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Quaternionic Reduction and Quaternionic Orbifolds

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0. Introduction

In recent years there has been a growth of interest in riemannian manifolds with $Sp_1{\cdot}Sp_n$ holonomy, or so-called quaternionic Kähler manifolds. One reason for this is that the classical σ-model, which consists of the energy functional for maps $f: \Sigma \longrightarrow X$ of a compact riemannian 4-manifold into a riemannian manifold X, admits locally an $N = 2$ supersymmetric extension only when X is a quaternionic Kähler space with non-zero scalar curvature.

From the classification of Berger [B] we know that there are very few holonomy groups possible on non-locally-symmetric, irreducible riemannian manifolds. The Kähler, hyperkähler and quaternionic Kähler manifolds constitute a large proportion of the interesting cases. Whenever a Kähler or hyperkähler manifold X carries a non-trivial connected group H of automorphisms, there is a process called the *Marsden-Weinstein reduction* for constructing a new Kähler (or hyperkähler) space [MW, H1].

The main part of this note is to present an analogous reduction process for quaternionic Kähler manifolds. We shall show that there exists in this case a certain "*momentum mapping*" which reduces to the classical one when the scalar curvature κ of the space is zero (and the metrics becomes hyperKähler). When $\kappa \neq 0$, this momentum mapping is uniquely defined for any compact group of automorphisms H. Its zero set $\mathcal{Z}_H$ is H-invariant and the quotient $\mathcal{Z}_H/H$ is again a quaternionic Kähler manifold at all regular points.

It is interesting that the momentum mapping has quite different features in the two cases: $\kappa = 0$ and $\kappa \neq 0$. (Since the metric on any quaternionic Kähler manifold is always Einstein, κ is constant). When $\kappa = 0$, the map is defined by integration. The constants of integration introduce ambiguities and for non-semi-simple Lie groups the map cannot always be made H-equivariant. When $\kappa \neq 0$, the map is defined by differentiation, and it is always H-equivariant. Moreover, only the zero-momentum level has global meaning in this case.

Beginning with the classical quaternionic Kähler manifolds, namely quaternionic projective space $\mathbb{P}^n_{\mathbb{H}}$, and considering S^1-actions, one can produce by this method many compact spaces with $Sp_1 \cdot Sp_n$ holonomy. These spaces are often non-trivial riemannian orbifolds. (By "non-trivial" we mean that they are not the quotients of the known spaces by finite automorphism groups.)

A riemannian orbifold is, by definition, a space whose local structure is that of a riemannian manifold divided by a finite group of isometries. The metric singularities on such spaces are of the simplest kind. (Note that even among 2-dimensional cones only those whose unit circle has length of the form $2\pi/n$ for some $n \in \mathbb{Z}^+$, are riemannian orbifolds.)

For each integer $k \geq 1$, we show that there exists an infinite family of distinct compact, simply-connected, quaternionic Kähler orbifolds of dimension $4k$. This result is of particular interest when $k = 1$. Here we have an infinite family of compact, simply-connected riemannian orbifolds of dimension 4 each of which is Einstein, self-dual and of positive scalar curvature. By a well-known theorem of Hitchin [H1], the only compact 4-dimensional manifolds admitting such metrics are the sphere S^4 and complex projective space $\mathbb{P}^2_{\mathbb{C}}$. However, if we allow these minor singularities, which regularize by "unfolding", we see that many examples exist. Each of these spaces has the topological type of the Thom-space of a complex line bundle over $\mathbb{P}^1_{\mathbb{C}}$. Every line bundle appears infinitely often in the family. These spaces are differentially equivalent, as orbifolds, to weighted projective spaces. It seems that the study of the σ-models for such spaces would be quite interesting.

The construction presented here is an abstraction to the general setting of the work done by the first named author [G1]. The case considered in [G1] is the most basic and already gives all the interesting examples.

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1. Definitions

We begin by recalling the basics of quaternionic Kähler geometry (cf. [I, A, S]). Let X be a smooth $4n$-dimensional manifold ($n > 1$). We say that X is almost quaternionic if there is a 3-dimensional subbundle $\mathcal{G} \subset Hom(TX, TX)$ with the following property: At each point $x \in X$ there is a basis $\{J_1, J_2, J_3\}$ of the fibre $\mathcal{G}_x$ such that:

$$J_i \circ J_j = -\delta_{ij}\mathrm{id} + \epsilon_{ijk} J_k \ . \tag{1.1}$$

In other words, $\mathbb{R} \cdot \mathrm{Id} \oplus \mathcal{G}$ is, at each point, a subalgebra isomorphic to the quaternions.

Suppose now that X carries a riemannian metric adapted to the quaternion structure in the sense that each $J \in \mathcal{G}$ is orthogonal, *i.e.,*

$$\langle JV, JW \rangle = \langle V, W \rangle \tag{1.2}$$

for all $J \in \mathcal{G}$ and all $V, W \in T_x X$ at all points $x \in X$. Adapted metric always exists. Given an adapted metric we obtain an isometric bundle embedding

$$\mathcal{G} \subset \Lambda^2 T^* X \tag{1.3}$$

which associates to each $J \in \mathcal{G}_x$ the non-degenerate 2-form ω defined by

$$\omega(V, W) = \langle JV, W \rangle. \tag{1.4}$$

for $V, W \in T_x X$.

Suppose $\{J_1, J_2, J_3\}$ are locally defined smooth sections of $\mathcal{G}$ which satisfy (1.1) at each point. Then these form an orthonormal frame field for $\mathcal{G}$ in the standard metric $\langle A, B \rangle = \frac{1}{2n} trace(A^t B)$ on $Hom(TX, TX)$. Let $\{\omega_i\}_{i=1,2,3}$ be the basis of 2-forms corresponding under (1.4). The associated exterior 4-form

$$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i \tag{1.5}$$

is invariant under change of frame field and thus globally defined on X. It is non-degenerate in the sense that Ω^n is nowhere vanishing on X. Thus, if $d\Omega = 0$ there are strong consequences in $H^*(X; \mathbb{R})$.

Definition 1.6: *The Riemannian manifold X $(n > 1)$ together with $\mathcal{G}$ is quaternionic Kähler if $\nabla \Omega = 0$ where ∇ denotes the Levi-Civita connection.*

This is equivalent to the hypothesis that the local holonomy group of the metric at point x is contained in the subgroup:

$$\{g \in SO_{4n}(T_x X) : \; g^* \Omega_x = \Omega_x\} \cong Sp_1 \times Sp_n / \mathbb{Z}_2 \stackrel{\text{def}}{=} Sp_1 \cdot Sp_n.$$

The hypothesis $\nabla \Omega = 0$ clearly implies that $d\Omega = 0$. It is a basic fact that if X is quaternionic Kähler it is Einstein. We suppose from this point on that X is quaternionic Kähler. Let $\{\omega_1, \omega_2, \omega_3\}$ be a local orthonormal frame field for $\mathcal{G} \subset \Lambda^2 T^* X$ as above. Then from (1.5) we have that

$$\sum_{i=1}^{3} (\nabla \omega_i) \wedge \omega_i = 0$$

from which it follows that

$$\nabla\omega_i = \sum_{j=1}^{3} \alpha_{ij} \otimes \omega_j, \tag{1.7}$$

where the α_{ij} are 1-forms which satisfy

$$\alpha_{ij} \equiv -\alpha_{ji} \qquad \forall i,j = 1,2,3\ . \tag{1.8}$$

This means in particular that the subspace $\Gamma(\mathcal{G}) \subset \Gamma(\Lambda^2 T^*X)$ is preserved by the riemannian covariant derivative. The matrix

$$A \equiv \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix} \tag{1.9}$$

is the connection 1-form with respect to the local frame field $\{\omega_1, \omega_2, \omega_3\}$. The curvature of this induced connection represents a component of the Riemann curvature tensor $\mathcal{R}$ and is given by

$$R = dA - A \wedge A.$$

Using the facts that $d\omega_i = \sum_{j=1}^3 \alpha_{ij} \wedge \omega_j$ and $d^2\omega_i = 0$, one deduces that

$$R_{ij} = d\alpha_{ij} - \sum_l \alpha_{il} \wedge \alpha_{lj} = \lambda \sum_k \epsilon_{ijk}\omega_k \tag{1.10}$$

for some constant λ. (One uses here the full structure of the forms $\{\omega_1, \omega_2, \omega_3\}$ and the fact that $n > 1$.) Note that this equation means that

$$\langle R(\omega_i), \omega_j \rangle = \lambda \sum_k \epsilon_{ijk}\omega_k \tag{1.10$'$}$$

for all i, j. Since $\mathcal{G}$ is an oriented 3-dimensional bundle, there is a canonical identification $\text{SkewEnd}(\mathcal{G}) \cong \mathcal{G}$ via the cross-product. Using this identification we can consider R as a map

$$R:\ \Lambda^2 TX \ \longrightarrow \text{SkewEnd}(\mathcal{G}) \cong \mathcal{G} \subset \Lambda^2 TX$$

and, as such, equation (1.10$'$) simply states that

$$R = \lambda\pi, \tag{1.11}$$

where π denotes pointwise orthogonal projection of $\Lambda^2 TX$ into $\mathcal{G}$.

By our assumption each of the subbundles in the orthogonal splitting

$$\Lambda^2 TX = \mathcal{G} \oplus \mathcal{G}^\perp$$

is preserved by the riemannian connection, and therefore also by the curvature tensor of $\Lambda^2 TX$

$$R: \ \Lambda^2 TX \longrightarrow \mathrm{SkewEnd}(\Lambda^2 TX)$$

which is given as follows. Let

$$\widetilde{(\,\cdot\,)}: \ \Lambda^2 TX \longrightarrow \mathrm{SkewEnd}(TX)$$

be the canonical identification and let

$$\mathcal{R}: \ \Lambda^2 TX \longrightarrow \mathrm{SkewEnd}(TX)$$

denote the Riemann curvature tensor of TX. Then for $\varphi, \psi \in \Lambda^2 TX$ we have that

$$R_\varphi(\psi) = [\mathcal{R}_\varphi \ , \ \widetilde{\psi}], \tag{1.12}$$

where $[\cdot \ , \ \cdot]$ denotes the commutator of endomorphisms. For any φ, R_φ maps $\mathcal{G}$ into $\mathcal{G}$ (and $\mathcal{G}^\perp$ into $\mathcal{G}^\perp$), that is, we have a decomposition

$$R = R_{\mathcal{G}} \oplus R_{\mathcal{G}^\perp},$$

where the first component is just the curvature tensor of $\mathcal{G}$ discussed above. Hence, equations (1.11) and (1.12) tell us that

$$[\mathcal{R}_\varphi \ , \ \widetilde{\psi}] = \lambda[\pi \ \varphi \ , \ \widetilde{\psi}] \qquad \forall \varphi \in \Lambda^2 TX \quad \text{and} \quad \forall \psi \in \mathcal{G}.$$

Therefore, for any $\varphi \in \Lambda^2 TX$, we have that

$$\mathcal{R}_\varphi = \lambda \pi \ \varphi + c_\varphi,$$

where $\widetilde{c_\varphi}$ commutes with the endomorphisms $\{J_1, J_2, J_3\}$ of $\mathcal{G}$. This means that

$$c_\varphi \in \mathfrak{sp}_n \subset \mathcal{G}^\perp,$$

where $\mathfrak{sp}_n$ denotes the skew-endomorphisms commuting with $\mathcal{G}$. Suppose now that $\varphi \in \mathcal{G}$ and $\psi = c_\varphi$. Then from the symmetry of the Riemann curvature tensor, we have that

$$\|\psi\|^2 = \langle \psi \ , \ \mathcal{R}_\varphi \rangle = \langle \mathcal{R}_\psi \ , \ \varphi \rangle = \langle \lambda \pi \ \psi \ , \ \varphi \rangle = 0.$$

Hence $c_\varphi = 0$, and we conclude that as a symmetric endomorphism $\mathcal{R}: \ \Lambda^2 TX \longrightarrow \ \Lambda^2 TX$, the Riemann curvature tensor has the property that

$$\mathcal{R} \mid_{\mathcal{G}} = \lambda Id_{\mathcal{G}} \ . \tag{1.13}$$

It is straightforward to show that λ is a positive multiple of the scalar curvature κ on X.

Consequently, if $\kappa = 0$, then $\mathcal{G}$ is flat and the metric is hyperKähler(Sp_n-holonomy). We are interested here in the case where $\kappa \neq 0$, for example, on quaternionic projective space $\mathbb{P}^n_{\mathbb{H}}$.

Note that in dimension 4, the condition that $\nabla\Omega = 0$ is trivially satisfied since Ω is the volume form. Nevertheless, a good extension of the concept of being quaternionic Kähler does exist for dimension 4. In this case the decomposition (1.12) corresponds exactly to the decomposition $\Lambda^2 TX = \Lambda_+ \otimes \Lambda_-$ into self-dual and anti-self-dual 2-forms.

DEFINITION 1.14: *An oriented riemannian 4-manifold is called quaternionic Kähler if condition (1.13) holds (where $\mathcal{G} = \Lambda_+$). This condition is equivalent to the assumption that X is Einstein and anti-self-dual.*

For further details on quaternionic Kähler manifolds, see [A, S, I, W].

2. The Quaternionic Momentum Mapping

We shall now describe an analogue of the Marsden-Weinstein reduction for the case of quaternionic Kähler manifolds. To do this we first consider the spaces $\Omega^p(\mathcal{G}) \equiv \Gamma(\Lambda^p T^*X \otimes \mathcal{G})$ of smooth exterior p-forms on X with values in $\mathcal{G}$. The connection given on $\mathcal{G}$ induces a "*de Rham*" sequence

$$\Omega^0(\mathcal{G}) \xrightarrow{d^\nabla = \nabla} \Omega^1(\mathcal{G}) \xrightarrow{d^\nabla} \Omega^2(\mathcal{G}) \xrightarrow{d^\nabla} \cdots \tag{2.1}$$

such that

$$d^\nabla \circ d^\nabla(f) = R(f) \tag{2.2}$$

for $f \in \Omega^0(\mathcal{G})$. (See [BL].)

Consider now the Lie group

$$Aut(X) \equiv \{g \in Isom(X) : \; g^*\Omega = \Omega\}$$

and its Lie algebra

$$aut(X) \equiv \{V \in K(X) : \; L_V\Omega = 0\}$$

embedded naturally in the space $K(X)$ of Killing vector fields on X. (Here L_V denotes the Lie derivative.) To each $V \in aut(X)$ we associate the $\mathcal{G}$-valued 1-form

$$\Theta_V \in \Omega^1(\mathcal{G})$$

defined in terms of a local frame $\omega_1, \omega_2, \omega_3$ by

$$\Theta_V \equiv \sum_i (i_V\omega_i) \otimes \omega_i \tag{2.3}$$

where i_V denotes contraction with V. Clearly Θ_V remains invariant under local change of frame field (*i.e.,* under local gauge transformations).

THEOREM 2.4: *Assume that the scalar curvature of X is not zero. Then to each $V \in aut(X)$ there corresponds a unique section $f_V \in \Omega^0(\mathcal{G})$ such that*

$$\nabla f_V = \Theta_V \ . \tag{2.5}$$

In fact, under the canonical bundle isometry $\sigma :\ SkewEnd(\mathcal{G}) \longrightarrow \mathcal{G}$, f_V is given explicitly by the formula

$$f_V = \frac{1}{\lambda}\sigma(\mathcal{L}_V - \nabla_V). \tag{2.6}$$

PROOF: In light of the explicit formula (2.6) the proof could be left as an exercise for the reader. However, there is an approach to this theorem which proceeds in analogy with the classical (Kähler) case. It was by this route that the authors were led to the result. We present it here because it gives insight into the problem and may be of use to people searching for other generalizations of the momentum map. Moreover, in this argument we shall clearly see exactly where the classical and non-classical cases diverge.

Consider the "de Rham" sequence (2.1). We are given $\Theta_V \in \Omega^1(\mathcal{G})$ and want to solve the equation $d^\nabla f_V = \Theta_V$. Differentiating this equation and using (2.2) we find that

$$R(f_V) = d^\nabla d^\nabla f_V = d^\nabla \Theta_V \ .$$

The main point now is that if $\lambda \neq 0$, then the map $R :\ \mathcal{G} \longrightarrow \Lambda^2 TX \otimes \mathcal{G}$ is an injective bundle map. Therefore equation (2.5) can be solved uniquely for f_V provided that $d^\nabla \Theta_V$ lies in the subbundle $R(\mathcal{G}) \subset \Lambda^2 TX \otimes \mathcal{G}$. This "compatibility condition" is guaranteed by the invariance of the form Ω under V.

Let us carry out this argument explicitly in terms of a local orthonormal frame field $\{\omega_1, \omega_2, \omega_3\}$ for $\mathcal{G}$. We write

$$f_V = \sum_i f_i \omega_i$$

and recall that

$$\nabla f_V = \sum_j (df_j + \sum_i f_i \alpha_{ij}) \otimes \omega_j \ .$$

Hence, equation (2.5) can be reexpressed as

$$df_j + \sum_i f_i \alpha_{ij} = i_V \omega_j. \tag{2.7}$$

Applying d to this equation and recalling that $R_{ij} = d\alpha_{ij} - \sum \alpha_{il} \wedge \alpha_{lj}$, we find that

$$\sum_i f_i R_{ij} = d(i_V \omega_j) - \sum_i (i_V \omega_i) \wedge \alpha_{ij} \stackrel{\text{def}}{=} \Psi_j$$

From (1.10) we deduce that for each j

$$\lambda \sum_{i,k} f_i \epsilon_{ijk} \omega_k = \Psi_j. \tag{2.8}$$

Since $\lambda \neq 0$ this equation can be solved uniquely for $\{f_i\}$ provided that $\{\Psi_j\}$ is of the form $\Psi_j = \sum c_{jk}\omega_k$ where $c_{jk} = -c_{kj}$, *i.e.*, provided that

$$\sum_j \omega_j \wedge \Psi_j = 0.$$

This last equation follows easily from the condition $\mathcal{L}_V(\sum \omega_j^2) = 0$.

To get the explicit form of the solution, note that

$$\mathcal{L}_V \omega_j = d(i_V \omega_j) + i_V(d\omega_j) = d(i_V\omega_j) + i_V(\sum_k \alpha_{jk} \wedge \omega_k) =$$

$$= \sum_k \alpha_{jk}(V)\omega_k + d(i_V\omega_j) - \sum_k \alpha_{jk} \wedge (i_V\omega_k) = \nabla_V \omega_j + \Psi_j.$$

Replacing Ψ_j by $(\mathcal{L}_V - \nabla_V)\omega_j$ in (2.8) and solving for f_V yields the formula (2.6). ∎

We observe now that by the uniqueness in Theorem 2.4, the map $V \longrightarrow f_V$ transforms naturally under the group of automorphisms. This means specifically that for $g \in Aut(X)$ and $V \in aut(X)$ we have

$$f_{g_*(V)} = g_*(f_V) \tag{2.9}$$

where $g_*(f_V)(x) \stackrel{\text{def}}{=} \tilde{g}(f_V(g^{-1}(x)))$ and where $\tilde{g}$ denotes the map induced by g on the bundle $\mathcal{G} \subset \Lambda^2 TX$. Note also that $g_* V = Ad_g(V)$. Hence, (2.9) means that the diagram

$$\begin{array}{ccc} aut(X) & \xrightarrow{Ad_g} & aut(X) \\ \downarrow f & & \downarrow f \\ \Omega^0(\mathcal{G}) & \xrightarrow{g_*} & \Omega^0(\mathcal{G}) \end{array}$$

commutes.

Suppose now that $H \subset Aut(X)$ is a compact Lie subgroup with corresponding Lie algebra $\mathfrak{h}$.

DEFINITION 2.10: *The* <u>momentum map</u> *associated to H is the section f of the bundle $\mathfrak{h}^* \otimes \mathcal{G} \cong Hom(\mathfrak{h}, \mathcal{G})$ whose value at a point x is the homomorphism $V \longrightarrow f_V(x)$.(Here $\mathfrak{h}$ denotes the trivial $\mathfrak{h}$-bundle over X).*

From the equivariance above we see immediately that the momentum map is H-equivariant. Since the action of H in the bundle $\mathfrak{h}^* \otimes \mathcal{G}$ is linear on the fibers, it preserves the zero section. Consequently the set

$$\mathcal{Z}_H \stackrel{\text{def}}{=} \{x \in X : \quad f(x) = 0\} \tag{2.11}$$

is H-invariant.

This momentum mapping (in the case when scalar curvature $\lambda \neq 0$), differs from the standard symplectic one in two essential ways. Here there are no constants of integration; in fact we solve by differentiation rather then by integration. This gives us the group invariance, which does not hold in general in the symplectic case. On the other hand, the only naturally defined "level set" of this mapping is the set $\mathcal{Z}_H$, where $f = 0$.

3. Quaternionic Reduction

The main result of this section is the following:

THEOREM 3.1: *Let X be a quaternionic Kähler manifold with scalar curvature $\kappa \neq 0$. Let $H \subset Aut(X)$ be a compact subgroup with momentum map f. Let $\mathcal{Z}_H^{\mathbf{X}}$ denote the H-invariant subset of $\mathcal{Z}_H = \{x \in X : \quad f(x) = 0\}$ where f intersects the zero section transversally and where H acts freely. Then $\mathcal{Z}_H^{\mathbf{X}}/H$ equipped with the submersed metric (i.e., the one which makes the projection $\mathcal{Z}_H^{\mathbf{X}} \longrightarrow \mathcal{Z}_H^{\mathbf{X}}/H$ a riemannian submersion), is again a quaternionic Kähler manifold.*

Corollary 3.2: *Let X be as above and suppose $H \cong S^1 \subset Aut(X)$ is a closed 1-parameter subgroup generated by a vector field $V \in aut(X)$. If $V_x \neq 0$ at all points $x \in \mathcal{Z}_H$, then $\mathcal{Z}_H/H$ is a compact quaternionic Kähler orbifold.*

PROOF OF 3.1: It will suffice to consider the case when $H \cong S^1$. The general case is quite similar. Let V be the Killing vector field generating the free S^1-action on $\mathcal{Z}_H^{\mathbf{X}}$. Then $V^\perp$ is the field of horizontal planes for the submersion $\mathcal{Z}_H^{\mathbf{X}} \stackrel{\pi}{\longrightarrow} \mathcal{Z}_H^{\mathbf{X}}/H$. Furthermore, at any point $x \in \mathcal{Z}_H^{\mathbf{X}}$, we have that

$$J_1 V, J_2 V \text{ and } J_3 V \quad \text{are normal to } \mathcal{Z}_H^{\mathbf{X}}, \tag{3.3}$$

where J_1, J_2, J_3 are a basis of $\mathcal{G}_x$ as in §1. To see this, let $\{\omega_i\}$ be a local frame field for $\mathcal{G}$ as above, and recall that $f_V = \sum f_i \omega_i$ where $\nabla f_i = i_V \omega_i$. Consequently

$$df_i = \sum_j \alpha_{ij} f_j + i_V \omega_i = i_V \omega_i \cong J_i V \tag{3.4}$$

at each point of the set where $f_V = 0$. This establishes (3.3) and furthermore shows that the subset of $\mathcal{Z}_H$ where f_V intersects the zero-section transversally is precisely the subset where $V \neq 0$.

Let $\Omega = \sum_i \omega_i \wedge \omega_i$ be the parallel 4-form on X where $\omega_i(V,W) \equiv \langle J_i V, W\rangle$, and let $\widetilde{\Omega}$ denotes the restriction of Ω to $\mathcal{Z}_H^{\mathbf{X}}$. From (3.3) we see that

$$i_V \widetilde{\Omega} = 0. \tag{3.5}$$

Hence, $\widetilde{\Omega}$ is a horizontal H-invariant 4-form, and so it descends to a form Ω_H on $\mathcal{Z}_H^{\mathbf{X}}/H$ with $\pi^*\Omega_H = \widetilde{\Omega}$.

Note from (3.3) that at any point $x \in \mathcal{Z}_H^{\mathbf{X}}$ we have an orthogonal decomposition $T_x X = V_x^{\perp} \oplus \mathbb{R}\cdot V_x \oplus span\{J_i V_x\}$. It follows that the plane field $V^{\perp}$ is $\{J_1, J_2, J_3\}$-invariant. This gives us a H-invariant almost quaternionic structure on $V^{\perp}$ which descends to an almost quaternion structure on $\mathcal{Z}_H^{\mathbf{X}}/H$, whose associated 4-form is Ω_H. Hence, it is only necessary to prove that

$$\nabla \Omega_H = 0.$$

To do this, we assume that $W_0, ..., W_4$ are smooth vector fields on $\mathcal{Z}_H^{\mathbf{X}}/H$ and we let $\widetilde{W}_0, ..., \widetilde{W}_4$ denote their unique horizontal lifts to $\mathcal{Z}_H^{\mathbf{X}}$. We want to show that

$$\Big(\nabla_{W_0}\Omega_H\Big)(W_1, ..., W_4) = 0\ .$$

This can be rewritten as

$$W_0\Big(\Omega_H(W_1, ..., W_4)\Big) = \sum_{1=i}^{4} \Omega_H(W_1, ..., \nabla_{W_0} W_i, ..., W_4)$$

which is equivalent to the equation

$$\widetilde{W}_0\Big(\widetilde{\Omega}(\widetilde{W}_1, ..., \widetilde{W}_4)\Big) = \sum_{i=1}^{4} \widetilde{\Omega}(\widetilde{W}_1, ..., \nabla_{W_0} W_i, ..., \widetilde{W}_4)\ . \tag{3.6}$$

Now it is a consequence of the O'Neill formulas [ON] that

$$\nabla_{W_0} W_i = \Big[\widetilde{\nabla}_{\widetilde{W}_0} \widetilde{W}_i\Big]^{\mathbf{H}},$$

where $\widetilde{\nabla}$ denotes the covariant derivative on $\mathcal{Z}_H^{\mathbf{X}}$ and $\Big(\cdot\Big)^{\mathbf{H}}$ denotes orthogonal projection onto $V^{\perp}$. Since the form $\widetilde{\Omega}$ is horizontal we can rewrite (3.6) as

$$\widetilde{W}_0\Big(\widetilde{\Omega}(\widetilde{W}_1, ..., \widetilde{W}_4)\Big) = \sum_{i=1}^{4} \widetilde{\Omega}(\widetilde{W}_1, ..., \widetilde{\nabla}_{\widetilde{W}_0} \widetilde{W}_i, ..., \widetilde{W}_4)$$

and we may clearly drop the tildas on the Ω's since $\widetilde{\Omega}$ is just the restriction of Ω to $\mathcal{Z}_H^{\mathbf{X}}$. Now it is easy to see from the fact that $\omega_i(W, W') = \langle J_i W, W'\rangle$, that the following is true. Fix $x \in \mathcal{Z}_H^{\mathbf{X}}$ and choose vectors $U_1, U_2, U_3 \in (V)_x^{\perp}$.Then for any $U \in T_x X$ we have that

$$\Omega(U_1, U_2, U_3, U) = \Omega(U_1, U_2, U_3, U^{\mathbf{H}}).$$

It is a fundamental fact that

$$\widetilde{\nabla}_{\widetilde{W}_0}\widetilde{W}_i = \left(\nabla_{\widetilde{W}_0}\widetilde{W}_i\right)^{\mathbf{T}},$$

where ∇ is the riemannian connection on X and $\left(\cdot\right)^{\mathbf{T}}$ denotes orthogonal projection onto $T_x\mathcal{Z}_H^{\mathbf{X}}$. Hence, we can rewrite (3.6) as:

$$\widetilde{W}_0\left(\Omega(\widetilde{W}_1, ..., \widetilde{W}_4)\right) = \sum_{i=1}^{4}\Omega(\widetilde{W}_1, ..., \nabla_{\widetilde{W}_0}\widetilde{W}_i, ..., \widetilde{W}_4)\ .$$

This equation is a direct consequence of the hypothesis that

$$\nabla\Omega \equiv 0$$

on X. This proves the theorem in dimensions greater than 4. In dimension 4 one could compute directly using fundamental equations for submersions and immersions. We shall present an alternative proof which uses the moving frame and works also in dimension four.

Set $X' = \mathcal{Z}_H^{\mathbf{X}}/H$ and let

$$i:\ \ \mathcal{Z}_H^{\mathbf{X}} \subset X \quad \text{and} \quad \pi:\ \ \mathcal{Z}_H^{\mathbf{X}} \longrightarrow\ X'$$

denote the inclusion and projection maps respectively. Our first observation here is that the standard restriction on 2-forms gives us a bundle map $\mathcal{G} \longrightarrow i^*\mathcal{G}$ which, along the submanifold $\mathcal{Z}_H^{\mathbf{X}}$ is a bundle isometry up to homothety, *i.e.,* an orthonormal basis $\{\omega_1, \omega_2, \omega_3\}$ of $\mathcal{G}$ at $x \in \mathcal{Z}_H^{\mathbf{X}}$ restricts to an orthogonal basis $\{i^*\omega_1, i^*\omega_2, i^*\omega_3\}$ with $\|i^*\omega_j\|^2 = (n-1)/n$ for all j (where $4n = dimX$). The forms $\{i^*\omega_j\}$ are horizontal for the submersion π, and by assumption the bundles $\mathcal{G}$ and therefore also $i^*\mathcal{G}$ are preserved by the S^1-action. Hence, there is a bundle $\mathcal{G}'$ of 2-forms which determines a topological quaternionic Kähler structure on X' and has the property that

$$\pi^*\mathcal{G}' = i^*\mathcal{G}. \tag{3.7}$$

(All of the above is easily verified using the relationship of the ω_j's with the J_j's.) Note that when $\dim(X') = 4$, we have $\mathcal{G}' \cong \Lambda^2_+$.

Fix a point $x \in X'$ and choose a smooth orthonormal basis $\{\omega_1', \omega_2', \omega_3'\}$ of $\mathcal{G}'$ in a closed neighborhood U of x. Pulling back by π^* and using the isomorphism (3.7) we see that there is a pointwise orthonormal basis $\{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3\}$ of sections of $\mathcal{G}$ defined <u>along</u> the set $\pi^{-1}(U) \subset \mathcal{Z}_H^{\mathbf{X}}$ such that $i^*\tilde{\omega}_j = \pi^*\omega_j'$ for each j. (We ignore the scale factor $\sqrt{n/(n-1)}$ here.) A standard and easy argument now shows that there exists an orthonormal basis $\{\omega_1, \omega_2, \omega_3\}$ of sections of $\mathcal{G}$ defined in a neighborhood of $\pi^{-1}(U)$ in X, such that

$$\pi^*\omega_j' = i^*\omega_j \quad \text{for} \ \ j = 1, 2, 3. \tag{3.8}$$

At this point we shall specialize to the remaining case where $\dim(X') = 4$. (The argument can be trivially generalized to all dimensions.)

The Levi-Civita connection on 2-forms preserves the sub-bundle and can be written as

$$\nabla\omega_j = \sum_{k=1}^{3} \alpha_{jk} \otimes \omega_k, \qquad j = 1, 2, 3,$$

where the α_{jk} are 1-forms which satisfy: $\alpha_{jk} = -\alpha_{kj}$. (See (1.7).) The sub-bundle $\mathcal{G}' = \Lambda^2_+ \subset \Lambda^2$ is also preserved by the Levi-Civita connection of X' and similarly expressed:

$$\nabla\omega'_j = \sum_{k=1}^{3} \alpha'_{jk} \otimes \omega'_k, \qquad j = 1, 2, 3,$$

where $\alpha'_{jk} = -\alpha'_{kj}$.

From the symmetry of the Levi-Civita connection we have the following fundamental identities

$$d\omega_j = \sum_{k=1}^{3} \alpha_{jk} \wedge \omega_k \quad \text{and} \quad d\omega'_j = \sum_{k=1}^{3} \alpha'_{jk} \wedge \omega'_k \tag{3.9}$$

for each j. Since both d and exterior multiplication commute with i^* and π^*, the equations (3.8) and (3.9) imply that

$$\sum_{k=1}^{3} (\pi^*\alpha'_{jk} - i^*\alpha_{jk}) \wedge \pi^*\omega'_k = 0 \quad \text{for} \quad j = 1, 2, 3. \tag{3.10}$$

We claim that equation (3.10) implies that

$$\pi^*\alpha'_{jk} = i^*\alpha_{jk} \quad \text{for all} \ \ j, k. \tag{3.11}$$

To begin we observe that the 1-forms $i^*\alpha_{jk}$ must be "horizontal", *i.e.,* their restrictions to the fibres of the map π must be zero. (This follows easily from (3.10) and the linear independence of the forms $\pi^*\omega'_j$.) Hence, at any point $\widetilde{x} \in \mathcal{Z}_H^{\mathbf{X}}$, all the forms appearing in (3.10) are supported in the horizontal 4-plane, and we are reduced to the following linear algebraic statement which can be formulated downstairs at $x = \pi(\widetilde{x})$. Let A_1, A_2, A_3 be 1-forms on T_xX'. We want to prove the following assertion.

$$\sum_{k,l} \epsilon_{jkl} A_l \wedge \omega'_k = 0 \ \text{ for } \ 1 \le j \le 3 \quad \Longrightarrow \quad A_1 = A_2 = A_3 = 0. \tag{3.12}$$

This is a system of 12 equations in 12 unknowns. Given a Kähler-type form $\omega(V, W) = \langle JV, W\rangle$ as above and a 1-form A, we note that: $*(A \wedge \omega) = A \circ J$ where $(A \circ J)(V) \equiv A(JV)$. Hence, the first equation in (3.12) can be rewritten in terms of the endomorphisms $\{J_1, J_2, J_3\}$ as

$$\mathcal{J}(A) \equiv \sum_{lk} \epsilon_{jkl} A_l \circ J_k = 0.$$

Writing out the coefficients of the A_j's in 3-successive blocks, the operator $\mathcal{J}$ can be expressed in block matrix form as:

$$\begin{pmatrix} 0 & J_3 & -J_2 \\ -J_3 & 0 & J_1 \\ J_2 & -J_1 & 0 \end{pmatrix}.$$

This is easily seen to have non-zero determinant, and our assertion (3.11) is proved.

The condition that X be quaternionic Kähler implies that

$$d\alpha_{jk} - \sum_l \alpha_{jl} \wedge \alpha_{lk} = \lambda \sum_l \epsilon_{jkl}\omega_l, \quad \text{for all} \quad j,k$$

for some constant λ. (See(1.10).) Restricting to $\mathcal{Z}_H^{\mathbf{X}}$ and using equations (3.8) and (3.11), we see that

$$d(\pi^*\alpha'_{jk}) - \sum_l \pi^*\alpha'_{jl} \wedge \pi^*\alpha'_{lk} = \lambda \sum_l \epsilon_{jkl}\pi^*\omega'_l, \quad \text{for all} \quad j,k.$$

Since π^* is an injection, this implies that

$$R'_{jk} \equiv d\alpha'_{jk} - \sum_l \alpha'_{jl} \wedge \alpha'_{lk} = \lambda \sum_l \epsilon_{jkl}\omega'_l, \quad \text{for all} \quad j,k$$

As observed in (1.13), this means that the Riemann curvature tensor $\mathcal{R}: \ \Lambda^2 \longrightarrow \ \Lambda^2$ has the property that $\mathcal{R}\mid_{\Lambda^2_+} = \lambda \mathrm{Id}_{\Lambda^2_+}$. This means precisely that X' is Einstein and anti-self-dual (or self-dual if one reverses the orientation of X). ∎

PROOF OF 3.2: It is an immediate consequence of (3.4) that the transversality hypothesis for f_V is satisfied at all points $x \in \mathcal{Z}_H$ where $V_x \neq 0$. Hence, the set

$$\mathcal{Z}'_H \stackrel{\text{def}}{=} \{x \in \mathcal{Z}_H : \quad V_x \neq 0\}$$

is a smooth, H-invariant submanifold. We are assuming here that $\mathcal{Z}'_H = \mathcal{Z}_H$. The action of $H \cong S^1$ on $\mathcal{Z}_H$ is therefore locally free, and the quotient $\mathcal{Z}_H/H$ is an orbifold with a smooth quaternionic Kähler metric at all regular points. ∎

In fact, this metric extends across the singularities to make $\mathcal{Z}_H/H$ a riemannian orbifold. This is a consequence of the following general fact:

PROPOSITION 3.13: *Suppose S^1 acts locally freely and by isometries on a riemannian manifold Z. Then the quotient space Z/S^1 inherits uniquely the structure of a riemannian orbifold with the property that at non-singular points the projection $\pi: \ Z \longrightarrow Z/S^1$ is a riemannian submersion.*

PROOF: Fix $p \in Z$ and let N_p denote the normal space to the orbit of S^1 through p. (Since the action is locally free, every orbit is a smoothly embedded curve.) Fix an orthonormal basis $\{v_1, ..., v_{n-1}\}$ of N_p and consider the map $\Psi : \mathbb{R}^{n-1} \longrightarrow Z$ given by

$$\Psi(x_1, ..., x_{n-1}) = exp_p(\sum_{j=1}^{n-1} x_j v_j),$$

where exp_p denotes the exponential map of Z at p. For $\epsilon > 0$ sufficiently small, this map gives a smooth embedding of the disc $D_\epsilon \equiv \{x \in \mathbb{R}^{n-1} : \ ||x|| < \epsilon\}$. At regular points p, *i.e.,* points where the isotropy group $G_p \equiv \{g \in S^1 : \ g(p) = p\}$ is trivial, the composition $\pi \circ \Psi : \ D_\epsilon \longrightarrow Z/S^1$ gives a smooth local coordinate chart on Z/S^1. If p is a singular point, *i.e.,* a point where $G_p \neq \{e\}$, then the embedded disk $\Psi(D_\epsilon)$ is G_p-invariant and the projection $D_\epsilon \longrightarrow \Psi(D_\epsilon)/G_p$ defines the orbifold structure at p.

It now remains to observe that the riemannian structure on Z/S^1 can be defined as follows. Let g' be the degenerate metric on Z obtained from the given metric g by "killing" the direction of the vector field V_0 which generates the S^1-action. Specifically, we set

$$g'(V, W) = g(V, W) - \frac{g(V, V_0)g(W, V_0)}{g(V_0, V_0)}.$$

This new pseudo-metric is still S^1-invariant and has the following property. For any two smooth embeddings transversal to the orbits $\Psi_1, \Psi_2 : \ D_\epsilon \longrightarrow Z$, such that $\pi \circ \Psi_1 = \pi \circ \Psi_2$, we have that

$$\Psi_1^* g' = \Psi_2^* g'.$$

These induced metrics are non-degenerate (since the maps are transversal to the orbits) and give us the desired metric structure on Z/S^1. This completes the proof of Proposition 3.13 and Corollary 3.2. ∎

4. Some Examples

In this section we shall examine a large family of examples of compact simply-connected riemannian orbifolds which arise from the process of quaternionic reduction. Many of the examples constructed will be mutually homeomorphic as topological spaces, but mutually distinct as orbifolds. We shall examine this phenomenon in detail. Using the global orbifold structure, we shall then prove that these quaternionic Kähler spaces are never locally symmetric.

Let $(u_o, ..., u_n)$ denote linear coordinates on the quaternionic vector space bbh^{n+1}, where scalar multiplication is defined from the right. We consider these to be "homogeneous coordinates" for the quaternionic projective space $\mathbb{P}^n_{\mathbb{H}} = (\mathbb{H}^{n+1} - \{0\})/\mathbb{H}^*$. When equipped with Sp_{n+1}-invariant metric

$$ds^2 = \frac{1}{||u||^2} \sum_\alpha d\overline{u}^\alpha \otimes du^\alpha - \frac{1}{||u||^4} \sum_{\alpha\beta} (\overline{u}^\alpha du^\alpha) \otimes (d\overline{u}^\beta u^\beta)$$

$\mathbb{P}^n_{\mathbb{H}}$ is a model example of a compact symmetric manifold with holonomy $Sp_n \cdot Sp_1$. For notational convenience we shall write our homogeneous coordinates as $u = (u_o, \mathbf{u})$ where $\mathbf{u} = (u_1, ..., u_n)$. For each pair of integers $p, q \in \mathbb{Z}^+$ with $(p, q) = 1$ and $0 < q/p \leq 1$, we shall consider the action on $\mathbb{P}^n_{\mathbb{H}}$ defined in homogeneous coordinates by

$$\varphi_t(u_o, \mathbf{u}) = (e^{2\pi i q t} u_o, \ e^{2\pi i p t} \mathbf{u}), \tag{4.1}$$

where $t \in [0, 1)$ if $(p+q)$ is odd and where $t \in [0, 1/2)$ if $(p+q)$ is even.

REMARK 4.2: The circle action described in (4.1) is an isometry of $\mathbb{P}^n_{\mathbb{H}}$ and it preserves its quaternionic structure, *i.e.*,

$$g^*\Omega = \Omega, \quad g \in S^1.$$

Note that all isometries of $\mathbb{P}^n_{\mathbb{H}}$ preserve its quaternionic structure, *i.e.*,

$$Aut(\mathbb{P}^n_{\mathbb{H}}) \equiv \{g \in Isom(\mathbb{P}^n_{\mathbb{H}}) : \ g^*\Omega = \Omega\} = Isom(\mathbb{P}^n_{\mathbb{H}}).$$

Consequently, we can consider the quaternionic momentum mapping for this action. The zero level set is given by an algebraic submanifold in $\mathbb{P}^n_{\mathbb{H}}$:

$$\mathcal{Z}_H = \{(u_o, \mathbf{u}) \in \mathbb{P}^n_{\mathbb{H}} : \ q\overline{u}_o i u_o + p\overline{\mathbf{u}} i \mathbf{u} = 0\}. \tag{4.3}$$

THEOREM 4.4: *The circle action given by (4.1) is locally free on $\mathcal{Z}_H$ for all $p, q \in \mathbb{Z}^+$, $q/p < 1$ and is free for $q = p = 1$.*

PROOF: This action on $\mathbb{H}^{n+1}$ is generated by the $\mathbb{H}^*$-invariant vector field

$$V(u_o, \mathbf{u}) = (iqu_o, ip\mathbf{u}).$$

The projection of this vector field on $\mathbb{P}^n_{\mathbb{H}}$ is zero at a point with homogeneous coordinates $(u_o, \mathbf{u})$ if and only if

$$V(u_o, \mathbf{u}) = (u_o\nu, \mathbf{u}\nu)$$

for some quaternion ν. Taking the inner product with $(u_o, \mathbf{u})$ and assuming that $|u_o|^2 + |\mathbf{u}|^2 = 1$ we find that

$$q\overline{u}_o i u_o + p\overline{\mathbf{u}} i \mathbf{u} = \nu$$

which, on $\mathcal{Z}_H$, implies that $\nu = 0$. However, this is impossible since $V \neq 0$ outside the origin. Hence, the projection of V is non-zero on $\mathcal{Z}_H$, and thus the action on $\mathcal{Z}_H$ is locally free as claimed. It is trivial to see that the action is free if $p = q = 1$. ∎

It follows that all conditions of Theorem 3.1-2 apply and that the quaternionic reduction gives a compact riemannian quaternionic orbifold $\mathcal{O}_{q,p}(n-1)$

$$\mathcal{O}_{q,p}(n-1) \stackrel{\text{def}}{=} \mathcal{Z}_H / S^1.$$

In the case where $p = q = 1$ we see that the set $\mathcal{Z}_H$ is invariant under transformations of the homogeneous coordinates (with matrix multiplication from the left). The resulting space is the Grassmann manifold

$$\mathcal{O}_{1,1}(n-1) = \frac{U(n+1)}{U(n-1)\times U(2)}$$

with its symmetric quaternionic Kähler metric.

For $q/p < 1$, $\mathcal{O}_{q,p}(n-1)$ is an orbifold with a quaternionic Kähler metric. Explicit calculation shows that as $q/p \ \to 1$ the metric on $\mathcal{O}_{q,p}(n-1)$ converges locally to the metric on $\mathcal{O}_{1,1}$. Similarly, as $q/p \ \to 0$ these metrics converge locally to a hyperKähler metric on $T\mathbb{P}^{n-1}_{\mathbb{C}}$ (constructedby Eguchi-Hanson for $n = 2$ and by Calabi in general dimensions). This limiting metric on $T\mathbb{P}^{n-1}_{\mathbb{C}}$ is not locally symmetric, and the convergence is uniform in three derivatives. Since the condition $\nabla R \equiv 0$ does not hold in the limit, it does not hold on $\mathcal{O}_{q,p}(n-1)$ for all q, p with q/p sufficiently small. This argument, which is presented in detail in [G1], shows that $\mathcal{O}_{q,p}(n-1)$ is not locally symmetric whenever q/p is sufficiently close to zero. We shall present here an alternative argument which shows that $\mathcal{O}_{q,p}(n-1)$ is not locally symmetric for all q, p with $0 < q/p < 1$. This argument will be based on an analysis of the large scale architecture of these spaces.

We shall now examine the global structure of $\mathcal{O}_{q,p}(n-1)$. We shall see that it has two connected singular sets: a "nut" and a "bolt", and we shall analyze the structure of $\mathcal{O}_{q,p}(n-1)$ in a neighborhood of each. To begin we introduce affine coordinates $\mathbf{w} = (w_1, ..., w_n)$ on the open set $U_0 \equiv \{[(u_o, \mathbf{u})] \in \mathbb{P}^n_{\mathbb{H}} : \quad u_o \neq 0\}$ by setting

$$w_\alpha = u_\alpha u_o^{-1}, \quad \text{for} \quad \alpha = 1, ..., n.$$

In these coordinates the action (4.1) becomes

$$\varphi_t(\mathbf{w}) = e^{2\pi i p t}\mathbf{w}e^{-2\pi i q t},$$

and the set $\mathcal{Z}_H$ is given as

$$\mathcal{Z}_H \cap U_0 \cong \{w \in \mathbb{H}^n : \quad iq + p\overline{\mathbf{w}}i\mathbf{w} = 0\}.$$

We then write

$$\mathbf{w} = \mathbf{w}_+ + j\overline{\mathbf{w}}_-$$

where $\mathbf{w}_+, \mathbf{w}_- \in \mathbb{C}^n$ and observe that

$$\varphi_t(w_+, w_-) = (e^{2\pi i(p-q)t}\mathbf{w}_+, \ e^{2\pi i(p+q)t}\mathbf{w}_-),$$

$$\mathcal{Z}_H \cap U_0 = \{(\mathbf{w}_+, \mathbf{w}_-) \in U_0 : \quad \|\mathbf{w}_-\|^2 - \|\mathbf{w}_+\|^2 = q/p, \quad \overline{\mathbf{w}}_- \cdot \mathbf{w}_+ = 0\}. \tag{4.6}$$

It is not difficult now to see that the action has non-trivial isotropy in $\mathcal{Z}_H \cap U_0$ exactly at the points where $\mathbf{w}_+ = 0$. This gives us a singular set $\Sigma_0 \subset \mathcal{O}_{q,p}(n-1)$ described explicitly as

$$\begin{aligned}\Sigma_0 &= \{(\mathbf{w}_+, \mathbf{w}_-) \in \mathcal{Z}_H : \quad \mathbf{w}_+ = 0\}/S^1 \\ &= \{(0, \mathbf{w}_-) \in \mathbb{C}^{2n} : \|\mathbf{w}_-\|^2 = 1\}/S^1 \cong \mathbb{P}^{n-1}_{\mathbb{C}}.\end{aligned} \tag{4.7}$$

One easily checks that the isotropy group $\Gamma_0 = \{t \in S^1 : \quad \varphi_t(x) = x\}$, for points x corresponding to Σ_0, is

$$\Gamma_0 \cong \begin{cases} \mathbb{Z}_{p+q}, & \text{if } (p+q) \text{ is odd;} \\ \mathbb{Z}_{\frac{p+q}{2}}, & \text{if } (p+q) \text{ is even.} \end{cases} \tag{4.8}$$

To describe the second singular set of $\mathcal{O}_{q,p}(n-1)$ we introduce affine coordinates $\mathbf{v} = (v_o, ... v_{n-1})$ on the open set

$$U_1 \equiv \{[(u_o, \mathbf{u})] \in \mathbb{P}^n_{\mathbb{H}} : \quad u_n \neq 0\}$$

by setting

$$v_o = \sqrt{\frac{p}{q}} u_o u_n^{-1}, \quad \text{and} \quad v_\alpha = u_\alpha u_n^{-1} \quad \text{for} \quad \alpha = 1, ..., n-1.$$

If we write $\mathbf{v} \equiv (v_1, ..., v_{n-1})$, the action becomes

$$\varphi_t(v_o, \mathbf{v}) = (e^{2\pi i q t} v_o e^{-2\pi i p t}, \ e^{2\pi i p t} \mathbf{v} e^{-2\pi i p t}).$$

If, as before, we write $\mathbf{v} = \mathbf{v}_+ + j \overline{\mathbf{v}}_-$, then $\mathcal{Z}_H$ is described here as:

$$\mathcal{Z}_H \cap U_1 \cong \{(v_+, v_-) \in \mathbb{C}^n \times \mathbb{C}^n : \quad \|\mathbf{v}_+\|^2 + 1 = \|\mathbf{v}_-\|^2 \ \text{ and } \ \overline{\mathbf{v}}_- \cdot \mathbf{v}_+ = 0\}.$$

It is now easy to see that the action has non-trivial isotropy in $\mathcal{Z}_H \cap U_1$ exactly at the points where $v_o = 0$. Hence, the remaining singular points form a set

$$\Sigma_1 \equiv \{[\mathbf{u}] \in \mathcal{Z}_H : \quad u_o = 0\}/S^1.$$

The action of S^1 on the hyperplane $\{u_o = 0\} \subset \mathbb{P}^n_{\mathbb{H}}$ is of type (1,1) and Σ_1 is just quaternionic reduction with respect to this action. Hence, using our observation above,

$$\Sigma_1 \cong \mathcal{O}_{1,1}(n-2) \cong \frac{U(n)}{U(n-2) \times U(2)}. \tag{4.9}$$

An easy calculation shows that the isotropy group

$\Gamma_1 = \{t \in S^1 : \quad \varphi_t(x) = x\}$ for points x above Σ_1, is

$$\Gamma_1 \cong \begin{cases} \mathbb{Z}_{2p}, & \text{if } (p+q) \text{ is odd;} \\ \mathbb{Z}_{p}, & \text{if } (p+q) \text{ is even.} \end{cases} \tag{4.10}$$

In summation we have the following. Let $\mathbb{G}_{k,n}$ denote the Grassmann manifold of complex k-planes in $\mathbb{C}^n$.

PROPOSITION 4.11: *The singular points of the orbifold $\mathcal{O}_{q,p}(n-1)$ are the two disjoint manifolds $\Sigma_0 \cong \mathbb{G}_{1,n-1}$ and $\Sigma_1 \cong \mathbb{G}_{2,n}$ described above, and the local group of the orbifold structure of Σ_k is Γ_k for each k.*

If we set $\mathcal{O}^0_{q,p}(n-1) \equiv \mathcal{O}_{q,p}(n-1) - \Sigma_1$ and $\mathcal{O}^1_{q,p}(n-1) \equiv \mathcal{O}_{q,p}(n-1) - \Sigma_0$, then our orbifold is the union

$$\mathcal{O}_{p,q}(n-1) = \mathcal{O}^0_{p,q}(n-1) \cup \mathcal{O}^1_{p,q}(n-1) \tag{4.12}$$

of these two open sets. Note that

$$\Sigma_k \subset \mathcal{O}^k_{q,p}(n-1) \quad \text{for } k = 0, 1.$$

We shall see that topologically $\mathcal{O}^k_{q,p}(n-1)$ is the finite quotient of a vector bundle over Σ_k by a fibre-wise linear action. The orbifold structure is, however, more subtle.

For simplicity of exposition we shall confine ourselves from this point on to the case where $n = 2$ ($dim\mathcal{O} = 4$) and where $(p+q)$ is odd. We write $\mathcal{O}^k_{q,p}(1) \equiv \mathcal{O}^k_{q,p}$. The simpler component of the singular set here is $\Sigma_1 =$ a point, and the orbifold structure in a neighborhood is straightforward.

LEMMA 4.13: *There is a smooth orbifold equivalence $\mathcal{O}^1_{q,p} \cong \mathbb{C}^2/\mathbb{Z}_{2p}$ where the action of $\mathbb{Z}_{2p}$ on $\mathbb{C}^2$ is generated by scalar multiplication by the complex number $e^{\pi i q/p}$.*

PROOF: We have from above that

$$\mathcal{Z}_H \cap U_1 \cong \{(\mathbf{v}_+, \mathbf{v}_-) \in \mathbb{C}^2 \times \mathbb{C}^2 : \quad \|\mathbf{v}_+\|^2 + 1 = \|\mathbf{v}_-\|^2 \text{ and } \overline{\mathbf{v}}_- \cdot \mathbf{v}_+ = 0\},$$

$$\varphi_t(v_{o+}, v_{1+}) = (e^{2\pi i(p-q)t} v_{o+},\ v_{1+}),$$

and

$$\varphi_t(v_{o-}, v_{1-}) = (e^{2\pi i(p+q)t} v_{o-},\ v_{1-}).$$

There is a map of the complex 2-disc transversal to the orbit at $v_o = 0$ in $\mathcal{Z}_H$ given by

$$\mathbf{v}_+ = (t, -\frac{t\overline{r}}{g}), \quad \mathbf{v}_- = (r, g), \qquad (t, r) \in \mathbb{C}^2,$$

where g is a real analytic function of t and r of the form $g = 1 + \epsilon(\|r\|, \|t\|)$.(The function g is determined by the equation $\|\mathbf{v}_+\|^2 + 1 = \|\mathbf{v}_-\|^2$.) Set $\omega = e^{\pi i/p}$. The image disc is invariant under the transformation φ_ω which generates the isotropy group at the point $\mathbf{v}_+ = (0,0)$, $\mathbf{v}_- = (0,1)$. The induced transformations of the disc in our (t, r)-coordinates is

$$(t, r) \longrightarrow (\omega^q t\ ,\ \omega^q r).$$

Hence, $\mathcal{O}^1_{q,p}$ is of the form $\mathbb{C}^2/\mathbb{Z}_{2p}$ as claimed. ∎

To understand the structure of the component $\mathcal{O}^0_{q,p} \supset \Sigma_0 \cong S^2$, we introduce the following spaces. Set $S^{2n-1} = \{\mathbf{z} \in \mathbb{C}^n : \ \ \|\mathbf{z}\| = 1\}$ and let r, s be a pair of relatively prime positive integers. Then we set

$$\mathcal{H}_{r,s} \equiv S^3 \times \mathbb{C}/\Phi^{r,s}, \tag{4.14}$$

where $\Phi^{r,s}$ is the action of S^1 on $S^3 \times \mathbb{C}$ given by

$$\Phi^{r,s}_\tau(\mathbf{z}, \alpha) = (\tau^r \mathbf{z},\ \tau^s \alpha).$$

The natural projection $\mathcal{H}_{r,s} \longrightarrow S^2 \cong S^3/S^1$ makes $\mathcal{H}_{r,s}$ a complex line bundle over S^2. When $r = 1$, this is one of standard definitions of the complex line bundle of Chern class 1 over S^2, so we take as given that $c_1\Big(\mathcal{H}_{1,s}\Big) = s$.

LEMMA 4.15: $c_1\Big(\mathcal{H}_{r,s}\Big) = s$.

PROOF: Consider the map $F : \ S^3 \times \mathbb{C} \longrightarrow S^3 \times \mathbb{C}$ given by $F(\mathbf{z}, \alpha) = (\mathbf{z}, \alpha^r)$, and note that

$$F \circ \Phi^{r,s}_\tau = \Phi^{1,s}_{\tau^r} \circ F$$

for all $\tau \in S^1$. It follows that F induces a smooth map $\mathcal{H}_{r,s} \longrightarrow \mathcal{H}_{1,s}$ which is easily shown to be a complex line bundle isomorphism. ∎

This lemma shows that the structure of $\mathcal{H}_{r,s}$ as a topological manifold is completely determined by the integer s. However, the structure of $\mathcal{H}_{r,s}$ as an orbifold (determined by the S^1-action) depends on both r and s.

THEOREM 4.16: *For $(p+q)$ odd, there is a smooth orbifold equivalence*

$$\mathcal{O}^0_{q,p} \cong \mathcal{H}_{p+q,2p}.$$

In particular as a topological space $\mathcal{O}^0_{q,p}$ is simply a complex line bundle over S^2 of Chern class $2p$.

PROOF: Let $(\mathbf{w}_+, \mathbf{w}_-) \in \mathbb{C}^2 \times \mathbb{C}^2$ be the affine coordinates on $\mathbb{P}^2_{\mathbb{H}}$ defined above. We make a slight change of these coordinates by setting

$$\mathbf{z}_+ = \mathbf{w}_+ \ \text{ and } \ \mathbf{z}_- = \sqrt{\Big(q/p + \|\mathbf{w}_+\|^2\Big)}\,\mathbf{w}_- \ .$$

From (4.6) we see that, in these coordinates, $\mathcal{Z}_H$ is given by

$$\mathcal{Z}_H \cap U_0 \cong \{(\mathbf{z}_+, \mathbf{z}_-) \in \mathbb{C}^2 \times \mathbb{C}^2 : \quad \|\mathbf{z}_-\|^2 = 1 \ \text{ and } \ \overline{\mathbf{z}}_- \cdot \mathbf{z}_+ = 0\}$$

and the action is given by

$$\varphi_\tau(\mathbf{z}_-, \mathbf{z}_+) = (\tau^{p+q}\mathbf{z}_-,\ \tau^{p-q}\mathbf{z}_+) \tag{4.17}$$

for $\tau \in S^1$. Consequently, we have that as an orbifold $\mathcal{O}^0_{q,p}$ is equivalent to the quotient of the set

$$\mathcal{Z}^0_H \equiv \{(\mathbf{z}_-, \mathbf{z}_+) \in S^3 \times \mathbb{C}^2 : \ \ \mathbf{z}_+ \perp \mathbf{z}_-\}$$

by the action (4.17). We define a map $f : S^3 \times \mathbb{C} \longrightarrow \mathcal{Z}^0_H$ by setting $f(\mathbf{z}, \alpha) = (\mathbf{z}, \alpha \mathbf{z}^\dagger)$ where if $\mathbf{z} = (z_o, z_1)$ then $\mathbf{z}^\dagger = (-\overline{z}_1, \overline{z}_o)$. Note that for any $\tau \in S^1$ we have the diagram commutativity:

$$\begin{array}{ccc} (\mathbf{z}, \alpha) & \longrightarrow & (\mathbf{z}, \alpha \mathbf{z}^\dagger) \\ \downarrow & & \downarrow \\ (\tau^{p+q}\mathbf{z}, \tau^{2p}\alpha) & \longrightarrow & (\tau^{p+q}\mathbf{z}, \tau^{p-q}\alpha \mathbf{z}^\dagger) \end{array}$$

i.e., we have that $f \circ \Phi^{p+q,2p}_\tau = \varphi_\tau \circ f$. Thus f is an S^1-equivariant diffeomorphism and therefore f induces a smooth equivalence of the quotient spaces as orbifolds. ∎

REMARK 4.19: If $(p+q)$ is even, analogous considerations show that

$$\mathcal{O}^0_{q,p} \cong \mathcal{H}_{\frac{p+q}{2},p} \quad \text{and} \quad \mathcal{O}^1_{q,p} \cong \mathbb{C}^2/\mathbb{Z}_p.$$

REMARK 4.20: In higher dimensions the analysis is not dissimilar. Note that the set

$$\mathcal{Z}^0_H \equiv \{(\mathbf{z}_-, \mathbf{z}_+) \in \mathbb{C}^n \times \mathbb{C}^n : \ \ \|\mathbf{z}_-\|^2 = 1 \ \text{ and } \mathbf{z}_+ \perp \mathbf{z}_-\}$$

divided by the action $\Psi_\tau(\mathbf{z}_-, \mathbf{z}_+) = (\tau \mathbf{z}_-, \ \tau \mathbf{z}_+)$ is just the tangent bundle of $\mathbb{P}^{n-1}_{\mathbb{C}}$. Using (4.6) we find that if $(p+q)$ is odd, then as a topological space

$$\mathcal{O}^0_{q,p}(n-1) \cong T\mathbb{P}^{n-1}_{\mathbb{C}}/\mathbb{Z}_{2p},$$

where $\mathbb{Z}_{2p}$ acts on $T\mathbb{P}^{n-1}_{\mathbb{C}}$ by scalar multiplication in the fibers by the $2p^{th}$-roots of unity. (The orbifold structure is more complicated.) Similarly, we have that

$$\mathcal{O}^1_{q,p}(n-1) \cong E/\mathbb{Z}_{2p},$$

where E is the canonical complex vector bundle of rank 2 over the Grassmannian $\mathbb{G}_{2,n}$.

In particular, we see that Σ_k is a deformation retract of $\mathcal{O}^k_{q,p}$. Thus, from the Van-Kampen Theorem we know the following.

THEOREM 4.21: *$\mathcal{O}_{q,p}(n)$ is simply-connected for all q, p and n.*

Using the detailed structure presented above we can now prove the following

THEOREM 4.22: *$\mathcal{O}_{q,p}(n)$ is not locally symmetric for any q, p, n with $q/p < 1$.*

PROOF: Consider first the case of dimension 4 ($n = 1$), with $p+q$ odd and > 4. Suppose that the metric on $\mathcal{O}_{q,p}(1)$ were locally symmetric for some relatively prime pair p, q with $q/p < 1$. Then from the classification (cf. [W] for example) we know that $\mathcal{O}_{q,p}(1)$ must be locally isometric to either S^4 or $\mathbb{P}^2_{\mathbb{C}}$ (with the standard metrics). In fact every point of $\mathcal{O}_{q,p}(1)$ must have a neighborhood which is isometric to a neighborhood of a point x_0 in X/Γ, where X is one of the two spaces above and where Γ is a finite group of isometries of X which fixes the point x_0. The action of Γ is entirely determined up to linear equivalence by the orbifold structure of $\mathcal{O}_{q,p}(1)$.

Consider the singular point Σ_1 and its neighborhood $\mathcal{O}^1_{q,p}$. By (4.13) we know that as a smooth orbifold, we have $\mathcal{O}^1_{q,p} = \mathbb{C}^2/\mathbb{Z}_{2p}$ where $\mathbb{Z}_{2p}$ is generated by scalar multiplication by $\omega = e^{i\pi/p}$ and where Σ_1 corresponds to the origin in $\mathbb{C}^2$. This linear $\mathbb{Z}_{2p}$-action is linearly equivalent to a unique conjugacy class of $\mathbb{Z}_{2p}$ subgroups in SO_4. Therefore, under our assumption that $\mathcal{O}_{q,p}(1)$ is locally symmetric, we conclude the following. There is a neighborhood U of Σ_1 in $\mathcal{O}_{q,p}(1)$ which is isometric to a neighborhood of x_0 in $X/\mathbb{Z}_{2p}$ where $X = S^4$ or $\mathbb{P}^2_{\mathbb{C}}$ and $\mathbb{Z}_{2p}$ acts on X by exponentiation of its orthogonal action on $T_{x_0}X$ described above.

We now know quite explicitly what our metric is in a neighborhood of Σ_1. Therefore, by the analyticity of the metric, we know it essentially everywhere. To make this precise, let $\tilde{\mathcal{O}}^1_{q,p}$ be the non-singular riemannian manifold with $\mathbb{Z}_{2p}$-action so that $\mathcal{O}^1_{q,p} = \tilde{\mathcal{O}}^1_{q,p}/\mathbb{Z}_{2p}$. Now $\tilde{\mathcal{O}}^1_{q,p}$ is locally isometric to the symmetric space X, and we have given an explicit $\mathbb{Z}_{2p}$-equivariant isometry defined in a neighborhood of the fixed point $\tilde{\Sigma}_1$. From the standard arguments of Cartan-Ambrose-Hicks, there is a uniquely defined "developing map" extending this isometry along arbitrary curves in $\tilde{\mathcal{O}}^1_{q,p}$. Since $\tilde{\mathcal{O}}^1_{q,p}$ is simply-connected we thereby obtain a uniquely defined isometric immersion

$$\tilde{\psi} \ : \ \tilde{\mathcal{O}}^1_{q,p} \longrightarrow X$$

which, because of its uniqueness, is $\mathbb{Z}_{2p}$-equivariant.

For each $r > 0$, consider the ball

$$B_r = \{v \in T_{\tilde{\Sigma}_1}\tilde{\mathcal{O}}^1_{q,p} : \ \ \|v\| < r\}$$

and identify this ball with the corresponding set $\{v \in T_{x_0}X \ : \ \ \|v\| < r\}$ via $\tilde{\psi}_*$. Then for all r sufficiently small we have a commutative diagram

$$\begin{array}{ccc} B_r & \xrightarrow{\exp} & \tilde{\mathcal{O}}^1_{q,p} \\ & \searrow & \downarrow \tilde{\psi} \\ & & X \end{array} \tag{4.23}$$

of $\mathbb{Z}_{2p}$-equivariant maps. For simplicity, let us renormalize our metric so that X has injectivity radius π. We then consider the number

$$\rho \equiv sup\{r \ \mid \ exp: \ B_r \longrightarrow \ \tilde{\mathcal{O}}^1_{q,p} \quad \text{is injective}\}.$$

The map $exp : B_\pi \longrightarrow X$ degenerates on ∂B_π, and the map $\tilde{\psi}$ is an immersion. Hence, by (4.23) we conclude that $\rho \leq \pi$. Since $exp : B_r \longrightarrow X$ is an embedding for $r \leq \pi$, we conclude that ρ is also the radius of the largest ball on which the immersion $\tilde{\psi}$ is invertible. From this it follows easily that there must exist some unit vector $v \in \overline{B_1}$ so that $exp(\rho v) \in \tilde{\mathcal{O}}^1_{q,p}$ is not defined, *i.e.,* so that the geodesic $\gamma = exp(tv)$ in $\tilde{\mathcal{O}}^1_{q,p}$ cannot be extended beyond the interval $(0, \rho)$ in the positive direction. Cosequently, $\gamma(t)$, $0 \leq t < \rho$, must be the lift of a minimal geodesic γ_0 in $\mathcal{O}^1_{q,p}$ joining Σ_1 to Σ_0.

Let $\tilde{\mathcal{C}} = \tilde{\mathcal{O}}^1_{q,p} - \{\tilde{\Sigma}_1\}$ and note that projection $\eta : \tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{C}}/\mathbb{Z}_{2p} = \mathcal{O}_{q,p}(1) - \{\Sigma_0 \cup \Sigma_1\}$ is the universal covering of the regular set $\mathcal{O}_{q,p}(1) - \{\Sigma_0 \cup \Sigma_1\}$. The preimage of the geodesic γ_0 in $\tilde{\mathcal{C}}$ consists exactly of the $2p$ translations $g\gamma$ for $g \in \mathbb{Z}_{2p}$. Let us suppose that $\rho < \pi$. Then the limits $e_g \equiv \lim_{t \to \rho} g\gamma(t)$ for $g \in \mathbb{Z}_{2p}$, correspond to distinct points on the boundary of the ball $\overline{exp(B_\rho)} \subset X$. In particular, no two of these geodesics will reconverge in $exp(B_\rho)$ as $t \to \rho$.

However, suppose we consider the orbifold structure of $\mathcal{O}_{q,p}(1)$ in a neighborhood U of the point $\sigma_0 = \gamma_0(\rho) \in \Sigma_0$. This neighborhood can be written as $\tilde{U}/\mathbb{Z}_{p+q}$ where $\tilde{U}$ is a neighborhood of a point $\tilde{\sigma}_0 \in X$ and where $\mathbb{Z}_{p+q} \subset Isom(X)$ is a cyclic group fixing a totally geodesic surface $\tilde{\Sigma}_0 \subset X$ which contains $\tilde{\sigma}_0$. For any choice of $\tilde{\sigma}_0 \in \tilde{\Sigma}_0 \subset X$, the normal plane $N_{\tilde{\sigma}_0}$ to $\tilde{\Sigma}_0$ at $\tilde{\sigma}_0$ has the property that $exp(N_{\tilde{\sigma}_0})$ is totally geodesic in X. Under this ramified covering $\tilde{U} \longrightarrow \tilde{U}/\mathbb{Z}_{p+q} = U$, the geodesic γ_0 lifts to $(p+q)$ geodesics which lie in the surface $exp(N_{\tilde{\sigma}_0})$ and converge to the point $\tilde{\sigma}_0$ in a regular pattern with successive angles $2\pi/(p+q)$. Now this desingularized geometry can be matched to the geometry of $exp(B_\rho) \subset X$ at the endpoint of the geodesic γ. Since $2\pi/(p+q) < \pi/2$, we see that there are at least two more geodesics which correspond to local lifts of γ_0 and which enter into the ball $exp(B_r)$. We recall that $exp(B_r - \{0\})$ is actually a subset of $\tilde{\mathcal{C}}$, and that $\tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{C}}/\mathbb{Z}_{2p}$ is the universal cover of $\mathcal{O}^1_{q,p} - \{\Sigma_0 \cup \Sigma_1\}$. It is clear then that these three geodesic rays near $\gamma(\rho)$ must descend to the minimal geodesic γ_0. hence, the geodesics $g\gamma$, $g \in \mathbb{Z}_{2p}$ in $\mathcal{C}$ do reconverge in contradiction with our conclusion above. Hence, we have proved that $\rho = \pi$.

Recall now that

$$X - exp(B_\pi) = \begin{cases} point, & \text{if } X = S^4, \\ \mathbb{P}^1_{\mathbb{C}}, & \text{if } X = \mathbb{P}^2_{\mathbb{C}}. \end{cases}$$

In either case it is clear that $\tilde{\mathcal{O}}^1_{q,p} \equiv exp(B_\pi)$ since the immersion $\tilde{\psi}$ cannot be extended further. Hence, there is an isometry

$$\mathcal{O}^1_{q,p} = (S^4 - point)/\mathbb{Z}_{2p} \;\text{ or }\; (\mathbb{P}^2_{\mathbb{C}} - \mathbb{P}^1_{\mathbb{C}})/\mathbb{Z}_{2p}$$

which immediately yields an isometry

$$\mathcal{O}_{q,p} = S^4/\mathbb{Z}_{2p} \;\text{ or }\; (\mathbb{P}^2_{\mathbb{C}} - \mathbb{P}^1_{\mathbb{C}})/\mathbb{Z}_{2p} \tag{4.24}$$

of the metric space completions. As before we easily see that the riemannian structure of the spaces (4.24) are not compatible with the given $\mathbb{Z}_{p+q}$ structure acting locally isometrically on S^4 or $\mathbb{P}^2_{\mathbb{C}}$. We conclude that $\mathcal{O}_{q,p}$ is not locally symmetric. The case where $(p+q)$

is even and > 8 is entirely similar. The remaining 10 cases can be handled in the same set up with a little extra work. The arguments in higher dimensions proceed analogously. ∎

As we have seen above, the space $\mathcal{O}_{q,p}(1)$ (for $(p+q)$ odd) can be obtained by gluing the cone on the standard lens space $L_{2p,1} = S^3/\mathbb{Z}_{2p}$ together with the line bundle of Chern class $2p$ over S^2. This space is not equivalent as an orbifold to $\mathbb{P}^2_{\mathbb{C}}/\mathbb{Z}_{2p}$. It is, however, equivalent to a so-called "weighted projective space". These are defined as follows. Fix positive integers p, q and r with $(p,q,r) = 1$ and define the weighted projective plane $\mathbb{P}^2_{p,q,r}$ to be the quotient of $S^5 \subset \mathbb{C}^3$ by the S^1-action:

$$(z_o, z_1, z_2) \longrightarrow (\tau^p z_o,\ \tau^q z_1,\ \tau^r z_2).$$

PROPOSITION 4.25: *For each pair of positive integers $q \leq p$ with $(p,q) = 1$ there is a smooth orbifold equivalence*

$$\mathcal{O}_{q,p}(1) \cong \begin{cases} \mathbb{P}^2_{2p,p+q,p+q}, & \text{if } (p+q) \text{ is odd,} \\ \mathbb{P}^2_{p,\frac{p+q}{2},\frac{p+q}{2}}, & \text{if } (p+q) \text{ is even.} \end{cases}$$

PROOF: We use the affine coordinates $(\mathbf{w}_+, \mathbf{w}_-) \in \mathbb{C}^2 \times \mathbb{C}^2$ and write $\mathcal{Z}_H \cap U_0$ as in (4.6). Let $V_0 \equiv \{\mathbf{z} \in S^5 : \ \|z_1\|^2 + \|z_2\|^2 \neq 0\}$ and map $V_0 \longrightarrow \mathcal{Z}_H \cap U_0$ by setting

$$\mathbf{w}_+ = \frac{1}{\|z_1\|^2 + \|z_2\|^2}(z_o\overline{z}_2,\ -z_o\overline{z}_1) \quad ; \quad \mathbf{w}_- = \frac{1}{\|z_1\|^2 + \|z_2\|^2}(z_1, z_2).$$

Note that

$$(\tau^{2p} z_o,\ \tau^{p+q} z_1,\ \tau^{p+q} z_2) \longrightarrow \Big((\tau^{p-q} w_+(z_1,z_2),\ \tau^{p+q} w_-(z_1,z_2)\Big)$$

for any $\tau \in S^1$, *i.e.*, this map is S^1-equivariant. It is straightforward to check that the induced map on quotients extends to give the orbifold equivalence desired. ∎

As a consequence, we have the following result.

THEOREM 4.26: *Each of the weighted projective planes $\mathbb{P}^2_{2p,p+q,p+q}$ for p and q as above and $(p+q)$ odd carries a self-dual Einstein orbifold metric with positive scalar curvature. (This is also true of the weighted projective planes $\mathbb{P}^2_{p,\frac{p+q}{2},\frac{p+q}{2}}$ with p, q as above and $(p+q)$ even.) At most finitely many of these metrics are locally symmetric. As $q/p \to 1$, these metrics converge locally to the Fubini-Study metric on $\mathbb{P}^2_{1,1,1}$. Hence, there are infinitely many which are mutually non-isometric even locally.*

REMARK 4.27: Choosing other weights for the S^1-action on $\mathbb{P}^2_{\mathbb{C}}$ gives similar metrics on $\mathbb{P}^2_{q,p,r}$ for other values of p, q and r.

Concluding our paper we would like to mention that the quaternionic reduction method described here can be applied to the case of non-compact quaternionic Kähler manifolds, and in particular to the non-compact dual of $\mathbb{P}^n_{\mathbb{H}}$. Unlike the compact case, one obtains here new *smooth* manifolds with $Sp_n \cdot Sp_1$ holonomy. Hitchin [H2] used the method to quotient quaternionic hyperbolic space by $\mathbb{R}$ and constructed $4n$-dimensional generalizations of the Pedersen metric [P]. However, it is possible to construct many other new non-compact quaternionic Kähler metrics which are not locally symmetric. All of them are given by quaternionic reduction in quaternionic hyperbolic space. In particular, in dimension 4 one obtains many new self-dual Einstein metrics with negative cosmological constant [G2].

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