Transverse Fano Structures and Einstein Metrics on Exotic Spheres

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“WHY?” and “WHAT?”

0. Introduction

(i) what’s known, some problems and questions

(ii) some answers and theorems

“HOW?”

1. Riemannian Geometry

(i) Sasakian manifolds: basic definitions and properties

(ii) fundamental foliations and transverse geometry

2. Differential Topology

(i) Sasakian geometry of links of isolated hypersurface singularities

(ii) Differential topology of the Brieskorn-Pham links

3. Global Analysis $\longrightarrow$ Algebraic Geometry

(i) Calabi Conjecture I

(ii) Calabi Conjecture II

(iii) Kähler-Einstein metrics on Brieskorn-Pham orbifolds

REFERENCES and CO-AUTHORS

1. J. Kollár, *Einstein metrics on connected sums of* $S^2 \times S^3$, preprint 2004.
2. C. P. Boyer, K. Galicki, *Einstein metric on rational homology spheres*, preprint 2003.
3. C. P. Boyer, K. Galicki, J. Kollár, *Einstein metric on spheres*, preprint 2003.
4. C. P. Boyer, K. Galicki, J. Kollár, E. Thomas, *Einstein metrics on exotic spheres in dimension 7,11, and 15*, preprint 2003.
5. C. P. Boyer, K. Galicki, M. Nakamaye, *On Sasakian-Einstein Geometry of 5-Manifolds*, **Mathematische Annalen 325** (2003), 485-524.
6. C. P. Boyer, K. Galicki, *New Einstein Manifolds in Dimension Five,* **Journal of Differential Geometry 57** (2001), 443-467.
7. J.-P. Demailly, J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, **Ann. Scient. Ec. Norm. Sup. Paris 34** (2001), 525-556.
8. C. P. Boyer, K. Galicki, *Sasakian Manifolds*, **Oxford University Press** (hopefully in 2005).

More at
http://www.math.unm.edu/~galicki.

• Any standard sphere S^n, $n > 1$, admits a metric of constant positive sectional curvature. These canonical metrics are $SO(n+1)$-homogeneous and Einstein, i.e., the Ricci curvature tensor is a constant positive multiple of the metric.

•**[1974]** The spheres S^{4m+3}, $m > 1$ are known to have another $Sp(m+1)$-homogeneous Einstein metric discovered by Jensen. The metric is obtained from the "quaternionic Hopf fibration" $S^3 \to S^{4m+3} \to \mathbb{HP}^m$. Since both base and fiber are Einstein spaces with positive Einstein constant we obtain two Einstein metrics in the canonical variation. The second metric is also called "squashed sphere" metric in some physics literature.

•**[1978]** In addition, S^{15} has a third Spin(9)-invariant homogeneous Einstein metric discovered by Bourguignon and Karcher in 1978. The existence of such metric has to do with the fact that S^{15}, in addition to fibering over $\mathbb{HP}^2$, also fibers over S^8 with fiber S^7. Thus 15-sphere admits 3 different homogeneous Einstein metrics.

•**[1982]** Ziller proved that these are the only homogeneous Einstein metrics on spheres.

•**[1998]** Böhm obtained infinite sequences of non-isometric Einstein metrics, of positive scalar curvature, on S^5, S^6, S^7, S^8, and S^9. Böhm's metrics are of cohomogeneity one and they are not only the first inhomogeneous Einstein metrics on spheres but also the first non-canonical Einstein metrics on even-dimensional spheres.

Remark: Even with Böhm's result Einstein metrics on spheres appear to be rather rare. But are they really?

Questions 1: Are there other inhomogeneous Einstein metrics on standard spheres?

Questions 2: Do some exotic spheres admit Einstein metrics of positive scalar curvature?

In odd dimensions we will provide some surprising answers to these questions.

Theorem 1 [Boyer-Galicki-Kollár'2003]: *On S^5 there are at least 68 inequivalent families of Sasakian-Einstein metrics. Some of these families admit non-trivial continuous Sasakian-Einstein deformations. The biggest constructed family has real dimension 10.*

Remarks:

• Our metrics differ from Böhm's. His have rather large isometry group while all ours (and this comment applies to all dimensions) are, by construction, of "maximal" cohomogeneity, i.e., they have **one-dimensional isometry group**.

• Böhm gets infinitely many non-isometric Einstein metrics on S^5 but he does not get any **continuous families** as is typically the case with our metrics.

• We note that the numbers "68" or "10" in Theorem 1 no special significance: more families and higher-dimensional moduli can most likely be obtained using the same basic techniques. Nonetheless, in each odd dimension, they are to illustrate the abundance of metrics we get: both the number of families as well as the dimension of the moduli of the largest family increases **doubly exponentially** in each next odd dimension.

Theorem 2 [Boyer-Galicki-Kollár'2003]: *Let* Σ_i^7, *be a homotopy 7-sphere corresponding to the element* $i \in bP_8 \simeq \mathbb{Z}_{28} \simeq \Theta_7$ *in the Kervaire-Milnor group.* Σ_i^7 *admits at least* n_i *inequivalent families of Sasakian-Einstein metrics, where* $(n_1, \ldots, n_{28})$ =(376, 336, 260, 294, 231, 284, 322, 402, 317, 309, 252, 304, 258, 390,409, 352, 226, 260, 243, 309, 292, 452, 307, 298, 230, 307, 264, 353), *giving a total of* 8610 *cases. In each oriented diffeomorphism class some of the families depend on a moduli. In particular, the standard 7-sphere* Σ_{28}^7 *admits an 82-dimensional family of inequivalent Sasakian-Einstein metrics.*

Theorem 3 [Boyer-Galicki-Kollár'2003]: *In dimensions* $(4n+1)$ *both the standard and the so-called Kervaire spheres admit many families of inequivalent Sasakian-Einstein metrics for each* $n \geq 2$.

Remarks:

• In dimension 9, for example, the Kervaire sphere is exotic, i.e., $bP_{10} = \mathbb{Z}_2$. A computer search yields more than $3 \cdot 10^6$ inequivalent families of Sasakian-Einstein metrics on both spheres.

• In dimension 13 the Kervaire sphere is diffeomorphic to the standard sphere as $bP_{14} = 0$. In sharp contrast with the fact that **the only Einstein metric known on** S^{13} **is the standard one** we find

Corollary: *The standard sphere* S^{13} *admits over* 10^9 *distinct families of Sasakian-Einstein metrics one of them depending on* **21,300,113,901,610** *parameters.*

Conjecture [Boyer-Galicki-Kollár'2003]: *In any odd dimension, all homotopy spheres which bound parallelizable manifolds admit families of Sasakian-Einstein metrics.*

Theorem 4 [Boyer-Galicki-Kollár-Thomas'2003]: *The conjecture is true in dimensions 11 and 15. More precisely, each homotopy sphere (992 possible oriented diffeomorphism types) in dimension 11 admits at least one Sasakian-Einstein metric and each homotopy 15-sphere which bounds a parallelizable manifold (8128 possible oriented diffeomorphism types) admits at least one Sasakian-Einstein metric.*

Remark:

• Theorem 4 required some considerable computing time. For example, using the same machines and the same codes would require about 3000 years to extend the result to the case of 19-spheres (bP_{20} has 130,816 elements).

Important Remark: The methods of proving such existence results apply to a **wide range of smooth contact manifolds**. In this talks I have chosen homotopy spheres to illustrate the power of the technique. To give another example here are two more recent theorems one can get in 5 dimensions:

Theorem 5 [Boyer-Galicki'03]: *Let M^5_k be a simply connected, smooth, oriented, compact, spin manifold such that $H_2(M^5_k, \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_k$. Then, for any k such that $\gcd(k, 3) = 1$, the rational homology 5-sphere M^5_k admits continuous families of Sasakian-Einstein metrics.*

Theorem 6 [Kollár'04]: *Let M be a simply connected, smooth, oriented, compact, spin 5-manifold with $b_2(M) = n$ and such that $H_2(M, \mathbb{Z}) = \mathbb{Z}^n$. Then for any $n \geq 6$, M admits infinitely many $(2n-2)$-dimensional families of Sasakian-Einstein metrics.*

Remarks:

• By Smale's theorem simply connected, smooth, oriented, compact, spin 5-manifold is determined by its second homology.

• Theorem 6, without the statement about the moduli, is also true for $n = 0, 1, \ldots, 5$.

1. RIEMANNIAN GEOMETRY

Sasakian Manifolds

Definition-Proposition: *A Riemannian manifold (M,g) is called* **Sasakian** *if any one, hence all, of the following equivalent conditions hold:*

(i) There exists a Killing vector field ξ of unit length on M so that the tensor field Φ of type $(1,1)$, defined by $\Phi(X) = \nabla_X \xi$, satisfies the condition

$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X,Y)\xi$$

for any pair of vector fields X and Y on M.

(ii) There exists a Killing vector field ξ of unit length on M so that the Riemann curvature satisfies the condition

$$R(X,\xi)Y = g(\xi, Y)X - g(X,Y)\xi,$$

for any pair of vector fields X and Y on M.

(iii) The metric cone on M $(C(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ is Kähler.

We refer to the quadruple $\mathcal{S} = (g, \eta, \xi, \Phi)$ as a **Sasakian structure** on M, where η is the 1-form dual vector field ξ. It is easy to see that η is a contact form whose Reeb vector field is ξ. In particular $\mathcal{S} = (g, \eta, \xi, \Phi)$ is a special type of *metric contact structure.*

The vector field ξ is nowhere vanishing, so there is a one-dimensional foliation $\mathcal{F}_\xi$ associated to any Sasakian structure, called the *characteristic foliation*. We will denote the space of leaves of this foliation by $\mathcal{Z}$. Each leaf of $\mathcal{F}_\xi$ has a holonomy group associated to it. The dimension of the closure of the leaves is called the *rank* of $\mathcal{S}$. We shall be interested in the case $\mathrm{rk}(\mathcal{S}) = 1$.

Definition: *When* $rk(\mathcal{S}) = 1$ *we say that Sasakian structure* $\mathcal{S}$ *is* **quasi-regular**. *If* $\mathcal{F}_\xi$ *defines a principal* S^1 *bundle, we say that* $\mathcal{S}$ *is* **regular**.

When $\mathcal{S}$ is compact and quasi-regular then then $\mathcal{Z}$ has a structure of a Riemannian orbifold (or a V-manifold). $\mathcal{Z}$ is a smooth manifold in the regular case.

Definition-Proposition: *A Sasakian manifold* (M, g) *is* **Sasakian-Einstein** *if the metric* g *is also Einstein. For any 2n+1-dimensional Sasakian manifold* $\mathrm{Ric}(X, \xi) = 2n\eta(X)$ *implying that any Sasakian-Einstein metric must have positive scalar curvature. Thus any complete Sasakian-Einstein manifold must have finite fundamental group. Furthermore the metric cone* $(\mathcal{C}(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ *is Kähler Ricci-flat (Calabi-Yau).*

Definition-Proposition: (M, g) *is 3-Sasakian if the metric cone* $(\mathcal{C}(M), \bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$ *on* M *is hyperkähler. In particular, any 3-Sasakian manifold is Einstein.*

The following slide presents some fundamental structure theorems for Sasakian and Sasakian-Einstein manifolds:

Characteristic Foliation
and
Transverse Kähler Geometry

Theorem: *Let (M,g) be a compact quasi-regular Sasakian manifold of dimension $2n+1$, and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation. Then*

- *(i) The leaf space $\mathcal{Z}$ is a Hodge orbifold with Kähler metric h and Kähler form ω which defines an integral class $[\omega]$ in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ so that $\pi:(M,g)\longrightarrow(\mathcal{Z},h)$ is an orbifold Riemannian submersion. The fibers of π are totally geodesic submanifolds of M diffeomorphic to S^1.*
- *(ii) (M,g) is Sasakian-Einstein iff $(\mathcal{Z},h)$ is Kähler-Einstein with scalar curvature $4n(n+1)$.*

Theorem: *Let $(\mathcal{Z},h)$ be a Hodge orbifold. Let $\pi:M\longrightarrow\mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $[\omega]$, and let η be a connection 1-form in M whose curvature is $2\pi^*\omega$. Then M with the metric $\pi^*h+\eta\otimes\eta$ is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the group of the bundle S^1, the total space M is a smooth Sasakian manifold.*

Standard Example:

$$
\begin{array}{ccc}
\mathbb{C}^n\setminus\{0\} & & \\
 & \searrow & \\
\Big\downarrow & & S^{2n-1} \\
 & \swarrow & \\
\mathbb{P}\mathbb{C}^{n-1} & &
\end{array}
$$

Definition: *A Sasakian manifold* (M, g) *is called* **positive** *if the transverse geometry* $\mathcal{Z}$ *is* **Fano**.

Remarks:

• A complex orbifold $\mathcal{Z}$ is Fano if suitably defined orbifold canonical bundle $K_{\mathcal{Z}^{orb}}$ is anti ample. In the case $\mathcal{Z}$ is well-formed, that is when orbifold singularities are of codimension at least 2, this is the same as saying that $K_{\mathcal{Z}}$ (viewed simply as a complex variety) is anti ample. However, in the presence of codimension 1 singularities one needs to be more careful as it is not true in general.

• Just as in the smooth case positivity can also be be expressed in terms of orbifold Chern classes of $\mathcal{Z}$ (first Chern class $c_1(\mathcal{Z}) > 0$), or in terms of basic Chern classes of the characteristic foliation $\mathcal{F}_\xi$ (first Chern class $c_1(\mathcal{F}_\xi) > 0$).

The following theorem is an **orbi-bundle version** of the famous **Kobayashi bundle construction** of Einstein metrics on bundles over positive Kähler-Einstein manifolds.

Theorem: *Let* $(\mathcal{Z}, h)$ *be a compact Fano orbifold with* $\pi_1^{orb}(\mathcal{Z}) = 0$ *and Kähler-Einstein metric* h*. Let* $\pi : M \longrightarrow \mathcal{Z}$ *be the* S^1 *V-bundle whose first Chern class is* $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$*. Suppose further that the local uniformizing groups of* $\mathcal{Z}$ *inject into* S^1*. Then with the metric* $g = \pi^* h + \eta \otimes \eta$*,* M *is a smooth compact simply connected Sasakian-Einstein manifold.*

$$
\begin{array}{ccc}
\mathcal{C}(M) & \hookleftarrow & M \\
 & & \Big\downarrow \pi \\
 & & \mathcal{Z}
\end{array}
$$

$\mathcal{C}(M)$: CONE GEOMETRY	M	$M/\mathcal{F}_\xi$: TRANSVERSE GEOMETRY
SYMPLECTIC	**CONTACT**	**SYMPLECTIC**
KÄHLER	**SASAKIAN**	**KÄHLER**
KÄHLER	**POSITIVE SASAKIAN**	**FANO**
CALABI-YAU	**SASAKIAN EINSTEIN**	**FANO KÄHLER-EINSTEIN**
HYPER-KÄHLER	**3-SASAKIAN**	**FANO, $\mathbb{C}$-CONTACT KÄHLER-EINSTEIN**

2. DIFFERENTIAL TOPOLOGY

Sasakian Structures on Links

Let $\mathbf{w} = (w_0, \ldots, w_n)$ and consider a weighted $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ given by $(z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n)$, where the **weights** w_j are all positive integers. We assume that $\gcd(w_0, \ldots, w_n) = 1$.

Definition *$f \in \mathbb{C}[z_0, \ldots, z_n]$ is a* **weighted homogeneous polynomial** *of weight* $\mathbf{w}$ *and* **degree** *d if*

$$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n).$$

Assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity.

Definition *$L_f = \{f = 0\} \cap S^{2n+1}$, where S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$ is called the* **link** *of f.*

• **Fact 1**: The well-known construction of Milnor for isolated hypersurface singularities shows that L_f is (n-2)-connected. Hence, he only non-vanishing integral homology groups are $H_{n-1}(L_f, \mathbb{Z})$ and $H_n(L_f, \mathbb{Z})$ and they, at least in principle, can be calculated using Milnor's fibration theorem.

• **Fact 2**: L_f is endowed with a quasi-regular Sasakian structure inherited as a Sasakian submanifold of the sphere S^{2n+1} with a "weighted" Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which in the standard coordinates $\{z_j = x_j + iy_j\}_{j=0}^{n}$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is determined by

$$\eta_{\mathbf{w}} = \frac{\sum_{i=0}^{n}(x_i dy_i - y_i dx_i)}{\sum_{i=0}^{n} w_i(x_i^2 + y_i^2)}, \qquad \xi_{\mathbf{w}} = \sum_{i=0}^{n} w_i(x_i\partial_{y_i} - y_i\partial_{x_i}),$$

and the standard Sasakian structure (ξ, η, Φ, g) on S^{2n+1}.

The quotient of S^{2n+1} by the "weighted S^1-action" generated by the vector field $\xi_{\mathbf{w}}$ is the weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$, and we have a commutative diagram:

$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are orbifold Riemannian submersions. L_f is the total space of the principal S^1 V-bundle over the orbifold $\mathcal{Z}_f$.

Proposition: *The orbifold $\mathcal{Z}_f$ is Fano iff $d - \sum w_i < 0$.*

Milnor Fibration Theorem

There is a fibration of $(S^{2n+1} - L_f) \longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n \vee \cdots \vee S^n$. The *Milnor number* μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $\mathbf{w}$ by the formula $\mu = \mu(L_f) = \prod_{i=0}^{n}\big(\frac{d}{w_i} - 1\big)$. The closure $\bar{F}$ of F has the same homotopy type as F and is a compact manifold with boundary precisely the link L_f. So the reduced homology of F and $\bar{F}$ is only non-zero in dimension n and $H_n(F, \mathbb{Z}) \approx \mathbb{Z}^\mu$. Using the Wang sequence of the Milnor fibration together with Alexander-Poincare duality gives the exact sequence

$$0 \to H_n(L_f, \mathbb{Z}) \to H_n(F, \mathbb{Z}) \xrightarrow{\mathbb{I} - h_*} H_n(F, \mathbb{Z}) \to H_{n-1}(L_f, \mathbb{Z}) \to 0$$

where h_* is the *monodromy* map (or characteristic map) induced by the $S^1_{\mathbf{w}}$ action. From this we see that $H_n(L_f, \mathbb{Z}) = \ker(\mathbb{I} - h_*)$ is a free Abelian group, and $H_{n-1}(L_f, \mathbb{Z}) = \mathrm{Coker}(\mathbb{I} - h_*)$ which in general has torsion, but whose free part equals $\ker(\mathbb{I} - h_*)$. So the topology of L_f is encoded in the monodromy map h_*. There is a well-known algorithm due to Milnor and Orlik for computing the free part of $H_{n-1}(L_f, \mathbb{Z})$ in terms of the characteristic polynomial $\Delta(t) = \det(t\mathbb{I} - h_*)$, namely the Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$.

Brieskorn-Pham Links

In his famous work, in 1966 Brieskorn considered special links often called Brieskorn-Pham links (or BP for short) $L(\mathbf{a})$ defined by

$$\sum_{i=0}^{n} |z_i^2| = 1, \qquad f_{\mathbf{a}}(\mathbf{z}) = z_0^{a_0} + \cdots + z_n^{a_n} = 0.$$

To the vector $\mathbf{a} = (a_0, \cdots, a_n) \in \mathbb{Z}_+^{n+1}$ one associates a graph $G(\mathbf{a})$ whose $n+1$ vertices are labeled by $a_0, \cdots, a_n$. Two vertices a_i and a_j are connected if and only if $\gcd(a_i, a_j) > 1$. Let C_{ev} denote the connected component of $G(\mathbf{a})$ determined by the even integers. Note that all even vertices belong to C_{ev}, but C_{ev} may contain odd vertices as well. Then we have the so-called **Brieskorn Graph Theorem**

THEOREM: *The link $L(\mathbf{a})$ (with $n \geq 3$) is homeomorphic to the (2n-1)-sphere if and only if either of the following conditions hold*

(i) *$G(\mathbf{a})$ contains at least two isolated points,*

(ii) *$G(\mathbf{a})$ contains one isolated point and C_{ev} has an odd number of vertices and for any distinct $a_i, a_j \in C_{ev}$,* $\gcd(a_i, a_j) = 2$.

Recall that by seminal work of Milnor, Kervaire and Milnor and Smale, for each $k \geq 5$, differentiable homotopy spheres of dimension k form an Abelian group Θ_k, where the group operation is connected sum. Θ_k has a subgroup bP_{k+1} consisting of those homotopy k-spheres which bound parallelizable manifolds V_{k+1}. Kervaire and Milnor proved that $bP_{2m+1} = 0$ for $m \geq 1$, $bP_{4m+2} = 0$, or $\mathbb{Z}_2$ and is $\mathbb{Z}_2$ if $4m + 2 \neq 2^i - 2$ for any $i \geq 3$. The most interesting groups are bP_{4m} for $m \geq 2$. These are cyclic of order

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1} - 1) \text{ numerator} \binom{4B_m}{m},$$

where B_m is the m-th Bernoulli number. Thus, for example $|bP_8| = 28, |bP_{12}| = 992, |bP_{16}| = 8128$ and $|bP_{20}| = 130,816$. In the first two cases these include all exotic spheres. The correspondence is given by

$$KM : \Sigma \mapsto \frac{1}{8}\tau(V_{4m}(\Sigma)) \text{mod} |bP_{4m}|,$$

where $V_{4m}(\Sigma)$ is any parallelizable manifold bounding Σ and τ is its signature.

In the case of BP links, by the Milnor Fibration Theorem we can take V to be the Milnor fiber which is diffeomorphic to $V_{\mathbf{a}}^{2n} = \{\mathbf{z} \in \mathbb{C}^{n+1} \mid f_{\mathbf{a}}(z_0, \cdots, z_n) = 1\}$. Brieskorn shows that the signature of $V_{\mathbf{a}}^{2n}$ can be written combinatorially as

$$\#\left\{\mathbf{x} \in \mathbb{Z}^{2m+1} \mid 0 < x_i < a_i \text{ and } 0 < \sum_{j=0}^{2m} \frac{x_i}{a_i} < 1 \bmod 2\right\}$$

$$-\#\left\{\mathbf{x} \in \mathbb{Z}^{2m+1} \mid 0 < x_i < a_i \text{ and } 1 < \sum_{j=0}^{2m} \frac{x_i}{a_i} < 2 \bmod 2\right\},$$

where $n = 2m$.

Example: Let us consider the Brieskorn-Pham link $L(6k-1, 3, 2, 2, 2)$. By Brieskorn Graph Theorem this is a homotopy 7-sphere. One can easily compute the signature using (2.14) to find out that $\tau(L(6k-1,3,2,2,2)) = 8k$. Hence $L(5,3,2,2,2) = \Sigma_1^7$ is an exotic 7-sphere and it is called Milnor generator (all others can be obtained from it by taking connected sums).

Question: *Suppose we take a link L_f so that the transverse geometry $\mathcal{Z}_f$ is Fano. Could one perhaps prove existence of an Kähler-Einstein metric on $\mathcal{Z}_f$? Every time this can successfully be done we automatically get a Sasakian-Einstein metric on the link L_f.*

J.-P. Demailly and J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, preprint AG/9910118, Ann. Scient. Ec. Norm. Sup. Paris 34 (2001), 525-556.

Calabi Yau Conjecture I

Let (M, g, J, ω_g) be a Kähler manifold. In a local complex chart the metric g is simply a Hermitian matrix $g_{i\bar{j}}$. It was Kähler who discovered that locally the Ricci tensor of a Kähler metric can be written as

$$R_{i\bar{j}} = -\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}\ln(\det(g_{i\bar{j}})).$$

Only much later Calabi realized the importance of this remarkable formula. If one defines the associated 2-form

$$\rho_\omega = -\frac{i}{2}R_{i\bar{j}}dz^i \wedge d\bar{z}^j$$

which is called the **Ricci form** one realizes that the cohomology class of ρ_ω **does not depend on the choice of the Kähler metric** within Kähler class. In fact, it is easy to see that Ricci form represents the cohomology class $2\pi c_1(M)$. Naturally Calabi asked if the converse was also true. More specifically 50 years ago he conjectured that

Calabi Conjecture I *Let (M, J, g, ω_g) be a compact Kähler manifold, $[\omega_g]$ the corresponding Kähler class and ρ_g the Ricci form. Consider any real $(1,1)$-form Ω on M such that $[\Omega] = 2\pi c_1(M)$. Then there exists a unique Kähler metric h such that $[\omega_h] = [\omega_g]$ and $\Omega = \rho_h$.*

Calabi was right and his conjecture in its full generality was eventually proved by Yau in 1976.

Monge-Ampère Problem

Let us reformulate the problem using the so-called global $i\partial\bar{\partial}$-lemma. We start with a given Kähler metric g on M in a Kähler class $[\omega_g]$. Since both Ω and ρ_g represent $2\pi c_1(M)$ there exists a globally defined function $f \in C^\infty(M, \mathbb{R})$ such that

$$\Omega - \rho_g = -i\partial\bar{\partial}f.$$

Appropriately, f may be called **a discrepancy potential function** for the Calabi problem.

Now, supposed the desired solution of the problem is a metric h with $[\omega_h] = [\omega_g]$ and $\rho_h = \Omega$. We know that the Kähler form of h can be written as

$$\omega_h = \omega_g + i\partial\bar{\partial}\phi,$$

for some smooth function $\phi \in C^\infty(M, \mathbb{R})$. In a local, complex chart (and with appropriate choice of constants)

$$\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\right)}{\det(g_{i\bar{j}})} = e^f.$$

Remarks

• This equation is called the Monge-Ampère equation.

• It is "folklore" that Calabi-Yau Conjecture is also true for compact orbifolds. In the context of Sasakian geometry and fundamental foliations the transverse space $\mathcal{Z}$ is typically a compact Kähler orbifold. In the context of foliations a **transverse Yau theorem** was proved by El Kacimi-Alaoui in 1990.

Calabi Yau Conjecture II

Kähler-Einstein Condition

Let (M, J, g, ω_g) be a compact Käher manifold. We would like to know if one can always find a Kähler-Einstein metric h such that $[\omega_h] = [\omega_g]$. Recall that on a Kähler-Einstein manifold

$$\rho_g = \lambda \omega_g,$$

where the metric can be normalized so that $\lambda = \pm 1$ or 0. Since $2\pi c_1(M) = \lambda[\omega_g]$ we have

- When $c_1(M) > 0$ we must have $\lambda = +1$.
- When $c_1(M) < 0$ we must have $\lambda = -1$.
- When $c_1(M) = 0$ we must have $\lambda = 0$.

Suppose there exists an Einstein metric h with Kähler from in the same cohomology class. We have globally defined functions $\phi, f \in C^\infty(M, \mathbb{R})$ such that

$$\rho_g - \lambda \omega_g = i\partial\bar{\partial} f, \qquad \omega_h - \omega_g = i\partial\bar{\partial}\phi.$$

After fixing appropriate normalization we get, in local coordinates, the following Monge-Ampère equation

$$\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right)}{\det(g_{i\bar{j}})} = e^{f - \lambda\phi}.$$

The case of $\lambda = -1$ is actually the simplest and was solved by Aubin and independently by Yau.

Theorem *Let (M, J, g, ω_g) be a compact Kähler manifold with $c_1(M) < 0$. Then there exists a unique Kähler metric h with $[\omega_h] = [\omega_g]$ such that $\rho_h = -\omega_h$.*

When $\lambda = +1$ the problem is much harder. It has been known for quite some time that there are actually non-trivial obstructions to the existence. Let $\mathfrak{h}(M)$ be the complex Lie algebra of all holomorphic vector fields on M. Matsushima proved that one a compact Kähler-Einstein manifold with $c_1(M) > 0$ $\mathfrak{h}(M)$ must be reductive, i.e., $\mathfrak{h}(M) = Z(\mathfrak{h}(M)) \oplus [\mathfrak{h}(M), \mathfrak{h}(M)]$.

Calabi Conjecture II *Let (M, g, J, ω_g) be a Kähler manifold with $c_1(M) > 0$. Suppose $\mathfrak{h}(M) = 0$. Then there exists an Einstein metric h with $[\omega_h] = [\omega_g]$.*

Remark This time Calabi was wrong. This conjecture is true for smooth del Pezzo surfaces but it already breaks down for del Pezzo surfaces with orbifold singularities. This was established by Ding and Tian in 1992. Later Tian showed that there exist smooth Fano 3-folds which have $\mathfrak{h}(M) = 0$ but do not admit any Kähler-Einstein metric.

Continuity Method

Here one tries to solve the Monge-Ampère equation

$$\frac{\det(g_{i\bar{j}} + \partial_i \bar{\partial}_{\bar{j}} \phi_t)}{\det(g_{i\bar{j}})} = e^{-t\phi_t + tf + C_t}, \quad g_{i\bar{j}} + \partial_i \bar{\partial}_{\bar{j}} \phi_t > 0$$

We start with ϕ_0 and $C_0 = 0$ and we try to reach $t = 1$, where the metric will be Kähler-Einstein. The so called "continuity method" sets out to show that the interval where solutions exist is both open and closed. **Openness follows from the Implicit Function Theorem**, but there are well-known **obstructions to closedness**. It turns out that the critical step is a 0th order estimate. That is, as the values of t for which the Monge-Ampère equation is solvable approach a critical value $t_0 \in [0, 1]$, a subsequence of the ϕ_t converges to a function ϕ_{t_0} which is the sum of a C^∞ and of a plurisubharmonic function. We only need to prove that

$$\int_{\mathcal{Z}} e^{-\gamma t \phi_{t_0}} \omega_0^n < +\infty$$

for some $\gamma \in (\frac{n}{n+1}, 1)$, where ω_0 is the Kähler form of $g_0 = g$.

This problem has been studied by many people; Yau, Tian, Siu, Nadel, and most recently by Demailly and Kollár who work in the orbifold category.

Kähler Einstein Metrics on Brieskorn-Pham Orbifolds

Consider a Brieskorn–Pham singularity

$$Y(\mathbf{a}) := (\sum_{i=0}^{n} z_i^{a_i} = 0) \subset \mathbb{C}^{n+1}.$$

Set $C = \mathrm{lcm}(a_i : i = 0, \ldots, n)$. $Y(\mathbf{a})$ is invariant under the $\mathbb{C}^*$-action

$$(z_0, \ldots, z_n) \mapsto (\lambda^{C/a_0} z_0, \ldots, \lambda^{C/a_n} z_n).$$

In the notation of a previous slide we have $w_i = C/a_i$ and $d = C$. Furthermore, the quotient $Y(\mathbf{a})/\mathbb{C}^*$ can be identified with the transverse space $\mathcal{Z}(\mathbf{a})$ of the associated BP link $L(\mathbf{a})$. One can easily see that $Y(\mathbf{a})/\mathbb{C}^*$ is a Fano orbifold iff

$$1 < \sum_{i=0}^{n} \frac{1}{a_i}.$$

More generally, we consider weighted homogeneous perturbations

$$Y(\mathbf{a}, p) := (\sum_{i=0}^{n} z_i^{a_i} + p(z_0, \ldots, z_n) = 0) \subset \mathbb{C}^{n+1},$$

where weighted degree$(p) = C$. The genericity condition we need, which is always satisfied by $p \equiv 0$ is: **The intersections of $Y(\mathbf{a}, p)$ with any number of hyperplanes $(z_i = 0)$ are all smooth outside the origin.**

The continuity methods produces the following sufficient conditions on $Y(\mathbf{a},p)/\mathbb{C}^*$ to admit a Kähler-Einstein metric:

Theorem *Let $\mathcal{Z}(\mathbf{a},p) = Y(\mathbf{a},p)/\mathbb{C}^*$ be the transverse space of the BP link $L(\mathbf{a})$. Then $\mathcal{Z}(\mathbf{a},p)$ is Fano and it has a Kähler-Einstein metric if*

(i) $1 < \sum_{i=0}^{n} \frac{1}{a_i}$,

(ii) $\sum_{i=0}^{n} \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_i\{\frac{1}{a_i}\}$, *and*

(iii) $\sum_{i=0}^{n} \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_{i,j}\{\frac{1}{b_i b_j}\}$.

In this case the link $L(\mathbf{a},p)$ admits a Sasakian-Einstein metric with one-dimensional isometry group.

Remark All but one theorems mentioned the introduction are special cases of the above result.

Example Consider sequences of the form $\mathbf{a} = (2, 3, 7, m)$. By explicit calculation, the corresponding link $L(\mathbf{a})$ gives a Sasakian-Einstein metric on S^5 if $5 \leq m \leq 41$ and m is relatively prime to at least two of $2, 3, 7$. This is satisfied in 27 cases.

Example The sequence $\mathbf{a} = (2, 3, 7, 35)$ is especially noteworthy. If $C(u, v)$ is any sufficiently general homogeneous septic polynomial, then the link of

$$x_1^2 + x_2^3 + C(x_3, x_4^5)$$

also gives a Sasakian-Einstein metric on S^5. The relevant automorphism group of $\mathbb{C}^4$ is

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, \alpha_3 x_3 + \beta x_4^5, \alpha_4 x_4).$$

Hence we get a $2(8 - 3) = 10$ real dimensional family of Sasakian-Einstein metrics on S^5.

Example The sequence $\mathbf{a} = (2, 3, 7, 43, 43 \cdot 31)$ gives a standard 7-sphere with a $2(43 - 2) = 82$-dimensional family of Sasakian-Einstein metrics on S^7.

Consider the sequence defined by the recursion relation

$$c_{k+1} = c_1 \cdots c_k + 1 = c_k^2 - c_k + 1, \qquad c_1 = 2.$$

It starts as

$$2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots$$

We have

$$\sum_{i=1}^{m} \frac{1}{c_i} = 1 - \frac{1}{c_{m+1} - 1} = 1 - \frac{1}{c_1 \cdots c_m}.$$

Example

(i) $\mathbf{a} = (2, 3, 7, 43, 1807, 3263443, 10650056950807, m)$

(ii) $L(\mathbf{a})$ is just the standard 13-sphere for any suitably chosen m.

(iii) If we choose $m = (10650056950807 - 2) \cdot 10650056950807$ we get $2(10650056950807 - 2)$-dimensional family of deformations.