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Geometry of the Scalar Couplings in N=2 Supergravity Models

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ABSTRACT: We describe the quaternionic geometry of various σ-model target spaces for the hypermultiplet couplings in 4-dimensional $N = 2$ conformal and Poincaré supergravity models. The process of eliminating the non-physical superconformal gauge fields of chiral $SU(2)$ and dilatations is examined in the language of the quaternionic associated bundle. The σ-model target space X parameterized by the scalar fields of the hypermultiplets in an $N = 2$ conformal supergravity-matter Lagrangian is shown to be the quaternionic associated bundle of the σ-model quaternionic Kähler target space M of the corresponding $N = 2$ Poincaré supergravity-matter system. The manifold X is a special hyperkähler manifold with an isometric $SU(2)$ action rotating the hyperkähler structure and M can be obtained as a certain $SU(2)$ reduction of X. This suggests possible generalizations of the superconformal tensor calculus and the Lagrangian constructed by de Wit et al. The quaternionic Kähler join and some new matter couplings are also discussed.

1. Introduction

Recently 4-dimensional $N = 2$ supergravity theories coupled to matter multiplets have received a lot of renewed attention. The focus of this attention is the geometry of the manifolds allowed as target spaces of locally $N = 2$ supersymmetric $(0, \frac{1}{2}, 1)$ vector multiplets [1]. Any such manifold M is Kähler, and it is characterized by the existence of a holomorphic $Sp(2n+2, \mathbb{R}) \otimes \mathbb{C}^*$ vector bundle over it with nowhere-vanishing holomorphic section Ω. The Kähler potential is then given by the logarithm of the $Sp(2n+2, \mathbb{R})$ invariant norm of Ω. The geometry of M is often called *special geometry* [2] and it is relevant to compactified string theories. For instance, it is well-known that both the moduli space of the Calabi-Yau threefolds and the moduli space of the $c = 9, (2, 2)$ conformal field theories are special manifolds.

On the other hand, 4-dimensional $N = 2$ supergravity-matter systems with locally supersymmetric $(0, \frac{1}{2})$ chiral multiplets are described by geometry of a different kind. Just as in the vector multiplet case, the invariance of the action under the local supersymmetry transformations puts certain restrictions on the geometry of the underlying σ-model

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manifold M parameterized by the scalar fields of the chiral multiplet. Here M must be a *(pseudo-)quaternionic Kähler* manifold. The requirement of the correct sign of the kinetic term further restricts M to be a quaternionic Kähler manifold ($Sp(n)\cdot Sp(1)$ holonomy, $\dim M = 4n$) of negative scalar curvature. Witten and Bagger explicitly constructed the most general form of such couplings in terms of the geometry of M [3]. (There are some additional global restrictions on M but those seem to be relevant only when M is smooth, compact, and of positive scalar curvature.)

The first two nonlinear matter couplings in $N = 2$ supergravity theory were derived by Breitenlohner and Sohnius [4]. These are σ-models with the scalar matter fields parameterizing non-compact symmetric homogeneous Wolf spaces: the quaternionic projective $4n$-ball $\mathbb{HH}^n = Sp(n,1)/Sp(n)\times Sp(1)$ and the complex Grassmannian $\tilde{X}^n = U(n,2)/U(n)\times U(2)$. Later de Wit et al. gave a very general form of the $N = 2$ supergravity Lagrangian coupled to an arbitrary number of Yang-Mills and scalar multiplets [5]. They pointed out that both of the above couplings can also be obtained in their formalism. However, as already noticed in [6], Witten and Bagger's Lagrangian is more general. It gives all interactions in terms of the geometric properties of an arbitrary quaternionic Kähler manifold M with negative scalar curvature. The Lagrangian of [5], although it includes interactions with an arbitrary number of vector multiplets, can describe only those chiral multiplet couplings which correspond to σ-models on quaternionic Kähler manifolds of a specific kind. These are the quaternionic Kähler manifolds that are quaternionic Kähler quotients of the quaternionic projective spaces $\mathbb{HP}^n = Sp(n+1)/Sp(n)\times Sp(1)$ or their non-compact and pseudo-Riemannian analogues (such as, for instance, $\mathbb{HH}^n$). The quaternionic Kähler quotients were first introduced by Galicki and Lawson [7,8] as a generalization of both the Marsden-Weinstein reduction for symplectic manifolds [9] and the hyperkähler reduction of Hitchin et al. [10] It is true that some of the Wolf spaces are indeed quaternionic Kähler quotients of the quaternionic projective spaces. The Wolf space $X^n = U(n+2)/U(n)\times U(2)$, for example, is obtained as the quaternionic Kähler quotient of $\mathbb{HP}^{n+1}$ by $U(1)\subset Sp(n+2)$ acting diagonally. Similarly, its noncompact dual $\tilde{X}^n = U(n,2)/U(n)\times U(2)$ may be obtained as a quotient of the pseudo-quaternionic Kähler manifold $Sp(n,2)/Sp(n,1)\times Sp(1)$ by the diagonal circle action of $U(1)\subset Sp(n,2)$. In our previous work (see [7]) we showed how to get $Y^n = O(n+4)/O(n)\times O(4)$ as the $Sp(1)\subset Sp(n+4)$ quotient of $\mathbb{HP}^{n+3}$. Similarly, its noncompact dual $\tilde{Y}^n = O(n+4)/O(n)\times O(4)$ can be constructed as a quaternionic Kähler quotient of $Sp(n,4)/Sp(n,3)\times Sp(1)$. Hence, both Y^n and $\tilde{Y}^n$ σ-models can also be coupled to the $N = 2$ supergravity using the approach of de Wit et al. However, not all manifolds with $Sp(n)\cdot Sp(1)$ holonomy are quaternionic Kähler quotients of the quaternionic projective space, quaternionic hyperbolic space or their pseudo-Riemannian analogues. In particular, it was observed by Swann [11] that this is the case with the five examples of quaternionic Kähler manifolds which are cosets of the exceptional Lie groups: $G_2/SO(4)$, $F_3/Sp(3)\times Sp(1)$, $E_6/SU(6)\times Sp(1)$, $E_7/Spin(12)\times Sp(1)$, and $E_8/E_7\times Sp(1)$. In fact, more is true: None of these five spaces is a quaternionic Kähler quotient of another.

The quaternionic Kähler quotient technique was successfully used to construct many new quaternionic Kähler manifolds of negative scalar curvature [6,12] (in particular, selfdual and Einstein metrics with negative cosmological constant when M is 4-dimensional), as well as compact quaternionic Kähler orbifold metrics of positive scalar curvature [8,13]. All of them may be used as σ-model target spaces for the hypermultiplet couplings in $N = 2$ supergravity and the methods of [5] can be applied to construct the whole invariant Lagrangian.

In this paper we want to examine the problem of possible generalizations of de Wit et al.'s Lagrangian to describe any $(0, \frac{1}{2})$ hypermultiplet coupling allowed in the 4-dimensional $N = 2$ supergravity. Such generalizations, if possible, would bridge the apparent gap between their Lagrangian and that of Bagger and Witten. First we find an elegant geometric description of the process of eliminating superconformal gauge fields. In the formalism of de Wit et al. some of the superconformal gauges decouple algebraically and, as a result, one obtains an $N = 2$ Poincaré supergravity theory. Let us describe the simplest case. One starts with a local $N = 2$ superconformally invariant theory coupled to the hypermultiplets given by $4r$ scalar fields A_i^α. The fields A_i^α can be thought of as global coordinates on the r-dimensional quaternionic vector space $\mathbb{H}^r$. If, in addition, we equip $\mathbb{H}^r$ with the standard flat (pseudo-)metric $<\cdot,\cdot>$ then $(X, d) = (\mathbb{H}^r, <\cdot,\cdot>)$ becomes a *(pseudo-)hyperkähler* manifold. The hyperkähler 2-forms are given by $\frac{1}{2}\mathrm{Im}(d\overline{u} \otimes du) = \omega^1 i + \omega^2 j + \omega^3 k$ where $\{i, j, k\}$ are the quaternionic units and $u = u(A)$ is the quaternionic coordinate on $\mathbb{H}^r$. The group of $N = 2$ superconformal transformations acts on the scalar fields A_i^α. The chiral $SU(2) \simeq Sp(1)$ generated by the gauge field $V_{\mu j}^i$ acts on $A_i^\alpha \in \mathbb{H}^r$ rotating the hyperkähler 2-forms and, together with the scale transformations of $\mathbb{R}^+$ generated by the dilatation gauge field b_μ, gives a homothetic $\mathbb{H}^* \simeq Sp(1) \times \mathbb{R}^+$ action on $\mathbb{H}^r$. One can verify that $\mathbb{H}^*$ acts on $\mathbb{H}^r$ by the quaternionic scalar multiplication from the right. Breaking the $SU(2)$ gauge invariance and fixing the scale takes us from the $N = 2$ conformal supergravity coupled to the scalar and vector multiplets to the corresponding $N = 2$ Poincaré supergravity-matter system. Geometrically, in the scalar multiplet sector, this is equivalent to taking the $\mathbb{H}^*$ quotient of $\mathbb{H}^r \setminus \{0\}$ which yields the quaternionic projective space $\mathbb{HP}^{r-1}$, its non-compact dual $\mathbb{HH}^{r-1}$ or their semi-Riemannian analogues, depending on the signature of the metric $<\cdot,\cdot>$. The projection map $\mathbb{H}^r \setminus \{0\} \xrightarrow{\mathbb{H}^*} \mathbb{HP}^{r-1}$ provides a geometric picture of how one gets an $N = 2$ Poincaré supergravity coupled to matter out of the $N = 2$ superconformal Lagrangian of [5].

One may ask the following question: Can $(\mathbb{H}^r, <\cdot,\cdot>)$ be replaced by some other non-flat manifold (X, d)? In other words, can one modify the superconformal tensor calculus and construct an $N = 2$ conformal supergravity coupled to $(0, \frac{1}{2})$ hypermultiplets such that the scalar fields A_i^α are local coordinates on some curved (pseudo-)Riemannian manifold X? If so, then what is the geometry of X allowed in such couplings? Or is the flat space $\mathbb{H}^r$ the only possibility?

It is easy to see that the answer to the last question is negative . There are other manifolds X which are allowed. In order to describe the geometry of X we will show how the fiberation $\mathbb{H}^r \setminus \{0\} \xrightarrow{\mathbb{H}^*} \mathbb{HP}^{r-1}$ generalizes in the case of other quaternionic Kähler manifolds. We will use some recent results of Swann [11,14]. He proved that over any (pseudo-)quaternionic Kähler manifold M of dimension $4k$ there exists a hyperkähler manifold $\mathcal{U}(M)$ of dimension $4(k+1)$ which admits a special homothetic $\mathbb{H}^*$ action with $Sp(1) \subset \mathbb{H}^*$ acting by isometries rotating the hyperkähler structure. This strongly suggest a possible general model for our X.

The paper is organized as follows: In Section 2 we briefly review the notation and formalism of de Wit et al. [5]. In Section 3 we describe the quaternionic geometry of the hypermultiplet matter couplings in the $N=2$ conformal and Poincaré supergravity theories which are constructed as quaternionic Kähler quotient. In Section 4 we give a global description of all hyperkähler manifolds which allow for invariant couplings to the $N=2$ conformal supergravity and can be derived in the framework of [5]. We discuss the quaternionic join of two quaternionic Kähler manifolds and close with some concluding remarks.

2. N=2 Supergravity-Matter Couplings

We only very briefly review the formalism of de Wit et al. [5] for the $N=2$ conformal supergravity coupled to an arbitrary number of scalar and vector multiplets. We use some of the results presented in our previous work [6]. We refer the interested reader to the original papers.

Using the superconformal tensor calculus one can couple the Weyl multiplet

$$\{ \; e_\mu^a, \; \psi_\mu^i, \; b_\mu, \; A_\mu, \; V_{\mu j}^i, \; T_{ab}^{ij}, \; \chi^i, \; D \; \} \tag{2.1}$$

to m Yang-Mills multiplets of some Lie group G

$$\{ \; X^I, \; \Omega_i^I, \; W_\mu^I, \; Y_{ij}^I \; \} \tag{2.2}$$

and $2r$ hypermultiplets

$$\{ \; A_i^\alpha, \; \xi^\alpha \}, \tag{2.3}$$

where we have introduced the following indices:

$\alpha = 1, ..., 2r$	— "matter" representation index of some Lie group G
$i, j = 1, 2$	— the chiral $SU(2)$ index
$I, J = 1, ..., m$	— G-group index in the adjoint representation
$\mu; a, b = 1, 2, 3, 4$	— spacetime indices (curved and flat)on E.

In the Weyl multiplet e_μ^a is the vierbein on some 4-dimensional coordinate spacetime E, ψ_μ^i is the gravitino $SU(2)$ doublet, b_μ is the gauge field for dilatations, A_μ and $V_{\mu j}^i$ are the gauge fields for chiral $U(1)$ and $SU(2)$ respectively, T_{ab}^{ij} is a real $SU(2)$ antisymmetric tensor, χ^i is a spinor doublet, and D is a real scalar. In the vector multiplet X^I is a complex scalar, Ω_i^I is a real spinor doublet W_μ^I is a gauge vector field, and Y_{ij}^I is a real $SU(2)$ triplet. In the hypermultiplet, ξ^α is a spinor and A_i^α are $2r$ complex scalar fields subject to the following reality condition

$$\overline{A_i^\alpha} = \epsilon^{ij} \rho_{\alpha\beta} A_j^\beta \stackrel{\text{def}}{=} A_\alpha^i \tag{2.4}$$

where ϵ^{ij} is the $SU(2)$ invariant antisymmetric tensor and $\rho_{\alpha\beta}$ is a skew symmetric matrix. The matrix ρ can always be put in the standard form

$$\rho = \begin{pmatrix} \mathbb{O} & \mathbb{I}_r \\ -\mathbb{I}_r & \mathbb{O} \end{pmatrix} . \tag{2.5}$$

It follows from (2.4) that A^i_α, viewed as a $2 \times 2r$ complex matrix, has the following form

$$\mathbb{A} = \begin{pmatrix} \Phi_- & \Phi_+ \\ -\Phi^*_+ & \Phi^*_- \end{pmatrix} , \tag{2.6}$$

where Φ_-, Φ_+ are r-dimensional complex vectors. Hence, the scalar fields A^α_i are global coordinates on r-dimensional quaternionic vector space $\mathbb{H}^r$. In order to make the quaternionic structure more explicit we introduce a quaternion-valued scalar vector field as follows

$$\mathbf{u} \stackrel{\text{def}}{=} \Phi_+ + j\Phi^*_-, \tag{2.7a}$$

or

$$u^\alpha = A^\alpha_2 + jA^{\alpha+r}_2, \quad \alpha = 1, ..., r, \tag{2.7b}$$

where j is another quaternionic unit. In order to construct an invariant coupling of the hypermultiplets to the Weyl multiplet we introduce the flat (pseudo-)Riemannian metric on $\mathbb{H}^r$ with quaternionic signature (p, q). The metric is represented by a diagonal matrix $\mathbf{h}$

$$\mathbf{h} = (h_{\alpha\beta}) = \begin{pmatrix} -\mathbb{I}_p & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_q \end{pmatrix} \tag{2.8a}$$

in the $\mathbf{u}$ coordinates or by a diagonal matrix $\mathbf{d}$

$$\mathbf{d} = (d^\beta_\alpha) = \begin{pmatrix} \mathbf{h} & \mathbb{O} \\ \mathbb{O} & \mathbf{h} \end{pmatrix} \tag{2.8b}$$

in the A^α_i coordinates. The scalar product associated with it will be denoted by

$$< \mathbf{u}, \mathbf{u} > \stackrel{\text{def}}{=} \sum_{\alpha,\beta=1}^{r} h_{\alpha\beta}(u^\alpha)^* u^\beta = \frac{1}{2} \sum_{\alpha,\beta=1}^{2r} \sum_{i=1,2} d_\alpha{}^\beta A^\alpha_i A^i_\beta \stackrel{\text{def}}{=} <\mathbb{A}, \mathbb{A}> \tag{2.9}$$

where $(u^\alpha)^*$ is the quaternionic conjugation. $(\mathbb{H}^*, \mathbf{d})$ is now a flat (pseudo-)hyper- kähler manifold with the symplectic 2-forms given by the imaginary part of the tensor product $\frac{1}{2} \sum_\alpha (du^\alpha)^* \otimes du^\alpha$. Both $\mathbb{A}$ and $\mathbf{u}$ are maps from the coordinate 4-manifold E into the vector space $\mathbb{H}^r$.

We will further require that the scalar fields transform under gauge transformations of some Lie group $K \subset Sp(p,q)$ of hyperkähler isometries of the metric $<\cdot,\cdot>$. Infinitesimally, we have

$$\delta_\lambda A_i^\alpha = g \sum_{I=1}^{dimH} \sum_{\alpha=1}^{2r} \lambda^I \tilde{T}_{I\beta}^\alpha A_i^\beta \tag{2.10a}$$

or

$$\delta_\lambda u^\alpha = g \sum_{I=1}^{dimH} \sum_{\alpha=1}^{r} \lambda^I T_{I\beta}^\alpha u^\beta, \tag{2.10b}$$

where g is a coupling constant and $\lambda = (\lambda_1, ..., \lambda_{\mathrm{dim}K}) \in \mathfrak{k}$ are coordinates on the Lie algebra of K. Both $\mathbb{T}_I$ and $\tilde{\mathbb{T}}_I$ are antihermitian generators taking values in the Lie algebra of K. They describe the same action of K in two different coordinate systems. $\tilde{\mathbb{T}}_I$ is a complex antihermitian $2r \times 2r$ matrix in the Lie algebra of K where K is viewed as a subgroup of $USp(2p, 2q)$. $\mathbb{T}_I$ is an antihermitian $r \times r$ quaternionic matrix taking values in the Lie algebra of K viewed as a subalgebra of $\mathfrak{sp}(p,q)$.

For simplicity we are going to assume that the gauge group G of m Yang-Mills multiplets is a product $G_0 \times K$. We shall assume that the scalar fields do not transform under G_0, *i.e.,* they do not couple to the vector multiplets of the group G_0. Furthermore, we will include the kinetic term in the Lagrangian for the gauge group G_0 only. All the vector multiplet fields associated with the group K, as a consequence, will be auxiliary and will decouple algebraically from the theory. We will also omit the kinetic terms for the gauge fields associated with the superconformal transformations. (Such a Lagrangian would be quadratic in second derivatives of the metric, *i.e.,* we would get a theory which classically has indefinite energy and quantum mechanically is not unitary.) As a consequence of this, some of the superconformal gauge fields will decouple algebraically to give an $N = 2$ Poincaré supergravity coupled to matter.

First let us begin with a Lagrangian $\mathcal{L}$ which is invariant under the whole superconformal group. We have

$$\mathcal{L} = \mathcal{L}_{sc} + \mathcal{L}_{G_0}. \tag{2.11}$$

Here $\mathcal{L}_{sc}$ describes the coupling of the $2r$ hypermultiplets to the Weyl multiplet and the vector multiplets of the group K. $\mathcal{L}_{G_0}$ is the Lagrangian describing the coupling of the vector multiplets of the group G_0 to the Weyl multiplet. There has to be at least one vector multiplet with the kinetic term in the Lagrangian as it must provide the spin one field for the gravitational multiplet of the $N = 2$ Poincaré supergravity. Let us write the part of the Lagrangian $\mathcal{L}_{sc}$ which is relevant to our discussion of the geometry of manifold X parameterized by the scalar fields A_i^α.

$$e^{-1}\mathcal{L}_\sigma = -g^{\mu\nu} d_\alpha{}^\beta \mathcal{D}_\mu A_i^\alpha \mathcal{D}_\nu A_\beta^i + \frac{1}{2} D d_\alpha{}^\beta A_i^\alpha A_\beta^i + g Y^{Iij} d_\alpha{}^\beta A_\beta^k \epsilon_{ki} T_{I\gamma}^\alpha A_j^\gamma, \tag{2.12a}$$

where

$$\mathcal{D}_\mu A_i^\alpha = \partial_\mu A_i^\alpha + \frac{1}{2}\mathcal{V}_{\mu i}{}^j A_j^\alpha - g W_\mu^I \tilde{T}_{I\beta}^\alpha A_i^\beta,$$

$g^{\mu\nu}$ is a (pseudo-)Riemannian metric on the spacetime E, and $\mathcal{V}^i_{\mu j} = \overline{\mathcal{V}_{\mu i}{}^j} = -\mathcal{V}_{\mu j}{}^i$.

One can rewrite the above Lagrangian using the quaternionic notation. First let us notice that $\mathcal{V}^i_{\mu j}$ is an $su(2)$ Lie algebra valued matrix (it is an antihermitian traceless 2×2 complex matrix) and under standard Lie algebra homomorphism $\tau : \mathfrak{su}(2) \longrightarrow \mathfrak{sp}(1)$ we can identify the matrix $\mathcal{V}^i_{\mu j}$ with a purely imaginary quaternion V_μ (which is exactly the Lie algebra of $Sp(1)$). The chiral $SU(2)$ acts on the fields A^α_i by $\delta_\Lambda A^\alpha_i = \Lambda^j_i A^\alpha_j$. In the language of the quaternionic vector $\mathbf{u}$, this action is given by the quaternionic scalar multiplication from the right by the group $Sp(1)$ of unit quaternions.

Similarly, the field Y^I is just a 2×2 complex symmetric matrix where $Y^I_{11} = \overline{Y^I_{22}}$ is complex and Y^I_{12} is imaginary. Hence, we can introduce a purely imaginary quaternionic vector $Y^I \sim (Y^I_{12} + jY^I_{11})$. Now we can rewrite eq. (2.12.a) in terms of $\mathbf{u}, V_\mu$ and Y^I as

$$
\begin{aligned}
e^{-1}\mathcal{L}_\sigma = -2g^{\mu\nu} &< \partial_\mu \mathbf{u} + \mathbf{u}V_\mu - gW^I_\mu \mathbb{T}_I \mathbf{u}\ ,\ \partial_\nu \mathbf{u} + \mathbf{u}V_\nu - gW^I_\nu \mathbb{T}_I \mathbf{u} > \\
&+ D < \mathbf{u}, \mathbf{u} > + gY^I < \mathbf{u}, \mathbb{T}_I \mathbf{u} > + g < \mathbf{u}, \mathbb{T}_I \mathbf{u} > Y^I. \qquad (2.12b)
\end{aligned}
$$

Noticed that we have used the fact that $< \mathbf{u}, \mathbb{T}_I \mathbf{u} >^* = - < \mathbf{u}, \mathbb{T}_I \mathbf{u} >$ is a purely imaginary quaternion since $\mathbb{T}_I$ is antihermitian with respect to the metric $< \cdot, \cdot >$. The Lagrangian (2.12.b), together with the remaining terms given by (2.11), is invariant under the full group of the $N = 2$ superconformal transformations. However, all the fields of the Weyl multiplet except for the graviton and the gravitino are auxiliary and they decouple algebraically.

The same is true of the Yang-Mills vector multiplets of the group K as we have not introduced the kinetic term for them. Their equations of motion are algebraic, too, and can be eliminated from the action.

We have a choice: One could first eliminate all the algebraic fields of the Weyl multiplet, fixing the superconformal gauges for chiral $SU(2)\times U(1)$, conformal boost, and dilatations. As demonstrated in [5], this leads to the $N = 2$ Poincaré supergravity theory coupled to vector and scalar multiplets with scalar fields parameterizing the (pseudo-)quaternionic Kähler manifold $Sp(q,p)/Sp(q,p-1)\times Sp(1)$. After that one could eliminate all the auxiliary vector multiplets of the group K which, according to [6], amounts to taking the quaternionic Kähler quotient of Galicki and Lawson [7,8]. Such a quotient yields a new (pseudo-)quaternionic Kähler manifold (often with singularities) and a new chiral matter coupling in the $N = 2$ supergravity theory.

However, we could first get rid of the auxiliary fields of the vector multiplets of the group K and only after that eliminate the non-physical fields of the Weyl multiplet. It is clear that in both cases we should obtain the same theory. After all, it does not matter in which order we are solving a set of algebraic equations of motion. In fact, as we shall see in the next chapter, the whole problem can be formulated rigorously in terms of the geometry of the associated quaternionic bundle introduced by Swann 11,14].

3. σ-Models, Quaternionic Geometry, and Quotients

In this chapter we shall describe some σ-model geometries associated with the Lagrangian (2.12.b). For simplicity, let us choose the Euclidean signature on $\mathbb{H}^r$ ($(q,p) = (0,r)$). We begin by examining the geometry of the process of eliminating nonpropagating fields of the Weyl multiplet.

As explained in [5], fixing the scale transformations, together with the field equation for the auxiliary field D, gives the following constraint on $\mathbf{u}$

$$\nu(\mathbf{u}) \stackrel{\rm def}{=} <\mathbf{u},\mathbf{u}> = -1/\kappa^2. \tag{3.1}$$

The level set $\nu^{-1}(-1/\kappa^2)$ describes a $(4r-1)$-dimensional sphere S^{4r-1} in $\mathbb{H}^r$. The chiral $SU(2) \simeq Sp(1)$ acts on this sphere and the quotient $\nu^{-1}(-1/\kappa^2)/Sp(1)$, which corresponds to fixing of the chiral $SU(2)$ gauge freedom, is the quaternionic projective space $\mathbb{HP}^{r-1}$. The map $\nu^{-1}(-1/\kappa^2) \stackrel{\pi}{\longrightarrow} \mathbb{HP}^{r-1}$ is the standard quaternionic Hopf fiberation. Together with the scale transformations $Sp(1)$ gives a free homothetic action of $\mathbb{H}^*$ on $\mathbb{H}^r \setminus \{0\}$ and the quaternionic projective space is just the quotient $\mathbb{H}^r \setminus \{0\}/\mathbb{H}^*$. Moreover, the metric calculated from the Lagrangian (2.12.b) is precisely the Fubini-Study $Sp(r)$ invariant quaternionic Kähler metric. To see this, one has to solve for the auxiliary chiral $Sp(1)$ field V_μ.

$$V_\mu = \kappa^2 {\rm Im} <\mathbf{u},\partial_\mu \mathbf{u}> \tag{3.2}$$

and our Lagrangian becomes

$$e^{-1}\mathcal{L}^P_\sigma = -2g^{\mu\nu}\Big(<\partial_\mu \mathbf{u}\ ,\ \partial_\nu \mathbf{u}> + \kappa^2 <\mathbf{u},\partial_\mu\mathbf{u}><\partial_\nu\mathbf{u},\mathbf{u}> + W^I_\mu V_\nu <\mathbf{u},\mathbb{T}_I\mathbf{u}> -$$

$$<\mathbf{u},\mathbb{T}_I\mathbf{u}> V_\mu W^I_\nu + gW^I_\mu(<\mathbf{u},\partial_\nu\mathbb{T}_I\mathbf{u}> - <\partial_\nu\mathbf{u},\mathbb{T}_I\mathbf{u}>) + g^2 W^I_\mu W^J_\nu <\mathbb{T}_I\mathbf{u},\mathbb{T}_J\mathbf{u}>\Big). \tag{3.3}$$

The first two terms of (2.15) describe the Fubini-Study metric on the quaternionic projective space $\mathbb{HP}^{r-1}$ in homogeneous coordinates $\mathbf{u}$. The remaining terms describe the gauging of the group $K \subset Sp(r)$ of the isometries of $\mathbb{HP}^{r-1}$. The Lagrangian $\mathcal{L}^P_\sigma$ is an $N=2$ Poincaré supergravity-matter system with scalar fields coupled to the gauge fields of K. At this point $\mathcal{L}^P_\sigma$ is still invariant under the local gauge transformations of K. However, the fields W^I_ν, as well as their supersymmetric partners, are all auxiliary and can be eliminated. We have two algebraic equations of motions for W^I_ν and Y^I respectively:

$$gW^J_\mu <\mathbb{T}_{(I}\mathbf{u},\mathbb{T}_{J)}\mathbf{u}> = <\mathbf{u},\overset{\leftrightarrow}{\partial}_\mu \mathbb{T}_I\mathbf{u}>, \quad I = 1,...,{\rm dim}K, \tag{3.4}$$

$$f(\mathbf{u}) \stackrel{\rm def}{=} <\mathbf{u},\mathbb{T}_I\mathbf{u}> = 0, \quad I = 1,...,{\rm dim}K. \tag{3.5}$$

In both of the above equations the fields $\mathbf{u}$ are understood to be the homogeneous coordinates on $\mathbb{HP}^{r-1}$. The 3dim$K$ equations of (3.5) are the quaternionic Kähler moment map equations of the action of K on the quaternionic projective space [6]. They define $f^{-1}(0) \subset \mathbb{HP}^{r-1}$ a submanifold in the projective space. The orbit space $\widehat{M} = f^{-1}(0)/K$ carries a natural quaternionic Kähler metric at all regular points. Depending on the action of K on $\mathbb{HP}^{r-1}$, $\widehat{M}$ is often singular. The metric on $\widehat{M}$ can be calculated by eliminating all the auxiliary fields of the vector multiplets of K. We get

$$\frac{1}{e}\mathcal{L}^P_{\hat{\sigma}} = -2g^{\mu\nu}\Bigg(<\partial_\mu \mathbf{u}, \partial_\nu \mathbf{u}> + \kappa^2 <\mathbf{u}, \partial_\mu \mathbf{u}><\partial_\nu \mathbf{u}, \mathbf{u}> + g^2 W^I_\mu W^J_\nu <\mathbb{T}_I \mathbf{u}, \mathbb{T}_J \mathbf{u}> \Bigg), \quad (3.6)$$

where $\mathbf{u}$ and W^I_μ are subject to the conditions (3.4-5). We can illustrate the process of eliminating the auxiliary fields of the Weyl multiplet and then the vector multiplets of K by the following diagram

$$\begin{array}{ccccc} & \mathcal{L}_\sigma & & \\ Sp(1)\times\mathbb{R}^+ & \downarrow & & \\ & \mathcal{L}^P_\sigma & \stackrel{K}{\longrightarrow} & \mathcal{L}^P_{\hat{\sigma}} \end{array} \quad (3.7)$$

The vertical arrow represents the fixing of the superconformal gauge transformations and the horizontal arrow represents the process of eliminating the vector multiplets associated with K, or, in the language of the quaternionic Kähler quotients, the reduction of $\mathbb{HP}^{r-1}$ by its isometry subgroup K.

As we already mentioned, we could start with the Lagrangian $\mathcal{L}_\sigma$ and eliminate the auxiliary fields of the vector multiplets of the group K first. This would yield the same equations (3.4-5), giving W^I_μ in terms of $\mathbf{u}$ and putting quadratic constraints on $\mathbf{u}$. But now we must think of $\mathbf{u}$ as a linear coordinate on the quaternionic vector space $\mathbb{H}^r$ (and <u>not</u> homogeneous quaternionic coordinates on the projective space as before). Observe that $f(\mathbf{u}) = <\mathbf{u}, \mathbb{T}_I \mathbf{u}> \in \mathfrak{k}^* \otimes \mathbb{R}^3$ is now just the momentum map equation of Hitchin et al. The equation (3.4) describes a subset of $\mathbb{H}^r$ given by the inverse image of the zero momentum. The group K is the isotropy group of the point $\vec{0} \in f^{-1}(0) \subset \mathbb{H}^r$. Let us assume that this action is free on $f^{-1}(0) \subset \mathbb{H}^r \setminus \{0\}$. Then the quotient $X \stackrel{\text{def}}{=} f^{-1}(0)/K$ is a hyperkähler manifold [10]. Noticed that in the hyperkähler quotient construction we have freedom to take any K invariant element ξ of the Lie co-algebra $\mathfrak{k}^*$ and consider the ξ-momentum level set $f^{-1}(\xi)$. However, in this case, only the zero level set is $Sp(1)$ invariant. As a consequence, X is not only hyperkähler. It also has an isometric action of $Sp(1)$ (or chiral $SU(2)$) rotating the hyperkähler structure. This action extends to a homothetic $\mathbb{H}^*$ action. Hence, we obtain a hypermultiplet $\hat{\sigma}$-model coupling of the scalar fields parameterizing X to the $N = 2$ Weyl multiplet. The Lagrangian $\mathcal{L}_{\hat{\sigma}}$ is invariant under the full group of the $N = 2$ superconformal transformations. But as the superconformal gauge fields decouple, we can eliminate them getting, as before, $\mathcal{L}^P_{\hat{\sigma}}$. Thus, we have the following commutative diagram

$$
\begin{array}{cccc}
 & \mathcal{L}_\sigma & \stackrel{K}{\longrightarrow} & \mathcal{L}_{\hat{\sigma}} & \\
Sp(1)\times\mathbb{R}^+ & \downarrow & & \downarrow & SO(3)\times\mathbb{R}^+ \\
 & \mathcal{L}^P_\sigma & \stackrel{K}{\longrightarrow} & \mathcal{L}^P_{\hat{\sigma}} &
\end{array}
\tag{3.8}
$$

Classically, all of the above Lagrangians describe the same physical interactions. In terms of the differential geometry of the various target spaces we can rewrite (3.8) as

$$
\begin{array}{ccc}
\mathbb{H}^r\setminus\{0\} & \stackrel{K}{\longrightarrow} & X=\mathcal{U}(\widehat{M}) \\
\downarrow \mathbb{H}^* & & \downarrow \mathbb{H}^*/\mathbb{Z}_2 \\
\mathbb{HP}^{r-1} & \stackrel{K}{\longrightarrow} & \widehat{M}
\end{array}
\tag{3.9}
$$

Here we introduced a special notation for X which will be defined later. The vertical arrows represent certain principal fiberations with fibers $\mathbb{H}^*$ and $\mathbb{H}^*/\mathbb{Z}_2$. The horizontal arrows describe the hyperkähler reduction of $\mathbb{H}^r\setminus\{0\}$ by the group K (with respect to the zero momentum level set) and the quaternionic Kähler reduction of $\mathbb{HP}^{r-1}$ by K respectively. Locally, $\widehat{M}$ is just a quotient of X by the action of $\mathbb{H}^*$. Globally, we have the fiberation $X\to\widehat{M}$ which is a special example of the associated quaternionic bundle introduced by Swann [11,14]. Swann shows that for any (pseudo-)quaternionic Kähler manifold M there exists a quaternionic associated line bundle $\mathcal{U}(M)=F\times_{Sp(k)\cdot Sp(1)}(\mathbb{H}^*/\mathbb{Z}_2)$ where F is the frame bundle of M, $\dim M=4k$, and $Sp(k)\cdot Sp(1)\stackrel{\mathrm{def}}{=}Sp(k)\times_{\mathbb{Z}_2}Sp(1)$. The bundle $\mathcal{U}(M)$ is a principal fiber bundle with fiber $\mathbb{H}^*/\mathbb{Z}_2$. It admits a (pseudo-)hyperkähler metric which is Riemannian when M has positive scalar curvature. It also admits a quaternionic Kähler metric in the same quaternionic class. Both of the metrics can be constructed explicitly. Swann also shows that the quaternionic Kähler quotient of Galicki and Lawson is just the hyperkähler quotient of Hitchin et al. in the associated bundle (and with respect to the zero momentum level). In other words the diagram

$$
\begin{array}{ccc}
X=\mathcal{U}(M) & \stackrel{K}{\longrightarrow} & \widehat{X}=\mathcal{U}(\widehat{M}) \\
\downarrow & & \downarrow \\
M & \stackrel{K}{\longrightarrow} & \widehat{M}
\end{array}
\tag{3.10}
$$

commutes for any quaternionic Kähler manifold M and for any quaternionic Kähler quotient $\widehat{M}$.

4. Hyperkähler Geometry in the Associated Bundle

As already mentioned in the introduction, not all the quaternionic Kähler σ-models allowed as couplings in the $N=2$ Poincaré supergravity theory can be constructed from de

Wit et al.'s Lagrangian. All the couplings in their formalism are of the form $\mathcal{L}^P_{\hat{\sigma}}$ as in the diagram (3.8). Geometrically, all such couplings are derived as some quaternionic Kähler quotients of the quaternionic projective space. This is the case because the Lagrangian $\mathcal{L}_\sigma$ starts with the flat quaternionic vector space $X = \mathbb{H}^r$ as the model for the hypermultiplet coupling to the Weyl multiplet and vector multiplets. Swann's theory of the quaternionic associated bundles over quaternionic Kähler manifolds and the commutativity of the diagram (3.10) strongly suggest that one could modify the superconformal tensor calculus to construct more general Lagrangians then that given by (2.12b). Clearly, in such a theory the fields should parameterize some (pseudo-)hyperkähler manifold (X, d) with non-flat metric $d = <\cdot,\cdot>$. The manifold X should be (at least locally) the associated bundle of some quaternionic Kähler manifold M describing the σ-model target space of the scalar couplings in the underlying $N = 2$ Poincarè supergravity coupled to matter. In fact, it is rather easy to construct the bosonic part of such a general Lagrangian. We also believe one can find the full $N = 2$ superconformally symmetric Lagrangian with hypermultiplets parameterizing some (pseudo-)hyperkähler manifold X and vector multiplets with gauge fields W^I_μ given by the hyperkähler Killing vectors on X. We are not going to derive it here. Instead let us describe the geometry of X. For simplicity we again assume the Riemannian signature. The generalization to the semi-Riemannian category is obvious.

First, the manifold (X, d) must be hyperkähler. That means that one has three parallel complex structures J^i such that

$$J^i \circ J^j = -\delta^{ij} + \epsilon^{ijk} J^k; \quad i, j, k = 1, 2, 3. \tag{4.1}$$

If the metric d is chosen to be Hermitian with respect to all three complex structures then we can define a trivial $\mathbb{R}^3$ bundle of symplectic 2-forms with a basis

$$\omega^i(Z, Y) \stackrel{\text{def}}{=} d(J^i Z, Y); \quad Z, Y \in TX, \quad i = 1, 2, 3. \tag{4.2}$$

We further require that X has $G = K \times Sp(1) \subset \mathrm{Isom}(X)$ as a subgroup of isometries. The group K acts by hyperkäher isometries, *i.e.*,

$$\mathcal{L}_V \omega^i = 0, \quad i = 1, 2, 3; \quad V \in \mathcal{K}, \tag{4.3}$$

where $\mathcal{L}_V$ is the Lie derivative and V is any Killing vector field for the action of K.

On the other hand, $Sp(1)$ acts on X by isometries rotating the hyperkähler 2-forms. If we choose $\{\xi^i\}_{i=1,2,3}$ to be a basis for the vector fields of the local $Sp(1)$ action with the Lie bracket $[\xi^i, \xi^j] = 2\epsilon^{ijk}\xi^k$ then we require that

$$\mathcal{L}_{\xi^i} \omega^j = 2\epsilon^{ijk} \omega^k. \tag{4.4}$$

This is not quite enough to describe the geometry of X we need. For instance, the moduli space $\mathcal{M}^k_o$ of charge k monopoles with fixed center is known to be a smooth $(4k-4)$-dimensional hyperkähler manifold with $SO(3)$ isometric action rotating the hyperkähler structure as in (4.4) [15]. But $\mathcal{M}_o^k$ turns out not to be an associated bundle of any quaternionic Kähler manifold (even locally) and, therefore, not the right model for our X. Another well-known example of a hyperkähler manifold with an isometric $SO(3)$ action rotating the hyperkähler structure is the $4k$-dimensional analogue of the Taub-NUT metric [12,16]. However, just as in the case of $\mathcal{M}_o^k$, it is not a correct model for the target space X of our σ-model coupling. We need another restriction on the hyperkähler geometry of X. As we noted in [17], (4.4) implies that

$$\xi^i \rfloor \omega^j = -d\nu\delta^{ij} + d\Phi^{ij} + \epsilon^{ijk}\eta^k, \tag{4.5}$$

where ν is an $Sp(1)$ invariant function on X, $d\Phi^{ij}$ is the traceless symmetric part of the decomposition $\mathfrak{sp}(1) \otimes \mathfrak{sp}(1) = \mathbb{I} \oplus Sym_0^2(\mathbb{R}^3) \oplus sp(1)$, and $d\eta^k = \omega^k$. Suppose that $Sp(1)$ (or $SO(3)$) acts freely on $\nu^{-1}(c)$. If not, we can always consider only the part of $\nu^{-1}(c)$ on which the actions is free. One can show that, although ν exists globally for any $Sp(1)$ (or $SO(3)$) action rotating the hyperkähler structure, there are obstructions for the quotient space $M = \nu^{-1}(c)/Sp(1)$ to carry a quaternionic Kähler structure [17]. These obstructions are described by Φ^{ij} and η^k. The quotient $M = \nu^{-1}(c)/Sp(1)$ is quaternionic Kähler if and only if Φ^{ij} is constant on the level sets $\nu^{-1}(c)$ and the one-forms η^k annihilate horizontal planes for the Riemannian submersion $\nu^{-1}(c) \longrightarrow M$ [17].

The model example of the above $Sp(1)$ quotient is given by the r-dimensional quaternionic vector space $\mathbb{H}^r$. Let $\mathbf{u} \in \mathbb{H}^r$ and let $\mathbf{u}^*$ be its quaternionic conjugate. We define the flat metric $g = Re\{d\mathbf{u}^* \otimes d\mathbf{u}\}$ and the hyperkähler structure $\omega = \frac{1}{2}\mathrm{Im}\{d\mathbf{u}^* \otimes d\mathbf{u}\} = \sum_{i=1}^3 \omega^a e_i$, where $\{e_i\}_{i=1,2,3} = \{i, j, k\}$ is the standard basis for the unit quaternions. If we introduce real coordinates $\mathbf{u} = \mathbf{t} + \mathbf{x}^i e_i$ then

$$\omega^i = \sum_{\alpha=1}^r \Big(dt_\alpha \wedge dx_\alpha^i - \frac{1}{2}\epsilon^{ijk} dx_\alpha^j \wedge dx_\alpha^k \Big). \tag{4.6}$$

Let us consider the action of $Sp(1)$ on $\mathbb{H}^r$ by the scalar multiplication from the right,

$$\mathbb{H}^r \ni \mathbf{u} \stackrel{\lambda}{\longmapsto} \mathbf{u}\lambda, \qquad \lambda \in Sp(1). \tag{4.7}$$

It is an isometric action rotating the quaternionic structure as in (4.5), that is $\omega \stackrel{\sigma}{\longmapsto} \overline{\lambda}\omega\lambda$. The action is free on $\mathbb{H}^r \setminus \{0\}$. Locally, it is generated by $\xi = \xi^i e_i = \mathrm{Im}(\mathbf{u}^{*t}\frac{\partial}{\partial \mathbf{u}})$ where $\frac{\partial}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{t}} + e_i \frac{\partial}{\partial \mathbf{x}^i}$. In real coordinates we have

$$\xi^i = \sum_{\alpha=1}^r \left(t_\alpha \frac{\partial}{\partial x_\alpha^i} - x_\alpha^i \frac{\partial}{\partial t_\alpha} - \frac{1}{2}\epsilon^{ijk} x_\alpha^j \frac{\partial}{\partial x_\alpha^k} \right). \tag{4.8}$$

With this choice of normalization $[\xi^i, \xi^j] = 2\epsilon^{ijk}\xi^k$. Easy calculation shows that $\nu = 2\mathbf{u}^{*t}\mathbf{u}$, $\phi \equiv 0$ and $\eta = \mathrm{Im}(\mathbf{u}^{*t} d\mathbf{u})$. In this case $M = \nu^{-1}(c)/Sp(1) \simeq \mathbb{HP}^{r-1}$ and the quotient metric is the standard one. The $Sp(1)$ action extends to a homothetic $\mathbb{H}^*$ action free on $\mathbb{H}^r \setminus \{0\}$ and $M = \nu^{-1}(c)/Sp(1) \simeq \mathbb{H}^r \setminus \{0\}/\mathbb{H}^*$.

The condition (4.4) and vanishing of the obstructions in (4.5) together imply that $Sp(1)$ extends to a homothetic action of $\mathbb{H}^* \simeq SU(2) \times \mathbb{R}^+$ (or $\mathbb{H}^*/\mathbb{Z}_2$) on X. This is necessary for the consistent coupling of the dilatation gauge fields in our sigma model Lagrangian $\mathcal{L}_\sigma(X)$ based on X.

The geometry of X described here is very restrictive. In fact, locally, the above conditions imply that X is the associated bundle of the quotient $M = \nu^{-1}(c)/Sp(1)$. Also, X cannot be a complete hyperkähler manifold unless it is the flat space. It always admits a function ν which is called the hyperkähler potential. If $\partial_J, \overline{\partial}_J$ denote the $\partial, \overline{\partial}$-operators in the complex coordinates with respect to J then

$$\omega^J = 2j\partial_J\overline{\partial}_J\nu \tag{4.9}$$

where J is any complex structure on the 2-sphere. The function ν is then an ordinary Kähler potential but with respect to any complex structure J on X.

There are many examples of hyperkähler manifolds with such properties. The quaternionic vector space $\mathbb{H}^r$ is the simplest one. Other include the moduli spaces $\mathcal{M}_0^k(\mathbb{R}^4)$ of based charge k instantons on $\mathbb{R}^4$ (see, for example, [18]) and the adjoint nilpotent orbits $\mathcal{O}$ of complex Lie groups with the hyperkähler structure introduced by Kronheimer [19]. We also have the associated bundles of the homogeneous quaternionic Kähler manifolds: the Wolf spaces [20] and the Alekseevskii spaces [21]. These were constructed explicitly by Swann [11]. Locally, $X/\mathbb{H}^* \simeq M$ is a quaternionic Kähler manifold and it describes a σ-model target space of the Poincaré supergravity-matter system $\mathcal{L}_\sigma^P(M)$.

Notice that, if X_1 and X_2 are both hyperkähler manifolds with the properties described above, then the product $X_1 \times X_2$ also has the same properties. In other words, $X_1 \times X_2$ can be a model for the target space of the hypermultiplet coupling in $N = 2$ conformal supergravity. Clearly,

$$\mathcal{L}_\sigma(X_1 \times X_2) = \mathcal{L}_\sigma(X_1) + \mathcal{L}_\sigma(X_2). \tag{4.10}$$

The equation (4.10) illustrates a very simple property of the $N = 2$ conformal supergravity-matter systems. If one has two different $N = 2$ conformal supergravity-matter Lagrangians with hypermultiplets A_i^α and B_i^α parameterizing two manifolds X_1 and X_2 respectively then the sum of these two Lagrangians is also invariant under the group of the $N = 2$ superconformal transformations. In this sense the $N = 2$ conformal supergravity is "linear" in hypermultiplet couplings. This is a familiar feature of the σ-models with global supersymmetry.

However, the same is not true for the $N = 2$ Poincaré supergravity matter-systems. If we consider two different models $\mathcal{L}_\sigma^P(M_1)$ and $\mathcal{L}_\sigma^P(M_2)$ with the target spaces based on quaternionic Kähler manifolds M_1 and M_2 then $\mathcal{L}_\sigma^P(M_1) + \mathcal{L}_\sigma^P(M_2)$ is no longer invariant.

It is easy to understand this phenomenon in the geometric language. The product $M_1 \times M_2$ is not a quaternionic Kähler manifold.

The eq. (4.10) suggests a very simple construction of new $N = 2$ Poincaré supergravity-matter couplings. Although we cannot just add $\mathcal{L}^P_\sigma(M_1)$ and $\mathcal{L}^P_\sigma(M_2)$ we can construct a new Lagrangian $\mathcal{L}^P_\sigma(\mathcal{J}(M_1, M_2))$ as follows. Let $X_1 = \mathcal{U}(M_1)$ and $X_2 = \mathcal{U}(M_2)$ be the associated bundles of M_1 and M_2 respectively. Then the product $X_1 \times X_2$ is a hyperkähler manifold with the $Sp(1)$ action satisfying all our conditions. Hence, there is an $N = 2$ conformal supergravity-matter Lagrangian modeled on $X_1 \times X_2$, and it is given by eq. (4.7). Eliminating the auxiliary fields of the Weyl multiplet we get

$$\begin{array}{c} \mathcal{L}_\sigma(X_1 \times X_2) \\ \downarrow \\ \mathcal{L}^P_\sigma \simeq \mathcal{L}^P_\sigma(\mathcal{J}(M_1, M_2)) \end{array} \tag{4.11}$$

a new $N = 2$ Poincaré supergravity model with scalar fields parameterizing some quaternionic Kähler manifold M. The geometry of M is determined by X_1 and X_2 (or M_1 and M_2). By construction, the product $X_1 \times X_2$ is the associated bundle of M. Swann calls $M \simeq \mathcal{J}(M_1, M_2)$ the *quaternionic join* of M_1 and M_2 [14]. Notice that if $\dim M_1 = 4k,\ \ \dim M_2 = 4r$ then $\dim \mathcal{J}(M_1, M_2) = 4k + 4r + 4$. This very elegant operation allows for construction of new couplings out of old ones. For example, let M^{4k} be a quaternionic Kähler manifold of positive scalar curvature and let $X = \mathcal{U}(M)$. We introduce $X \times \mathbb{H}^*$ equipped with the product metric of signature $(4k + 4, 4)$. The product $X \times \mathbb{H}^*$ has a homothetic $\mathbb{H}^*$ action and locally $\tilde{M}^{4k+4} \simeq (X \times \mathbb{H}^*)/\mathbb{H}^*$ is a quaternionic Kähler manifold of negative scalar curvature. This way one obtains new σ-model couplings with very interesting target space geometries. $\tilde{M}$ is usually not complete.

Although we do not derive the full $N = 2$ conformal supergravity-matter Lagrangian $\mathcal{L}_\sigma(X)$ with hypermultiplets parameterizing some hyperkähler manifold $X = \mathcal{U}(M)$, where X has all the properties described in this chapter, it is rather evident that such a Lagrangian can be indeed constructed. In fact, it should be possible to express all the interactions between the Weyl multiplet and the hypermultiplets in terms of the hyperkähler potential function ν on X. Then the Lagrangian (2.11) would depend on just two functions F and ν (both subject to some restrictions) describing the local interactions for the vector multiplets and the hypermultiplets respectively. We will address this problem in our future work.

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