MATH327 – HOMEWORK SOLUTIONS

HOMEWORK #5

Section 2.2: Problems 1, 4, 6, 8, 9
Section 2.3: Problems 3,4,13,17

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Problem 2.2.1.

a) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 3x + 5$ is clearly both injective and surjective. Hence, the inverse exists and it is $g(x) = \frac{1}{3}(x - 5)$, where $g : \mathbb{Q} \to \mathbb{Q}$.

b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 2$ is bijective. Hence, the inverse exists and it is $g(x) = (x + 2)^{1/3}$, where $g : \mathbb{R} \to \mathbb{R}$.

c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x|x|$ is clearly injective and surjective. Hence, the inverse exists and it is

$$g(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x \leq 0, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$.

d) $\beta : (3/4, \infty) \to \mathbb{R}$ defined by $\beta(x) = \log_2(3x - 4)$ is bijective. Hence, the inverse exists and it is $g(x) = \frac{1}{3}(2^x + 4)$, where $g : \mathbb{R} \to (3/4, \infty)$.

Problem 2.2.4. Let $S = \{1, 2, 3, 4, 5\}$ and $T = \{1, 2, 3, 8, 9\}$ and define $f : S \to T$ and $g : S \to S$ by

$$f = \{(1, 8), (3, 9), (4, 3), (2, 1), (5, 2)\},$$

$$g = \{(1, 2), (3, 1), (2, 2), (4, 3), (5, 2)\}.$$

a) $f \circ g : S \to T$ and we get $f(g(1)) = f(2) = 1$, $f(g(2)) = f(2) = 1$, $f(g(3)) = f(1) = 8$, $f(g(4)) = f(3) = 9$, $f(g(5)) = f(2) = 1$. Hence,

$$f \circ g = \{(1, 1), (2, 1), (3, 8), (4, 9), (5, 1)\}.$$

$g \circ f$ is not defined since the domain of g and the range of f are different. $f \circ f$ is not defined for the same reason. Finally, $g \circ g : S \to S$ and we get $g(g(1)) = g(2) = 2$, $g(g(2)) = g(2) = 2$, $g(g(3)) = g(1) = 2$, $g(g(4)) = g(3) = 1$, $g(g(5)) = g(2) = 2$. Hence,

$$g \circ g = \{(1, 2), (2, 2), (3, 2), (4, 1), (5, 2)\}.$$

b) f is both injective and surjective. g is neither: $g(1) = g(2)$ and $\{4, 5\}$ are not in its range.

c) Since f is both injective and surjective, its inverse exists and

$$f^{-1} = \{(8, 1), (9, 3), (3, 4), (1, 2), (2, 5)\}.$$

d) Since g is not injective it does not have an inverse.

Problem 2.2.6. Let $S = \{1, 2, 3, 4\}$ and $f, g : S \to S$ be

$$f = \{(1, 3), (2, 2), (3, 4), (4, 1)\},$$

$$g = \{(1, 4), (2, 3), (3, 1), (4, 2)\}.$$

Both functions are bijective and

$$f^{-1} = \{(3,1),(2,2),(4,3),(1,4)\},$$

$$g^{-1} = \{(4,1),(3,2),(1,3),(2,4)\}.$$

We have

a) $g^{-1} \circ f \circ g(1) = g^{-1} \circ f(4) = g^{-1}(1) = 3$, $g^{-1} \circ f \circ g(2) = g^{-1} \circ f(3) = g^{-1}(4) = 1$, $g^{-1} \circ f \circ g(3) = g^{-1} \circ f(1) = g^{-1}(3) = 2$, $g^{-1} \circ f \circ g(4) = g^{-1} \circ f(2) = g^{-1}(2) = 4$,

$$g^{-1} \circ f \circ g = \{(1,3),(2,1),(3,2),(4,4)\}.$$

b) $f \circ g^{-1} \circ g = f$.

c) $g \circ f \circ g^{-1}(1) = g \circ f(3) = g(4) = 2$, $g \circ f \circ g^{-1}(2) = g \circ f(4) = g(1) = 4$, $g \circ f \circ g^{-1}(3) = g \circ f(2) = g(2) = 3$, $g \circ f \circ g^{-1}(4) = g \circ f(1) = g(3) = 1$,

$$g \circ f \circ g^{-1} = \{(1,2),(2,4),(3,3),(4,1)\}.$$

d) $g \circ g^{-1} \circ f = f$.

e) We have

$$f^{-1} \circ g^{-1} \circ f \circ g(1) = f^{-1} \circ g^{-1} \circ f(4) = f^{-1} \circ g^{-1}(1) = f^{-1}(3) = 1,$$

$$f^{-1} \circ g^{-1} \circ f \circ g(2) = f^{-1} \circ g^{-1} \circ f(3) = f^{-1} \circ g^{-1}(4) = f^{-1}(1) = 4,$$
$$f^{-1} \circ g^{-1} \circ f \circ g(3) = f^{-1} \circ g^{-1} \circ f(1) = f^{-1} \circ g^{-1}(3) = f^{-1}(2) = 2,$$
$$f^{-1} \circ g^{-1} \circ f \circ g(4) = f^{-1} \circ g^{-1} \circ f(2) = f^{-1} \circ g^{-1}(2) = f^{-1}(4) = 3,$$
$$f^{-1} \circ g^{-1} \circ f \circ g = \{(1,1),(2,4),(3,2),(4,3)\}.$$

Problem 2.2.8. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x + 2, \quad g(x) = \frac{1}{x^2+1}, \quad h(x) = 3.$$

We have

$$f^{-1}(x) = x - 2.$$

Then

$$g \circ f(x) = \frac{1}{(x+2)^2+1},$$

$$
\begin{aligned}
f \circ g(x) &= \frac{1}{x^2+1} + 2, \\
h \circ g \circ f(x) &= 3, \\
g \circ h \circ f(x) &= \frac{1}{10}, \\
g \circ f^{-1} \circ f(x) &= g(x), \\
f^{-1} \circ g \circ f(x) &= \frac{1}{(x+2)^2+1} - 2.
\end{aligned}
$$

Problem 2.2.9. Let $f, g, h : \mathbb{R}^+ \to \mathbb{R}$ be

$$
f(x) = \frac{x}{x+1}, \quad g(x) = \frac{1}{x}, \quad h(x) = x + 1.
$$

Then

$$
\begin{aligned}
g \circ f(x) &= \frac{x+1}{x}, \\
f \circ g(x) &= \frac{1}{x+1}, \\
h \circ g \circ f(x) &= \frac{x+1}{x} + 1, \\
f \circ g(x) \circ h &= \frac{1}{x+2}.
\end{aligned}
$$

Problem 2.3.3.

a) $f = \{(x, 14), (y, -3), (\{a, b, c\}, t)\}$ is one of the 6 examples of such correspondences.

b) $f : 2\mathbb{Z} \to 17\mathbb{Z}, \quad f(2k) = 17k$, where k is any integer.

c) $f : \mathbb{N} \times \mathbb{N} \to \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{N}\}$ defined by $f(a, b) = a + bi$.

d) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{Z}$, where

$$
f(a, b) = \begin{cases} (a, k-1) & \text{if } b = 2k, \quad k \in \mathbb{N} \\ (a, -k) & \text{if } b = 2k-1, \quad k \in \mathbb{N} \end{cases}
$$

e) $f : \mathbb{N} \to \{m/n \mid m \in \mathbb{N}, n = 1, 2\}$, where

$$
f(a) = \begin{cases} k & \text{if } a = 2k, \quad k \in \mathbb{N} \\ k/2 & \text{if } a = 2k-1, \quad k \in \mathbb{N} \end{cases}
$$

Problem 2.3.4. This is false as shown by the example $2\mathbb{Z} \subset \mathbb{Z}$. The inclusion is proper but the sets have the same cardinality. The following theorems are true.

Theorem 1: Let $A \subsetneq B$ and let B be finite, that is cardinality $|B| < \infty$. Then $|A| < |B|$.

Proof: Since B is finite we can identify B with an N-element set $\{1, 2, ..., N\}$ for some N and then $|B| = N$. Since A is a proper subset of B, A will be in one-to-one correspondence with a proper subset of $\{1, 2, ..., N\}$ and therefore $|A| < N$. This proves the theorem. ∎

Theorem 2: Let $A \subseteq B$. Then $|A| \leq |B|$.

Proof: The inclusion $A \subseteq B$ is an injective map (but not surjective, unless $A = B$). By definition, $|A| \leq |B|$ if there exists any injective function $f : A \to B$. ∎

Problem 2.3.13.

a) The set $S = \{x \in \mathbb{R} \mid 1 < x < 2\}$ is uncountable as it is in one-to-one correspondence with the open interval $(0, 1)$, via the function $f(x) = x - 1$. The latter is uncountable via the diagonal argument.

b) The set $S = \{x \in \mathbb{Q} \mid 1 < x < 2\}$ countably infinite. It is infinite as $1 + \frac{1}{n+1} \in S$ for any $n \in \mathbb{N}$. On the other hand S is a proper subset of $\mathbb{Q}$, so it must be countable by the fact that $\mathbb{Q}$ is countable (see Theorem 2 above).

c) The set $S = \{m/n \mid m, n \in \mathbb{N},\ \ m < 100,\ \ 5 < n < 105\}$ is finite as $|S| \leq 99 \cdot 100 = 9900$.

d) The set $S = \{m/n \mid m, n \in \mathbb{Z},\ \ m < 100,\ \ 5 < n < 105\}$ is infinite as, every negative integer is an element of S (simply take $m = -100k, n = 100$). Furthermore, S is a proper subset of $\mathbb{Q}$, so it must be countable by the fact that $\mathbb{Q}$ is countable.

e) Note that $\{a + ib) \in \mathbb{C} \mid a, b \in \mathbb{N}\} = \mathbb{N} \times \mathbb{N}$ and the latter set is countable.

f) The set $S = \{(a, b) \in \mathbb{Q} \times \mathbb{Q}\ \ |a + b = 1\}$ is in one-to-one correspondence with the set $\mathbb{Q}$ via mapping $f(a, b) = a$. Hence it is infinitely countable.

g) The set $S = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid b = \sqrt{1 - a^2}\}$ is in one-to-one correspondence with the set $(-1, 1)$, via the map $f(a, b) = a$. On the other hand, the open interval $(-1, 1)$ is in is in one-to-one correspondence with the open interval $(0, 1)$ via the map $g(x) = \frac{x+1}{2}$. Hence, S is uncountable.

Problem 2.3.17. Theorem: If A is countable so is $A \times A$.

Proof: By definition, if A is countable then either A is finite, or there exists a bijection $f : A \to \mathbb{N}$. in the first case the Cartesian product is also finite, hence, countable. In the second case, observe that the map $F : A \times A \to \mathbb{N} \times \mathbb{N}$, defined by

$$F(a_1, a_2) = (f(a_1), f(a_2))$$

is also a bijection. Hence,

$$|A \times A| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| = \aleph_0. \blacksquare$$