_1(B(\Omega))$ which vanishes by Lemma 4.6. But then by the Hurewich Theorem $\pi_2(B(\Omega)) = H_2(B(\Omega))$ which again vanishes by Lemma 4.6. So $B(\Omega)$ is 2-connected. Then by applying the long exact sequence in homotopy to the fibration 4.2 shows that $N(\Omega)$ is also 2-connected. ∎

We continue with the proof of Theorem 4.1 assuming Lemma 4.6. Applying the long exact homotopy sequence to the principal fibration $T^{k-1}\longrightarrow N(\Omega)\longrightarrow H(\mathbf{p},\Omega_1)$ and using Corollary 4.7 gives (i) and (ii) of the Theorem. To prove (iii) and (iv) we apply Corollary 4.7 together with the long exact homotopy sequence of Haefliger and Salem [HS] to the orbifold bundle $T^k(\Omega)\longrightarrow N(\Omega)\longrightarrow\mathcal{S}(\Omega)$. This gives $\pi_1^{orb}(\mathcal{S}(\Omega))=0$ and $\pi_2^{orb}(\mathcal{S}(\Omega))\simeq\mathbb{Z}^k$. But then (i) and Corollary 5.9 of [HS] proves that $\mathcal{S}$ is simply connected proving (iii). To finish one easily sees from the definition 2.14 that the Universal Coefficient Theorem and Hurewicz Theorem apply equally well to the orbifold groups. So (iv) follows. This proves Theorem 4.1. ∎

Proof of Lemma 4.6: We analyze the Leray spectral sequence associated to the map 4.3 with the stratification 4.4. This is analogous to the spectral sequence in [BGMR], but now we need $\mathbb{Z}$ coefficients and must compute some differentials.

Define $Y_0=Q_0(\Omega)$, $Y_1=Q_0(\Omega)\cup Q_1(\Omega)$, and $Y_2=Q(\Omega)$. Then, we filter $B(\Omega)$ by $X_i=\pi^{-1}(B_i)$ to obtain the increasing filtration

$$X_0=B_0(\Omega),\quad X_1=B_0(\Omega)\cup B_1(\Omega),\quad\text{and}\quad X_2=B(\Omega).$$

The Leray spectral sequence associated to this filtration has E^1 term given by

$$E^1_{s,t}\;\cong\;H_{s+t}(X_t,X_{t-1};\mathbb{Z})$$

with differential $d_1\,:\,H_{s+t}(X_t,X_{t-1};\mathbb{Z})\longrightarrow H_{s+t-1}(X_{t-1},X_{t-2};\mathbb{Z})$, where we use the convention that $X_{-1}=\emptyset$.

To compute these E^1 terms notice that all the pairs (X_t,X_{t-1}) are relative manifolds so that one can apply the Alexander-Poincaré duality theorem. Hence, by Lemma 4.4,

$$\begin{aligned}H_s(X_0;\mathbb{Z})&\cong H_s(\sqcup_{k+2}T^k;\mathbb{Z});\\ H_s(X_1,X_0;\mathbb{Z})&\cong H^{k+2-s}(\sqcup_{k+2}T^{k+1};\mathbb{Z});\\ H_s(X_2,X_1;\mathbb{Z})&\cong H^{k+4-s}(T^{k+2};\mathbb{Z}),\end{aligned}$$

where $\sqcup_jT^l$ means the disjoint union of j copies of T^l. Hence, the $E^1_{s,t}$ term of the spectral sequence is described by the diagram

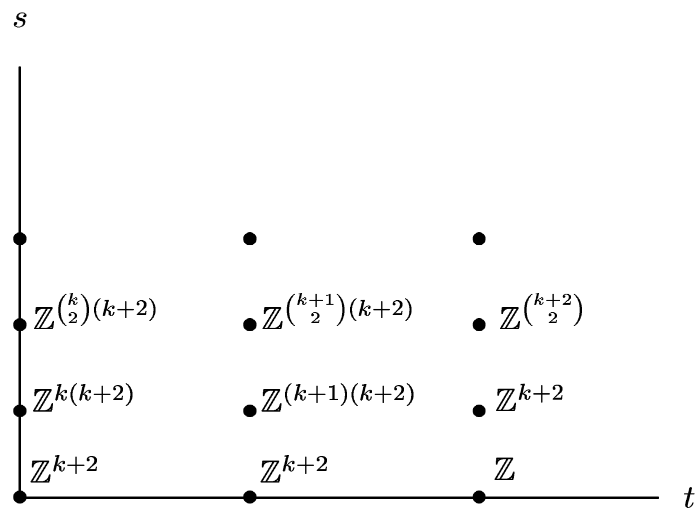

Diagram 4.8

Next we compute the d_1 differentials of this spectral sequence. In order to do so, we need some more details from [BGMR]. First, we mention that, although the equations for the moment map 3.4 have coefficients in $\mathbb{Z}$, we can pass to the field of fractions $\mathbb{Q}$ and consider a normalized Ω' such that $\mu_\Omega^{-1}(0) = \mu_\Omega'^{-1}(0)$, with

4.9
$$\Omega' = \begin{pmatrix} 1 & 0 & \dots & 0 & f^1 & g^1 \\ 0 & 1 & \dots & 0 & f^2 & g^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & f^k & g^k \end{pmatrix},$$

where $f_i, g_i \in \mathbb{Q} - \{0\}$. Recall [BGMR] that at most one quaternionic coordinate u_α can vanish, and this determines the vertices of the polygon. Suppose $u_{k+2} \neq 0$ (if this is the coordinate that vanishes interchange the role of u_{k+1} and u_{k+2}). We analyze the moment map equations 3.4 by considering a slice of the $Sp(1)$ action. In terms of complex coordinates $u_\alpha = z_\alpha + w_\alpha j$, we choose a part of the slice by putting $w_{k+2} = 0$. (Note that there is still an S^1's worth of freedom). With this choice the edges of the polygon are determined by the vanishing of precisely one complex coordinate in each pair (z_α, w_α) for $\alpha = 1, \cdots, k+2$. Exactly which edge corresponds which configuration of the vanishing of z_α or w_α's is determined by the relative sizes of the 2 by 2 minor determinants $f_i g_j - f_j g_i$. For convenience of exposition we make the choice $f_i = 1$ for all $i = 1, \cdots, k$, and $0 < g_1 < \cdots < g_k$ as the general case is just a renaming. Then the moment map equations 3.4 become

4.10
$$|z_j|^2 - |w_j|^2 + |z_{k+1}|^2 - |w_{k+1}|^2 + g_j |z_{k+2}|^2 = 0,$$
$$\bar{w}_j z_j + \bar{w}_{k+1} z_{k+1} = 0,$$

for $j = 1, \cdots, k$. An analysis of these equations shows that one must have the ordering of vertices and edges as indicated in Diagram 4.11 below. We can assign a sign pattern to each edge as follows: On a given edge we assign to each quaternionic coordinate u_α a + sign if $z_\alpha \neq 0$ and $w_\alpha = 0$, and a $-$ sign if $z_\alpha = 0$ and $w_\alpha \neq 0$. Then we have the following assignments: $e_j \sim (+,\cdots,+,-,\cdots,-,+)$ for $j = 1,\cdots,k$, where the "$\cdots$" indicate j plus signs and $k+1-j$ minus signs; $e_{k+1} \sim (-,\cdots,-,+,+)$, and $e_{k+2} \sim (-,\cdots,-,-,+)$.

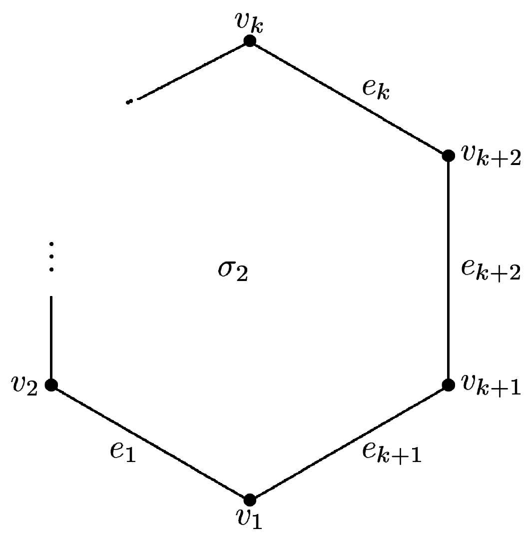

Diagram 4.11.

The d_1 differential in the bottom row of Diagram 4.8 can now be easily computed. They are just the geometrical boundary of the relevant simplices. (As in [BGMR] the labeling of the filtration in the spectral sequence has been shifted so that the d_1 differentials move horizontally from right to left.) Letting σ_2 denote the 2 simplex, we see that the boundary map gives:

$$\partial(\sigma_2) = e_1 + \cdots + e_{k+2}, \quad \partial(e_i) = v_{i+1} - v_i \;\text{ for } i = 1\cdots,k-1,$$

4.12 $$\partial(e_k) = v_{k+2} - v_k, \quad \partial(e_{k+1}) = v_1 - v_{k+1}, \quad \partial(e_{k+2}) = v_{k+1} - v_{k+2}.$$

We see easily from this that the bottom row of the E^2 term is as shown in diagram 4.14 below.

The computation of the d_1 differential on the next row ($s = 1$) is much more tedious. The homology classes on the first row are of the form $x_j \otimes v_\alpha$ at $t = 0$, $y_j \otimes e_\alpha$ at $t = 1$, and $z_j \otimes \sigma_2$ at $t = 2$, where $x_j \in H_1(T^k,\mathbb{Z})$, $y_j \in H_1(T^{k+1},\mathbb{Z})$, and $z_j \in H_1(T^{k+2},\mathbb{Z})$. The boundary maps are given by

$$\partial(z_j \otimes \sigma_2) = \rho_2(z_j) \otimes (e_1 + \cdots + e_{k+2}), \qquad \partial(x_j \otimes v_\alpha) = 0,$$

4.13 $$\partial(y_j \otimes e_\alpha) = \rho_1(y_j) \otimes \partial(e_\alpha),$$

where $\partial(e_\alpha)$ is given by 4.12, and $\rho_2(z_j), \rho_1(y_j)$ are the corresponding images in homology under the fibre mappings

$$T^{k+2} \xrightarrow{\rho_2} T^{k+2}/S^1 \simeq T^{k+1} \xrightarrow{\rho_1} T^{k+2}/T^2 \simeq T^k.$$

The explicit expressions for $\rho(z_j)$ and $\rho(y_j)$ depend on these fibre mappings and on the sign assignments given above. After a somewhat lengthy computation, one shows that the map $\mathbb{Z}^{k+2} \longrightarrow \mathbb{Z}^{(k+1)(k+2)}$ is injective while the map $\mathbb{Z}^{(k+1)(k+2)} \longrightarrow \mathbb{Z}^{k(k+2)}$ is surjective. This implies that the $E^1_{1,t=0,1,2}$ are all zero. Similarly one shows that the map from $\mathbb{Z}^{\binom{k+1}{2}(k+2)} \longrightarrow \mathbb{Z}^{\binom{k}{2}(k+2)}$ is surjective. Thus, the $E^2_{s,t}$ term is

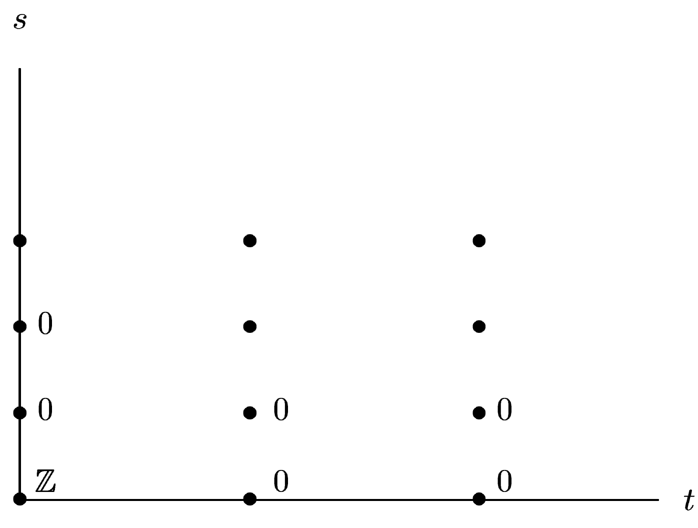

Diagram 4.14

Now $E^2_{s,t} = E^\infty_{s,t}$ which converges to $H_{s+t}(B(\Omega), \mathbb{Z})$, so this proves the lemma. ∎

Now by combining Theorem 4.1, Theorem 3.13, and Proposition 1.12 we arrive at Theorem A of the Introduction. ∎

Given a simply connected hypercomplex circle bundle $H(\mathcal{S})$, we can follow the construction of section 5 of [BGM2] to obtain hypercomplex circle bundles with cyclic fundamental group. Let θ denote the action of S^1 on $H(\mathcal{S})$ generated by the vector field Ξ, and consider the action θ^l of S^1 on $H(\mathcal{S}) \times S^1$, where the action on the first factor is θ and the action on the second is the standard map of degree $l \in \mathbb{Z}^+$. If this action is free, then, since $H(\mathcal{S})$ is simply connected, the quotient $H(\mathcal{S})_l$ has fundamental group $\mathbb{Z}_l$. For example, let $H(\mathbf{p}, \Omega_1)$ be one of the smooth hypercomplex 8-manifolds of Theorem A and suppose further that the associated $\mathcal{S}$ is smooth. Then the action θ^l is free, since θ is a free action. Then the quotient $H(\mathbf{p}, \Omega_1, l)$ is a hypercomplex manifold with $\pi_1 = l$, and $b_2 = k$. This gives Corollary B in the Introduction. As in [BGM2] even when $\mathcal{S}(\Omega)$ is an orbifold it is possible to choose l such that the quotient $H(\mathbf{p}, \Omega_1, l)$ is a smooth manifold.

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Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131
email: cboyer@math.unm.edu, galicki@math.unm.edu mann@math.unm.edu

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Max-Planck-Institut für Mathematik
Gottfried Claren Strasse 26
53225 Bonn, Germany
email: galicki@mpim-bonn.mpg.de