MATH327 – HOMEWORK SOLUTIONS

HOMEWORK #2

Section 1.1: Problems 1, 2, 4, 9
Section 1.2: Problems 1, 2, 5, 6, 12, 22

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Problem 1.1.1.

a) $\{-\sqrt{5}, \sqrt{5}\}$.
b) $\{+1, -1, +3, -3, +5, -5, +15, -15\}$.
c) $\{0, -3/2\}$.
d) $\{-1, 0, 1, 2, 3\}$.
e) $\emptyset$.

Problem 1.1.2.

a) $\{-1, 1, 0, i, -i\}$.
b) $\{1 - 2\sqrt{2}, 2 - 2\sqrt{2}, 3 - 2\sqrt{2}, 4 - 2\sqrt{2}, 5 - 2\sqrt{2}\}$.
c) $\{3/4, -3/4, 4/3, -4/3, 0\}$.
d) $\{2, 3, 5, 6, 9\}$.

Problem 1.1.4.

a) TRUE
b) FALSE
c) TRUE
d) FALSE
e) FALSE
f) FALSE because 1 is odd but $a + 2b$ is even if a is even.
g) FALSE because $a + b\sqrt{2} = 0$ and $b \neq 0$ implies that $\sqrt{2} = -a/b$. As $a, b \in \mathbb{Q}$ this contradicts the fact that $\sqrt{2}$ is irrational.

Problem 1.1.9.

a) 4 – $A = \{1, 2\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
b) 8 – $A = \{1, 2, 3\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
c) 2^n. This follows from the fact that one can choose k out of n elements in $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ ways and from the binomial expansion

$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Problem 1.2.1.

a)

$$A = \{1, 2, 3, 4, 5, 6\}, \quad B = \{-1, 0, 1, 2, 3, 4, 5\}, \quad C = \{-2, 0, 2\}.$$

b)

$$A \cup C = \{-2, 0, 1, 2, 3, 4, 5, 6\}, \quad B \cap C = \{0, 2\},$$

$$B \setminus C = \{-1, 1, 3, 4, 5\}, \quad A \oplus B = \{-1, 0, 6\},$$
$$C \times (B \cap C) = \{(-2, 0), (-2, 2), (0, 0), (0, 2), (2, 0), (2, 2)\},$$
$$(A \setminus B) \setminus C = \{6\}, \quad A \setminus (B \setminus C) = \{2, 6\}, (B \cup \emptyset) \cap \emptyset = \emptyset.$$

c)
$$T = \{(1, 2), (2, 2)\}.$$
$$S = \{(1, -1), (2, 0), (3, 1), (4, 2), (5, 3), (6, 4)\}.$$

Problem 1.2.2. If $S = \{2, 5, \sqrt{2}, 25, \pi, 5/2\}$ and $T = \{4, 25, \sqrt{2}, 6, 3/2\}$ then

a)
$$S \cap T = \{\sqrt{2}, 25\}, \quad S \cup T = \{2, 5, \sqrt{2}, 25, \pi, 5/2, 4, 6, 3/2\},$$
$$T \times (S \cap T) =$$
$$\{(4,\sqrt{2}),(4,25),(\sqrt{2},\sqrt{2}),(\sqrt{2},25),(25,\sqrt{2}),(25,25),(6,\sqrt{2}),(6,25),(\tfrac{3}{2},\sqrt{2}),((\tfrac{3}{2},25)\}$$

b)
$$\mathbb{Z} \cap (S \cup T) = \{2, 5, 25, 4, 6\}$$
$$(\mathbb{Z} \cap S) \cup (\mathbb{Z} \cap T) = \{2, 5, 25\} \cup \{4, 25, 6\} = \{2, 4, 5, 6, 25\}$$

We notice that
$$\mathbb{Z} \cap (S \cup T) = (\mathbb{Z} \cap S) \cup (\mathbb{Z} \cap T),$$
as they should be.

c)
$$\mathbb{Z} \cup (S \cap T) = \mathbb{Z} \cup \{\sqrt{2}, 25\} = \{\sqrt{2}, 0, \pm 1, \pm 2, \pm 3, ...\}$$
$$(\mathbb{Z} \cup S) \cap (\mathbb{Z} \cup T) = \{\sqrt{2}, \pi, 5/2, 0, \pm 1, \pm 2, \pm 3, ...\} \cap \{\sqrt{2}, 3/2, 0, \pm 1, \pm 2, \pm 3, ...\} =$$
$$= \{\sqrt{2}, 0, \pm 1, \pm 2, \pm 3, ...\}.$$

We notice that
$$\mathbb{Z} \cup (S \cap T) = (\mathbb{Z} \cup S) \cap (\mathbb{Z} \cup T),$$
as they should be.

Problem 1.2.5. Let $A = \{a, b, c, \{a, b\}\}$. We have

a) $A \setminus \{a, b\} = \{c, \{a, b\}\}$.

b) $\{\emptyset\} \setminus \mathcal{P}(A) = \emptyset$.

c) $A \setminus \emptyset = A$.

d) $\emptyset \setminus A = \emptyset$.

e) $\{a, b, c\} \setminus A = \emptyset$.

f) $(\{a, b, c\} \cup \{A\}) \setminus A = \{A\}$.

Problem 1.2.6. Let $U = \mathbb{R}$. We have

a) $A = (1, \infty) \cup (-\infty, -2]$, $\quad A^c = (-2, 1]$.

b) $A = (-3, \infty) \cap (-\infty, 4]$, $\quad A^c = (-\infty, -3] \cup (4, \infty)$.

c) $A = \{x \in \mathbb{R} \mid x^2 \leq -1\} = \emptyset$, $\quad A^c = \mathbb{R}$.

Problem 1.2.12. Let $A_n = \{a \in \mathbb{Z} \mid a \leq n\}$. We have

a) $A_3 \cup A_{-3} = A_3$ as $A_{-3} \subset A_3$.

b) $A_3 \cap A_{-3} = A_{-3}$ for the same reason.

c) $A_3 \cap (A_{-3})^c = \{-2, -1, 0, 1, 2, 3\}$.

d) $\bigcap_{i=0}^{4} A_i = A_0 \cap A_1 \cap A_2 \cap A_3 \cap A_4 = A_0$ as $A_0 \subset A_1 \subset A_2 \subset A_3 \subset A_4$.

Problem 1.2.22.

a) $A \cup B = A \cup C$ does not imply $B = C$. This can be seen with $A = \{1, 2\}$, $B = \{1\}$, and $C = \{2\}$.

b) $A \cap B = A \cap C$ does not imply $B = C$. This can be seen with $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1, 3\}$.

c) $A \oplus B = A \oplus C \Longrightarrow B = C$.

Proof. We prove it by contradiction. Suppose $B \neq C$. Then there are four possible cases:

Case 1. $x \in B$ and $x \notin C$ and $x \in A$. In this case $x \notin A \oplus B$ but $x \in A \oplus C$ which contradicts the hypothesis.

Case 2. $x \in B$ and $x \notin C$ and $x \notin A$. In this case $x \in A \oplus B$ but $x \notin A \oplus C$ which contradicts the hypothesis.

Case 3. $x \notin B$ and $x \in C$ and $x \in A$. In this case $x \in A \oplus B$ but $x \notin A \oplus C$ which contradicts the hypothesis.

Case 4. $x \notin B$ and $x \in C$ and $x \notin A$. In this case $x \notin A \oplus B$ but $x \in A \oplus C$ which contradicts the hypothesis.

In each case we get the contradiction which shows that $B = C$. ∎

d) $A \times B = A \times C$ and $A \neq \emptyset \Longrightarrow B = C$.

Proof. We prove it by contradiction. Suppose $B \neq C$. Then there are two cases:

Case 1. Either there is an element $x \in B$ which is not in C or

Case 2. There is an element $y \in C$ which is not in B.

In Case 1 choose any element $a \in A$ (here we need the assumption that A is not empty). Then the pair $(a, x) \in A \times B$ but $(a, x) \notin A \times C$ which contradict the hypothesis.

In Case 2 choose any element $a \in A$. Then the pair $(a, y) \in A \times C$ but $(a, y) \notin A \times B$ which contradict the hypothesis again. ∎