Sasakian Geometry

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CHAPTER 1

Structures on Manifolds

A unifying viewpoint for doing differential geometry involving different structures is that of a G-structure where G can stand for 'geometric' or more appropriately a 'Lie group'. A G-structure is just a reduction of the bundle of frames of a manifold from the general linear group to a subgroup. For example, from the point of view of G-structures a Riemannian metric on a manifold corresponds to a reduction of the frame bundle with group $GL(n,\mathbb{R})$ to the orthonormal frame bundle with subgroup, the orthogonal group $O(n,\mathbb{R})$. Many other examples will be given below.

1.1. Principal and Associated Bundles

We begin by considering principal fibre bundles. The main reference here is the classic text of Kobayashi and Nomizu[**129**].

Definition 1.1.1: *A* **principal bundle** *P over M with Lie group G consists of:*
(1) A pair of smooth manifolds P and M together with a smooth surjection $\pi : P \to M$.
(2) A smooth free action of the Lie group G on P, $R : P \times G \to P$ given by the "right" action $(u,a) \mapsto R_a u = ua$ such that the quotient is M, i.e. $M = P/G$.
(3) P is locally trivial. More precisely, every point $p \in M$ has a neighborhood U and a diffeomorphism $\chi : \pi^{-1}(U) \to U \times G$ defined by $\chi(u) = (\pi(u), \phi(u))$ where $\phi : \pi^{-1}(U) \to G$ satisfies the compatibility condition $\phi(ua) = \phi(u)a$.

As usual the manifold P is called the *total space*, the manifold M is called the *base space*, and for $p \in M$, $\pi^{-1}(p)$ is called the *fibre* at p. If we fix a point $u_0 \in \pi^{-1}(p)$ the map sending $a \in G$ to $u_0 a \in \pi^{-1}(p)$ identifies $\pi^{-1}(p)$ with G. Moreover, one easily sees that this map is a diffeomorphism from G to $\pi^{-1}(p)$ whose inverse is ϕ. When we want to emphasize the base space and the group we write $P(M,G)$ for P, or we refer to P as a principal G-bundle over M. Notice that for any open subset $U \subset M$, $\pi^{-1}(U)$ is a principal G-bundle over U. Another example is the "trivial G-bundle" when P is just the product $M \times G$.

Now let $\{U_\alpha\}$ be an open cover of M such that for each α the map $\chi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$ is a diffeomorphism, and satisfies the compatibility condition of Definition 1.1.1. Then if $u \in \pi^{-1}(U_\alpha \cap U_\beta)$ we see that $\phi_\beta(ua)(\phi_\alpha(ua))^{-1} = \phi_\beta(u)(\phi_\alpha(u))^{-1}$ for all $a \in G$. Hence, we can define a smooth map $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow G$ by

$$\psi_{\beta\alpha}(\pi(u)) = \phi_\beta(u)(\phi_\alpha(u))^{-1}. \tag{1.1.1}$$

The smooth maps $\psi_{\beta\alpha}$ are called *transition functions* for the principal bundle $P(M,G)$, and one easily sees that for each $p \in U_\alpha \cap U_\beta \cap U_\gamma$ they satisfy the 'cocycle condition'

$$\psi_{\gamma\alpha}(p) = \psi_{\gamma\beta}(p)\psi_{\beta\alpha}(p). \tag{1.1.2}$$

The appellation 'cocycle condition' comes from the fact that equivalence classes of transition functions are elements of the sheaf cohomology set $H^1(M, \mathcal{G})$ where $\mathcal{G}$ denotes the sheaf of germs of maps from opens sets of M to G. Here two transition functions $\psi_{\beta\alpha}$ and $\psi'_{\beta\alpha}$ are equivalent if there are maps $\psi_\alpha : U_\alpha \longrightarrow G$ such that $\psi'_{\beta\alpha}(p) = \psi_\beta(p)\psi_{\beta\alpha}(p)(\psi_\alpha(p))^{-1}$, and the inverse element in G is given by $\psi_{\alpha\beta}(p) = (\psi_{\beta\alpha}(p))^{-1}$. We then have the well known

Theorem 1.1.2: *Let $\{U_\alpha\}$ be an open cover on the smooth manifold M and let G be a Lie group. Suppose further that on each nonempty intersection $U_\alpha \cap U_\beta$ there exist smooth maps $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow G$ satisfying 1.1.2. Then there is a principal G-bundle $P(M, G)$ with transition functions $\psi_{\beta\alpha}$. Moreover, there is a bijective correspondence between isomorphism classes of principal G-bundles and elements of $H^1(M, \mathcal{G})$.*

The first statement in the theorem says that the transition functions determine the principal bundle, and the last statement says that the set $H^1(M, \mathcal{G})$ classifies principal G-bundles over M. The definition of an isomorphism of principal bundles will be given below. It is only in the case that G is Abelian, for example, $G = S^1$, the circle, that $H^1(M, \mathcal{G})$ has itself the structure of an Abelian group, and more standard techniques such as the exponential sequence can be used to relate this to the integral cohomology of M. We will discuss this later in more detail.

Perhaps the most important example of a principal bundle is:

EXAMPLE 1.1.3: **The linear frame bundle** $L(M)$. Recall that a frame $u = (X_1, \cdots, X_n)$ at a point $p \in M$ is a basis of the tangent space T_pM. We let $L(M)$ denote the set of all frames at all points of M. Then $\pi : L(M) \longrightarrow M$ is a principal bundle on M with group $GL(n, \mathbb{R})$. The action of $GL(n, \mathbb{R})$ on $L(M)$ is just given by matrix multiplication from the right, that is $(u, A) \mapsto uA$ for $u \in L(M)$ and $A \in GL(n, \mathbb{R})$. The local triviality can be seen by choosing a local coordinate chart $(U, x^1, \cdots, x^n)$ on M and writing the vectors X_i of the frame u in local coordinates as

$$X_i = \sum_j X_i^j \frac{\partial}{\partial x_j}$$

where the X_i^j are the components of a non-singular matrix (X_i^j) of smooth functions on U. Then the diffeomorphism $\psi : \pi^{-1}(U) \longrightarrow U \times GL(n, \mathbb{R})$ is given by $\psi(v) = (\pi(v), (X_i^j(v)))$. We remark that a point $u \in L(M)$ can be viewed as a linear map $u : \mathbb{R}^n \longrightarrow T_pM$ with $p = \pi(u)$ by $u(\mathbf{e}_i) = X_i$ where $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ denotes the standard basis of $\mathbb{R}^n$.

EXERCISE 1.1: Let G be a Lie group and H a closed Lie subgroup. Show that the Lie group G can be viewed as a principal H-bundle $G(G/H, H)$ over the homogeneous manifold G/H.

Now suppose we are given a principal G-bundle $P(M, G)$ over M, and a principal H-bundle $Q(N, H)$ on N. A *homomorphism of principal bundles* consists of a smooth map $\hat{f} : Q \longrightarrow P$ together with a Lie group homomorphism $h : H \longrightarrow G$ such that for all $u \in Q$ and $a \in H$, $\hat{f}$ satisfies $\hat{f}(ua) = \hat{f}(u)h(a)$. This condition implies that $\hat{f}$ maps fibres to fibres; hence, there is a smooth map $f : N \longrightarrow M$ such that the diagram

$$
\begin{array}{ccc}
Q & \xrightarrow{\hat{f}} & P \\
\Big\downarrow{\scriptstyle \pi_Q} & & \Big\downarrow{\scriptstyle \pi_P} \\
N & \xrightarrow{f} & M
\end{array} \tag{1.1.3}
$$

commutes. If in addition $\hat{f}$ is an embedding of smooth manifolds and h is a group monomorphism then we say that $\hat{f} : Q \longrightarrow P$ is an *embedding of principal bundles.* This implies that the map $f : N \longrightarrow M$ is also an embedding. In the case that $N = M$ and f is the identity map on M, then $Q(M, H)$ is called a *subbundle* or a *reduction* of $P(M, G)$ to the group H. We also say that $Q(M, H)$ is a *reduced subbundle* of $P(M, G)$. It may not be possible to reduce a principal bundle $P(M, G)$ to a given subgroup H. In general, there are topological obstructions to doing so. We shall see many examples of this below. If also h is an isomorphism then $Q(M, H)$ and $P(M, G)$ are *isomorphic* as principal bundles. We also say that $P(M, G)$ is *trivial* if it is isomorphic to the product $M \times G$. Observe that the local triviality conditions says that every point $p \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is trivial.

Concerning the transition functions we have

Proposition 1.1.4: *A principal G-bundle $P(M, G)$ can be reduced to a Lie subgroup H if and only if there is an open cover $\{U_\alpha\}$ of M with transition functions $\psi_{\beta\alpha}$ taking their values in H.*

PROOF. Suppose that $P(M, G)$ can be reduced to a Lie subgroup H. Then there is a principal H-bundle $Q(M, H)$ together with a smooth bundle map $f : Q \longrightarrow P$ such that the diagram 1.1.3 commutes and $f(ua) = f(u)a$ for all $a \in H \subset G$. On the open set $\pi_Q^{-1}(U_\alpha) \subset Q(M, H)$ the maps ϕ_α^Q and ϕ_α^P are related by $\phi_\alpha^Q = \phi_\alpha^P \circ f$. It follows that for any $v \in P$ there exists an $a \in G$ and $u \in Q$ such that $v = f(u)a$ and $\phi_\alpha^P(v) = \phi_\alpha^Q(u)a$. Hence, the transition functions $\psi_{\beta\alpha}^P$ satisfy

$$
\psi_{\beta\alpha}^P(v) = \phi_\beta^P(v)(\phi_\alpha^P(v))^{-1} = \phi_\beta^Q(u)a \cdot a^{-1}(\phi_\alpha^Q(u))^{-1} = \phi_\beta^Q(u)(\phi_\alpha^Q(u))^{-1}, \tag{1.1.4}
$$

and thus, have their values in H.

Conversely, given transition functions $\psi_{\beta\alpha}^P : U_\alpha \cap U_\beta \longrightarrow G$ which take their values in the Lie subgroup H, a standard result says that $\psi_{\beta\alpha}^P$ is smooth as a map into H. Thus, by Proposition 1.1.2 there is a principal H-bundle $Q(M, H)$ with transition functions $\psi_{\beta\alpha}^P$. To construct the bundle map f we define maps $f_\alpha : \pi_Q^{-1}(U_\alpha) \longrightarrow \pi_P^{-1}(U_\alpha)$ by putting $f_\alpha = \chi_P^{-1} \circ \chi_Q$. One easily sees that $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$, and so defines a global bundle map $f : Q \longrightarrow P$ with the requisite properties. □

Let $P(M, G)$ be a principal bundle and F a G-manifold, that is, a manifold together with a smooth action of the Lie group G. We denote this action by 'left multiplication' as $x \mapsto a^{-1}x$ where $x \in F$ and $a \in G$. Then we have the "product action" on $P \times F$ defined by $(u, x) \mapsto (ua, a^{-1}x)$. The quotient space $(P \times F)/G$ by this action is a smooth fibre bundle $E(M, F, G, P)$ called the *bundle associated to $P(M, G)$ with fibre F.* Note that the projection map π_E is defined as follows: let $[u, x]$ denote equivalence class of the pair $(u, x) \in P \times F$ where (u, x) is equivalent to (u', x') if there is an $a \in G$ such that $(u', x') = (ua, a^{-1}x)$. Then define the projection map $\pi_E : E \longrightarrow M$ by $\pi_E([u, x]) = \pi_P(u)$. One easily checks that this is well defined. In the case that F is a vector space, say $\mathbb{R}^k$, and $G = GL(k, \mathbb{R})$ the associated bundle $E = E(M, \mathbb{R}^k, GL(k, \mathbb{R}), P)$ is called a *vector bundle* associated to the principal bundle $P(M, GL(k, \mathbb{R}))$.

EXERCISE 1.2: Use π_E to define a differential structure on $E(M, F, G, P)$ that makes it a smooth fibre bundle with projection map π_E and fibre F.

EXERCISE 1.3: Show that the tangent bundle TM of M is a vector bundle associated to the principal bundle $L(M)$. More generally show that the tensor bundles $T^r_s M$, exterior bundles $\Lambda^p M$, and symmetric bundles $S^p M$ are vector bundles associated to the principal bundle $L(M)$.

One often describes structures on manifolds by tensor fields. Examples are complex structures, symplectic structures, Riemannian structures. These arise as smooth sections of certain bundles over M which are associated to the frame bundle $L(M)$ or its subbundles. Recall that a *section* of a fibre bundle $\pi_E : E \longrightarrow M$ is a (smooth) map $s : M \longrightarrow E$ such that $\pi_E \circ s = \mathrm{id}_M$. Unless otherwise stated sections will be smooth. Vector bundles always have sections (e.g. the zero section); whereas, a principal bundle has a section if and only if it is trivial. This is easy to verify directly, but also follows from Therem 1.1.5 below. By a *local section* of E we shall mean a section of the bundle $\pi_E^{-1}(U)$ for some open set U in M.

Theorem 1.1.5: *A principal G-bundle $P(M, G)$ is reducible to a closed subgroup H if and only if the associated bundle $E(M, G/H, G, P)$ admits a section $s : M \longrightarrow E$. Moreover, there is a bijective correspondence between such sections s and subbundles $Q(M, H)$ of $P(M, G)$.*

PROOF. Suppose $P(M, G)$ is reducible to a closed subgroup H and let $f : Q \longrightarrow P$ be the subbundle. The associated bundle E can be identified with the quotient space P/H by mapping the equivalence class $[u, gH] \in E$ to the equivalence class $[ug] \in P/H$. Let $\rho : P \longrightarrow E = P/H$ denote the natural projection. Then it is easy to see that $\rho \circ f$ is constant on the fibres of Q. Thus, we can define a section $s : M \longrightarrow E$ by setting $s(x) = \rho(f(u))$ where $\pi_P(f(u)) = x$.

Conversely, let $s : \longrightarrow E = P/H$ be a section, and consider P as a principal H-bundle over $E = P/H$ with transition functions $h_{\beta\alpha} : V_\alpha \cap V_\beta \longrightarrow H$ where $\{V_\alpha\}$ is an open cover of E. Then we can define transition functions for $P(M, G)$ with values in H by setting $\psi_{\beta\alpha} = h_{\beta\alpha} \circ s$. Thus, by Proposition 1.1.2 $P(M, G)$ is reducible to the subgroup H. The correspondence between sections and H-subbundles can be seen to be 1-1. $\square$

1.2. Connections in Principal and Associated Vector Bundles

In this section we briefly review the fundamentals of the theory of connections. As the reader undoubtably knows there are various settings and definitions for a connection. We begin with the most general as well as most abstract, namely that of connections in a principal fibre bundle. We then discuss connections in associated vector bundles, and the relationship to the former. Both formulations have their advantages and will be used in the sequel. This brief discussion will also allow us to set our notation and terminolgy for the remainder of the text. Again the standard reference for much of this material is the classic text of Kobayashi and Nomizu [**129**].

Let P be a principal bundle over M with group G. Let $u \in P$ and consider the tangent space T_uP at u. Let G_u be the *vertical* subspace of T_uP consisting of all vectors that are tangent to the fibre $\pi^{-1}(\pi(u))$. The differential of the restriction of the map ψ to the fibre through u identifies G_u with the tangent space T_eG at the identity $e \in G$, and thus with the Lie algebra $\mathfrak{g}$ of G.

Definition 1.2.1: *A* **connection** Γ *in* P *is an assignment to each* $u \in P$ *a complimentary subspace* H_u *to* $G_u \subset T_uP$*, called the* **horizontal** *subspace, that for all* $u \in P$ *satisfies:*

(i) $T_uP = G_u \oplus H_u$.
(ii) $H_{ua} = (R_a)_*H_u$ *for all* $a \in G$.
(iii) H_u *depends smoothly on* u.

In words a connection is a G-equivariant choice of compliment to the vertical that varies smoothly with u. If $X \in T_uP$ then vX and hX denotes the vertical and horizontal components of X, respectively. In "fancier"terminology definition 1.2.1 can be given as follows: Consider for each $u \in P$ the exact sequence of G-modules

$$0 \longrightarrow G_u \longrightarrow T_uP \longrightarrow H_u \longrightarrow 0.$$

Then a connection in P is a splitting of Eq.(1) as G-modules which varies smoothly with u. We can also consider the global objects, the vertical subbundle $\mathcal{V}P = \bigsqcup_{u\in P} G_u$, and the horizontal subbundle $\mathcal{H}P = \bigsqcup_{u\in P} H_u$. A connection then gives a decomposition of the tangent bundle TP as G-modules, viz.

$$TP = \mathcal{V}P \oplus \mathcal{H}P.$$

For a Lie group G acting smoothly on a manifold M there is well known homomorphism from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on M, and if G acts effectively (which we assume) this is a monomorphism. Given $A \in \mathfrak{g}$ we let A^* denote its image in $\mathcal{X}(M)$. If we specialize to the case of the right action of G on a principal G-bundle P, the vector field A^* is tangent to the fibres $\pi^{-1}(\pi(u))$ at each point $u \in P$, and is called the *fundamental vertical* vector field on P associated to $A \in \mathfrak{g}$. Evaluation of A^* at a point $u \in P$ gives a vector space isomorphism of the Lie algebra $\mathfrak{g}$ with the vertical tangent space G_u at u.

Given a connection Γ on P, we define the *connection form* associated to Γ to be the 1-form ω on P with values in the Lie algebra $\mathfrak{g}$ by setting $\omega(X)$ equal to the unique $A \in \mathfrak{g}$ such that $(A^*)_u$ is the vertical component of X. Clearly, ω satisfies $\omega(X) = 0$ if X is horizontal. We have

Proposition 1.2.2: *The connection form* ω *satisfies*

(i) $\omega(A^*) = A$ *for all* $A \in \mathfrak{g}$.
(ii) $R_a^*\omega = ad_{a^{-1}}\ \omega$ *for all* $A \in \mathfrak{g}$.

Conversely, given a $\mathfrak{g}$ *valued 1-form* ω *on* P *which satisfies conditions (1) and (2), there is a unique connection* Γ *in* P *whose connection form is* ω.

We shall often refer to a connection 1-form ω as simply a connection. By localizing and using partitions of unity one has

Theorem 1.2.3: *Every principal bundle* $P(M, G)$ *admits a connection.*

Let $P(M,G)$ be a principal bundle and V be a finite dimensional vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the quaternions[1] $\mathbb{Q}$. Let $\Omega^r(P,V)$ denote the set of smooth r-forms on P with values in V, that is the set of smooth sections of the bundle $\Lambda^r(P)\otimes V$. This set has some natural algebraic structures. It is an infinite dimensional vector space over $\mathbb{F}$ as well as a $C^\infty(P)\otimes_{\mathbb{R}}\mathbb{F}$ module. Consider a representation $\rho : G \to Aut\ V$ on V. A *pseudotensorial form* of degree r and of type (ρ, V) is r-form on P with values in V such that

$$R_a^*\phi = \rho(a^{-1})\cdot\phi.$$

ϕ is called *tensorial* if ϕ is pseudotensorial and satisfies $\phi(X_1,\cdots,X_r) = 0$ if at least one of the vector fields $X_1,\cdots,X_r$ is vertical. In particular, a pseudotensorial 0-form, which is automatically tensorial, is just a smooth G-equivariant map from P to V. We denote by $\underline{\mathcal{T}^r(P,V)}$ the closed subspace of tensorial r-forms. Now suppose that Γ is a connection in $P(M,G)$. Then the connection form ω of Γ is pseudotensorial of type $(ad,\mathfrak{g})$, but not tensorial. Notice, however, Proposition 1.2.2 implies that the difference $\omega' - \omega$ of any two connections is tensorial. Given any pseudotensorial form ϕ we can define a tensorial form by taking its horizontal projection, i.e. $\phi h(X_1,\cdots,X_r) = \phi(hX_1,\cdots,hX_r)$. Generally $d\phi$ is only pseudotensorial, even if ϕ is tensorial. However, the *exterior covariant derivative* $D\phi$ defined as the horizontal projection

$$D\phi(X_1,\cdots,X_{r+1}) = d\phi(hX_1,\cdots,hX_{r+1}) \tag{4}$$

is always tensorial. In particular, if ω is a connection 1-form, $D\omega$ is a tensorial 2-form of type $(ad,\mathfrak{g})$ called the *curvature 2-form* of ω and usually denoted by Ω. It is a smooth section of $\Lambda^2(P)\otimes\mathfrak{g}$, and satifies the famous Cartan structure equations

$$\Omega = D\omega = d\omega + \tfrac{1}{2}[\omega,\omega], \tag{5}$$

as well as the well known Bianchi identities

$$D\Omega = 0. \tag{6}$$

The meaning of the bracket in Eq.5 is $[\omega,\omega](X,Y) = [\omega(X),\omega(Y)]$. Generally, the covariant derivative of any tensorial form is

$$D\phi = d\phi + [\omega,\phi].$$

Tensorial forms can be described alternatively in terms of associated bundles. Let $E = P\times_G V$ be the $\mathbb{F}$-vector bundle on M with standard fibre V and associated to the $\mathbb{F}$-representation ρ of G. Then there is a 1-1 correspondence between tensorial r-forms $\underline{\mathcal{T}^r(P,V)}$ of type (ρ,V) and smooth sections of the bundle $\Lambda^r(M)\otimes E$ as follows: By fixing a point $u\in P$ the natural projection $P\times V \longrightarrow E$ gives an $\mathbb{F}$-linear isomorphism from the vector space V to the fibre $E_{\pi(u)}$. Then if $X_1,\ldots X_r$ are tangent vectors at $\pi(u)\in M$ we define the section $\tilde\phi$ of $\Lambda^r(M)\otimes E$ by

$$\tilde\phi(X_1,\cdots,X_r) = u\phi(X_1^*,\cdots,X_r^*) \tag{8}$$

where X^* denotes any vector on P that projects to X on M. One checks that $\tilde\phi$ is independent of the choices made. In particular, smooth G-equivariant functions $P\longrightarrow V$ correspond to smooth sections of E.

[1]We shall need all three types of vector spaces in the sequel. Some care must be taken when working with quaternionic vector spaces, due to the noncommutativity of $\mathbb{Q}$. For example, quaternionic vector spaces are considered by multiplication from the left.

Now a connection Γ in P induces a connection ∇ in the vector bundle E, and more generally in the vector bundles $\Lambda^r(M) \otimes E$. We denote the $C^\infty(M)$-module of smooth sections of $\Lambda^r(M) \otimes E$ by $\underline{\Lambda^r(M, E)}$.

Definition 1.2.4: *A (Koszul)* **connection** ∇ *on an $\mathbb{F}$-vector bundle E is an $\mathbb{F}$-linear map on sections: $\nabla : \Gamma(E) \to \Lambda^1(M, E)$ satisfying the Leibnitz rule $\nabla(fs) = f\nabla s + df \otimes s$, where $f \in C^\infty(M)$ and s is a smooth section of E.*

Given a connection Γ on $P(M, G)$ and an associated vector bundle $E = P \times_G V$, we can define a connection ∇ on E by setting

$$\nabla = uDu^{-1}. \tag{9}$$

Suppose conversely we have an $\mathbb{F}$-vector bundle E with associated principal bundle $P(M, G)$ and $\mathbb{F}$-representation ρ of G on the standard fibre V so that $E = P \times_G V$. Then given a connection ∇ on E, we obtain a connection Γ on P as follows: Let $\mathbf{s} = (s_1, \cdots, s_k)$ be a G-frame of local sections of E. If $X \in T_xM$, then the subspace H_u of P_u with $\pi(u) = x$ defined by

$$H_u = \{\mathbf{s}_* X | \nabla_X \mathbf{s} = 0\} \tag{10}$$

is a G-equivariant complement to the vertical subspace G_u. This defines the connection Γ in P.

As in Proposition 1.2.2 it is convenient to express the connection ∇ in terms of a 1-form. However, this can only be done locally in terms of a local trivialization of the vector bundle E. Given a local trivialization $E|U \simeq U \times V$, we can write

$$\nabla|U = d + A_U, \tag{11}$$

where the exterior derivative d represents the flat connection on U and A_U is a $\mathfrak{g}$-valued 1-form on U. Given another such trivialization on the open set $W \subset M$, and transition functions $g : U \cap W \longrightarrow G$ relating the two trivializations in the overlap, one obtains the relation

$$A_W = g^{-1} A_U g + g^{-1} dg. \tag{12}$$

Conversely, given a cover of M by open sets $\{U\}$ with $\mathfrak{g}$-valued 1-forms A_U on each open set in the cover satisfying Eq.12 in the overlaps, one can reconstruct the connection ∇ on E. This formulation is essentially E. Cartan's definition of connection, whereas, the formulation given in Proposition 1.2.2 is due to his student C. Ehresmann.

There is a natural extension of Eq.11 to the bundles $\Lambda^r(M) \otimes E$. We shall use the notation in [DK] and write the "covariant derivative"as a differential operator

$$d_A = d + A : \underline{\Lambda^r(M, E)} \longrightarrow \underline{\Lambda^{r+1}(M, E)}. \tag{13}$$

Here the symbol A denotes a family of $\mathfrak{g}$-valued 1-forms on open sets of M satisfying Eq.12 which act linearly via the representation $\rho_* : \mathfrak{g} \longrightarrow End\, V$ on the local sections of E obtained from a local trivialization of E. The curvature form $\Omega = D\omega$ on P corresponds by the isomorphism u to the smooth section $F^A = d_A A$ of $\Lambda^2(M) \otimes \mathfrak{g}$. The Cartan structure equations Eq.5 then take the form

$$F^A = d_A A = dA + A \wedge A, \tag{14}$$

where here again we follow the convention in [DK] using the wedge product in lieu of the brackets to emphasize A as an endomorphism of V via local trivializations. In a local coordinate chart (U, x) on M we write the $\mathfrak{g}$-valued curvature 2-form

$F^A = \sum F_{ij} dx^i \wedge dx^j$ and the connection 1-form as $A = \sum A_i dx^i$, in which case Eq.14 becomes

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j]. \tag{15}$$

1.3. G-Structures

In this section we describe structures on manifolds from the unifying viewpoint of G-structures. There are several texts where this point of view is expounded [**127, 159, 185, 197**]. Here we consider only first order G-structures, that is, reductions of the bundle of linear frames on M. Here is the precise definition.

Definition 1.3.1: *Let $G \subset GL(n,\mathbb{R})$ be a subgroup, then a* **G-structure** *on M is a reduction of the frame bundle $L(M)$ to the subgroup G. The G-structure is said to be* **integrable** *if every point of M has a local coordinate chart (U, x) such that local section*

$$\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right)$$

of $L(M)$ is a local section of the reduced bundle $P(M,G)$. Such a coordinate chart is called **admissible**.

Let us return to the bundle of linear frames $L(M)$ of a smooth manifold M. On $L(M)$ there is a canonical $\mathbb{R}^n$-valued 1-form θ defined as follows: as seen in 1.1.3 we can view any point $u \in L(M)$ as a vector space isomorphism $u : \mathbb{R} \longrightarrow T_{\pi(u)}M$. So θ can be defined by

$$< \theta, X >= u^{-1}\pi_* X \tag{1.3.1}$$

where $X \in T_u L(M)$ and $< \cdot, \cdot >$ denotes the natural pairing between the tangent bundle to the bundle of linear frames $TL(M)$ and its dual cotangent bundle $T^*L(M)$. If $P(M,G)$ is a G-structure on M, i.e. a subbundle of $L(M)$, then we can restrict θ to $P(M,G)$ giving a canonical 1-form on the G-structure. The canonical 1-form on a G-structure $P(M,G)$ behaves functorially under the action of the general linear group $GL(n,\mathbb{R})$ on $L(M)$. Indeed we have

Lemma 1.3.2: *For any $a \in G \subset GL(n,\mathbb{R})$ the transformation rule holds;*

$$R_a^* \theta = a^{-1}\theta.$$

Proof. For any vector field X on $P(M,G)$, we have

$$< R_a^*\theta, X >=< \theta, R_{a*}X >= (ua)^{-1}\pi_* R_{a*}X = a^{-1}u^{-1}\pi_* X =< a^{-1}\theta, X > .$$

□

Generally, there are topological obstructions to the existence of G-structures. To see that such a reduction does not always exist let $G = \{e\}$ the identity group. Then an $\{e\}$-structure on M is nothing but a global frame or parallelism of M. But it is well known that global frames do not always exist, that is that M may not be parallelizable. For example, the 2-sphere S^2 does not even have one nowhere vanishing vector field let alone a global frame. Even if there is a G-structure on M there may not be an integrable G-structure. For example, as seen below every manifold M admits many $O(n,\mathbb{R})$-structures, but a compact M admits an integrable $O(n,\mathbb{R})$-structure only if M is covered by a torus (see 1.3.5 below).

On the other hand every $GL(n, \mathbb{R})$ structure on M is integrable. The following proposition is evident.

Proposition 1.3.3: *A G-structure $P(M, G)$ is integrable if and only if there is an atlas of coordinate charts $\{(U_\alpha, x_\alpha)\}$ on M whose Jacobian matrices $(\frac{\partial x^i_\alpha}{\partial x^j_\tau})_{i,j}$ lie in G at all points of $U_\alpha \cap U_\beta$.*

We now want to put Theorem 1.1.5 to work by seeing how certain natural tensor fields define G-structures on M. Suppose that T_0 is an element of the tensor algebra $\mathcal{T}(\mathbb{R}^n)$ over $\mathbb{R}^n$ and that G is the largest closed Lie subgroup of $GL(n, \mathbb{R})$ that leaves T_0 invariant. Viewing a point $u \in L(M)$ as a vector space isomorphism $u : \mathbb{R}^n \longrightarrow T_{\pi(u)}M$, we obtain an induced isomorphism $u^* : \mathcal{T}(\mathbb{R}^n) \longrightarrow \mathcal{T}(T_{\pi(u)}M)$ of the tensor algebras. Then the image $T = u_* T_0$ is a section of the tensor algebra bundle $\mathcal{T}(M)$. If T_0 is a tensor of type (r, s) then T is a section of the tensor bundle $T^r_s M$. In any case because of the invariance of T_0 under G, the tensor field T is constant along the fibres of the G-structure $P(M, G) \subset L(M)$, and thus defines a section of the associated bundle $L(M)/G$. In this case we say that $P(M, G)$ is a *G-structure defined by the tensor T_0*.

Proposition 1.3.4: *Let $P(M, G)$ be a G-structure defined by the tensor T_0. Then $P(M, G)$ is integrable if and only if there exists a coordinate chart (U, x) of M such that the corresponding tensor field $T = u_* T_0$ has constant components on U.*

Proof. $(\Rightarrow)$ Suppose $P(M, G)$ is integrable, then the frame $u = (\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n})$ belongs to $P(M, G)$. Let $\{e_i\}$ denote the standard basis for $\mathbb{R}^n$ and $\{e^j\}$ its dual basis. Then $\frac{\partial}{\partial x^i} = u(e_i)$, and $dx^j = u(e^j)$. So if

$$T_0 = t^{i_1 \cdots i_r}_{j_1 \cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$

is a tensor of type (r, s), we have

$$T = u_* T_0 = t^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

Hence, T has constant components on U.

$(\Leftarrow)$ Conversely, suppose that (U, x) is a coordinate chart such that $T = u_* T_0$ has constant components. So T is a constant section of the associated bundle $L(U)/G$. Then there is a linear transformation of coordinates on U to new coordinates $(y^1, \cdots, y^n)$ such that the frame $u = (\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n})$ belongs to $P(M, G)$. Thus, the G-structure $P(M, G)$ is integrable. $\square$

Generally, this 'integrability condition' can be very restrictive, and we will discuss various 'levels of integrability'. Our first example is a good example where non-integrable G-structures are of more interest than the integrable ones.

Example 1.3.5: **Riemannian metrics.** We consider a reduction of the frame bundle $L(M)$ to the orthogonal group

$$O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid A^t A = \mathbb{I}_n\}.$$

The reduced bundle $O(M) \subset L(M)$ is called the *orthonormal frame bundle* of M. Theorem 1.1.5 says that such a reduction is equivalent to a choice of section of the associated bundle $E(M, GL(n, \mathbb{R})/O(n, \mathbb{R}), L(M)) = L(M)/O(n, \mathbb{R})$. We show that such a section is just a Riemannian metric on M. As mentioned previously each point $u \in L(M)$ gives an isomorphism of the standard vector space $\mathbb{R}^n$ with the tangent space $T_{\pi(u)}M$. Let $< \cdot, \cdot >$ denote the Euclidean inner product on $\mathbb{R}^n$, and let $X, Y \in T_{\pi(u)}M$. Then

$$g_u(X,Y) =< u^{-1}X, u^{-1}Y >$$

defines an inner product on $T_{\pi(u)}M$. Furthermore, if $a \in O(n, \mathbb{R})$ we have

$$\begin{aligned} g_{ua}(X,Y) &=< (ua)^{-1}X, (ua)^{-1}Y >=< a^{-1}u^{-1}X, a^{-1}u^{-1}Y > \\ &=< u^{-1}X, u^{-1}Y >= g_u(X,Y) \end{aligned}$$

where the second to the last equality holds by the invariance of $< \cdot, \cdot >$ under $O(n, \mathbb{R})$. This shows that g_u is constant along the fibres of $O(M)$, and thus, is a section of $L(M)/O(n, \mathbb{R})$. Thus, the associated bundle $L(M)/O(n, \mathbb{R})$ can be identified with a subbundle of the vector bundle $\mathrm{Sym}^2 T^*M$ of symmetric covariant 2-tensors on M. The choice of u modulo $O(n, \mathbb{R})$ corresponds to a choice of Riemannian metric on M. Since manifolds are paracompact the standard partition of unity argument shows that such sections and hence, such reductions always exist.

When are $O(n)$-structures integrable? According to Proposition 1.3.4 this occurs when the metric tensor g has constant components in some coordinate chart (U, x), and then by a change of coordinate, say (V, y) the metric can be brought to the form

$$g = \sum_i (dy^i)^2$$

in V. By the second fundamental theorem of Riemannian geometry this happens precisely when the Riemann curvature tensor vanishes. Thus, if M is compact it is either a torus or a Klein bottle depending on whether it is orientable or not.

Similarly, a reduction to the group $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap GL^+(n, \mathbb{R})$ corresponds to oriented Riemannain geometry.

In this case the non-integrable structures are much more interesting the integrable ones. In order to study these further one needs to study certain invariants of the G-structure. Generally, these invariants involve Spencer cohomology to which we refer the reader to the literature [**86, 140, 191, 197**]. In the case of $O(n, \mathbb{R})$-structures the relevant Spencer cohomology group consists of the Riemann curvature tensor. In our next example the integrable case is very rich.

EXAMPLE 1.3.6: **Almost complex structures.** A **complex structure** on a real vector space V is an endomorphism J of V that satisfies $J^2 = -\mathbb{I}$ where $\mathbb{I}$ denotes the identity endomorphism of V. We can give V the structure of a complex vector space by defining scalar multiplication with scalars in $\mathbb{C}$ by

$$(a + ib)v = av + bJv.$$

Conversely, if V is a complex vector space we define the endomorphism J on V by

$$Jv = iv.$$

Let $\{v_1, \cdots, v_n\}$ be basis for V as a complex vector space, then it is easy to check that $\{v_1, \cdots, v_n, Jv_1, \cdots, Jv_n\}$ is a basis for V as a real vector space. In particular, this implies that any real vector space with a complex structure has even real dimension. For example, by writing $z^j = x^j + iy^j$ for $j = 1, \cdots, n$, the complex vector $(z^1, \cdots, z^n)$ in the complex vector space $\mathbb{C}^n$ is identified with the real vector $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ in the real vector space $\mathbb{R}^{2n}$. The **canonical complex**

structure J_0 on V is given in block form with respect to the standard basis of $\mathbb{R}^{2n}$ by

$$J_0 = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$$

where $\mathbb{I}_n$ denotes the $n \times n$ identity matrix. We can identify the complex linear group $GL(n, \mathbb{C})$ with the subgroup of matrices in $GL(2n, \mathbb{R})$ that commute with J_0. Thus, the complex $n \times n$ matrix $A + iB$ is identified with the real $2n \times 2n$ matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

On a manifold M an **almost complex structure** is an endomorphism J_x of the tangent space T_xM that varies smoothly with x and satisfies $J_x^2 = -\mathrm{id}_x$ at each point $x \in M$, or in other words, a smooth section J of the endomorphism bundle End TM satisfying $J^2 = -\mathrm{id}$. As in the Riemannian case such a section can be obtained from the canonical complex structure J_0 on $\mathbb{R}^{2n}$ by identifying $u \in L(M)$ with a linear map $u : \mathbb{R}^{2n} \longrightarrow T_{\pi(u)}M$. Thus, we define

$$J_u = uJ_0u^{-1}.$$

The invariance of J_0 under the subgroup $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ shows that J_u is constant along the fibres of the bundle $\mathbb{C}L(M)$ of complex linear frames on M and thus, gives a section of the associated bundle $E(M, GL(2n, \mathbb{R})/GL(n, \mathbb{C}), L(M)) = L(M)/GL(n, \mathbb{C})$ which is the subbundle of End TM consisting of endomorphisms that satisfy $J^2 = -\mathrm{id}$. Unlike the Riemannian case it is not always possible to find such a section, or equivalently to obtain a reduction of the real frame bundle $L(M)$ to the complex linear group $GL(n, \mathbb{C})$. In general, this is a fairly deep topological question. The existence of an almost complex structure J on M is equivalent to the splitting of the complexified tangent bundle $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$. If one has such a splitting, J can be defined to be multiplication by i on $T^{1,0}M$ and multiplication by -1 on $T^{0,1}M$. Conversely, given J one easily sees that it is diagonalizble over $TM \otimes \mathbb{C}$ with eigenvalues $\pm i$.

By Proposition 1.3.3 a $GL(n, \mathbb{C})$-structure is integrable when there is an atlas of coordinate charts $(U_\alpha, x_\alpha^1, \cdots, x_\alpha^n, y_\alpha^1, \cdots, y_\alpha^n)$ whose Jacobian matrix lies in $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$. This corresponds to the complex coordinates $z_\alpha^j = x_\alpha^j + iy_\alpha^j$ being independent of the complex conjugate coordinates $\bar{z}_\beta^i$, that is the transition functions are holomorphic or equivalently satisfy the Cauchy-Riemann equations. An integrable almost complex structure is called a **complex structure**. The structure tensor of the G-structure that measures the failure of integrability is the Nijenhuis torsion tensor defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

The vanishing of this tensor field is equivalent to the integrability of the almost complex structure. If one assumes that M is real analytic this follows from a Frobenius integrability argument (cf. [**129**], vol II, Appendix 8). However, in the C^∞ case this is a much deeper result due to Newlander and Nirenberg [**168**].

Following up on the remark at the end of the second paragraph of Example 1.3.6, one can ask why a partition of unity argument doesn't work in the case of almost complex structures. The point is that one cannot guarentee the condition $J^2 = -\mathbb{I}$ at every point of the manifold by using partitions of unity. A better understanding is obtained from the point of view of G-structures. An alternative proof of the existence of Riemannian metrics on any manifold is as follows: by Theorem 1.1.5 giving a Riemannian metric is equivalent to giving a reduction of the frame bundle to the orthogonal group $O(n,\mathbb{R})$. This reduction can be accomplished (generally over any paracompact space) since as topological spaces the general linear group $GL(n,\mathbb{R})$ decomposes as $O(n,\mathbb{R}) \times \mathbb{R}^{\frac{n(n+1)}{2}}$, the so-called Iwasawa decomposition. Thus, $GL(n,\mathbb{R})$ is homotopy equivalent to $O(n,\mathbb{R})$. Then paracompactness allows one to construct this homotopy equivalence at each point of M giving the reduction. However, in the case of reducing $L(M)$ to the complex linear group $GL(n,\mathbb{C})$ there is no such homotopy equivalence. Indeed, $GL(2n,\mathbb{R})$ is homotopy equivalent to $O(2n,\mathbb{R})$, whereas, again the Iwasawa decomposition for $GL(n,\mathbb{C})$ says that $GL(n,\mathbb{C})$ is homotopy equivalent to the unitary group $U(n)$, that is, the subgroup of $GL(n,\mathbb{C})$ that leaves invariant the standard Hermitian form $< \cdot, \cdot >$ on $\mathbb{C}^n$. It is well known that $O(2n,\mathbb{R})$ and $U(n)$ are not homotopy equivalent, e.g. they have different homology groups. So, generally, there are obstructions to such reductions. A general theory of obstructions is expounded in the classic text of Steenrod [**196**]. A well known example of an even dimensional manifold which admits no almost complex structure is the 4-sphere S^4. This can be shown either by obstruction theory or using the Atiyah-Singer index theorem. Another interesting example is S^6 which is known to admit an almost complex structure, but it is still an open question whether S^6 has an integrable almost complex structure, i.e. a complex structure.

EXERCISE 1.4: Let $G = GL^+(n,\mathbb{R})$ be the subgroup of matrices in $GL(n,\mathbb{R})$ whose determinant is positive. Show that M admits such a G-structure if and only if M is orientable. Show that a $GL^+(n,\mathbb{R})$-structure on M coincides with a choice of orientation of M. What about integrability?

EXERCISE 1.5: Show that an almost complex structure on M determines an orientation on M; hence, any almost complex manifold is orientable.

Notice that an oriented Riemannian manifold gives a reduction of $L(M)$ to the special orthogonal group $SO(n) = \{A \in O(n) \mid \det(A) = 1\}$, and that $SO(n) = GL^+(n,\mathbb{R}) \cap O(n)$. We will frequently use G-structures that occur as a combination of other G-structures, or alternatively a G-structure defined by more than one tensor. Another important example of this is:

EXAMPLE 1.3.7: **Almost Hermitian structures**. Let $G = U(n)$ the unitary group defined by

$$U(n) = \{A \in GL(n,\mathbb{C}) \mid \bar{A}^t A = \mathbb{I}_n\}.$$

This is precisely the group that leaves the standard Hermitian form $< u, v >= \bar{u_1} v_1 + \cdots + \bar{u_n} v_n$ in $\mathbb{C}^n$ invariant. Since $U(n) = GL(n,\mathbb{C}) \cap O(2n)$, a $U(n)$-structure or almost Hermitian structure consists of an almost complex structure J and a Hermitian metric h on the complexified tangent space $T_pM \otimes \mathbb{C}$ at each point $p \in M$. Moreover, for all complex vector fields, the compatibility condition $h(JX, JY) = h(X, Y)$ holds.

A $U(n)$-structure is integrable if both the almost complex structure and the Riemannian structures are integrable. In this case there are weaker notions of integrable, the most interesting being the integrability of the almost complex structure J. In this case one has an Hermitian structure.

These ideas can also be applied to indefinite or pseudo-Riemannian and pseudo-Hermitian metrics[2]. For example,

EXAMPLE 1.3.8: **Pseudo-Riemannian structures**. Let $O(p,q)$ denote the subgroup of $GL(n,\mathbb{R})$ that leaves the quadratic form $u_1^2+\cdots+u_p^2-u_{p+1}^2-\cdots-u_{p+q}^2$ invariant. Then an $O(p,q)$-structure on M is the same as a choice of pseudo-Riemannian metric g on M of signature (p,q). If p or q equals 1, then this is called a Lorentzian structure and g a Lorentzian metric. Similarly, one has almost pseudo-Hermitian structures by considering the pseudo-unitary groups $U(p,q)$ which is the group that leaves invariant an indefinite Hermitian form of signature (p,q). Clearly, $U(p,q)=GL(p+q,\mathbb{C})\cap O(2p,2q)$. The integrability question is similar to that of Example 1.3.5.

Unlike Riemannian metrics (structures), pseudo-Riemannian metrics do not always exist on M. See for example [**108**] for the existence of Lorentzian metrics on M.

EXERCISE 1.6: Define the group

$$CO(n)=\{A\in GL(n,\mathbb{R})\mid A^tA=c\mathbb{I}_n, c>0\}.$$

Show that a $CO(n)$-structure on M coincides with a conformal class of Riemannian metrics on M. Recall that a conformal class is an equivalence class of Riemannian metrics where the Riemannian metrics g and g' are equivalent if there exists a positive function λ on M such that $g'=\lambda g$.

A $CO(n)$-structure on M is called a *conformal structure* on M; however, $CO(n)$ is usually not called the conformal group, a name that is usually reserved for the automorphism group of a conformal structure, and this groups is strictly larger than $CO(n)$. This is related to prolongations of G-structures. Conformal structures can be defined similarly in the case of pseudo-Riemannian structures.

Our next example will also be of fundamental importance to us.

EXAMPLE 1.3.9: **Almost symplectic structures**. Consider the nondegenerate antisymmetric bilinear form

$$\omega_0(\mathbf{u},\mathbf{v})=u_1v_{n+1}+\cdots+u_nv_{2n}-u_{n+1}v_1-\cdots-u_{2n}v_n$$

on the vector space $\mathbb{R}^{2n}$. Notice that ω_0 can also be written in terms of the standard Euclidean metric $<\cdot,\cdot>$ and canonical complex structure J_0 as

$$\omega_0(\mathbf{u},\mathbf{v})=<\mathbf{u},J_0\mathbf{v}>.$$

The subgroup of $GL(2n,\mathbb{R})$ that leaves ω_0 invariant is called the **real symplectic group** and denoted[3] by $Sp(n,\mathbb{R})$. It is easy to see that this group can be defined by

$$Sp(n,\mathbb{R})=\{A\in GL(2n,\mathbb{R})\mid A^tJ_0A=J_0\}.$$

An **almost symplectic structure** on a manifold M is then given by transporting the antisymmetric form ω_0 to the manifold as before by the linear isomorphism $u:\mathbb{R}^{2n}\longrightarrow T_pM$, viz. for $X,Y\in T_pM$ we define

$$\omega(X,Y)=\omega_0(u^{-1}X,u^{-1}Y).$$

[2]One should cautioned that the terminology pseudo-Hermitian structure is used in an entirely different sense in [**220**]

[3]The notation for this group is by no means standard. It is sometimes written as $Sp(2n,\mathbb{R})$, and is written as $Sp(2n)$ in [**151**]. We shall employ the notation $Sp(n)$ for the 'compact symplectic group' which is different.

ω is a section of the bundle $E(M, GL(2n,\mathbb{R})/Sp(n,\mathbb{R}), L(M)) = L(M)/Sp(n,\mathbb{R})$ which can be identified with the subbundle of the exterior bundle $\Lambda^2 M$ bundle consisting of nondegenerate 2-forms. By Theorem 1.1.5 this corresponds to a reduction of $L(M)$ to the real symplectic group $Sp(n,\mathbb{R})$. The appearance of the tensor J_0 suggests that there may be a connection between almost complex structures and almost symplectic structures. This is indeed the case for $Sp(n,\mathbb{R}) \cap O(2n,\mathbb{R}) = U(n) = GL(n,\mathbb{C}) \cap O(2n,\mathbb{R})$ Since, an $O(2n,\mathbb{R})$-structure exists on any $2n$-dimensional manifold, we see that M^{2n} admits an almost symplectic structure if and only if it admits an almost complex structure. Of course, the two structures are different.

By Proposition 1.3.4 an $Sp(n,\mathbb{R})$-structure is integrable if and only if the tensor field ω has constant components with respect to some local coordinates. In this case there are local coordinates $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ on M such that ω takes the form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

A well known theorem of Darboux says that an $Sp(n,\mathbb{R})$-structure is integrable if and only if the tensor field ω is closed, that is $d\omega = 0$. An integrable $Sp(n,\mathbb{R})$-structure on M is called a **symplectic** structure.

From the discussion above about Lie groups, it should be clear that an almost Hermitian structure gives naturally an almost symplectic structure. Indeed, given an Hermitian metric h and almost complex structure on M satisfying the compatibility condition discussed in Example 1.3.7, we can define a 2-form by

$$\omega(X,Y) = h(X, JY),$$

and one easily sees that this is nondegenerate and anti-symmetric. This is usually called the *Kähler form*, and when it is closed and J is integrable, the structure is called a *Kähler structure*. These will be of much interest to us later.

EXERCISE 1.7: Define the conformal symplectic group by

$$CSp(n,\mathbb{R}) = \{A \in GL(2n,\mathbb{R}) \mid A^t J_0 A = cJ_0, c \in \mathbb{R}^*\},$$

where $\mathbb{R}^*$ denotes $\mathbb{R} - \{0\}$. Show that a $CSp(n,\mathbb{R})$-structure on M coincides with a choice of equivalence class of nondegenerate 2-forms, where the 2-forms ω and ω' are equivalent if there is a nowhere zero function λ on M such that $\omega' = \lambda\omega$. Such a structure can be called a conformal almost symplectic structure. Show that if the $CSp(n,\mathbb{R})$-structure is integrable, then the nondegenerate 2-forms are closed and λ is a constant.

The next example will be of fundamental importance to us.

EXAMPLE 1.3.10: ***p*-dimensional distribution** A p-dimensional distribution on an n-dimensional manifold M is a choice of p-dimensional subbundle E of the tangent bundle TM. The group G that stabilizes E is given by

$$GL(p,q;\mathbb{R}) = \{\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \mid A \in GL(p,\mathbb{R}), B \in GL(q,\mathbb{R}), C \in \mathcal{M}^{q,p}\},$$

where $\mathcal{M}^{p,q}$ denotes the p by q real matrices, and $p + q = n$. An integrable $GL(p,q;\mathbb{R})$-structure on M is just a **foliation** $\mathcal{F}$ of M, that is, every point of M has a 'foliated coordinate chart' (U,ϕ) with coordinates $(x^1, \cdots, x^p; y^1, \cdots, y^q)$ such that $(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^p})$ spans $E|U$, and the Jacobian matrix of the change of foliated charts lies in $GL(p,q;\mathbb{R})$. Globally, this gives a partition of M into disjoint

p-dimensional immersed submanifolds $\mathcal{L}_\alpha$, called the **leaves** of $\mathcal{F}$. By the classical Frobenius Theorem, a p-dimensional distribution is integrable if and only if it it involutive, that is the Lie bracket of any sections of the subbundle E is also a section of E. This has a dual formulation as follows: let E^0 denote the subbundle of T^*M that annihilates E. Then the Frobenius Theorem says that E is integrable if and only if the ideal $\mathcal{I}(E^0)$ generated by the sections of E^0 is closed under exterior differentiation, i.e. $\mathcal{I}(E^0)$ is a differential ideal.

Alternatively, a foliation can be described by patching together local submersions. This is the point of view advanced by Haefliger [**104**]. Let $\{(U_\alpha, \phi_\alpha)\}$ be a foliated atlas with coordinates $\phi_\alpha = (x^1_\alpha, \cdots, x^p_\alpha; y^1_\alpha, \cdots, y^q_\alpha)$. Then there are local submersions $f_\alpha : U_\alpha \longrightarrow \mathbb{R}^q$ defined to be the composition of $\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q$ with projection onto the second factor. The inverse image $f^{-1}(y)$ of a point $y \in \mathbb{R}^q$ defines a **plaque** of $\mathcal{F}$, that is a component of $U_\alpha \cap \mathcal{L}_\alpha$. In the overlap of foliated charts $U_\alpha \cap U_\beta$ the local submersions f_α, f_β are related by

$$f_\beta = \tau_{\beta\alpha} f_\alpha,$$

where the transition functions $\tau_{\beta\alpha} : f_\alpha(U_\alpha \cap U_\beta) \longrightarrow f_\beta(U_\alpha \cap U_\beta)$ are diffeomorphisms that satisfy Haefliger's cocyle conditions

$$\tau_{\alpha\beta} = \tau^{-1}_{\beta\alpha} \text{ on } U_\alpha \cap U_\beta, \qquad \tau_{\gamma\alpha} = \tau_{\gamma\beta}\tau_{\beta\alpha} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

This point of view has particular interest in regard to tranverse G-structures which are defined below in Definition 1.5.13.

A (global) submersion (hence, a fibre bundle) is a special case of a foliation. It is often called also a simple foliation. More development of foliation theory is given below in section 1.5.

There are various special cases of Example 1.3.10 of interest.

EXAMPLE 1.3.11: **Almost product structures** This is Example 1.3.10 with $C = 0$. In this case the tangent bundle to M splits as a direct sum of two subbundles. By Proposition 1.3.3 an integrable almost product correspondence to a product structure, i.e. M is the product of two manifolds, one of dimension p and the other of dimension q.

We are especially interested in the case when $q = 1$ and $p = 2m$ which is related to almost contact structures treated in detail in Chapter 3. Unfortunately, the terminology with regard to contact and almost contact structures is very misleading. For example, a contact structure is generally not an almost contact structure in contrast to the usual use of this terminology; moreover both of these structures are defined in [**100**]! This was noted and fixed in [**198**], at least for topological purposes, but we would like consistency with geometric structures as well. We shall refer to our structure as an *almost contact G-structure* to distinguish it from an almost contact structure whose definition is by now too well ensconced in the literature.

EXAMPLE 1.3.12: **Almost contact G-structure**. Consider the group G defined as follows:

$$G = \{ \begin{pmatrix} A & 0 \\ \mathbf{c} & b \end{pmatrix} \mid A \in CSp(m, \mathbb{R}), b \in GL(1, \mathbb{R}), \mathbf{c} \in \mathcal{M}^{1,2m} \}$$

This G-structure picks out a codimension 1 subbundle with a conformal almost symplectic structure on it. Here an integrable G-structure is related to what could be called a **conformal cosymplectic structure**.

In the case of Example 1.3.12 we shall be more interested in the non-integrable case. In Chapter 3 we shall describe its relation with almost contact and contact structures.

Another structure that is related to both Examples 1.3.6 and 1.3.10 is a CR-structure. Here the concept of an almost CR-structure is not standard. Indeed the definition of a CR-structure varies as well. Some authors require that a CR-structure be defined by a codimension one subbundle, cf. [**119**], the idea being that of modeling a real hypersurface in $\mathbb{C}^{n+1}$. We shall be more general.

EXAMPLE 1.3.13: **Almost CR-structures** An almost CR-structure of codimension q is a G-structure with G given by

$$G = \{\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \mid A \in GL(m,\mathbb{C}), B \in GL(q,\mathbb{R}), C \in \mathcal{M}^{q,2m}\}$$

where $GL(m,\mathbb{C})$ is realized as a subgroup of $GL(2m,\mathbb{R})$ as described in Example 1.3.6. This G picks out a $2m$-dimensional subbundle E of TM with an almost complex structure, that is a smooth section J of the endomorphism bundle End E satisfying $J^2 = -\mathbb{I}_E$. We say that (E, J) defines a **CR-structure** on M if for any smooth sections X, Y of E the vector field $[JX, Y] + [X, JY]$ is a section of E and the Nijenhuis tensor N_J, as defined in Example 1.3.6, vanishes. We are interested in the case of codimension one, i.e. $q = 1$. If M is orientable then there exists a 1-form η on M such that $E = \ker\eta$. The symmetric bilinear form $L(X, Y)$ on E defined by $L(X, Y) = d\eta(X, JY)$ is called the **Levi form** of the CR-structure. The CR-structure is said to be **strictly pseudoconvex** if L is either positive or negative definite.

I will add more on quaternionic G-structures later.

1.4. Pseudogroup Structures

We have seen that the notion of G-structure is a unifying concept for studying geometric structures. Another unifying concept is the idea of a pseudogroup of transformations. Roughly speaking a pseudogroup is 'group of transformations' where there may be some problems with domains of definition. The concept goes back to S. Lie and E. Cartan, but the precise notion was defined more recently by C. Ehresmann [**72**]. The ideas were then developed much further by Kodaira and Spencer among others. cf. [**103, 131, 132, 195**]. Here is the definition following [**126**].

Definition 1.4.1: *A* **pseudogroup of transformations** *on* $\mathbb{R}^n$ *is a set* Γ *of local diffeomorphisms which satisfy*

(i) *Each* $f \in \Gamma$ *is a diffeomorphism of an open subset of* $\mathbb{R}^n$*, called the domain of* f*, onto an open subset of* $\mathbb{R}^n$*, called the range of* f*.*
(ii) *If* $U = \cup_i U_i$ *where each* U_i *is an open subset of* $\mathbb{R}^n$*, then a diffeomorphism* f *with domain* U *belongs to* Γ *if and only if its restriction to each* U_i *belongs to* Γ*.*

(iii) *For every open subset U of $\mathbb{R}^n$, the identity transformation on U belongs to Γ.*
(iv) *If $f \in \Gamma$ then so is f^{-1}.*
(v) *If $f : U \longrightarrow V$ and $f' : U' \longrightarrow V'$ are elements of Γ and $V \cap U' \neq \emptyset$, then the diffeomorphism $f' \circ f : f^{-1}(V \cap U') \longrightarrow f'(V \cap U')$ belongs to Γ.*

Γ is said to be **transitive** *if for every pair of points $p, q \in \mathbb{R}^n$ there is an $f \in \Gamma$ such that $f(p) = q$.*

More generally one can replace $\mathbb{R}^n$ by a different 'model space' [**126**], but we have no need for the more general situation. The importance of pseudogroups for our purpose lies in its ability to describe different types of atlases of coordinate charts.

Definition 1.4.2: *Let Γ be a transitive pseudogroup, and M a second countable, Hausdorff topological space. A* **Γ-atlas** *for M is a family of pairs (U_α, ϕ_α), called* **charts**, *such that*

(i) *The collection $\{U_\alpha\}$ of open sets covers M, i.e. each U_α is an open subset of M, and $\cup_\alpha U_\alpha = M$.*
(ii) *Each ϕ_α is a homeomorphism of U_α onto an open subset $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) : \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

belongs to Γ.

A Γ-atlas on M is said to be **maximal** *if it is not properly contained in any other Γ-atlas on M.*

Clearly, if Γ is the pseudogroup of all local diffeomorphisms of $\mathbb{R}^n$, this definition coincides with the usual definition of an atlas of coordinate charts for a differentiable manifold. As in the usual case, it is easily seen that every Γ-atlas is contained in a unique maximal Γ-atlas. Thus, a given Γ-atlas defines uniquely a *Γ-structure* on M. The relation between transitive pseudogroups and G-structure should now be evident from Proposition 1.3.3.

Proposition 1.4.3: *Let G be a Lie subgroup of $GL(n, \mathbb{R})$, and let Γ_G be the pseudogroup of local diffeomorphisms of $\mathbb{R}^n$ whose Jacobian matrix lies in G at each point of its domain. Then the Γ_G-structures on M are in $1-1$ correspondence with the integrable G-structures on M.*

It should be clear that the pseudogroup Γ_G is transitive since it contains the translations. Notice that not every Γ-structure is a Γ_G-structure for some $G \subset GL(n, \mathbb{R})$. We will see an important example of this later. Of course, as with integrable G-structures, or just G-structures, for a given Γ there may not exist any Γ-structure at all on M. That is, generally, there are topological obstructions to the existence of a Γ-structure. Two important Γ_G-structures are for $G = GL(n, \mathbb{C})$ and $G = Sp(\frac{n}{2}, \mathbb{R})$. It is clear from the two examples 1.3.6 and 1.3.9 that the $\Gamma_{GL(n,\mathbb{C})}$-structures are just the complex structures, while the $\Gamma_{Sp(\frac{n}{2},\mathbb{R})}$-structures are the symplectic structures. The example of the six sphere S^6 is interesting in this context. As mentioned in the paragraph after Example 1.3.6 S^6 is known to admit an almost complex structure, i.e. a $GL(3, \mathbb{C})$-structure, but it is not known whether it has a complex structure, i.e. a $\Gamma_{GL(3,\mathbb{C})}$-structure. From the symplectic point of view, S^6 also admits an almost symplectic structure, i.e. an $Sp(3, \mathbb{R})$-structure, since it admits an almost complex structure; however, it cannot admit a symplectic structure, i.e. a $\Gamma_{Sp(3,\mathbb{R})}$-structure, since the closed 2-form ω would provide a non-vanishing element of $H^2(S^6,\mathbb{R})$ by Stokes Theorem.

Following [**191**] we relate pseudogroups to collections of vector fields. Let $\{X_\alpha\}$ be a collection of smooth vector fields each defined on an open set $U_\alpha \subset M$. Each X_α generates a local1-parameter group $\phi_\alpha(t)$ of local diffeomorphisms defined for small t and for some open subset of U_α. The collection $\{\phi_\alpha\}$ defines a family of local diffeomorphisms and thus generates a pseudogroup $\mathcal{P}$ on M. Fix a point $x \in M$ and consider the subpseudogroup $\mathcal{P}_0$ of $\mathcal{P}$ generated by those vector fields in the collection that vanish at x. Then any local 1-parameter group $\phi_\alpha(t) \in \mathcal{P}_0$ satisfies $\phi_\alpha(t)(x) = x$ for all t in its domain. The differential $\phi_\alpha(t)_*$ of any such local 1-parameter group is a linear map on the vector space T_xM. Running through all such local 1-parameter groups in $\mathcal{P}_0$, determines a Lie subalgebra $\mathfrak{g}_0$ of $gl(n,\mathbb{R})$. We call $\mathfrak{g}_0$ the *linear isotropy algebra* of the pseudogroup $\mathcal{P}$. This definition differs a bit from the one in [**191**], but they should be equivalent. The connected subgroup G_0 of $GL(n,\mathbb{R})$ determined by the Lie algebra $\mathfrak{g}_0$ is called the *linear isotropy group* of $\mathcal{P}$.

1.5. Foliations

The theory of foliations is very well developed, there being many books on the subject [**58, ?, ?, ?, 160, 180, ?, 210**]. The modern theory began with the work of Ehresmann and Reeb in the 1940's, but as with much in mathematics its roots go back to the 19th century. It is the purpose of this section to gather together those well known facts concerning foliations that we shall need in the sequel. Much of this material can be found in the books of Reinhart [**180**], Molino [**160**] and Tondeur [**210**].

Examples

We give here several examples of foliations which should provide insight later on. Only those examples which illustrate concepts pertinent to our development are given.

EXAMPLE 1.5.1: **Submersions** As mentioned previously a submersion $f : M \longrightarrow N$, where M and N are smooth manifolds, is a very special case of a foliation, called a **simple foliation**. A fibre bundle is a special case of a submersion; however, not every submersion satisfies the local triviality condition. If one removes a point from a fibre of a fibre bundle, one still has a submersion, but local triviality fails. For a more interesting examples of submersions that are not locally trivial see [**58**] pages 6-7. The Ehresmann Fibration Theorem states that if f is proper then a submersion is a fibre bundle.

The next example illustrates phenomena that will reoccur in our work.

EXAMPLE 1.5.2: **Linear flows on tori**. For simplicity we consider the two dimensional torus T^2 defined as the quotient space by the integer lattice, viz. $\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z})$. A linear flow is the flow generated by the vector fields

$$X = a^1\frac{\partial}{\partial x^1} + a^2\frac{\partial}{\partial x^2}$$

where $a^i \in \mathbb{R}$ for $i = 1, 2$ are not both zero. For each such pair $(a^1, a^2) \in \mathbb{R}^2 - \{(0, 0)\}$ the vector field X generates a subbundle E of TT^2, and thus defines a 1-dimensional foliation $\mathcal{F}$ on T^2. (Frobenius integrability is automatic for 1-dimensional distributions). The leaf passing through the point $(x_0^1, x_0^2) \in T^2$ is given by the image of the flow $\phi : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by

$$\phi(t) = (x_0^1 + a^1 t, x_0^2 + a^2 t).$$

We assume that $a^1 \neq 0$. There are two cases to consider:

(i) The ratio $\frac{a^2}{a^1}$ is rational. In this case the vector field X is periodic and the leaves of the foliation are circles. The foliation $\mathcal{F}$ is simple, and in fact describes T^2 as an S^1 bundle over S^1.

(ii) The ratio $\frac{a^2}{a^1}$ is irrational. In this case the leaves are diffeomorphic to $\mathbb{R}$, and each leaf is dense in T^2. Hence, the topology of the quotient space $T^2/\mathcal{F}$ is not even T_1.

EXERCISE 1.8: Prove the assertions (i) and (ii) of Example 1.5.2.

Our next example generalizes a well known result in Lie theory.

EXAMPLE 1.5.3: **A locally free action of a Lie group**. Let G be a Lie group which acts locally freely on a smooth manifold M, that is there is a smooth map $G \times M \longrightarrow M$ such that the isotropy subgroup at any point is a discrete subgroup of G. Let $\mathfrak{g}$ denote the Lie algebra of G. The action induces a monomorphism $\psi : \mathfrak{g} \longrightarrow \Gamma(TM)$ of Lie algebras, and since this action is locally free the image $\psi(\mathfrak{g})$ generates a subbundle E of TM of dimension (dim $\mathfrak{g}$). Furthermore, since $\psi(\mathfrak{g})$ is subalgebra of $\Phi(TM)$, the subbundle E is integrable and defines a foliation $\mathcal{F}$ on M. In the next chapter we show that if the action of G is also proper and there is a Riemannian metric on M with isometry group G, then the space of leaves M/G is an orbifold such that the natural projection $\pi : M \longrightarrow M/G$ is an orbifold Riemannian submersion. This generalizes to the case of locally free action the well known theorem of Lie theory that says that if G acts properly and freely on M, then the the space of leaves of $\mathcal{F}$, i.e. the orbit space M/G of the action is a smooth manifold such that M is the total space of a principal G-bundle over M/G.

Leaf Holonomy and the Holonomy Groupoid

In this subsection we describe an important invariant of foliations, the holonomy groupoid. We begin by describing the leaf holonomy of a given leaf, a concept which generalizes the classical 'first return map' of Poincaré from dynamical systems theory. Let $(M, \mathcal{F})$ be a foliated manifold with dimension of M equal to n, and the dimension of $\mathcal{F}$ equal to p. A submanifold $N \overset{\iota}{\hookrightarrow} M$ is called a *transversal* or *transverse submanifold* if at each point $x \in N$ we have

$$T_{\iota(x)}M = T\mathcal{L}_x + T_xN.$$

In order to the construct leaf holonomy, we now describe, following [**160**], a procedure known as "sliding along the leaves". Consider two points x_0 and x_0' lying on the same leaf $\mathcal{L}_0$. Choose transversals N and N' through x_0 and x_0', respectively, and a piecewise smooth path $\gamma : [0,1] \longrightarrow \mathcal{L}_0$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_0'$. Subdivide the interval $[0,1]$ into subintervals $[t_{\alpha-1}, t_\alpha]$ with $0 = t_0 < t_1 < \cdots < t_k = 1$ such that each $\gamma([t_{\alpha-1}, t_\alpha])$ lies in a simple open set U_α of a foliated atlas. Moreover, since $\gamma([t_{\alpha-1}, t_\alpha])$ is connected it lies on a unique plaque of $\mathcal{L}_0 \cap U_\alpha$. Let N_α be transversals at $\gamma(t_\alpha)$. Then since $\gamma(t_{\alpha-1})$ lies on a single plaque in $U_{\alpha-1} \cap U_\alpha$, for each $\alpha = 1, \cdots, k$ there is a diffeomorphism ϕ_α from an open neighborhood $V_{i-1} \subset N_{\alpha-1}$ about $\gamma(t_{\alpha-1})$ onto an open neighborhood $V_\alpha \subset N_\alpha$ about $\gamma(t_\alpha)$

which is constructed from the local submersions $f_\alpha : U_\alpha \longrightarrow \mathbb{R}^q$ by the commutative diagram

$$
\begin{tikzcd}
V_{\alpha-1} \arrow[rr, "\phi_\alpha"] \arrow[dr] & & V_\alpha \arrow[dl] \\
& \mathbb{R}^q &
\end{tikzcd}
\tag{1.5.1}
$$

where the diagonal arrow are $f_{\alpha-1}|V_{\alpha-1}$ and $f_\alpha|V_\alpha$, respectively. By looking at the local foliated coordinate charts $(U_{\alpha-1}, x^1_{\alpha-1}, \cdots, x^p_{\alpha-1}, y^1_{\alpha-1}, \cdots, y^q_{\alpha-1})$ and $(U_\alpha, x^1_\alpha, \cdots, x^p_\alpha, y^1_\alpha, \cdots, y^q_\alpha)$ one can easily identify the coordinate representative of ϕ_α with the Haefliger cocycle $\tau_{(\alpha-1)\alpha}$. It follows that the composition $\phi_{k1} = \phi_k \circ \cdots \circ \phi_1 : V_0 \longrightarrow V_k$ is a diffeomorphism that depends on the subdivision and the intermediate transversals only through their domains. It does, however, depend on the choice of foliated atlas, and the transversals at x_0 and x_0'. The set of all local diffeomorphisms obtained in this way form a pseudogroup $\Gamma_{\mathcal{U},N,N'}$, called the *holonomy pseudogroup* of $\mathcal{F}$. This is a bit of a misnomer; however, a different choice of foliated atlas, and different choices of transversals give pseudogroups that are locally conjugate, so it makes sense to speak of 'the' holonomy pseudogroup.

Since the diffeomorphism ϕ_{k1} described above depends on the path, the subdivision, and intermediate transversals through their domains, it makes sense to pass to germs. We denote the germ of the diffeomorphism ϕ_{k1} at x_0 by h_γ. This is now independent of the subdivision and transversals. Notice also that if γ' is another path with the same endpoints and $\Gamma : [0,1] \times [0,1] \longrightarrow \cup_\alpha U_\alpha$ is a homotopy between γ and γ', then h_γ only depends on the homotopy class $[\gamma]$ of γ (cf. [**58**]). Now suppose that $x_0' = x_0$ in which case we can take $N' = N$. Then h_γ defines the germ of a local diffeomorphism of N which leaves x_0 fixed. We let $\mathcal{G}^{x_0}_{x_0}$ denote the set of all such germs obtained by sliding along leaves. (The notation will be clear shortly). This set forms a group under composition. Notice that if γ and γ' are two loops at x_0, then $h_{\gamma' \circ \gamma} = h_{\gamma'} \circ h_\gamma$, so one gets a group epimorphism

$$h_{x_0} : \pi_1(\mathcal{L}_0, x_0) \longrightarrow \mathcal{G}^{x_0}_{x_0} \hookrightarrow \mathfrak{Diff}_{x_0} \tag{1.5.2}$$

where $\mathfrak{Diff}_{x_0}$ denotes the group of germs of diffeomorphisms of N that fix x_0. It can be identified with the group of germs of local diffeomorphisms of $\mathbb{R}^q$ which fix the origin.

Definition 1.5.4: *The group $\mathcal{G}^{x_0}_{x_0}$ is called the* **holonomy group** *of the leaf $\mathcal{L}_0$ at x_0.*

If a different tranversal N' is chosen at x_0, then one can identify the germ of the transversal N' with the germ of the transversal N. So the leaf holonomy group is well defined up to this identification. Furthermore, if $x_1 \in \mathcal{L}_0$ is another point of the same leaf, any path $\gamma : [0,1] \longrightarrow \mathcal{L}_0$ satisfying $\gamma(0) = x_0$ and $\gamma(1) = x_1$ induces an isomorphism

$$\gamma^* : \mathcal{G}^{x_0}_{x_0} \longrightarrow \mathcal{G}^{x_1}_{x_1},$$

where $\gamma^*(h_{x_0}) = h_\gamma \circ h_{x_0} \circ h_{\gamma^{-1}}$. Thus, it makes sense to talk about the *holonomy group* $\mathrm{Hol}(\mathcal{L})$ of a leaf $\mathcal{L}$. A simple but important fact is

Proposition 1.5.5: *Let $\mathcal{F}$ be a simple foliation. Then $Hol(\mathcal{L})$ is trivial for all leaves $\mathcal{L}$ of $\mathcal{F}$.*

Proof. Since the foliation is simple the local submersions in diagram 1.5.1 are the restrictions of a global submersion f. Thus, after idenifying $V_{\alpha-1}$ with V_α by translation along the leaves, the map ϕ_α in 1.5.1 is the identity, so sliding along the leaves produces the germ of the identity diffeomorphism. $\square$

The kernel of the homomorphism h_x of Equation 1.5.2 is a normal subgroup of the fundamental group of the leaf called the *holonomy kernel*, and denote by $\mathrm{Holker}(\mathcal{L})$. We consider several examples regarding holonomy. The first is a continuation of Example 1.5.2.

EXAMPLE 1.5.6: **Linear flows on tori.** Again for simplicity we consider only T^2. In both the irrational and rational cases the holonomy groups are all trivial, but for different reasons. In the rational case the foliation is simple, so the holonomy is trivial. In the irrational case the leaves are simply connected, so the holonomy is again trivial. However, in this latter case the holonomy pseudogroup is nontrivial.

The next example is more interesting and plays an important role in the sequel.

EXAMPLE 1.5.7: **Linear flows on odd dimensional spheres.** Consider the unit sphere in $\mathbb{C}^{n+1}$ defined by

$$S^{2n+1} = \{\mathbf{z} = (z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \cdots + |z_n|^2 = 1\}.$$

By linear flow we mean linear in the complex Cartesian coordinates $\mathbf{z} = (z_0, \cdots, z_n)$. Again for simplicity we consider the case $n = 1$. Let w_0 and w_1 be two non-zero real numbers, and consider the real vector field on $\mathbb{C}^2$

$$\xi = i\big(w_0 z_0 \frac{\partial}{\partial z_0} + w_1 z_1 \frac{\partial}{\partial z_1} - c.c\big).$$

Restricted to S^3 this vector field is everywhere tangent to S^3, and defines a nowhere vanishing vector field on S^3. Thus, ξ generates a 1-dimensional foliation on S^3 whose flow is given by

$$(z_0, z_1) \mapsto (e^{2\pi i w_0 t} z_0, e^{2\pi i w_1 t} z_1).$$

Again there are two cases to consider depending on whether the ratio $\frac{w_1}{w_0}$ is rational or irrational. If the ratio is rational by a reparameterization and considering the complex conjugate coordinates if necessary, we can take the pair (w_0, w_1) to be coprime positive integers. In this case all the leaves are circles. The leaf through $x_0 = (1, 0)$ is a circle which returns to x_0 at $t = \frac{1}{w_0}$, whereas, the nearby leaves with $z_1 \neq 0$ take w_0 times as long to return. So the holonomy group $\mathrm{Hol}(\mathcal{L}_{x_0}) \approx \mathbb{Z}_{w_0}$. Similarly we see for $x_1 = (0, 1)$ the holonomy through the leaf $\mathcal{L}_{x_1}$ is $\mathrm{Hol}(\mathcal{L}_{x_1}) \approx \mathbb{Z}_{w_1}$. We also see that $\mathrm{Holker}(\mathcal{L}_i) \approx w_i\mathbb{Z}$. All other leaves have trivial holonomy.

Now consider the case when the ratio $\frac{w_1}{w_0}$ is irrational. Consider the leaf through a point with $z_0 z_1 \neq 0$, say the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. This leaf is non-compact; in fact, it is the irrational flow on the torus T^2 in S^3 defined by $|z_0|^2 = |z_1|^2 = \frac{1}{2}$. So this leaf extends for all values of $t \in \mathbb{R}$, and is not closed in S^3. Its closure is just the 2-torus T^2. The same phenomenon occurs for any point $(z_0, z_1) \in S^3$ with $z_0 z_1 \neq 0$. However, the two leaves through $x_0 = (1, 0)$ and $x_1 = (0, 1)$ are circles for they return after a time $t = \frac{1}{w_0}$ and $t = \frac{1}{w_1}$, respectively.

We are now ready for [**180, 214**]:

Proposition 1.5.8 (Reinhart/Winkelnkemper)**:** *Let $(M, \mathcal{F})$ be a foliated manifold. The collection of all triples $(x, y, [\gamma])$ where $x, y \in M$ lie on the same leaf $\mathcal{L}$, γ is a piecewise smooth path from x to y, and $[\gamma]$ denotes the holonomy equivalence class of γ, i.e. $\gamma \sim \gamma'$ if $\gamma'\gamma^{-1} = e \in \mathcal{G}^y_y$, is a Lie groupoid of dimension $n + p$ called* *the* **holonomy groupoid**[4] *of* $(M, \mathcal{F})$ *and denoted by* $\mathcal{G} = \mathcal{G}(M, \mathcal{F})$. *Furthermore, if every holonomy germ is conjugate to an analytic germ,* $\mathcal{G}(M, \mathcal{F})$ *is Hausdorff.*

PROOF. In Appendix A.1 we give a brief review of groupoids. Clearly, the set of objects G_0 is M and this is a smooth Hausdorff manifold, while G_1 is the set of triples $\{(x, y, [\gamma])\}$. We show that G_1 is a smooth manifold by following Connes [**180**] and exhibiting a system of coordinates on it. We then check that the structure maps are smooth. Write the coordinates as (v, w) and (v', w) having the same transverse coordinate w at the beginning and end of a path, respectively. Here v and v' are coordinates along the leaf. Then (v', v, w) is a coordinate system on G_1, and a change of coordinates is of the form

$$F(v', v, w) = (f'(v', w), f(v, w), g(w))$$

where $(v, w) \mapsto (f(v, w), g(w))$ is a change of coordinates on M. This gives G_1 a locally Euclidean topology of dimension $n + p$ that is not necessarily Hausdorff. This topology is also 2nd countable (see [**180**], pg 137). Moreover, G_1 is Hausdorff if and only if each holonomy germ at $x \in M$ which is the identity on some open set whose closure contains x is the identity germ at x [**180, 214**]. In particular, if every holonomy germ is conjugate to an analytic germ, the groupoid $\mathcal{G}(M, \mathcal{F})$ will be Hausdorff.

The canonical source and target maps $s : \mathcal{G} \longrightarrow M$ and $t : \mathcal{G} \longrightarrow M$, are defined by

$$s(x, y, [\gamma]) = x, \qquad t(x, y, [\gamma]) = y, \tag{1.5.3}$$

while the unit is $u(x) = (x, x, 0)$ where 0 denotes the constant loop. Furthermore, the associative multiplication on $\mathcal{G}$ is defined when the range of one element coincides with the source of the other, specifically in our case we have

$$(x, y, [\gamma]) \cdot (x', y', [\gamma']) = (x, y', [\gamma \circ \gamma']) \quad \text{if } y = x' \tag{1.5.4}$$

The inverse map is defined by $\iota(x, y, [\gamma]) = (y, x, [\gamma^{-1}])$. To see that the structure maps are smooth, we write in coordinates:

$$s(v', v, w) = (v, w), \qquad t(v', v, w) = (v', w), \qquad u(v, w) = (v, v, w),$$
$$\iota(v', v, w) = (v, v', w), \qquad (v'', v', w) \cdot (v', v, w) = (v'', v, w).$$

So the structure maps are smooth. Furthermore, both s and t are submersions. This completes the proof. $\square$

The fact that the holonomy groupoid $\mathcal{G}(M, \mathcal{F})$ of a foliation is, in general, not Hausdorff is of very little consequence for us, since it it known to be Hausdorff when the foliation $\mathcal{F}$ is Riemannian (see Definition 1.5.15 and Proposition 1.5.20 below) which is the case of interest to us. Notice that $\mathcal{F}$ induces a foliation of dimension $2p$ on $\mathcal{G}(M, \mathcal{F})$. Continue later.

Haefliger [**?**] has defined another important groupoid, closely related to, but different from $\mathcal{G}(M, \mathcal{F})$. which he called the transverse holonomy groupoid. Here is the definition.

Definition 1.5.9: *Let N be a complete transversal. The* **transverse holonomy groupoid** *associated to N is the subgroupoid $\mathcal{G}_N^N$ of $\mathcal{G}(M, \mathcal{F})$.*

[4] This is called the *graph* of the foliation in [**214**].

Here complete means that N intersects each leaf at least once. Complete transversals always exist since we do not require them to be connected. Of course, $\mathcal{G}_N^N$ depends on N, but Haefliger [?] shows that $\mathcal{G}_N^N$ is independent of N up to Morita equivalence. In fact

Theorem 1.5.10 (Haefliger)**:** *Let $\mathcal{F}$ be a p-dimensional foliation on a manifold of dimension $p+q$, and let N be any complete transversal. Then the transverse holonomy groupoid $\mathcal{G}_N^N$ is an étale Lie groupoid of dimension q that is Morita equivalent to the holonomy groupoid $\mathcal{G}(M, \mathcal{F})$.*

Proof. $\mathcal{G}^N = t^{-1}(N)$ is a submanifold of G_1 such that

$$\text{codim}(\mathcal{G}^N \hookrightarrow G_1) = \text{codim}(N \hookrightarrow G_0) = p.$$

Since G_1 has dimension $2p+q$ we see that dim $\mathcal{G}^N = p+q =$ dim G_0. Now consider the source map $s_N = s|_{\mathcal{G}^N}$ restricted to $\mathcal{G}^N$. If we show that this map $s_N : \mathcal{G}^N \longrightarrow G_0$ is an immersion, it will be a local diffeomorphism. Then $\mathcal{G}_N^N = s_N^{-1}(N)$ will be a submanifold of $\mathcal{G}^N$ of dimension q. So the restriction of s_N to $\mathcal{G}_N^N$ will be a local diffeomorphism. Thus, $\mathcal{G}_N^N$ will be étale. To see that the differential ds_n is injective at a morphism $(g : x \longrightarrow y) \in \mathcal{G}^N \subset G_1$, we consider a tangent vector $v \in T_v G_1$. It will be tangent to $\mathcal{G}^N$ if $(dt)_g(v) \in T_y N$. Since $s : G_1 \longrightarrow G_0$ is a submersion, it defines a p-dimensional foliation $\tilde{\mathcal{F}}$ on G_1 whose leaves are the connected components of the fibres of the submersion. Thus, if $ds_N(v) = 0$ with v tangent to $\mathcal{G}^N$, then v is both tangent to the fibres of the source submersion and $dt_g(v)$ is tangent to N. But $(dt)_g$ maps the tangent space to the leaf $\tilde{\mathcal{L}}_g$ of $\tilde{\mathcal{F}}$ at g to the tangent space to the leaf $\mathcal{L}_y$ of $\mathcal{F}$ at y. Thus, $v = 0$ which proves the injectivity of ds_N, or equivalently, that s_N is a local diffeomorphism.

To prove the last statement we consider the smooth functor $F : \mathcal{G}_N^N \longrightarrow \mathcal{G}(M, \mathcal{F})$ defined by $F(x, y, [g]) = (\iota_N(x), \iota_N(y), [g])$. We see from the fact that the transversal N is complete that s_N is surjective. This shows that the map $s \circ \pi_2 : N \times_{G_0} G_1 \longrightarrow G_0$ on pairs (x, g) satisfying $\iota_N(x) = t(g)$ is surjective. Moreover, one easily checks that corresponding square is a pullback. Thus, $F : \mathcal{G}_N^N \longrightarrow \mathcal{G}(M, \mathcal{F})$ is an essential equivalence, and hence $\mathcal{G}_N^N$ and $\mathcal{G}(M, \mathcal{F})$ are Morita equivalent. $\square$

In particular, Theorem **??** says that the étale groupoid $\mathcal{G}_N^N$ is independent of N up to Morita equivalence. We shall see later that Morita equivalent groupoids have homotopy equivalent classifying spaces. We will consider $\mathcal{G}_N^N$ further in the chapter 2.

We are most interested in the case where the foliation arises from a locally free action of a Lie group 2.1.9. So we recall the *action groupoid* from the Appendix (Definition A.1.5). For G a Lie group and M a smooth manifold, we consider the action groupoid $M \rtimes G$. According to Rinehart (book pg 138, no proof) the holonomy groupoid of a foliation coming from a locally free action of a Lie group G is just $M \times G$. This is clear when the action is free, but generally it is hard to see how to handle the isotropy. If one can measure distances I see how to do it, but not generally.

As in that example let G act locally freely on M, and assume for simplicity that G is compact. In this case the isotropy subgroups will be finite.

Basic Cohomology

We begin with a brief review of the basic cohomology of a foliation following [**210**]. Let $\mathcal{F}$ be a foliation of a smooth manifold M. A differential r-form ω on M is said to be *basic* if for all vector fields V on M that are tangent to the leaves of $\mathcal{F}$ the following conditions hold:

$$V \rfloor \omega = 0, \qquad \pounds_V \omega = 0. \tag{1.5.5}$$

In a local foliated coordinate chart (U, ϕ) with $\phi = (x^1, \cdots, x^p; y^1, \cdots, y^q)$ a basic r-form ω takes the form

$$\omega = \sum \omega_{i_1 \cdots i_r}(y^1, \cdots, y^q) dy^{i_1} \wedge \cdots \wedge dy^{i_r}, \tag{1.5.6}$$

where the sum is taken over all repeated indices.

Let $\mathit{\Omega}^r_B(\mathcal{F})$ denote sheaf of germs of basic r-forms on M, and $\Omega^r_B(\mathcal{F})$ the set of its global sections. The direct sum $\Omega_B(\mathcal{F}) = \oplus_r \Omega^r_B(\mathcal{F})$ is closed under addition and exterior multiplication, and so is a subalgebra of the algebra of exterior differential forms on M. Furthermore, exterior derivation takes basic forms to basic forms, that is,

$$\pounds_V d\omega = d\pounds_V \omega = 0, \qquad \text{and} \qquad V \rfloor d\omega = \pounds_V \omega - d(V \rfloor \omega) = 0. \tag{1.5.7}$$

Thus, the subalgebra $\Omega_B(\mathcal{F})$ forms a subcomplex of the de Rham complex, and its cohomology ring $H^*_B(\mathcal{F}) = H^*_B(\mathcal{F}, d_B)$ is called the *basic cohomology ring* of $\mathcal{F}$. Here d_B denotes the restriction of the exterior derivative d to the basic forms $\Omega_B(\mathcal{F})$.

The groups $H^r_B(\mathcal{F})$ are defined for all $0 \leq r \leq q$ where q is the codimension of the foliation, but generally they may be infinite dimensional for $r \geq 2$. However, they are finite dimensional for Riemannian foliations on compact manifolds which is mainly the case of interest to us. Let us look at the groups $H^r_B(\mathcal{F})$ for $r = 0, 1$. The set $\Omega^0_B(\mathcal{F})$ is just the set of functions that are constant along the leaves of $\mathcal{F}$, so the cocyles with respect to d_B are just the constant functions. Thus, if M is connected $H^0_B(\mathcal{F}) \approx \mathbb{R}$. Let us now consider $H^1_B(\mathcal{F})$. We claim that $H^1_B(\mathcal{F})$ injects into $H^1(M, \mathbb{R})$ induced by the natural inclusion $\Omega_B(\mathcal{F}) \longrightarrow \Omega(M)$. Let ω be a 1-cocycle, i.e. $d_B \omega = 0$ where ω is a smooth section of $\mathit{\Omega}^1_B(\mathcal{F})$. So if $\omega = df$ for some smooth function on M, then we have $Vf = V \rfloor df = V \rfloor d\omega = 0$ for any vector field V tangent to the leaves of $\mathcal{F}$. But this says that $f \in \Omega^1_B(\mathcal{F})$, so ω is a coboundary with respect to d_B. This proves the claim. We have arrived at

Proposition 1.5.11: *For any foliation $\mathcal{F}$ on a connected manifold M,*

(i) $H^0_B(\mathcal{F}) \approx H^0(M, \mathbb{R}) \approx \mathbb{R}$.
(ii) $H^1_B(\mathcal{F})$ *is a subgroup of* $H^1(M, \mathbb{R})$.

The appropriate notion of basic for vector fields is

Definition 1.5.12: *A vector field X on a foliated manifold $(M, \mathcal{F})$ is said to be* **foliate** *with respect to $\mathcal{F}$ if for every vector field V tangent to the leaves of $\mathcal{F}$, the Lie bracket $[X, V]$ is tangent to the leaves of $\mathcal{F}$.*

The set of foliate vector fields forms a Lie subalgebra of the Lie algebra of all vector fields on M. It is the normalizer of the subalgebra of the vector fields tangent to the leaves of $\mathcal{F}$. Equivalently, its local one parameter group preserves the foliation. In terms of local foliated coordinates $(x^1, \cdots, x^p; y^1, \cdots, y^q)$ a foliate vector field X takes the form

$$X = \sum_{a=1}^{q} A^a(y^1, \cdots, y^q)\frac{\partial}{\partial y^a} + \sum_{i=1}^{p} B^i(x^1, \cdots, x^p; y^1, \cdots, y^q)\frac{\partial}{\partial x^i}, \tag{1.5.8}$$

where A and B are smooth functions of the indicated variables.

Transverse Geometry

We are also interested in foliations with a 'transverse structure'. These were introduced by Conlon [**?**] and further developed by Molino [**?, 160**] in a slightly different guise. Here we follow [**160**], but our notation is somewhat different. Let $\mathcal{F}$ be a p-dimensional foliation on the manifold M with integrable subbundle E. There is an exact sequence of vector bundles, viz.

$$0 \longrightarrow E \longrightarrow TM \xrightarrow{\pi_Q} Q \longrightarrow 0. \tag{1.5.9}$$

The quotient bundle Q is called the *normal bundle* of the foliation. A *transverse frame* at the point $x \in M$ is a basis $(Y_1, \cdots, Y_q)$ for the fibre Q_x of Q at x. (Y_i is not a tangent vector at x, but an equivalence class of tangent vectors modulo vectors in E_x.) The set $L_T(M, \mathcal{F})$ of all transverse frames at all points of M forms a principal $GL(q, \mathbb{R})$ bundle on M, called the *transverse frame bundle* on $(M, \mathcal{F})$. Let $G \subset GL(q, \mathbb{R})$ be a Lie subgroup and let $\pi_T : P_T(M, G, \mathcal{F}) \longrightarrow M$ denote the corresponding principal G subbundle of $L_T(M, \mathcal{F})$. As with the ordinary frame bundle a point $z \in L_T(M, \mathcal{F})$ can be viewed as the linear map $z : \mathbb{R}^q \longrightarrow Q_{\pi_T(z)}$ that assigns to the standard basis of $\mathbb{R}^q$ the frame $z = (Y_1, \cdots, Y_q)$. Also as in the usual case $L_T(M, \mathcal{F})$ and its subbundles $P_T(M, G, \mathcal{F})$ have a canonical $\mathbb{R}^q$-valued 1-form θ_T defined by

$$< \theta_T, X >= z^{-1}\pi_Q \pi_{T*} X. \tag{1.5.10}$$

The foliation $\mathcal{F}$ on M lifts to a foliation $\mathcal{F}_T$ on $L_T(M, \mathcal{F})$ as follows: we define a foliation on $L_T(M, \mathcal{F})$ by the p-dimensional subbundle $E_T \subset TL_T(M, \mathcal{F})$ generated by the vectors in $T_z L_T(M, \mathcal{F})$ that satisfy

$$E_{T_z} = \{X_z \in T_z L_T(M, \mathcal{F}) \mid X_z \rfloor \theta_T = X_z \rfloor d\theta_T = 0 \text{ for all } z \in L_T(M, \mathcal{F})\}. \tag{1.5.11}$$

We now have

Proposition 1.5.13: *Let $(M, \mathcal{F})$ be a foliated manifold and let $L_T(M, \mathcal{F})$ be its transverse frame bundle. The subbundle E_T defined by equation 1.5.11 is integrable, and thus defines a foliation $\mathcal{F}_T$ on $L_T(M, \mathcal{F})$. The leaves of this foliation project under π_T to the leaves of the foliation $\mathcal{F}$.*

PROOF. Let X, Y be vector fields on $L_T(M, \mathcal{F})$ that are sections of the subbundle E_T. Then

$$0 = d\theta_T(X, Y) = X\theta_T(Y) - Y\theta_T(X) - \theta_T([X, Y]) = -\theta_T([X, Y]),$$

so $[X, Y] \rfloor \theta_T = 0$. But also we have for any section X of E_T that

$$\pounds_X \theta_T = d(X \rfloor \theta_T) + X \rfloor d\theta_T = 0.$$

Thus,

$$0 = [\pounds_X, \pounds_Y]\theta_T = \pounds_{[X,Y]}\theta_T = d([X, Y] \rfloor \theta_T) + [X, Y] \rfloor d\theta_T = [X, Y] \rfloor d\theta_T$$

which implies that E_T is integrable by Frobenius' Theorem.

Now suppose that $\mathcal{L}_T$ is a leaf of $\mathcal{F}_T$. Let U be a simple open set of M such that the quotient map of the foliation restricted to U is a submersion $f_U : U \longrightarrow \mathbb{R}^q$.

Then on $E_T|U = \pi_T^{-1}(U)$, the canonical 1-form θ_T is just the pullback $f_U^*\theta$ where θ is the canonical 1-form on the frame bundle $L(\mathbb{R}^q)$ on $\mathbb{R}^q$. So in local coordinates $(x^1, \cdots, x^p; y^1, \cdots, y^q; A_1^1, \cdots, A_1^q, \cdots, A_q^q)$ on E_T, equation 1.5.11 implies that the tangent vectors to E_T that are tangent to the leaves of the foliation $\mathcal{F}_T$ are spanned by the $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial A_j^i}$. Thus, these vectors project under π_T to vectors that are tangent to the leaves of the foliation $\mathcal{F}$ on M. But since π_T is surjective this implies that the leaf $\mathcal{L}_T$ of $\mathcal{F}_T$ projects to a leaf of $\mathcal{F}$. $\square$

The foliation $\mathcal{F}_T$ is called the *lifted foliation.* We are now ready for

Definition 1.5.14: *Let G be a Lie subgroup of $GL(q, \mathbb{R})$, and let $P_T(M, G, \mathcal{F})$ be a principal G subbundle of $L_T(M, \mathcal{F})$. We say that $P_T(M, G, \mathcal{F})$ is a* **transverse G-structure** *if at each point $z \in P_T(M, G, \mathcal{F})$ the tangent space $T_zP_T(M, G, \mathcal{F})$ contains the subspace E_{T_z} tangent to the leaves of the lifted foliation $\mathcal{F}_T$.*

Important examples for us will be when $G = O(q, \mathbb{R})$, $G = GL(m, \mathbb{C})$, and $G = U(m)$ where $q = 2m$. These correspond to tranverse Riemannian structures, transverse almost complex structures, and transverse almost Hermitian structures, respectively. The following result [**160**] gives a characterization of transverse G-structures in terms of the local submersions.

Proposition 1.5.15: *Let $P_T(M, G, \mathcal{F})$ be a principal G subbundle of $L_T(M, \mathcal{F})$. Then $P_T(M, G, \mathcal{F})$ is a transverse G-structure if and only if for every simple open set U, the bundle $P_T(M, G, \mathcal{F})|_U$ is the pullback by $\pi_T : L_T(U, \mathcal{F}|_U) \longrightarrow L(\mathbb{R}^q)$ of a G-structure $P(\mathbb{R}^q, G)$ on $\mathbb{R}^q$.*

PROOF. The argument is purely local. $\square$

Transverse Riemannian Geometry

Definition 1.5.16: *A* **Riemannian foliation** *is a foliation with a transverse Riemannian structure, i.e. a transverse G-structure with $G = O(q, \mathbb{R})$.*

A Riemannian foliation $(M, \mathcal{F})$ defines a *transverse Riemannian metric* g_T on M by

$$g_T(X, Y) = < z^{-1}\pi_Q X, z^{-1}\pi_Q Y > \tag{1.5.12}$$

where $X, Y \in TM_{\pi_T(z)}$ and π_Q is the projection in the exact sequence 1.5.9. Notice that g_T defines a nonnegative symmetric bilinear form on TM, and viewed as a section of $\mathrm{Sym}^2 TM$ has kernel $E \otimes TM + TM \otimes E$, and thus defines a Riemannian metric on the quotient bundle Q.

Proposition 1.5.17: *Let $\mathcal{F}$ be a codimension q foliation on M. There is a 1-1 correspondence between transverse $O(q, \mathbb{R})$-structures on M and transverse Riemannian metrics on M.*

The second definition is that of a bundle-like metric which is due to Reinhart [**179**].

Definition 1.5.18: *A Riemannian metric g is said to be* **bundle-like** *with respect to a foliation $\mathcal{F}$ if for any foliate horizontal vector fields X, Y, the function $g(X, Y)$ is basic, i.e. for any vector field V along the leaves of $\mathcal{F}$ the equation $Vg(X, Y) = 0$.*

The following is well known [**160**]

Proposition 1.5.19: *If g is a bundle-like metric with respect to a foliation $\mathcal{F}$, then $\mathcal{F}$ is a Riemannian foliation. Conversely, if $\mathcal{F}$ is a Riemannian foliation* *with transverse metric g_T, then there exist bundle-like metrics g whose associated transverse metric is g_T.*

PROOF. Let g be a bundle-like metric on M. This metric splits the exact sequence 1.5.9 as

$$TM = E \oplus E^{\perp}, \tag{1.5.13}$$

where $E^{\perp}$ denotes the subbundle of TM consisting of all vectors perpendicular to the vectors of E with respect to the Riemannian metric g. Thus, any vector field X on M can be decomposed as

$$X = X^{\top} + X^{\perp}$$

where $X^{\top}$ is a section of E and $X^{\perp}$ is a section of $E^{\perp}$. Define a transverse metric on $(M, \mathcal{F})$ by $g_T(X,Y) = g(X^{\perp}, Y^{\perp})$. (Finish later). □

For a Riemannian foliation on a compact manifold we have [**160**]

Theorem 1.5.20: *Let $(M, \mathcal{F})$ be a Riemannian foliation on a compact connected manifold M. Then the leaves of F all have the same universal cover.*

Concerning the holonomy groupoid we have

Proposition 1.5.21: *Let $\mathcal{F}$ be a Riemannian foliation on the manifold M. Then*

(i) *The holonomy groupoid $\mathcal{G}(M, \mathcal{F})$ of $\mathcal{F}$ is Hausdorff.*
(ii) *There is a unique Riemannian metric $g_{\mathcal{G}}$ on $\mathcal{G}(M, \mathcal{F})$ such that both s and t are Riemannian submersions.*
(iii) *The inverse map ι is an isometry with respect to $g_{\mathcal{G}}$.*

PROOF. (i): Since the transverse geometry of $\mathcal{F}$ is orthogonal, so are the leaf holonomy groups. Thus, the holonomy is linear, and hence, analytic. So the result follows by Proposition 1.5.5. □

Next we discuss the relation between the curvature of (M, g) and the curvature of the transverse metric g_T, where g is a bundle-like metric whose transverse component is g_T. Accordingly we split TM as in 1.5.13. This relationship is essentially that of O'Neill for Riemannian submersions [**169, 180, 210**]. We first define the induced connection ∇^T in the bundle $E^{\perp}$.

$$\nabla^T_X Y = \begin{cases} \pi_{E^{\perp}}(\nabla_X Y) & \text{if } X \text{ is a smooth section of } E^{\perp}, \\ \pi_{E^{\perp}}[V,Y] & \text{if } X = V \text{ is a smooth section of } E, \end{cases} \tag{1.5.14}$$

where Y is a smooth section of E. It is left as an exercise to mimic the standard proof of the first fundamental theorem of Riemannian geometry that ∇^T is the unique torsion free connection such that the transverse metric g_T is parallel, i.e. $\nabla^T_X g_T = 0$.

EXERCISE 1.9: Show that the connection ∇^T is the unique torsionfree metric connection with respect to the transverse metric g_T.

We let R^T denote the Riemannian curvature tensor field with respect to g_T. Then if Z, W are any vector fields on M, and X is a vector field in $E^{\perp}$, we have

$$R^T(Z,W)X = \nabla^T_Z \nabla^T_W X - \nabla^T_W \nabla^T_Z X - \nabla^T_{[Z,W]} \tag{1.5.15}$$

Proposition 1.5.22: *For any smooth section V of E the following hold:*

(i) $\pounds_V \nabla^T = 0$.
(ii) $V \rfloor R^T = 0$.

(iii) $\pounds_V R^T = 0$.

PROOF. (1) follows as usual from the fact that ∇^T is determined uniquely in term the transverse metric g_T and $\pounds_V g_T = 0$. (2): If W is a vector field on M and X is a smooth section of $E^\perp$ we have

$$\begin{aligned} R^T(V,W)X &= \nabla^T_V\nabla^T_W X - \nabla^T_W\nabla^T_V X - \nabla^T_{[V,W]}X \\ &= \pounds_V\nabla^T_W X - \nabla^T_W \pounds_V X - \nabla^T_{\pounds_V W}X = (\pounds_V\nabla^T)_W(X) = 0. \end{aligned}$$

(We leave the proof of (3) for later.) □

Let us fix some common notation [**22, 210**]: we let E_i denote any vector fields on M, while we let U, V, W denote vector fields in the integrable subbundle E, and X, Y, Z be vector fields in $E^\perp$. We also let $\pi : TM \longrightarrow E$, and $\pi^\perp : TM \longrightarrow E^\perp$ denote the corresponding natural projections. Following O'Neill [**169**] (see also Gray [**?**]) we introduce the type $(2,1)$ tensor field

$$T_{E_1}E_2 = \pi(\nabla_{\pi(E_1)}\pi^\perp(E_2)) + \pi^\perp(\nabla_{\pi(E_1)}\pi(E_2)). \tag{1.5.16}$$

EXERCISE 1.10: Show that T defined by 1.5.16 is a tensor field of type $(2,1)$.

Clearly, T only depends of projection of E_1 along the leaves, so

$$T_X U = T_X Y = 0. \tag{1.5.17}$$

We also have

$$T_U V = \pi^\perp(\nabla_U V), \qquad T_U X = \pi(\nabla_U X), \tag{1.5.18}$$

$$T_U V = T_V U, \tag{1.5.19}$$

$$T \text{ is anti-symmetric, i.e. } g(T_U V, X) = -g(V, T_U X). \tag{1.5.20}$$

The symmetry property Equation 1.5.19 follows by taking U and V as coordinate vector fields along the leaves of the foliation $\mathcal{F}$ (valid since T is a tensor field), and using the fact that ∇ is torsionfree, while the anti-symmetry property 1.5.20 follows since ∇ is a metric connection. Notice that by Equation 1.5.20 T vanishes if and only if $T_U V = 0$ for all vector fields tangent to the leaves of $\mathcal{F}$. Since any leaf L is an immersed submanifold of M, we see from Equation 1.5.18 that L is totally geodesic if and only if T vanishes along L. If $T_U V = 0$ for all U and V then all leaves of $\mathcal{F}$ are totally geodesic. In this case we say that the foliation $\mathcal{F}$ is a *totally geodesic foliation*. Summarizing we have

Proposition 1.5.23: *A foliation $\mathcal{F}$ is totally geodesic if and only if $T = 0$.*

Reversing the roles of the subbundles E and $E^\perp$ in Equation 1.5.16 we obtain

$$A_{E_1}E_2 = \pi(\nabla_{\pi^\perp(E_1)}\pi^\perp(E_2)) + \pi^\perp(\nabla_{\pi^\perp(E_1)}\pi(E_2)). \tag{1.5.21}$$

EXERCISE 1.11: Show that A defined by 1.5.21 is a tensor field of type $(2,1)$.

For O'Neill's A tensor field, we have the following

$$A_U V = 0, \qquad A_U X = 0, \tag{1.5.22}$$

$$A_X U = \pi^\perp(\nabla_X U), \qquad A_X Y = \pi(\nabla_X Y), \tag{1.5.23}$$

$$A_X Y = -A_Y X, \tag{1.5.24}$$

$$A \text{ is anti-symmetric, i.e. } g(A_X U, Y) = -g(U, A_X Y). \tag{1.5.25}$$

Equations 1.5.22 and 1.5.23 are obvious from the definition. Equation 1.5.24 will be proved in Proposition 1.5.23 below, while again Equation 1.5.25 follows since $\nabla g = 0$.

Proposition 1.5.24: *For any horizontal vector fields X, Y we have*

$$A_X Y = \frac{1}{2}\pi([X, Y]).$$

PROOF. We compute

$$\pi([X, Y]) = \pi(\nabla_X Y - \nabla_Y X) = A_X Y - A_Y X.$$

The result will now follow if we establish Equation 1.5.24. It suffices to show that $A_X X = 0$. Without loss of generality we may take the horizontal vector field X to be foliate. We have for arbitrary vertical U

$$\begin{aligned} g(U, A_X X) &= g(U, \pi(\nabla_X X)) = g(U, \nabla_X X) = -g(\nabla_X U, X) \\ &= -g(\nabla_U X, X) - g([X, U], X) = -\frac{1}{2} U g(X, X) = 0. \end{aligned}$$

The last expression vanishes since $g(X, X)$ is basic when X is foliate. □

Notice that Equation 1.5.25 implies that A vanishes precisely when $A_X Y$ vanishes for all horizontal vector fields X and Y, and Proposition 1.5.23 says that $A_X Y$ vanishes for all X, Y if and only if the subbundle $E^\perp$ is integrable. Next let us see how the covariant derivative on M decomposes. We let $\hat{\nabla}$ be the induced connection along the leaves. Then we have Gauss' formula for the leaves

$$\nabla_U V = \hat{\nabla}_U V + T_U V. \tag{1.5.26}$$

We also have

$$\nabla_U X = T_U X + \pi^\perp(\nabla_U X), \tag{1.5.27}$$

$$\nabla_X U = \pi(\nabla_X U) + A_X U, \tag{1.5.28}$$

$$\nabla_X Y = A_X Y + \nabla^T_X Y. \tag{1.5.29}$$

Concerning the covariant derivatives of A and T we have the following:

Lemma 1.5.25: *The following hold:*

(i) $(\nabla_U A)_V = -A_{T_U V}, \qquad (\nabla_X T)_Y = -T_{A_X Y},$
(ii) $(\nabla_X A)_V = -A_{A_X V}, \qquad (\nabla_V T)_Y = -T_{T_V Y},$
(iii) $g((\nabla_U A)_X V, W) = g(T_U V, A_X W) - g(T_U W, A_X V),$
(iv) $g((\nabla_{E_1} A)_X Y, V)$ *is anti-symmetric in* X, Y,
(v) $g((\nabla_{E_1} T)_U V, X)$ *is anti-symmetric in* U, V,
(vi) $\mathfrak{S} g((\nabla_Z A)_X Y, V) = \mathfrak{S} g(A_X Y, T_V Z)$ *where* $\mathfrak{S}$ *denotes the cyclic sum taken over* X, Y, Z.

PROOF. We establish the first identity of (2), and leave the rest as an exercise. For any vector field F we have

$$(\nabla_X A)_V F = \nabla_X (A_V F) - A_{\nabla_X V} F - A_V(\nabla_X F) = -A_{A_X V} F.$$

□

EXERCISE 1.12: Finish the proof of Lemma 1.5.24.

Theorem 1.5.26: *Let $R, \hat{R}$, and R^T denote the Riemann curvature associated to the connections $\nabla, \hat{\nabla}$, and ∇^T, respectively Then the following curvature identities hold:*

$$
\begin{aligned}
g(R(U,V)W,W') &= g(\hat{R}(U,V)W,W') + g(T_UW,T_VW') - g(T_VW,T_UW');\\
g(R(U,V)W,X) &= g((\nabla_UT_V)W,X) - g((\nabla_VT_U)W,X);\\
g(R(X,U)Y,V) &= g(T_UX,T_VY) - g((\nabla_XT_U)V,Y)\\
&\quad - g(A_XU,A_YV) - g((\nabla_UA)_XY,V);\\
g(R(U,V)X,Y) &= g((\nabla_VA)_XY,U) - g((\nabla_UA)_XY,V)\\
&\quad + g(A_XV,A_YU) - g(A_XU,A_YV)\\
&\quad + g(T_UX,T_VY) - g(T_VX,T_UY);\\
g(R(X,Y)Z,U) &= g(A_YZ,T_UX) + g(A_ZX,T_UY)\\
&\quad + g((\nabla_ZA)_XY,U) - g(A_XY,T_UZ);\\
g(R(X,Y)Z,Z') &= g(R^T(X,Y)Z,Z') + 2g(A_XY,A_ZZ')\\
&\quad + g(A_XZ,A_YZ') - g(A_XZ',A_YZ)
\end{aligned}
$$

PROOF. The first equation is just Gauss' Equation along the leaves viewed as immersed submanifolds, while the second equatin is the Codazzi equation along the leaves. We leave the proof for the other equations to the reader. □

We now have an easy corollary for the various sectional curvatures.

Corollary 1.5.27: *Let X, Y, U, V be linearly independent unit vector fields with respect to the metric g, and let $K, K^T, \hat{K}$ denote the sectional curvatures with respect to the metrics $g, g_T, g|_{\mathcal{L}}$, respectively. Then one has*

$$
\begin{aligned}
K(U,V) &= \hat{K}(U,V) + |T_UV|^2 - g(T_UU,T_VV),\\
K(X,U) &= g((\nabla_XT)_UU,X) - |T_UX|^2 + |A_XU|^2,\\
K(X,Y) &= K^T(X,Y) - 3|A_XY|^2.
\end{aligned}
$$

We would like expressions for the Ricci and scalar curvatures similar to those of Theorem 1.5.25. First we need some preliminary development which we lift from Besse [**22**]. In what follows we let $\{X_i\}_i$ and $\{U_a\}_a$ denote local orthonormal frames for the horizontal subspace $E^\perp$ and vertical subspace E, respectively. The following identities hold:

$$g(A_X,A_Y) = \sum_i g(A_XX_i,A_YX_i) = \sum_a g(A_XU_a,A_YU_a), \tag{1.5.30}$$

$$g(A_X,T_U) = \sum_i g(A_XX_i,T_UX_i) = \sum_a g(A_XU_a,T_UU_a), \tag{1.5.31}$$

$$g(AU,AV) = \sum_i g(A_{X_i}U,A_{X_i}V), \tag{1.5.32}$$

$$g(TX,TY) = \sum_a g(T_{U_a}X,T_{U_a}Y). \tag{1.5.33}$$

Next for any tensor field $\mathcal{T}$ we define the projected divergence operators,

$$\delta^T\mathcal{T} = -\sum_i (\nabla_{X_i}\mathcal{T})_{X_i}, \tag{1.5.34}$$

$$\hat{\delta}\mathcal{T} = -\sum_a (\nabla_{U_a}\mathcal{T})_{U_a}, \tag{1.5.35}$$

which satisfy $\delta = \hat{\delta} + \delta^T$ with respect to the ordinary divergence operator δ. We define the *mean curvature vector field* N along the leaves of $\mathcal{F}$ by

$$N = \sum_a T_{U_a}U_a \tag{1.5.36}$$

which vanishes if and only if each leaf is a minimal submanifold of M. We are now ready for the relations between the Ricci curvatures. We let $\mathrm{Ric}, \mathrm{Ric}_T$, and $\widehat{\mathrm{Ric}}$ denote the Ricci curvatures of g, g_T, and the family of metrics $g|_{\mathcal{L}}$, respectively.

Theorem 1.5.28: *The Ricci curvatures* $\mathrm{Ric}, \mathrm{Ric}_T$, *and* $\widehat{\mathrm{Ric}}$ *satisfy*

$$\mathrm{Ric}(U,V) = \widehat{\mathrm{Ric}}(U,V) - g(N,T_UV) + g(AU,AV) + \sum_i g((\nabla_{X_i}T)_UV, X_i),$$

$$\mathrm{Ric}(X,U) = g((\hat{\delta}T)U,X) + g(\nabla_UN,X) - g((\delta^TA)X,U) - 2g(A_X,T_U),$$

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}_T(X,Y) - 2g(A_X,A_Y) - g(TX,TY) + \frac{1}{2}(g(\nabla_XN,Y) + g(\nabla_YN,X)).$$

Defining the quantities

$$|A|^2 = \sum_i g(A_{X_i}, A_{X_i}) = \sum_a g(AU_a, AU_a), \tag{1.5.37}$$

$$|T|^2 = \sum_i g(TX_i, TX_i) = \sum_a g(T_{U_a}, T_{U_a}), \tag{1.5.38}$$

we can give the relation between the various scalar curvatures, viz.

Corollary 1.5.29: *Letting* $s, s_T, \hat{s}$ *denote the scalar curvatures of the metrics* $g, g_T, g|_{\mathcal{L}}$, *respectively, we have*

$$s = s_T + \hat{s} - |A|^2 - |T|^2 - |N|^2 - 2\delta^TN.$$

Transverse Complex Geometry

Next we consider the characteristic foliation $\mathcal{F}_\xi$ of an almost contact structure. First we notice that the endomorphism Φ endows $\mathcal{F}_\xi$ with a *transverse almost complex structure* $J = \Phi|_{\mathcal{D}}$. Let $\mathcal{D}_{\mathbb{C}}$ denote the complexification of $\mathcal{D}$, and decompose it into its eigenspaces with respect to J, that is, $\mathcal{D}_{\mathbb{C}} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. Similarly, we get a splitting of the complexification of the set Ω^1_B of basic one forms on M, namely

$$\Omega^1_B \otimes \mathbb{C} = \Omega^{1,0}_B \oplus \Omega^{0,1}_B,$$

and we let $\Omega^{p,q}_B$ denote the set of basic forms of type (p,q). As in the usual case there is a splitting

$$\Omega^r_B \otimes \mathbb{C} = \bigoplus_{p+q=r} \Omega^{p,q}_B, \tag{1.5.39}$$

as well as the *basic Dolbeault complex*

$$\Omega^{p,0}_B \xrightarrow{\bar{\partial}} \Omega^{p,1}_B \xrightarrow{\bar{\partial}} \cdots \longrightarrow \Omega^{p,n}_B \longrightarrow 0, \tag{1.5.40}$$

together with its *basic Dolbeault cohomology groups* $H_B^{p,q}(\mathcal{F}_\xi)$.

Riemannian Flows

We shall be interested in 1-dimensional Riemannian foliations.

Definition 1.5.30: *A one dimensional oriented Riemannian foliation is called a* **Riemannian flow**.

Before we begin our study of Riemannian flows, we briefly mention one dimensional foliations that are never Riemannian, the so-called Anasov flows [**?**]. Any oriented 1-dimensional foliation on a smooth manifold M is determined by a nowhere vanishing vector field V. The flow ϕ_t determined by V is said to be *Anasov* if the tangent bundle TM splits as $TM = L_V \oplus E^s \oplus E^u$ where L_V is the trivial line bundle generated by V and $E^s(E^u)$ denote the *stable (unstable)* subbundles defined as follows: there exist constants $c \geq 1$ and $a > 0$ such that for all $t \geq 0$ the estimates hold:

(i) $|(\phi_t)_*X| \leq ce^{-at}|X|$ for every smooth section X of E^s,
(ii) $|(\phi_t)_*X| \geq c^{-1}e^{at}|X|$ for every smooth section X of E^u.

Here the norm $|\cdot|$ is taken with respect to some Riemannian metric on M, but when M is compact the condition of being Anasov is actually independent of the metric. An important example of an Anasov flow is the geodesic flow on the unit tangent bundle over a compact manifold of constant negative curvature [**?**]. See also [**210**].

For a given Riemannian foliation $(M, \mathcal{F})$ one is interested in whether the leaves are closed in the topology of M. This is of particular interest to us in the case of Riemannian flows. In order to study this in more detail we consider certain types of "singular foliations". Here we follow Molino [**160**].

Definition 1.5.31: *Let M be a smooth manifold. A* **singular Riemannian foliation** $\mathcal{SF}$ *is a partition of M into connected immersed submanifolds, called* **leaves** *that satisfy:*

(i) *The Lie subalgebra of vector fields tangent to the leaves of $\mathcal{SF}$ is transitive on each leaf.*
(ii) *There exists a Riemannian metric g on M such that every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.*

It is not difficult to see that any ordinary Riemannian foliation satisfies these conditions, and thus is a special case of a singular Riemannian foliation. Furthermore, Molino shows a converse in the following sense. If $\mathcal{F}$ is a foliation on M (in the usual sense) and the conditions of Definition 1.5.30 hold, then $\mathcal{F}$ is a Riemannian foliation with respect to g. For us the importance of singular Riemannian foliations is given by

Proposition 1.5.32 (Molino)**:** *Let $(M, \mathcal{F})$ be a Riemannian foliation on a compact connected manifold M. Then the closures $\bar{\mathcal{F}}$ of the leaves of $\mathcal{F}$ form a singular Riemannian foliation for which the leaves are embedded submanifolds of M.*

PROOF. See [**160**] Proposition 6.2 and Theorem 5.1. □

We are interested in the case when the Riemannian flow is an isometry, and then many simplifications occur. Indeed the following theorem of Carrière [**60**] (see also Appendix A of [**160**] by Carrière) becomes much easier to prove when the flow is an isometry.

Theorem 1.5.33 (Carrière)**:** *Let $\mathcal{F}$ be Riemannian flow on a compact manifold M. Then the leaf closure $\bar{\mathcal{L}}$ of a leaf $\mathcal{L}$ of $\mathcal{F}$ is diffeomorphic to a torus T^k, and $\mathcal{F}$ restricted to $\bar{\mathcal{L}}$ is conjugate to a linear flow.*

PROOF. Here we give the proof assuming that the flow is an isometry, but the theorem holds more generally. See [**60**] for the proof of the general case. Let g be a bundle-like Riemannian metric for $(M, \mathcal{F})$ and let ϕ denote the one parameter subgroup of the isometry group $\mathfrak{I}(M, g)$ generated by $\mathcal{F}$. Since M is compact, so is $\mathfrak{I}(M, g)$, and the closure $\bar{\phi}$ is a torus $\mathfrak{T} \subset \mathfrak{I}(M, g)$. For any $x \in M$ the closure $\overline{\phi x}$ of the orbit ϕx coincides with the orbit $\bar{\phi} x$ of the closure $\bar{\phi}$. So if ϕ_x denotes the isotropy subgroup at x, we see that $\bar{\phi}/\bar{\phi}_x$ is a torus T^k. Hence, the leaf closure $\bar{\mathcal{L}}$ is a torus. Furthermore, ϕ is a subgroup of the torus, so it is conjugate to a linear flow. □

Note that the dimension $k = k(\mathcal{L})$ of the torus in general depends on the leaf $\mathcal{L}$ and is clearly bounded by the dimension of the manifold. This leads to

Definition 1.5.34: *The* **toral rank** $\mathrm{rk}(M, \mathcal{F})$ *of a Riemannian flow $\mathcal{F}$ on a compact manifold M is defined by*

$$\mathrm{rk}(M, \mathcal{F}) = \max_{\mathcal{L} \in M} k(\mathcal{L}).$$

It is clear that $1 \leq \mathrm{rk}(M, \mathcal{F}) \leq \dim M$. The $\mathrm{rk}(M, \mathcal{F}) = 1$ case will be of great interest to us. In this case the leaves are all circles, and the space of leaves will be a compact orbifold. See Chapter 2 for details.

Definition 1.5.35: *A one dimensional foliation $\mathcal{F}$ on M is said to be* **isometric** *if there is a Riemannian metric g on M such that the flow is an isometry of g.*

We have

Proposition 1.5.36: *An isometric one dimensional foliation is Riemannian.*

PROOF. Let V be a nowhere vanishing vector field on M generating a one dimensional foliation $\mathcal{F}$ that is isometric with respect to a Riemannian metric g, and let X, Y be any horizontal foliate vector fields on M. Then

$$Vg(X, Y) = (\pounds_V g)(X, Y) + g([V, X], Y) + g(X, [V, Y]). \tag{1.5.41}$$

The first term vanishes since V is an infinitesimal isometry (Killing vector field), and the last two terms vanish since X, Y are both horizontal and foliate. Thus, g is a bundle-like metric for $(M, \mathcal{F})$. So $\mathcal{F}$ is Riemannian by Proposition 1.5.18. □

The converse is not true in general; however, see Proposition 1.5.39 below. Carrière has related the isometry of a one dimensional foliation to the orbits being geodesics. The following result is well-known [**210**].

Lemma 1.5.37: *Let $\mathcal{F}$ be a one dimensional foliation on a manifold M, and let V be a nowhere vanishing vector field generating $\mathcal{F}$. Then the following are equivalent:*
(i) There exists a Riemannian metric g on M such that V has unit length and the orbits of V are geodesics.
(ii) There exists a 1-form η on M such that $\eta(V) = 1$ and $\pounds_V \eta = 0$.
(iii) There exists a 1-form η on M such that $\eta(V) = 1$ and $V \rfloor d\eta = 0$.

PROOF. Choose a metric g on M such that V has unit length and define η to be the dual of V, that is $\eta = g(V, \cdot)$. Then the equivalence of (ii) and (iii) follows immediately from the well known formula $\pounds_X = X\rfloor \circ d + d \circ X\rfloor$. Now let ∇ denote any affine connection on M. Then, the orbits of V are geodesics with respect to the connection ∇ if and only if the *mean curvature vector field* $(\nabla_V V)^\perp$ vanishes, where $\perp$ indicates the component perpendicular to V. Equivalently, this means that the dual *mean curvature 1-form* $\kappa = g((\nabla_V V)^\perp, \cdot)$ vanishes. But for any vector field X perpendicular to V, we have

$$\begin{aligned}\kappa(X) &= g(\nabla_V V, X) = Vg(V,X) - \nabla_V g(V,X) - g(V, \nabla_V X)\\ &= (\pounds_V \eta)(X) + \eta([V,X]) - \nabla_V g(V,X) - \eta(\nabla_V X)\\ &= (\pounds_V \eta)(X) + \eta(\nabla_V X) - \eta(\nabla_X V) + \eta(T(V,X)) - \nabla_V g(V,X) - \eta(\nabla_V X)\\ &= (\pounds_V \eta)(X) + \frac{1}{2}(\nabla_X g)(V,V) - (\nabla_V g)(V,X) + \eta(T(V,X))\end{aligned}$$

where T denotes the torsion tensor of ∇. In particular, choosing ∇ to be the Levi-Civita connection proves the result. $\square$

Now we have

Proposition 1.5.38 (Carrière)**:** *A Riemannian flow whose orbits are geodesics is isometric. Conversely, the orbits of a one dimensional isometric foliation are geodesics.*

Proof. Let $\mathcal{F}$ be a Riemannian flow generated by the nowhere vanishing vector field V whose orbits are geodesics, and let g be a bundle-like metric such that V has unit length. Consider Equation 1.5.41 for horizontal vector fields X, Y. Since this equation is tensorial we can take X, Y to be foliate as well. So the last two terms on the right hand side vanish as does the left hand side since g is bundle-like. Thus, $(\pounds_V g)(X,Y) = 0$ for X, Y horizontal. On the other hand if η is the1-form dual to V, then by (iii) of Lemma 1.5.36 we have for X horizontal

$$0 = d\eta(V,X) = V\eta(X) - X\eta(V) - \eta([V,X]) = -\eta([V,X]),$$

which implies that $(\pounds_V g)(V,X) = 0$. Also $(\pounds_V g)(V,V) = 0$ since V has constant length. Thus, V is a Killing vector field and the flow is isometric.

For the converse let $\mathcal{F}$ be a one dimensional isometric foliation. Then $\mathcal{F}$ is generated by a nowhere vanishing vector field V, and there is Riemannian metric g on M such that V is a Killing vector field. Let η be the 1-form dual to V with respect to g. Then condition (ii) of Lemma 1.5.36 is satisfied. Thus, the orbits of V are geodesics. $\square$

The following is an immediate consequence of the proof of Theorem 1.5.32.

Proposition 1.5.39: *Let $(M,\mathcal{F})$ be an isometric Riemannian flow with toral rank* $\mathrm{rk}(M,\mathcal{F}) = k$. *Then the isometry group $\mathfrak{I}(M,g)$ contains a torus subgroup $\mathfrak{T}^k$ of dimension k.*

Molino (See [**160**] and references therein) has studied the leaf closures for general Riemannian foliations $\mathcal{F}$ and has shown the existence of a sheaf of local transverse Killing vector fields $C(M,\mathcal{F})$ that he calls the *commuting sheaf* of the foliation. Proposition 1.5.38 says that for isometric Riemannian flows the sheaf $C(M,\mathcal{F})$ admits precisely k global sections. In fact on a compact manifold M a result of Molino and Sergiescu [**161**] says that the commuting sheaf admits a global trivialization if and only if $\mathcal{F}$ is isometric. Furthermore, Proposition 5.5 of [**160**] says that for a Riemannian foliation on a compact simply connected manifold, $C(M,\mathcal{F})$ admits a global trivialization. Thus,

Proposition 1.5.40: *On a simply connected compact manifold every Riemannian flow is isometric.*

We are now ready for a Theorem of Wadsley [**?**] that will be of great importance to us in the sequel.

Theorem 1.5.41 (Wadsley)**:** *Let $\mathcal{F}$ be a one dimensional foliation on M such that all the leaves of $\mathcal{F}$ are circles. Then there is a smooth action $A : S^1 \times M \longrightarrow M$ whose orbits are precisely the leaves of $\mathcal{F}$ if and only if there exists a Riemannian metric on M such that the leaves of $\mathcal{F}$ are geodesics.*

Next we discuss an obstruction to the existence of Riemannian flows noticed by Carrière. This comes from a seminal paper of Gromov [**102**] in which he defines two important invariants of smooth manifolds. The first, called the *minimal volume* is defined by

$$\mathrm{MinVol}(M) = \inf_{|K(g)|\leq 1} \mathrm{Vol}(M,g) \tag{1.5.42}$$

where $\mathrm{Vol}(M,g)$ denotes the volume of M with respect to the Riemannian metric g, and the infimum is taken over all complete metrics whose sectional curvatures $K(g)$ are bounded in absolute value by 1. One easily sees that, for example, for compact oriented 2-manifolds M^2 one has $\mathrm{MinVol}(M^2) = 2\pi|\chi(M^2)|$ where χ denotes the Euler characteristic. On the other hand Gromov noticed that if a smooth manifold M admits a locally free S^1 action, then by scaling the metrics along the orbits of the S^1, one sees that $\mathrm{MinVol}(M) = 0$, and Carrière [**60**] noticed that this argument generalizes to arbitrary Riemannian flows.

The second invariant of Gromov is intuitively less geometrical, but computationally more useful. It is the so-called *simplicial volume* of M defined[5] as follows: consider the singular chain complex C_* of all finite combinations $c = \sum_i r_i\sigma_i$ of singular simplices σ_i in M with real coefficients r_i. Define the *simplicial norm* of c by $||c|| = \sum_i |r_i|$. This gives a pseudo-norm on the singular homology $H_*(M,\mathbb{R})$ by $||[z]|| = \inf_z ||[z]||$ where the infimum is taken over all singular homology cycles representing $[z]$. If M is an orientable manifold, its *simplicial volume* $||M||$ (also known as the Gromov invariant of M) is defined to be the simplicial norm of the fundamental class of M. If M is not orientable, then $||M|| =\frac{1}{2}||\tilde{M}||$ where $\tilde{M}$ is the 2-fold cover of M. The simplicial volume enjoys the following important properties:

(i) If $f : M \longrightarrow M'$ is a proper map of degree d, then $||M|| \geq d||M'||$, and if f is a d-sheeted covering map, $||M|| = d||M'||$.

(ii) If M_1 is compact and M_2 is arbitrary, then there is a positive constant C depending only on the dimension of $M_1 \times M_2$ such that

$$C||M_1||||M_2|| \geq ||M_1 \times M_2|| \geq C^{-1}||M_1||||M_2||.$$

(iii) If dim $M_1 =$ dim $M_1 \geq 3$, then $||M_1\#M_2|| = ||M_1|| + ||M_2||$.

Furthermore, Gromov proves the following estimate for the simplicial and minimal volumes

$$||M|| \leq (n-1)^n n!\mathrm{MinVol}(M). \tag{1.5.43}$$

[5]Here M can be any topological space

We mention that if M is closed (compact without boundary), then $||M||$ is completely determined by the fundamental group. In particular, compact simply connected manifolds have vanishing Gromov invariant. (More generally $||M||$ vanishes for any compact manifold whose fundamental group is amenable).

Summarizing we have

Theorem 1.5.42 (Carrière)**:** *If a compact manifold M admits a Riemannian flow, then* $\mathrm{MinVol}(M) = 0$*. In particular, $||M|| = 0$ and all the Pontrjagin numbers of M vanish.*

We are interested in seeing which manifolds M have nonvanishing Gromov invariant $||M||$. In [**102**] Gromov proves a theorem which he attributes to Thurston which says that a complete Riemannian manifold of finite volume whose sectional curvatures are pinched between $-k$ and -1 for some $k > 1$ has nonvanishing Gromov invariant $||M||$. At about the same time Inoue and Yano [**?**] proved that any compact Riemannian manifold whose sectional curvatures is bounded from above by $-\delta$ for $\delta > 0$ has nonvanishing Gromov invariant. We state the precise results as:

Theorem 1.5.43: *Let (M, g) be a n-dimensional Riemannian manifold.*

(i) *Suppose that (M, g) is complete with finite volume* $0 < \mathrm{Vol}(M, g) < \infty$*, and that the sectional curvatures of g are pinched between $-k$ and -1 for some real number $k > 1$. Then the estimate*

$$Vol(M, g) \leq C_n ||M||$$

holds where C_n is a positive constant depending only on n.

(ii) *Let (M, g) be compact without boundary whose sectional curvatures are bounded from above by $-\delta$ for some $\delta > 0$. Then there exists a positive constant C_n such that*

$$||M|| \geq C_n \delta^{n/2}\, Vol(M, g).$$

Hence, in both cases $||M|| \neq 0$.

It follows immediately that no manifold satisfying either of the two conditions of this theorem can admit a Riemannian flow.

CHAPTER 2

Fundamentals of Orbifolds

2.1. Orbifolds, Orbisheaves, and V-bundles

The notion of orbifold was introduced under the name V-manifold by Satake [**188**] in 1956, and subsequently he developed Riemannian geometry on V-manifolds [**189**] ending with a proof of the Gauss-Bonnet theorem for V-manifolds. Contemporaneously, Baily introduced complex V-manifolds and generalized both the Hodge decomposition theorem [**?**], and Kodaira's projective embedding theorem [**?**] to V-manifolds. Somewhat later in the late 1970's and early 1980's Kawasaki generalized various index theorems [**121, 122, 123**] to the category of V-manifolds. It was about this time that Thurston [**208**] rediscovered the concept of V-manifold, under the name of orbifold, in his study of the geometry of 3-manifolds, and defined the orbifold fundamental group π_1^{orb}. By now orbifold has become the accepted term for these objects and we shall follow suit. However, we do use the name V-bundle for fibre bundles in this category.

Orbifolds arise naturally as spaces of leaves of Riemannian foliations with compact leaves, and we are particularly interested in this point of view. Conversely, every orbifold can be realized in this way. In fact, given an orbifold $\mathcal{O}$, we can construct on it the V-bundle of orthonormal frames whose total space P is a smooth manifold with a locally free action of the orthogonal group $O(n)$ such that $\mathcal{O} = P/O(n)$. Thus, every orbifold can be realized as the quotient space by a locally free action of a Lie group. We are not certain of the history of this connection, but it was surely well understood by Haefliger [**?**] in 1982 who developed the basic techniques for studying the topology of orbifolds.

The definition given here is close to Satake's original definition [**188, 189**] (cf. [**12**]), but with more modern terminology as in Moerdijk and Pronk [**158**].

Definition 2.1.1: *Let X be a paracompact Hausdorff space. An* **orbifold chart** *or* **local uniformizing system** *on X is a triple $\{\tilde{U}, \Gamma, \varphi\}$, where $\tilde{U}$ is connected open subset of $\mathbb{R}^n$ containing the origin, Γ is a finite group of diffeomorphisms of $\tilde{U}$, and $\varphi : \tilde{U} \longrightarrow U$ is a continuous map onto an open set $U \subset X$ such that $\varphi \circ \gamma = \varphi$ for all $\gamma \in \Gamma$ and the induced natural map of $\tilde{U}/\Gamma$ onto U is a homeomorphism. An* **injection** *or* **embedding** *between two such charts $\{\tilde{U}, \Gamma, \varphi\}$ and $\{\tilde{U}', \Gamma', \varphi'\}$ is a smooth embedding $\lambda : \tilde{U} \longrightarrow \tilde{U}'$ such that $\phi' \circ \lambda = \phi$. An* **orbifold atlas** *on X is a family $\mathcal{U} = \{\tilde{U}_i, \Gamma_i, \varphi_i\}$ of orbifold charts such that*

(i) $X = \cup_i \phi_i(\tilde{U}_i)$,

(ii) *given two charts $(\tilde{U}_i, \Gamma_i, \varphi_i)$ and $(\tilde{U}_j, \Gamma_j, \varphi_j)$ with $U_i = \phi_i(\tilde{U}_i)$ and $U_j = \phi_j(\tilde{U}_j)$ and a point $x \in U_i \cap U_j$, there exist an open neighborhood U_k of x, and a chart $(\tilde{U}_k, \Gamma_k, \varphi_k)$ such that there are injections $\lambda_{ik} : (\tilde{U}_k, \Gamma_k, \varphi_k) \longrightarrow (\tilde{U}_i, \Gamma_i, \varphi_i)$ and $\lambda_{jk} : (\tilde{U}_k, \Gamma_k, \varphi_k) \longrightarrow (\tilde{U}_j, \Gamma_j, \varphi_j)$.*

An atlas $\mathcal{U}$ is said to be a refinement of an atlas $\mathcal{V}$ if there exists an injection of every chart of $\mathcal{U}$ into some chart of $\mathcal{V}$. Two orbifold atlases are said to be **equivalent** *if they have a common refinement. A smooth* **orbifold** *(or* **V-manifold***) is a paracompact Hausdorff space X with an equivalence class of orbifold atlases. We denote the orbifold by $\mathcal{X} = (X, \mathcal{U})$. If the finite group Γ consists of orientation preserving diffeomorphisms and there is an atlas such that all the injections are orientation preserving then the orbifold is* **orientable**.

REMARKS 2.1.1: 1) The finite groups Γ_i are called *local uniformizing groups,* and we can always take them to be finite subgroups of the orthogonal group $O(n) \subset GL(n, \mathbb{R})$ and in the orientable case $SO(n)$. 2) Every orbifold atlas is contained in a unique maximal atlas satisfying the required properties. Thus, we shall often think of the orbifold $\mathcal{X} = (X, \mathcal{U})$ where $\mathcal{U}$ is the unique maximal atlas. This justifies the notation. 3) Notice that any $\gamma \in \Gamma$ satisfies $\phi(\gamma \cdot x) = \phi(x)$ for any $x \in \tilde{U}$. Thus, every γ defines an injection by $x \mapsto \gamma \cdot x$. 4) Similarly, a *complex orbifold* can be defined by making the obvious changes, namely replacing $\mathbb{R}^n$ by $\mathbb{C}^n$ and smooth by holomorphic. In this case the local uniformizing groups can be taken as finite subgroups of $U(n)$.

The following lemma was proved by Satake in [**189**] under the added assumption that the fixed point set has codimension at least two[1]. However, this assumption was removed recently in [**158**].

Lemma 2.1.2 (Satake/Moerdijk-Pronk)**:** *Let $\lambda_1, \lambda_2 : (\tilde{U}, \Gamma, \varphi) \longrightarrow (\tilde{U}', \Gamma', \varphi')$ be two injections. Then there exists a unique $\gamma' \in \Gamma'$ such that $\lambda' = \gamma' \circ \lambda$.*

The notions of structure preserving maps and bundle theory, or more generally sheaf theory, are fundamental to doing geometry on any object. The standard notions of smooth maps between orbifolds as well as V-bundles on orbifolds have been given in an analogous manner to manifolds (see [**188, 189, 12, 13**]). However, it wasn't realized until recently (see for example [**62, 158**]) that certain problems arise, namely it does not in general induce morphisms of sheaves or V-bundles. This led to the notion of "good map" in [**62**]; however, we follow the development in [**158**] and [**157**].

Definition 2.1.3: *Let $\mathcal{X} = (X, \mathcal{U})$ and $\mathcal{Y} = (Y, \mathcal{V})$ be orbifolds. A map $f : X \longrightarrow Y$ is said to be* **smooth** *if for every point $x \in X$ there are orbifold charts $(\tilde{U}, \Gamma, \phi)$ about x and $(\tilde{V}, \Phi, \psi)$ about $f(x)$ such that f maps $U = \phi(\tilde{U})$ into $V = \psi(\tilde{V})$ and there exist local lifts $\tilde{f} : \tilde{U} \longrightarrow \tilde{V}$ satisfying $\psi \circ \tilde{f} = f \circ \phi$.*

We shall often write $f : \mathcal{X} \longrightarrow \mathcal{Y}$ for a smooth map of orbifolds and refer to this as an orbifold map. It is easy to see that the composition of smooth maps is smooth, so the collection of orbifolds with smooth maps forms a category. Thus, we also have

Definition 2.1.4: *Two orbifolds $\mathcal{X} = (X, \mathcal{U})$ and $\mathcal{Y} = (Y, \mathcal{V})$ are said to be* **diffeomorphic** *or* **equivalent** *if exist smooth maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = \mathbb{I}_Y$ and $g \circ f = \mathbb{I}_X$ where $\mathbb{I}_X, \mathbb{I}_Y$ are the respective identity maps.*

Notice that in particular a diffeomorphism between orbifolds gives a homeomorphism of the underlying topological spaces. We now wish to describe sheaves and bundles over orbifolds. Following [**158**] we have

[1]The codimension two condition occured as part of Satake's original definiton of V-manifold.

Definition 2.1.5: *Let $\mathcal{X} = (X, \mathcal{U})$ be an orbifold. A* **sheaf** *$\mathcal{F}$ on $\mathcal{X}$ or* **orbisheaf** *on $\mathcal{X}$ consists of a sheaf $\mathcal{F}_{\tilde{U}}$ over $\tilde{U}$ for each orbifold chart $(\tilde{U}, \Gamma, \phi)$ such that for each injection $\lambda : (\tilde{U}, \Gamma, \phi) \longrightarrow (\tilde{V}, \Phi, \psi)$ there is an isomorphism of sheaves $\mathcal{F}(\lambda) : \mathcal{F}_{\tilde{U}} \longrightarrow \lambda^* \mathcal{F}_{\tilde{V}}$ which is functorial in λ.*

Let $\mathcal{F}$ be an orbisheaf on $\mathcal{X}$, and $\mathcal{U} = (\tilde{U}, \Gamma, \phi)$ an orbifold chart. By 3) of Remarks 2.1.1 each $\gamma \in \Gamma$ defines an injection λ_γ, so there is a sheaf map $\mathcal{F}(\gamma) : \mathcal{F}_{\tilde{U}} \longrightarrow \lambda_\gamma^* \mathcal{F}_{\tilde{U}}$, and if $s \in \mathcal{F}_{tU,x}$ is a point in the stalk over $x \in \tilde{U}$, then $\mathcal{F}(\gamma)(s)$ lies in the stalk over $g \cdot x$. This defines an action of Γ on the sheaf $\mathcal{F}_{\tilde{U}}$, which says that $\mathcal{F}_{\tilde{U}}$ is a Γ-equivariant sheaf on $\tilde{U}$. So every orbisheaf $\mathcal{F}$ is equivariant under the local uniformizing groups Γ_i. We now have

Definition 2.1.6: *The* **structure sheaf** *$\mathcal{O}_{\mathcal{X}}$ of an orbifold $\mathcal{X}$ is the orbisheaf defined by the structure sheaf $\mathcal{O}_{\tilde{U}}$ on each orbifold chart $(\tilde{U}, \Gamma, \phi)$.*

$\mathcal{O}_{\mathcal{X}}$ is well defined since each injection $\lambda : \tilde{U}, \Gamma, \phi) \longrightarrow (\tilde{U}', \Gamma', \phi')$ induces an isomorphism $\mathcal{O}_{\tilde{U}} \approx \lambda^* \mathcal{O}_{\tilde{U}'}$ by sending $f \in \mathcal{O}_{\tilde{U},x}$ to $f \circ \lambda^{-1} \in (\lambda^* \mathcal{O}_{\tilde{U}'})_x$. The structure sheaves that are of most interest to us are the smooth structure sheaf when $\mathcal{O}_{\tilde{U}}$ is the sheaf of C^∞ functions on $\tilde{U}$ for real orbifolds, the holomorphic structure sheaf when $\mathcal{O}_{\tilde{U}}$ is the sheaf of holomorphic functions on $\tilde{U}$ for complex orbifolds, and the regular structure when $\mathcal{O}_{\tilde{U}}$ is the sheaf of regular functions on $\tilde{U}$.

Definition 2.1.7: *A* **morphism of orbisheaves** *$\alpha : \mathcal{F} \longrightarrow \mathcal{F}'$ is a family of sheaf maps $\alpha_{\tilde{U}} : \mathcal{F}_{\tilde{U}} \longrightarrow \mathcal{F}'_{\tilde{U}}$, one for each orbifold chart, that is compatible with the injections λ in the sense that for each injection the diagram*

$$\begin{tikzcd}
\mathcal{F}_{\tilde{U}} \arrow[r, "\alpha_{\tilde{U}}"] \arrow[d, "\mathcal{F}(\lambda)"] & \mathcal{F}'_{\tilde{U}} \arrow[d, "\mathcal{F}'(\lambda)"] \\
\lambda^* \mathcal{F}_{\tilde{V}} \arrow[r, "\lambda^* \alpha_{\tilde{V}}"] & \lambda^* \mathcal{F}'_{\tilde{V}}
\end{tikzcd}$$

commutes.

So the collection of all sheaves of sets, Abelian groups, rings, etc. forms a category[2] $\tilde{\mathcal{X}}$ that we call the *category of orbisheaves* on $\mathcal{X}$. It is important to realize that $\tilde{\mathcal{X}}$ is quite different from the category of sheaves on X unless the orbifold structure is trivial, i.e. all the local uniformizing groups Γ_i are the identity. Any map $\tilde{f} : \tilde{\mathcal{X}} \longrightarrow \tilde{\mathcal{Y}}$ of orbisheaves sends $\mathcal{O}_{\mathcal{X}}$ to $\mathcal{O}_{\mathcal{Y}}$, and thus induces a morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$. However, the converse is not true. An orbifold map as defined by Definition 2.1.3 does not in general induce a morphism of orbisheaves. Nevertheless, the following does hold [**158**]:

Proposition 2.1.8: *A diffeomorphism of orbifolds $\mathcal{X} \approx \mathcal{Y}$ induces an equivalence of orbisheaves $\tilde{\mathcal{X}} \approx \tilde{\mathcal{Y}}$.*

Proof. The proof is technical. See [**158**] for details. □

The structure sheaf $\mathcal{O}_{\mathcal{X}}$ of an orbifold $\mathcal{X}$ is a sheaf of commutative rings, and we can construct an orbisheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{X}}$ modules on $\mathcal{X}$ by giving a sheaf $\mathcal{M}_{\tilde{U}}$ of $\mathcal{O}_{\tilde{U}}$

[2]Actually this is a special type of topos called a smooth étendue by Grothendieck and Verdier [**10**], but we shall not use this terminology

modules on each orbifold chart $(\tilde{U}, \Gamma, \phi)$. As usual for a sheaf $\mathcal{F}_{\tilde{U}}$ on $\tilde{U}$ we denote by $(\mathcal{F}_{\tilde{U}})^r$ the direct sum $\mathcal{F}_{\tilde{U}} \oplus \overset{r\ times}{\cdots} \oplus \mathcal{F}_{\tilde{U}}$.

Definition 2.1.9: *An orbisheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{X}}$ modules on $\mathcal{X}$ is said to be* **locally V-free** *or just* **locally free** *if each point $x \in X$ has an orbifold chart such that $\mathcal{M}_{\tilde{U}} \approx (\mathcal{O}_{\tilde{U}})^r$ for some positive integer r, called the* **rank** *of $\mathcal{M}$. An orbisheaf of rank one is called* **invertible** *orbisheaf or a V-invertible sheaf.*

We use the terms locally free and locally V-free interchangably when referring to orbisheaf. The terms V-free and V-invertible are used mainly for emphasis since a sheaf may be an orbisheaf which is a different structure. Many of the usual differential geometric concepts that hold for smooth or complex analytic manifolds also hold in the orbifold category, in particular the important notion of a fiber bundle.

Definition 2.1.10: *A* **V-bundle** *or* **orbibundle** *over an orbifold X consists of a bundle $B_{\tilde{U}}$ over $\tilde{U}$ for each orbifold chart $(\tilde{U}_i, \Gamma_i, \varphi_i)$ with Lie group G and fiber F (independent of $\tilde{U}_i$) together with a homomorphism $h_{\tilde{U}_i} : \Gamma_i \longrightarrow G$ satisfying:*
i) If b lies in the fiber over $\tilde{x}_i \in \tilde{U}_i$ then for each $\gamma \in \Gamma_i$, $bh_{\tilde{U}_i}(\gamma)$ lies in the fiber over $\gamma^{-1}\tilde{x}_i$.
ii) If $\lambda_{ji} : \tilde{U}_i \longrightarrow \tilde{U}_j$ is an injection, then there is a bundle map $\lambda^*_{ij} : B_{U_j}|\lambda_{ji}(\tilde{U}_i) \longrightarrow B_{\tilde{U}_i}$ satisfying the condition that if $\gamma \in \Gamma_i$, and $\gamma' \in \Gamma_j$ is the unique element such that $g_{ji} \circ \gamma = \gamma' \circ g_{ji}$, then $h_{\tilde{U}_i}(\gamma) \circ \lambda^*_{ji} = \lambda^*_{ji} \circ h_{\tilde{U}_j}(\gamma')$, and if $\lambda_{kj} : \tilde{U}_j \longrightarrow \tilde{U}_k$ is another such injection then $(\lambda_{kj} \circ \lambda_{ji})^* = \lambda^*_{ji} \circ \lambda^*_{kj}$.

If the fiber F is a vector space and G acts on F as linear transformations of F, then the V-bundle is called a **vector** *V-bundle. Similarly, if F is the Lie group G with its right action, then the V-bundle is called a* **principal** *V-bundle.*

The *total space* of a V-bundle over X is an orbifold E with local uniformizing systems $\{B_{\tilde{U}_i}, \Gamma^*_i, \varphi^*_i\}$. By choosing the local uniformizing neighborhoods of X small enough, we can always take $B_{\tilde{U}_i}$ to be the product $\tilde{U}_i \times F$ which we shall heretofore assume. There is an action of the local uniformizing group Γ_i on $\tilde{U}_i \times F$ given by sending $(\tilde{x}_i, b) \in \tilde{U}_i \times F$ to $(\gamma^{-1}\tilde{x}_i, bh_{\tilde{U}_i}(\gamma))$, so the local uniformizing groups Γ^*_i can be taken to be subgroups of Γ_i. We are particularly interested in the case of a principal bundle. In the case the fibre is the Lie group G, so the image $h_{\tilde{U}_i}(\Gamma^*_i)$ acts freely on F. Thus the total space P of a principal V-bundle will be smooth if and only if $h_{\tilde{U}_i}$ is injective for all i.

Remark 2.1.1: We shall often denote a V-bundle by the standard notation $\pi : P \longrightarrow X$ and think of this as an “orbifold fibration”. It must be understood, however, that an orbifold fibration is not a fibration in the usual sense. Shortly, we shall show that it is a fibration rationally. Again the standard notions of smooth maps between V-bundles, and isomorphism classes of V-bundles can be given in the usual manner. We let this description to the reader. An *absolute* V-bundle resembles a bundle in the ordinary sense, and corresponds to being able to take $h_{\tilde{U}} = \mathrm{id}$, for all local uniformizing neighborhoods $\tilde{U}$. In particular, the trivial V-bundle $X \times F$ is absolute. Another important notion introduced by Kawasaki [**122**] is that of proper. A V-bundle E is said to be *proper* if the local uniformizing groups Γ^*_i of E act effectively on X when viewed as subgroups of the local uniformizing groups Γ_i on X. Any V-bundle with smooth total space is clearly proper. The Kawasaki index theorems such as his Riemann-Roch Theorem used in section 3.2 require the V-bundles to be proper.

The notion of sections of bundles works just as well in the orbifold category.

Definition 2.1.11: *Let E be a V-bundle over an orbifold $\mathcal{X}$. Then a section σ of E over an open set $W \subset X$ is a section $\sigma_{\tilde{U}}$ of the bundle $B_{\tilde{U}}|_W$ for each orbifold chart $(\tilde{U}, \Gamma, \varphi)$ such that $U \subset W$ and for any $x \in \tilde{U}$ we have*
(i) For each $\gamma \in \Gamma$ $\sigma_{\tilde{U}}(\gamma^{-1}x) = h_{\tilde{U}}(\gamma)\sigma_{\tilde{U}}(x)$.
(ii) If $\lambda : (\tilde{U}, \Gamma, \varphi) \longrightarrow (\tilde{U}', \Gamma', \varphi')$ is an injection, then $\lambda^* \sigma_{\tilde{U}'}(\lambda(x)) = \sigma_{\tilde{U}}(x)$.

If each of the local sections $\sigma_{\tilde{U}}$ is continuous, smooth, holomorphic, etc., we say that σ is continuous, smooth, holomorphic, etc., respectively. Given local sections $\sigma_{\tilde{U}}$ of a vector V-bundle we can always construct Γ-invariant local sections by "averaging over the group", i.e., we define $\sigma^I_{\tilde{U}} = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \sigma_{\tilde{U}} \circ \gamma$. The standard notions of tangent bundle, cotangent bundle, and all the associated tensor bundles all have V-bundle analogues [**12, 188, 189**]. Thus, one easily constructs Riemannian metrics, symplectic 2-forms, connections, etc. that are invariant under the local uniformizing groups. Integration theory also goes through. In particular, if V is an open subset of $\varphi(\tilde{U})$ then the integral of an n-form (measurable) σ is defined by $\int_V \sigma = \frac{1}{|\Gamma|}\int_{\varphi^{-1}(V)} \sigma_{\tilde{U}}$. So all of the standard integration techniques, such as Stokes' theorem, hold on V-manifolds. The usual relation between locally free sheaves and vector bundles also holds for orbifolds. We consider both real and complex vector V-bundles and refer to them as $\mathbb{F}$-vector V-bundles where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. First we need

Definition 2.1.12: *A* **V-bundle map** *$\alpha : E \longrightarrow E'$ consists of a family of bundle maps $\alpha_{\tilde{U}} : B_{\tilde{U}} \longrightarrow B'_{\tilde{U}'}$, one for each orbifold chart, that is compatible with the injections $\lambda : (\tilde{U}, \Gamma, \phi) \longrightarrow (\tilde{V}, \Phi, \psi)$ and $\lambda' : (\tilde{U}', \Gamma', \phi') \longrightarrow (\tilde{V}', \Phi', \psi')$ in the sense that the diagram*

$$\begin{array}{ccc} B_{\tilde{V}} & \xrightarrow{\alpha_{\tilde{V}}} & B'_{\tilde{V}'} \\ \Big\downarrow{\scriptstyle \lambda^*} & & \Big\downarrow{\scriptstyle \lambda'^*} \\ B_{\tilde{U}} & \xrightarrow{\alpha_{\tilde{U}}} & B'_{\tilde{U}'} \end{array}$$

commutes.

By refining the orbifold charts if necessary we can take $\tilde{U}$ such that $B_{\tilde{U}} \approx \tilde{U} \times F$. Then by identifying $B_{\tilde{U}}$ with $\tilde{U} \times F$ and letting $\alpha^1_{\tilde{U}}$ denote the map $\alpha_{\tilde{U}}$ on the first factor, we see that Definition 2.1.12 implies $\alpha^1_{\tilde{V}} \circ \lambda = \lambda' \circ \alpha^1_{\tilde{U}}$.

Proposition 2.1.13: *There is a one-to-one correspondence between isomorphism classes of $\mathbb{F}$-vector V-bundles on $\mathcal{X}$ and isomorphism classes of locally V-free orbisheaves on $\mathcal{X}$.*

2.2. Orbifolds and Groupoids

An alternative definition of orbifold given by Haefliger [**?**] can be obtained as follows: Let $G_{\Gamma,g}$ denote the groupoid of germs of diffeomorphisms generated by the germs of elements in Γ_i and the germs of the diffeomorphisms g_{ji} described above. Let $\tilde{U} = \sqcup_i \tilde{U}_i$ denote the disjoint union of the $\tilde{U}_i$. Then $x, y \in \tilde{U}$ are equivalent if there is a germ $\gamma \in G_{\Gamma,g}$ such that $y = \gamma(x)$. The quotient space $X = \tilde{U}/G_{\Gamma,g}$ defines an orbifold (actually an isomorphism class of orbifolds). In the case that an orbifold X is given as the space of leaves of a foliation $\mathcal{F}$ on a smooth manifold, the groupoid $G_{\Gamma,g}$ is the transverse holonomy groupoid of $\mathcal{F}$.

The following result relating to foliations, which is given in Molino, is fundamental to our work:

Theorem 2.2.1: *([**160**] Proposition 3.7) Let $(M, \mathcal{F}, g)$ be a Riemannian foliation of codimension q with compact leaves and bundle-like metric g. Then the space of leaves $M/\mathcal{F}$ admits the structure of a q-dimensional orbifold such that the natural projection $\pi : M \longrightarrow M/\mathcal{F}$ is an orbifold submersion.*

Let X be an orbifold and choose a local uniformizing system $\{U, \Gamma, \varphi\}$. Let $x \in X$ be any point, and let $p \in \varphi^{-1}(x)$, then up to conjugacy the isotropy subgroup $\Gamma_p \subset \Gamma$ depends only on x, and accordingly we shall denote this isotropy subgroup by Γ_x. A point of X whose isotropy subgroups $\Gamma_x \neq \mathrm{id}$ is called a *singular* point. Those points with $\Gamma_x = \mathrm{id}$ are called *regular* points. The subset of regular points is an open dense subset of X. The isotropy groups give a natural stratification of X by saying that two points lie in the same stratum if their isotropy subgroups are conjugate. Thus, the dense open subset of regular points forms the principal stratum. In the case that X is the space of leaves of a foliation, the isotropy subgroup Γ_x is precisely the leaf holonomy group of the leaf x. An orbifold X is a smooth manifold or in the complex analytic category a complex manifold if and only if $\Gamma_x = \mathrm{id}$ for all $x \in X$. In this case we can take $\Gamma = \mathrm{id}$ and $\varphi = \mathrm{id}$, and the definition of an orbifold reduces to the usual definition of a smooth manifold.

2.3. Orbifold Homology, Cohomology, and Homotopy Groups

Since an orbifold fibration is not a fibration in the usual sense, the usual techniques in topology for fibrations do not apply directly. However, Haefliger [**?**] has defined orbifold homology, cohomology, and homotopy groups which do have an analogue in the standard theory. Let X be an orbifold of dimension n and let P denote the bundle of orthonormal frames on X. It is a smooth manifold on which the orthogonal group $O(n)$ acts locally freely with the quotient X. Let $EO(n) \longrightarrow BO(n)$ denote the universal $O(n)$ bundle. Consider the diagonal action of $O(n)$ on $EO(n) \times P$ and denote the quotient by BX. Now there is a natural projection $p : BX \longrightarrow X$ with generic fiber the contractible space $EO(n)$, and Haefliger defines the orbifold cohomology, homology, and homotopy groups by

$$H^i_{orb}(X, \mathbb{Z}) = H^i(BX, \mathbb{Z}), \quad H_i^{orb}(X, \mathbb{Z}) = H_i(BX, \mathbb{Z}), \quad \pi_i^{orb}(X) = \pi_i(BX).$$

This definition of π_1^{orb} is equivalent to Thurston's better known definition [**208**] in terms of orbifold deck transformations, and when X is a smooth manifold these orbifold groups coincide with the usual groups. It should be noted that generally these groups are not topological invariants of X, but invariants of the orbifold structure only. Rationally, however, the orbifold groups coincide with the usual groups, and thus are topological invariants. Indeed from the Leray spectral sequence for the map p we have

Proposition 2.3.1: [**?**] *The map $p : BX \longrightarrow X$ induces an isomorphism*

$$H^i_{orb}(\mathcal{S}, \mathbb{Z}) \otimes \mathbb{Q} \simeq H^i(\mathcal{S}, \mathbb{Z}) \otimes \mathbb{Q}.$$

Now with this in hand for the orbifold category, the circle V-bundles over $\mathcal{S}$ are classified [**106**] by $H^2_{orb}(\mathcal{S},\mathbb{Z})$. Of course, rationally there is no difference by Proposition A.2.1. The rational Gysin sequence for orbifold sphere bundles whose generic fibres are spheres also holds. Haefliger's theory also applies to the following situation. Let G be a compact Lie group acting locally freely on an orbifold Y with quotient X. This gives rise to a fibration $EO(n)\times G\longrightarrow BY\longrightarrow BX$, which induces the long exact homotopy sequence

$$(2.3.1)\qquad \cdots\longrightarrow\pi_i(G)\longrightarrow\pi_i^{orb}(Y)\longrightarrow\pi_i^{orb}(X)\longrightarrow\pi_{i-1}(G)\longrightarrow\cdots.$$

This was used by Haefliger and Salem [**106**] in their study of torus actions on orbifolds.

We are particularly interested in the case of circle V-bundles. Using the exponential exact sequence one sees as the usual case that $H^2_{orb}(X,\mathbb{Z})$ classifies equivalence classes of circle V-bundles over an orbifold X. Furthermore, in [**106**] it is shown that $H^2(X,\mathbb{Z})$ classifies circle V-bundles up to local equivalence. This gives a monomorphism $H^2(X,\mathbb{Z})\longrightarrow H^2_{orb}(X,\mathbb{Z})$ which is an isomorphism rationally.

In [**34**] we introduced the set $\mathrm{Pic}^{orb}(X)$ of equivalence classes holomorphic line V-bundles over a complex orbifold X and one easily sees [**34**]:

Lemma 2.3.2: *$Pic^{orb}(X)$ forms an Abelian group. Furthermore, there is a monomorphism $Pic(X)\longrightarrow Pic^{orb}(X)$ which is an isomorphism rationally.*

Riemannian metrics also exist by the standard partition of unity argument, and we shall always work with Γ-invariant metrics. Moreover, all the standard differential geometric objects involving curvature and metric concepts, such as the Ricci tensor, Hodge star operator, etc., hold equally well. On a complex orbifold there is a Γ-invariant tensor field J of type $(1,1)$ which describes the complex structure on the tangent V-bundle TX. The almost complex structure J gives rise in the usual way to the V-bundles $A^{r,s}$ of differential forms of type (r,s). The standard concepts of Hermitian and Kähler metrics hold equally well on V-manifolds, and all the special identities involving Kähler, Einstein, or Kähler-Einstein geometry hold. In particular, the standard Weizenböck formulas hold.

Finally, there is associated to every compact orbifold X an integer m_0 called the *order* of X and defined to be the least common multiple of the orders of the local uniformizing groups.

We want to have the analogue of an "atlas of charts" on BX. Let $\{\tilde{U}_i,\Gamma_i,\phi_i\}$ be a cover of uniformizing charts for the orbifold X, and let P_i denote the linear frame bundle over U_i. The group $GL(n)$ acts locally freely on P_i with isotropy group Γ_i fixing the frames over the center $a_i\in U_i$. So we have homeomorphisms $P_i/GL(n)\approx\tilde{U}_i/\Gamma_i\approx U_i$. Thus, we can cover BX by neighborhoods of the form $\tilde{U}_i\times_{\Gamma_i}EGL(n)$ where Γ_i is viewed as a subgroup of $GL(n)$. In fact Γ_i can always be taken as a subgroup of $O(n)$. Now by refining the cover if necessary we have injection maps $g_{ji}:\tilde{U}_i\longrightarrow\tilde{U}_j$ and these induce the change of "charts" maps

$$G_{ji}:\tilde{U}_i\times_{\Gamma_i}EGL(n)\longrightarrow\tilde{U}_j\times_{\Gamma_j}EGL(n)$$

given by $G_{ji}([\tilde{x}_i,e])=[g_{ji}(\tilde{x}_i),e]$. This is well-defined since the unique element γ_j discussed in Remark is identified with γ_i under the identification of Γ_i as a subgroup of Γ_j. This will allow us to construct local data on $\tilde{U}_i\times_{\Gamma_i}EGL(n)$ by considering smooth (or holomorphic) data on the $\tilde{U}_i$ and continuous data on $EGL(n)$ which is invariant under the Γ_i action. Since the g_{ji} are diffeomorphisms (or biholomorphisms) this local data will then patch to give global data on BX. For example we denote by $\mathcal{E}$ the sheaf of germs of complex-valued functions on BX that are smooth in $\tilde{U}_i$ and continuous in $EGL(n)$. We shall call global (local) sections of the sheaf $\mathcal{E}$ smooth functions on BX. Similarly we refer to "smooth" maps from BX. For example, if $f : X_1 \longrightarrow X_2$ is a smooth map of orbifolds, then this induces a smooth map $F : BX_1 \longrightarrow BX_2$. Similarly, because of the commutative diagram

$$\begin{array}{ccc} \tilde{U}_i \times EGL(n) & \longrightarrow & \tilde{U}_i \\ \big\downarrow & & \big\downarrow \phi_i \\ \tilde{U}_i \times_{\Gamma_i} EGL(n) & \xrightarrow{p_i} & U_i. \end{array}$$

the map $p : BX \longrightarrow X$ is smooth, since its local covering maps p_i are smooth.

Next we give a characterization of V-bundles.

Theorem 2.3.3: *Let X be an orbifold. There is a one-to-one correspondence between isomorphism classes of V-bundles on X with group G and generic fiber F and isomorphism classes of bundles on BX with group G and fiber F.*

Proof. A V-bundle on X is a bundle on $\tilde{U}_i$ for each local uniformizing neighborhood $\tilde{U}_i$ together with a group homomorphism $h_{\tilde{U}_i} \in Hom(\Gamma_i, G)$ that satisfy the compatibility conditions of Definition 2.1.10. This gives an action of Γ_i on $B_{\tilde{U}_i}$. Now cover BX by neighborhoods of the form $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ where we make use of the fact that the local uniformizing groups Γ_i can be taken as subgroups of $O(n) \subset GL(n)$. There is an action of Γ_i on $\tilde{U}_i \times F \times EGL(n)$ given by $(\tilde{x}_i, u, e) \mapsto (\gamma^{-1}\tilde{x}_i, uh_{U_i}(\gamma), e\gamma)$, and this gives a G-bundle over $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ with fiber F for each i. Moreover, the compatibility condition (ii) of Definition 2.1.10 guarantees that these bundles patch together to give a G-bundle on BX with fiber F.

Conversely, given a G-bundle on BX with fiber F, restricting to $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ gives a G-bundle there. Since for each i $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ is the Eilenberg-MacLane space $K(\Gamma_i, 1)$, there is a one-to-one correspondence between isomorphism classes of G-bundles on $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ and conjugacy classes of group homomorphisms $Hom(\Gamma_i, G)$. The fact that these G-bundles come from a global G-bundle on BX implies that the compatibility conditions (ii) of Definition 2.1.10 are satisfied. The correspondence can be seen to be bijective. □

In what follows we shall not distinguish between V-bundles on X and bundles on BX. Thus, we have

Proposition 2.3.4: *The the isomorphism classes of V-bundles on X with group G are in one-to-one correspondence with elements of the non-Abelian cohomology set $H^1(BX, \mathcal{G}) = H^1_{orb}(X, \mathcal{G})$ where $\mathcal{G}$ is the sheaf of germs of maps to the group G.*

We now turn to the Abelian case with coefficients in a sheaf. Let $\mathcal{E}$ denote sheaf of germs of smooth complex-valued functions on BX and $\mathcal{E}^*$ the sub-sheaf of no-where vanishing complex valued smooth functions. The isomorphism classes of complex line bundles on BX, hence complex line V-bundles on X, are in one-to-one correspondence with the elements of the cohomology group $H^1_{orb}(X, \mathcal{E}^*)$. Now BX is an infinite paracompact CW complex, and $\mathcal{E}$ is a fine sheaf, so the exponential sequence gives an isomorphism

$$H^1_{orb}(X, \mathcal{E}^*) \xrightarrow{c_1} H^2_{orb}(X, \mathbb{Z}),$$

and we define the *Chern class* of a line V-bundle $\mathcal{L}$ to be the image $c_1(\mathcal{L}) \in H^2_{orb}(X, \mathbb{Z})$. More generally one can define the Chern classes of complex vector V-bundles by using the splitting principle.

We now want to describe the holomorphic line V-bundles. Let $\mathcal{Z}$ be a compact complex orbifold. In [BG1] the authors introduced the group $\mathrm{Pic}^{orb}(\mathcal{Z})$ of holomorphic line V-bundles on $\mathcal{Z}$. We now describe this group as a certain sheaf cohomology group. We consider the classifying space $B\mathcal{Z}$ as covered by "charts" of the form $\tilde{U}_i \times_{\Gamma_i} EGL(n)$. In the complex category the local uniformizing neighborhoods $\tilde{U}_i$ are identified with $\mathbb{C}^n$ with the origin as the fixed point of Γ_i, and the diffeomorphisms g_{ji} of Definition **??** are biholomorphisms. We now define the sheaf $\mathcal{A}$ of germs of holomorphic functions on $B\mathcal{Z}$ to be the sub-sheaf of $\mathcal{E}$ consisting of functions that are holomorphic in the $\tilde{U}_i$. Similarly, $\mathcal{A}^*$ denotes the sub-sheaf of germs of nowhere vanishing functions. As in Proposition 2.3.4 we have $\mathrm{Pic}^{orb}(\mathcal{Z}) \simeq H^1_{orb}(\mathcal{Z}, \mathcal{A}^*)$. Of course, to get the analogue of Theorem **??** in the holomorphic category, one must use Grauert's more difficult theorem that holomorphic bundles over $\tilde{U}_i$ are trivial, but everything else goes through in the same fashion. Then the exponential exact sequence gives the exact sequence

$$\cdots \longrightarrow H^1_{orb}(\mathcal{Z}, \mathcal{A}) \longrightarrow \mathrm{Pic}^{orb}(\mathcal{Z}) \xrightarrow{c_1} H^2_{orb}(\mathcal{Z}, \mathbb{Z}) \longrightarrow H^2_{orb}(\mathcal{Z}, \mathcal{A}) \longrightarrow \cdots.$$

In the case of interest to us when $\mathcal{Z}$ is a Fano variety, it seems likely that there is a vanishing theorem on $B\mathcal{Z}$ so that c_1 should be an isomorphism in the holomorphic case as well.

Now let $\mathcal{Z}$ be a compact complex Fano orbifold, that is, the anti-canonical line V-bundle $K^{-1}_{\mathcal{Z}}$ is ample. In this case $c_1(K^{-1}_{\mathcal{Z}}) > 0$.

Definition 2.3.5: *The* **index** *of a Fano orbifold $\mathcal{Z}$ is the largest positive integer m such that $\frac{c_1(K^{-1}_{\mathcal{Z}})}{m}$ is an element of $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. The index of $\mathcal{Z}$ is denoted by $Ind(\mathcal{Z})$.*

In view of the fact that the requisite vanishing theorem is lacking in the singular case, we shall need

Lemma 2.3.6: *Let $\mathcal{Z}$ be a Fano orbifold with $Ind(\mathcal{Z}) = m$. Then there is a holomorphic line V-bundle $\mathcal{L} \in Pic^{orb}(\mathcal{Z})$ such that $\mathcal{L}^m = K^{-1}_{\mathcal{Z}}$.*

Proof. The idea of the proof is simple, but since we are working on $B\mathcal{Z}$ we write out the details. First on $B\mathcal{Z}$ we define the following sheaf $\mathcal{E}^{p,q}$ of "differential forms": Let $(z_1, \cdots, z_n)$ be complex coordinates on $\tilde{U}_i$. Then using the standard multi-index notation, we construct the sheaf $\mathcal{E}^{p,q}$ whose stalks are spanned by elements of the form $f_{IJ}(z,e)dz_I \wedge d\bar{z}_J$ where $I = i_1 \cdots i_p$ and $J = j_1 \cdots j_q$ are the usual multi-indices, and f is a smooth function on $\tilde{U}_i \times EGL(2n)$ satisfying $f(\gamma^{-1}z, e\gamma) = f(z,e)$ for $\gamma \in \Gamma_i$. We have, as usual, differential operators $\bar{\partial}$ and the Dolbeault complex

$$\cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+1} \xrightarrow{\bar{\partial}} \cdots.$$

Notice that $\mathcal{E}^{0,0} = \mathcal{E}$ and $\ker(\bar{\partial} : \mathcal{E}^{0,0} \longrightarrow \mathcal{E}^{0,1}) = \mathcal{A}$.

By the isomorphism 2.3.4 there is a complex line V-bundle $\mathcal{L}$ such that $\mathcal{L}^m = K^{-1}$. The transition function h_{ij} for $\mathcal{L}$ are nowhere vanishing local sections of $\mathcal{E}$

over $\tilde{U}_i \times_{\Gamma_i} EGL(2n) \cap \tilde{U}_j \times_{\Gamma_j} EGL(2n)$ and the transition functions for $\mathcal{L}^m = K^{-1}$ are h_{ij}^m. But since K^{-1} is holomorphic $\bar{\partial} h_{ij}^m = 0$; hence, $\bar{\partial} h_{ij} = 0$ implying that $\mathcal{L}$ is holomorphic. □

In order to better understand the orbifold cohomology groups, we study the Leray spectral sequence of the map $p : B\mathcal{Z} \longrightarrow \mathcal{Z}$. For any sheaf $\mathcal{F}$ on $B\mathcal{Z}$ we let $R^q p(\mathcal{F})$ denote the derived functor sheaves, that is the sheaves associated to the presheaves $U \mapsto H^q(p^{-1}(U), \mathcal{F})$. Then Leray's theorem [Bre] says that there is a spectral sequence $E_r^{p,q}$ with E_2 term given by

$$E_2^{p,q} = H^p(\mathcal{Z}, R^q p(\mathcal{F}))$$

converging to $H_{orb}^{p+q}(\mathcal{Z}, \mathcal{F})$.

Lemma 2.3.7: *Let $\mathcal{Z}$ be a compact complex orbifold with only cyclic singularities. Then the derived functor sheaves are*

$$R^q p(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0, \\ \mathcal{F}^q & \text{for } q > 0 \text{ even}, \\ 0 & \text{for } q \text{ odd}, \end{cases}$$

where $\mathcal{F}^q$ denotes the sheaf whose support is contained in the singular set Σ of $\mathcal{Z}$ with stalks of the form $\mathbb{Z}_{m_i}$ at $x_i \in \Sigma$.

Proof. If $(\tilde{U}, \Gamma)$ is a local uniformizing system for the open set $U \subset \mathcal{Z}$ then $p^{-1}(U) = \tilde{U} \times_\Gamma EO(n)$ which is a $K(\Gamma, 1) = B\Gamma$. Thus, $H^q(p^{-1}(U), \mathbb{Z})$ is the group cohomology $H^q(\Gamma, \mathbb{Z})$. Now for Γ a cyclic group $\mathbb{Z}_m$ it is well-known that

$$H^q(\mathbb{Z}_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0, \\ \mathbb{Z}_m & \text{for } q > 0 \text{ even}, \\ 0 & \text{for } q \text{ odd}. \end{cases}$$

So for $q > 0$ the stalks vanish at regular points and is $\mathbb{Z}_{m_i}$ at the singular point x_i. □

Next we have from Lemma 2.3.7:

Lemma 2.3.8: *The E_2 terms satisfies $E_2^{p,q} = 0$ for q odd and for $p = 1, q = 0$, whereas for $q > 0$ even it is a product of cyclic groups.*

One now easily sees that by tensoring with certain rings the spectral sequence collapses and we recover a result of Haefliger [Hae], namely

Corollary 2.3.9: *$H_{orb}^q(\mathcal{Z}, A) \simeq H^q(\mathcal{Z}, A)$ for $A = \mathbb{Q}$ or $\mathbb{Z}_p$ where p is relatively prime to $Ord(\mathcal{Z})$.*

2.4. Weighted Projective Spaces

In this section we discuss weighted projective spaces both as singular (or non-singular) algebraic varieties, and as complex orbifolds. It is important to regard orbifolds as "added structure", and distinquish the orbifold from its underlying algebraic variety. In this sense an orbifold is a special type of Deligne-Mumford stack. We begin with algebraic geometric treatment of weighted projective spaces. References here are [**64, 67, 78**].

Given a sequence $\mathbf{w} = (w_0, \ldots, w_n)$ of ordered positive integers one can form the graded polynomial ring $S(\mathbf{w}) = \mathbb{C}[z_0, \ldots, z_n]$, where z_i has *weight* w_i. Following [**67**] we have

Definition 2.4.1: *The* **weighted projective space** *with weights* $\mathbf{w} = (w_0, \cdots, w_n)$ *is defined to be the projective scheme Proj*$(S(\mathbf{w}))$*, and is denoted by* $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$.

CHAPTER 3

Almost Contact and Contact Geometry

In this chapter we describe the basics of contact and almost contact geometry. Unfortunately, the terminology is not completely standard although preference has been given to defining a contact structure as a codimension one subbundle of the tangent bundle that is as far as possible from being integrable (see ? below). However, what is worse, is that preference has also been given to defining an almost contact manifold [**100**] in terms of an equivalence class of reductions of the orthonormal frame bundle to the group $U(n)\times 1$, and with this convention a contact manifold is not necessarily an almost contact manifold! Although this, in the words of R.E. Stong [**198**], is "terrible", it seems to be too well ensconced in the literature to be rectifiable. In fact for our purposes this problem of terminology is mute, since the confusion only arises when the contact line bundle is not trivial, and the contact structures of interest to us always have trivial contact line bundle. Nevertheless, we take some pains to set things up correctly in the beginning. We shall then revert to the usual convention.

3.1. Contact Structures

Contact Transformations in $\mathbb{R}^{2n+1}$.

Contact transformations arose in the theory of Analytical Mechanics developed in the 19th century by Hamilton, Jacobi, Lagrange, and Legendre. But its first systematic treatment was given by Sophus Lie [Lie] under the German name Behrührungstransformation. Consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x^1,\cdots,x^n;y^1,\cdots,y^n;z)$, together with a 1-form α given by

$$\alpha = dz-\sum_i y^i dx^i. \tag{3.1.1}$$

It is easy to see that α satifies

$$\alpha\wedge(d\alpha)^n\neq 0 \tag{3.1.2}$$

on $\mathbb{R}^{2n+1}$. A 1-form that satisfies this equation is called a **contact form**.

REMARK 3.1.1: The choice of α in Equation 3.1.1 is not unique. Other choices are probably just as reasonable, for example,

$$\alpha' = dz+\frac{1}{2}\sum_{i=0}^n (x^i dy^i - y^i dx^i).$$

This clearly also satifies 3.1.2, and differs from α by an exact form, namely, $\frac{1}{2}d(\mathbf{x}\cdot\mathbf{y})$.

We shall mainly use the form α of Equation 3.1.1 which we call the *standard 1-form* on $\mathbb{R}^{2n+1}$. There is local converse to this, namely, the contact version of Darboux's theorem.

Theorem 3.1.1 (Darboux)**:** *Let α be a 1-form on $\mathbb{R}^{2n+1}$ that satisfies equation 3.2. Then there is an open set $U \subset \mathbb{R}^{2n+1}$ and local coordinates $(x^1, \cdots, x^n; y^1, \cdots, y^n; z)$ such that α has the form 3.1 in U.*

Darboux's theorem says that locally all contact forms look the same. A proof can be found in most books on Analytical Mechanics, e.g. [AM,LM]. The modern proof is due to Moser [Mo] and Weinstein [Wei]

Definition 3.1.2: *A diffeomorphism $\phi : U \longrightarrow V$ where U, V are open sets of $\mathbb{R}^{2n+1}$ is called a* **contact transformation** *if there is a nowhere vanishing smooth function f such that*

$$\phi^*\alpha = f\alpha.$$

If $f \equiv 1$ on U, then ϕ is called a **strict contact transformation**.

The collection $\Gamma^{\mathfrak{C}}$ of all such contact transformations forms a pseudogroup, called the *contact pseudogroup*; it is one of the transitive infinite, simple pseudogroups of Lie and Cartan. In fact $\Gamma^{\mathfrak{C}}$ is an example of a transitive pseudogroup which does not come from an integrable G-structure. It is, however, related to a non-integrable G-structure as we shall see.

Exercise 3.1: Show that the subset of strict contact transformations forms a subpseudogroup $\Gamma^{s\mathfrak{C}}$ of $\Gamma^{\mathfrak{C}}$.

Next we describe the Lie algebra of infinitesimal contact transformations. By taking ϕ_t to be a local 1-parameter group of transformations generated by a vector field X on U, it is easy to see that the infinitesimal version of Definition 3.1.2 is:

Definition 3.1.3: *An* **infinitesimal contact transformation** *is a local vector field X defined on an open set $U \subset \mathbb{R}^{2n+1}$ that satisfies*

$$\pounds_X\alpha = g\alpha$$

where g is a smooth function on U. If g vanishes on U, then X is called a **infinitesimal strict contact transformation**. *Let $s\mathfrak{c}(U)$ and $\mathfrak{c}(U)$ denote the subsets of all vector fields on U consisting of infinitesimal strict contact transformations and infinitesimal contact transformations, respectively.*

Exercise 3.2: Show that $\mathfrak{c}(U)$ forms a Lie subalgebra of the Lie algebra of all vector fields on U. Show that $s\mathfrak{c}(U)$ is a Lie subalgebra of $\mathfrak{c}(U)$.

It is easy to see that the vector space $\mathfrak{c}(U)$ of all infinitesimal contact transformations on U forms a Lie algebra under the Lie bracket of vector fields. In order to see that the contact pseudogroup $\Gamma^{\mathfrak{C}}$ is transitive, we shall compute explicitly the Lie algebra of infinitesimal contact transformations. Using the well known identity $\pounds_X\alpha = d(X\rfloor\alpha) + X\rfloor d\alpha$, one sees that X must take the form

$$X = \sum_{i=1}^{n}\big(y^i\frac{\partial F}{\partial z} + \frac{\partial F}{\partial x^i}\big)\frac{\partial}{\partial y^i} + \sum_{i=1}^{n}\frac{\partial F}{\partial y^i}\frac{\partial}{\partial x^i} + \big(F - \sum_{i=1}^{n} y^i\frac{\partial F}{\partial y^i}\big)\frac{\partial}{\partial z} \tag{3.1.3}$$

where F is an arbitrary smooth function on U known as a *Hamiltonian* function for the infinitesimal contact transformation X.

Consider the subalgebra $\mathfrak{h}$ of $\mathfrak{sc}(\mathbb{R}^{2n+1})$ spanned by the vector fields

$$Z = \frac{\partial}{\partial z}, \qquad X_i = \frac{\partial}{\partial x^i}, \qquad Y_i = \frac{\partial}{\partial y^i} + x^i \frac{\partial}{\partial z} \tag{3.1.4}$$

which satisfy the Lie bracket identities

$$[X_i, Y_j] = \delta_{ij} Z, \qquad [X_i, Z] = [Y_i, Z] = 0.$$

$\mathfrak{h}$ is known as the *Heisenberg algebra*, and the Lie group $\mathfrak{H}$ that it generates is the *Heisenberg group*. Here we have realized $\mathfrak{H}$ as a Lie subgroup of the pseudogroup $\Gamma^{s\mathfrak{C}}$ of strict contact transformations on $\mathbb{R}^{2n+1}$.

EXERCISE 3.3: Show that the Heisenberg group $\mathfrak{H}$ is realized as a subgroup of $\Gamma^{s\mathfrak{C}}$ by the transformations (using vector notation)

$$(\mathbf{x}, \mathbf{y}, z) \mapsto (\mathbf{x} + \mathbf{a}, \mathbf{y} + \mathbf{b}, z + c + \mathbf{x} \cdot \mathbf{b}).$$

Notice that as manifolds $\mathfrak{H} = \mathbb{R}^{2n+1}$ whose group law is given by $(\mathbf{a}, \mathbf{b}, c) \cdot (\mathbf{x}, \mathbf{y}, z) =$ the right hand side of the above equation. It is clear that $\mathfrak{H}$ acts transitively on $\mathbb{R}^{2n+1}$; hence, we have arrived at

Proposition 3.1.4: $\Gamma^{s\mathfrak{C}}$ *and* $\Gamma^{\mathfrak{C}}$ *are transitive pseudogroups.*

Basic Definitions.

We begin with the most general definition of a contact manifold. We give a more restricted definition below, which is the one we concentrate on in this monograph.

Definition 3.1.5: *A* $2n+1$ *dimensional manifold* M *with a* $\Gamma^{\mathfrak{C}}$*-structure is called a* **contact manifold**. *If* M *has a* $\Gamma^{s\mathfrak{C}}$*-structure, then it is called a* **strict contact manifold**.

The definition of contact manifold appears to be due to Gray [Gr]. This has also been given the somewhat cumbersome name *contact structure in the wider sense*, but we shall not use this terminology. A contact structure on M gives rise to a real line bundle $\mathcal{L}$ called the *contact line bundle* as follows. In the chart (U_i, ϕ_i) we have the contact form $\phi_i^*\alpha$, and in the overlap $U_i \cap U_j$ there exists a nowhere vanishing function ρ_{ij} such that

$$\phi_i^*\alpha = \rho_{ij}\phi_j^*\alpha. \tag{3.3}$$

The $\{\rho_{ij}\}$ define a cocycle for the sheaf cohomology group $H^1(M, \mathcal{G}\ell(1,\mathbb{R}))$ where $\mathcal{G}\ell(1,\mathbb{R})$ is the sheaf of nowhere vanishing real functions on M, and thus a real line bundle $\mathcal{L}$. The cocycle $\{\rho_{ij}\}$ is trivial or a coboundary if for all i, j the ρ_{ij} can be taken to be positive, that is local sections of the sheaf $\mathcal{G}\ell(1,\mathbb{R})^+$ of strictly positive functions. In this case there are positive functions f_i such that for all i, j we have $f_i\phi_i^*\alpha = f_j\phi_j^*\alpha$. Thus, we have a nowhere vanishing global section η of $\mathcal{L}$ defined on each U_i by $\eta|U_i = f_i\phi_i^*\alpha$. This trivializes $\mathcal{L}$ in which case M admits a global contact 1-form. In the sequel we shall deal almost exclusively with contact manifolds with trivial contact line bundle $\mathcal{L}$. In fact it is quite common to take this as the definition of a contact manifold. We shall often consider a contact structure $\Gamma^{\mathfrak{C}}$ on a strict contact manifold M. In this case the maximal atlas defining the strict contact structure is a subatlas of the corresponding maximal atlas defining $\Gamma^{\mathfrak{C}}$. In this case the contact structure can be defined as an equivalence class of globally defined contact 1-forms. Thus, we are led to our second slightly more restrictive definition of a contact manifold.

Definition 3.1.6: *A $2n+1$ dimensional manifold M is a (strict)* **contact manifold** *if there exists a 1-form η, called a* **contact 1-form**, *on M such that*

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A **contact structure** *on M is an equivalence class of such 1-forms where $\eta' \sim \eta$ if there is a nowhere vanishing function f on M such that $\eta' = f\eta$.*

For purposes of this section only we shall refer to a contact structure in the sense of Definition 3.1.6 as a *strict* contact structure. Clearly for any real line bundle $\mathcal{L}$ the tensor product with itself $\mathcal{L}^2$ is trivial. This gives rise to the following result attributed to Sasaki [**28**]:

Proposition 3.1.7: *Let M^{2n+1} be a contact manifold which is not strict, then its 2-fold cover is a strict contact manifold. In particular, every simply connected contact manifold is a strict contact manifold.*

A contact structure also gives rise to a codimension one subbundle of TM. On U_i we define $\mathcal{D}_i = \ker\phi_i^*\alpha$, and on $U_i \cap U_j$ equation 3.3 holds implying that $\mathcal{D}_i = \mathcal{D}_j$. So we get a well defined subbundle $\mathcal{D}$ of TM, called the *contact subbundle*, or *contact distribution* which is 'as far from being integrable as possible'. This terminology stems from the definition of integrability in the sense of the classical Frobenius Theorem. Let $\mathcal{D}$ be a codimension one subbundle of TM so there is an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{D} \longrightarrow TM \longrightarrow Q \longrightarrow 0.$$

Dualizing we see that Q^* is just the contact line bundle $\mathcal{L}$, so we have the exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow T^*M \longrightarrow \mathcal{D}^* \longrightarrow 0.$$

Now $\mathcal{D}$ is integrable if and only if there are local sections α_i of $\mathcal{L}$ that satisfy $\alpha_i \wedge d\alpha_i = 0$ for all i. Hence, we say that a subbundle $\mathcal{D}$ of TM defined as the kernel of the local 1-forms α_i is *as far from being integrable as possible* if the local 1-forms satisfy $\alpha_i \wedge (d\alpha_i)^n \neq 0$. Thus, the contact subbundle actually characterizes the contact structure, that is we have

Proposition 3.1.8: *Let M be a $2n+1$-dimensional manifold. Then a codimension one subbundle $\mathcal{D}$ of TM defines a contact structure on M if and only if $\mathcal{D}$ is as far from being integrable as possible.*

Definition 3.1.9: *If the vector bundle $\mathcal{D}$ is oriented we say that the contact manifold M is* **co-oriented**.

The following is a result of Stong [**198**] which corrects an error in Proposition 2.2.1 and Theorem 2.2.2 of [**100**]. This error was propogated in [**126**].

Proposition 3.1.10: *Let M be a $2n+1$-dimensional contact manifold with contact bundle $\mathcal{D}$. Then*

(i) *If n is odd, then M is oriented.*
(ii) *If n is even, then M is co-oriented. Thus, in this case M has a strict contact structure if and only if M is oriented.*

PROOF. There is a cover $\{U_i\}$ of M together with 1-forms α_i on U_i defining the contact structure. In the overlap $U_i \cap U_j \neq \emptyset$ there are nonvanishing functions

f_i, f_j such that $f_i\alpha_i = f_j\alpha_j$. Then on $U_i \cap U_j \neq \emptyset$ we have

$$(d\alpha_i)^n|_{\mathcal{D}} = \big(\frac{f_j}{f_i}\big)^n (d\alpha_j)^n|_{\mathcal{D}}, \qquad \alpha_i \wedge (d\alpha_i)^n = \big(\frac{f_j}{f_i}\big)^{n+1} \alpha_j \wedge (d\alpha_j)^n.$$

Thus, if n is odd, we get a well-defined orientation; whereas, if n is even we get an orientation on $\mathcal{D}$. In the latter case M is orientable if and only if we can choose the f_i all positive, that is, if and only if the contact line bundle is trivial. $\square$

Now we have an immediate

Corollary 3.1.11: *Let M be a strict contact manifold. Then M is both oriented and co-oriented.*

Examples.

Our first example is the most obvious.

EXAMPLE 3.1.12: $M = \mathbb{R}^{2n+1}$. The global 1-form given by equation 3.1.1 defines a strict contact structure on $\mathbb{R}^{2n+1}$. The contact subbundle $\mathcal{D}$ is spanned by the vector fields $\partial_{x^i} + y^i\partial_z$ and ∂_{y^i}. This is called the **standard contact structure** on $\mathbb{R}^{2n+1}$.

The next example will be fundamental in the sequel.

EXAMPLE 3.1.13: $M = S^{2n+1}$, the unit $(2n+1)$-sphere. Consider the 1-form on $\mathbb{R}^{2n+2}$ defined in the standard Cartesian coordinates $(x^0, \cdots, x^n, y^0, \cdots, y^n)$ by $\sum_{i=0}^n (x^i dy^i - y^i dx^i)$. Let η denote its restriction to the unit sphere. We claim that $\eta \wedge (d\eta)^n \neq 0$ everywhere on S^{2n+1}. A straightforward computation shows that

$$\eta \wedge (d\eta)^n = 2^n n! \big(\sum_{i=0}^n x^i dx^0 \wedge dy^0 \wedge \cdots \wedge \widehat{dx^i} \wedge dy^i \wedge \cdots \wedge dx^n \wedge dy^n$$

$$- \sum_{i=0}^n y^i dx^0 \wedge dy^0 \wedge \cdots \wedge dx^i \wedge \widehat{dy^i} \wedge \cdots \wedge dx^n \wedge dy^n\big)|_{S^{2n+1}},$$

where as usual $\widehat{dx}$ means the dx is removed. Now the 1-form $\beta = \sum_{i=0}^n (x^i dx^i + y^i dy^i)$ is normal to the sphere S^{2n+1} and an easy computation gives

$$\beta \wedge \eta \wedge (d\eta)^n = 2^n n! \sum_{i=0}^n \big((x^i)^2 + (y^i)^2\big) dx^0 \wedge dy^0 \wedge \cdots \wedge dx^n \wedge dy^n.$$

This shows that $\eta \wedge (d\eta)^n$ vanishes nowhere on S^{2n+1}. This defines the **standard contact structure** on S^{2n+1}. Notice also that η is invariant under the reflection $(x^i, y^i) \mapsto (-x^i, -y^i)$ for all $i = 0, \cdots, n$. Thus, we get an induced contact structure on the real projective space $\mathbb{RP}^{2n+1}$ as well.

There is a straightforward generalization of the example above due to Gray [**100**] (see also [**28, 224**]) that shows that any immersed hypersurface in $\mathbb{R}^{2n+2}$ such that no point of the tangent space to the hypersurface contains the origin $0 \in \mathbb{R}^{2n+2}$ admits a strict contact structure. Here is the construction:

Proposition 3.1.14: *Let M be an immersed hypersurface in $\mathbb{R}^{2n+2}$ such that no tangent space of M contains the origin of $\mathbb{R}^{2n+2}$. Then M has a strict contact structure.*

PROOF. Again consider the 1-form in Cartesian coordinates $(x^0,\cdots,x^{2n+1})$ of $\mathbb{R}^{2n+2}$ defined by

$$\alpha=\sum_{i=0}^{n}(x^idx^{n+1+i}-x^{n+1+i}dx^i).$$

Choose a point $\mathbf{x}_0\in M$ together with a basis $(v_1,\cdots,v_{2n+1})$ of $T_{\mathbf{x}_0}M$. Consider the vector $\mathbf{w}$ whose components are $w^i=(\star dx^i)(v_1,\cdots,v_{2n+1})$ where $\star$ denotes the Hodge star operator with respect to the Euclidean metric on $\mathbb{R}^{2n+2}$. Now $\mathbf{w}$ is perpendicular to $T_{\mathbf{x}_0}M$ viewed as a subspace of $\mathbb{R}^{2n+2}$. Regarding $\mathbf{x}_0$ as a vector, a simple computation gives

$$\alpha\wedge(d\alpha)^n(v_1,\cdots,v_{2n+1})=\mathbf{x}_0\cdot\mathbf{w}.$$

Thus, the right hand side of this equation vanishes if and only if the tangent space $T_{\mathbf{x}_0}M$ contains the origin. So under the hypothesis of the proposition, the restriction of α to M defines a strict contact structure on M. □

Another variation on the same theme, again due to Gray, uses the cotangent bundle T^*N of a manifold N. On T^*N there is a tautological or canonical 1-form γ defined as follows: if $v\in T_\alpha T^*N$ at the covector $\alpha\in T^*N$, then γ is defined by

$$\gamma(v)=<\alpha,\pi_*v> \tag{3.1.5}$$

where $\pi:T^*N\longrightarrow N$ is the natural projection and $<\cdot,\cdot>$ denotes the natural pairing between T_xN and T^*_xN. It is not difficult to show that $(d\gamma)^{n+1}\neq 0$. Thus, the cotangent bundle of any manifold has a natural symplectic structure.

Proposition 3.1.15: *Let M be an immersed submanifold of T^*N with its canonical 1-form γ. Then the restriction of γ to M defines a contact structure on M if and only if the following conditions hold:*

(i) *dim $(M\cap\pi^{-1}(x))=n$ for all $x\in N$.*
(ii) *$M\cap N=\emptyset$ where N is identified with the zero section of T^*N.*
(iii) *No tangent space to $M\cap\pi^{-1}(x)$ in $\pi^{-1}(x)$ contains the origin in $\pi^{-1}(x)$.*

Furthermore, if the conditions are satisfied the contact structure is strict.

PROOF. The proof is similar to that of Proposition 3.1.14. If $(x^0,\cdots,x^n)$ are local coordinates on an open set $U\subset N$, and $(y_0,\cdots,y_n)$ are coordinates in the fibres, then $(x^0,\cdots,x^n,y_0,\cdots,y_n)$ are coordinates in $\pi^{-1}(U)$. In these coordinates γ takes the form $\gamma=\sum_{i=0}^n y_idx^i$. Clearly, γ vanishes along the zero section of T^*N, so condition (2) is necessary. As in the previous case let $(v_1,\cdots,v_{2n+1})$ be a local frame for $M\cap\pi^{-1}(U)$. Define a tangent vector w to the fibres $\pi^{-1}(x)$ whose components with respect to the coordinate frame in $\pi^{-1}(x)$ are

$$w^i=dy_0\wedge dx^0\wedge\cdots\wedge\widehat{dy_i}\wedge dx^i\wedge\cdots dy_n\wedge dx^n(v_1\cdots,v_{2n+1}),$$

where only the dy's are removed. Then a simple computation gives

$$\gamma\wedge(d\gamma)^n(v_1,\cdots,v_{2n+1})=\sum_{i=0}^n y_iw^i. \tag{3.1.6}$$

Now the proposition follows as before. □

As an application of this proposition we have immediately

EXAMPLE 3.1.16: The unit sphere bundle T_1^*N of a manifold N with respect to some Riemannian metric. One easily checks that T_1^*N satifies the three conditions of Proposition 3.1.15. Thus, T_1^*N has a (strict) contact structure.

Proposition 3.1.15 suggests the following

Definition 3.1.17: *Let (N,ω) be a symplectic manifold. A hypersurface $M \overset{\iota}{\hookrightarrow} N$ is said to be of* **contact type** *if there is a 1-form α on M such that $\iota^*\omega = d\alpha$.*

Next is an example of a contact structure due to Gray [**100**] which shows (ironically) that (ii) of Proposition 2.2.1 as well as Theorem 2.2.2 of [**100**] and Theorem 7.3 of [**126**] are incorrect. This was first noticed in [**198**].

EXAMPLE 3.1.18: $M = \mathbb{R}^{n+1} \times \mathbb{RP}^n$. Let x^i for $i = 0, \cdots, n$ denote the Cartesian coordinates on $\mathbb{R}^{n+1}$ and t_i for $i = 0, \cdots, n$ denote the homogeneous coordinates on the real projective space $\mathbb{RP}^n$. Let $U_i \subset \mathbb{R}^{n+1} \times \mathbb{RP}^n$ be the affine neighborhood defined by $t_i \neq 0$. Clearly $\{U_i\}_{i=0}^n$ cover M. We define the contact structure by a sequence of 1-forms α_i defined in U_i by

$$\alpha_i = dx^i + \sum_{j\neq i} \frac{t_j}{t_i} dx^j = \sum_j \frac{t_j}{t_i} dx^j.$$

In the overlap $U_i \cap U_j$ we see that

$$\alpha_j = \frac{t_i}{t_j}\alpha_i,$$

and this defines the contact line bundle $\mathcal{L}$ which is non-trivial since it is induced by the tautological line bundle on $\mathbb{RP}^n$. Hence, there is no globally defined contact 1-form on M which defines the contact structure, that is, the contact structure is not strict. An easy calculation also shows that

$$\alpha_j \wedge (d\alpha_j)^n = \big(\frac{t_i}{t_j}\big)^{n+1} \alpha_i \wedge (d\alpha_i)^n,$$

so M is orientable if and only if n is odd, and in this case M is not co-orientable.

Stong [**198**] noticed the following generalization of this example:

EXERCISE 3.4: Consider the projectivised cotangent bundle $\pi : M = \mathbb{P}(T^*N) \longrightarrow N$ where N has dimension $n+1$. Let U be the open set of the chart defined by the local coordinates $(x^0, \cdots, x^n)$, and let $(y^0, \cdots, y^n)$ denote the homogeneous coordinates in the fibres of $\pi^{-1}(U) \subset T^*N$. Let $U_i \subset M$ be the open set of $\pi^{-1}(U)$ defined by setting $y^i \neq 0$. On U_i define the 1-form $\alpha_i = (y^i)^{-1} \sum_{j=0}^n y^j dx^j$. Show that the α_i's patch together to define a contact structure on M.

For further examples see Blair's recent book [**28**]. Of particular interest is the fundamental theorem of Martinet in dimension three, viz.

Theorem 3.1.19 (Martinet)**:** *Every orientable 3-manifold admits a contact structure.*

Following Blair [**28**] we give an explicit contact structure on the 3-torus T^3.

EXAMPLE 3.1.20: $M = T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Let (x, y, z) denote standard Cartesian coordinates in $\mathbb{R}^3$, and consider the 1-form

$$\eta = \sin y dx + \cos y dz.$$

Then we find $\eta \wedge d\eta = -dx \wedge dy \wedge dz$. The contact subbundle $\mathcal{D}$ is spanned by the vector fields

$$\{\frac{\partial}{\partial y}, \cos y \frac{\partial}{\partial x} - \sin y \frac{\partial}{\partial z}\}.$$

In particular, Blair discusses the so-called overtwisted contact structures in dimension 3 of Eliashberg. These seem to have no analogues in higher dimensions. Lutz [?] proved the existence of contact structures on principal T^2-bundles over 3-manifolds. In particular, the 5-torus T^5 admits a contact structure.

EXERCISE 3.5: Let $(x_1, \cdots, x_5)$ denote the standard coordinates on $T^5 = \mathbb{R}^5/\mathbb{Z}^5$. Show that the 1-form

$$\eta = (\sin x_1 \cos x_3 - \sin x_2 \sin x_3)dx_4 + (\sin x_1 \sin x_3 + \sin x_2 \cos x_3)dx_5$$
$$+ \sin x_2 \cos x_2 dx_1 - \sin x_1 \cos x_1 dx_2 + \cos x_1 \cos x_2 dx_3$$

defines a contact structure on T^5.

Global Contact Transformations

We begin by describing some elementary properties of the group of contact transformations. Let M be a strict contact manifold, and let $\mathfrak{C}(M, \mathcal{D})$ denote the group of contact transformations, *i.e.,* the subgroup of the group Diff(M) of diffeomophisms of M that leaves the contact distribution $\mathcal{D}$ invariant. If we fix a contact form η such that $\mathcal{D} = \ker\, \eta$, then $\mathfrak{C}(M, \mathcal{D})$ can be characterized as the subgroup of diffeomorphisms $\phi : M \longrightarrow M$ that satisfy $\phi^*\eta = f\eta$ for some nowhere vanishing $f \in C^\infty(M)$. With the 1-form η fixed we are interested in the subgroup $\mathfrak{C}(M, \eta)$ of *strict contact transformations* defined by the condition $\phi^*\eta = \eta$. Likewise, we denote the "Lie algebras" of $\mathfrak{C}(M, \mathcal{D})$ and $\mathfrak{C}(M, \eta)$ by $\mathfrak{c}(M, \mathcal{D})$ and $\mathfrak{c}(M, \eta)$, respectively. They can be characterized as follows:

$$\mathfrak{c}(M, \mathcal{D}) = \{X \in \Gamma(M) | \mathcal{L}_X \eta = g\eta \text{ for some } g \in C^\infty(M)\},$$
$$\mathfrak{c}(M, \eta) = \{X \in \Gamma(M) | \mathcal{L}_X \eta = 0\},$$

where $\Gamma(M)$ denotes the Lie algebra of smooth vector fields on M. Clearly, $\mathfrak{c}(M, \eta)$ is a Lie subalgebra of $\mathfrak{c}(M, \mathcal{D})$. In contrast to the symplectic case, every infinitesimal contact transformation is Hamiltonian. More explicitly,

Proposition 3.1.21: *The contact 1-form η induces a Lie algebra isomorphism Φ between the Lie algebra $\mathfrak{c}(M, \mathcal{D})$ of infinitesimal contact transformations and the Lie algebra $C^\infty(M)$ of smooth functions with the Jacobi bracket defined by*

$$\Phi(X) = \eta(X).$$

Furthermore, under this isomorphism the subalgebra $\mathfrak{c}(M, \eta)$ is isomorphic to the subalgebra $C^\infty(M)^\xi$ of functions in $C^\infty(M)$ that are invariant under the flow generated by the Reeb vector field ξ. In particular, $\xi = \Phi^{-1}(1)$.

The function $\eta(X)$ is known as a *contact Hamiltonian function.*

Similarly, on $C(M)$ we let $\mathfrak{S}(C(M), \omega)$ denote the group of symplectomorphisms of $(C(M), \omega)$, and $\mathfrak{S}_0(C(M), \omega)$ the subgroup of $\mathfrak{S}(C(M), \omega)$ that commutes with homotheties, *i.e.,* the automorphism group of the symplectic Liouville structure. The corresponding Lie algebras are denoted by $\mathfrak{s}(C(M), \omega)$ and $\mathfrak{s}_0(C(M), \omega)$, respectively. Given a vector field $X \in \mathfrak{s}_0(C(M), \omega)$, the fact that X commutes with the Liouville vector field Ψ implies that X is projectable to a vector field X_M on M and one easily sees that [LM]

Proposition 3.1.22: *The maps $X \mapsto X_M \mapsto \eta(X_M)$ induce Lie algebra isomorphisms $\mathfrak{s}_0(C(M),\omega) \approx \mathfrak{c}(M,\eta) \approx C^\infty(M)^\xi$. Furthermore, ξ is in the center of $\mathfrak{c}(M,\eta)$.*

Add more later. In particular, Gray's Theorem.

3.2. Almost Contact Structures

Definitions and Topological Properties

Argueably the most efficient way to define almost structures is through G-structures which we shall employ. Our first definition of an almost contact structure completely non-standard, but at least it is terminologically consistent. As mentioned previously the standard definition of almost contact structures [**27, 28, 224**] is too restrictive. For example, the contact manifold $M = \mathbb{R}^{n+1} \times \mathbb{RP}^n$ of Example 3.1.18 is *not* an almost contact manifold in the usual sense. Recall the almost contact G-structure of Example 1.3.12, then we have

Theorem 3.2.1: *A contact structure on a manifold M is a non-integrable almost contact G-structure.*

Proof. Let us look at the contact pseudogroup $\Gamma^{\mathfrak{C}}$ at the level of the tangent bundle. Let $\phi \in \Gamma^{\mathfrak{C}}$, then its differential ϕ_* leaves the contact subbundle $\mathcal{D}$ invariant. Choose a local 1-form α_i representing the contact structure. Then $d\alpha_i|_{\mathcal{D}}$ defines a symplectic structure on $\mathcal{D}$ over U_i. Fix a point $x \in U_i$, and let $\Gamma^{\mathfrak{C}}_x$ denote the subpseudogroup generated by the vector fields in $\mathfrak{c}(U_i)$ that vanish at x. The differentials of the elements of $\Gamma^{\mathfrak{C}}_x$ span the linear isotropy algebra of $\mathfrak{c}$ which determines the G-structure. Let $\phi \in \Gamma^{\mathfrak{C}}_x$, then ϕ^* leaves the symplectic form $d\alpha_i|_{\mathcal{D}}$ invariant up to a multiplicative factor. So the linear isotropy group is precisely the group G of Example 1.3.12.

Furthermore, since α_i and the symplectic form $d\alpha_i|_{\mathcal{D}}$ satisfy Equation 3.1.2, this G-structure is not integrable. □

A consistent definition for an almost contact structure in terms of G-structures is:

Definition 3.2.2: *An* **almost contact structure** *on M is a reduction of the frame bundle to the group G defined in 1.3.12, that is, an almost contact G-structure.*

Thus, an almost contact structure picks out a codimension one subbundle $\mathcal{D}$ of TM with a conformal symplectic structure on it. By reducing to maximal compact subgroups we have

Theorem 3.2.3: *A manifold M of dimension $2n+1$ admits an almost contact structure if and only if its frame bundle can be reduced to the group $U(n)\times\mathbb{Z}_2\times\mathbb{Z}_2$, where the first $\mathbb{Z}_2$ corresponds to the sign of the conformal factor in $CSp(n,\mathbb{R})$ and the second $\mathbb{Z}_2$ corresponds to the sign of the determinant. Moreover, if M is orientable, there is a reduction to the group $U(n)\times\mathbb{Z}_2$, which further reduces to the group $U(n)\times 1$ if and only if the subbundle $\mathcal{D}$ of TM is orientable. Furthermore, if M is non-orientable, but $\mathcal{D}$ is orientable, then there is a reduction to the subgroup $U(n)\times\mathbb{Z}_2$.*

Proof. The group $CSp(n,\mathbb{R})$ is homotopy equivalent to $\mathbb{Z}_2 \times Sp(n,\mathbb{R})$ where the group $\mathbb{Z}_2$ is identified with the sign of the conformal factor, and $Sp(n,\mathbb{R})$ is homotopy equivalent to its maximal compact subgroup $U(n)$. So the group G of Example 1.3.12 is homotopy equivalent to the block matrices of the form

$$G = \{\begin{pmatrix} C & 0 \\ 0 & b \end{pmatrix} \mid C \in H,\ b \in \mathbb{Z}_2\}$$

where $H \approx U(n) \times \mathbb{Z}_2$ is realized as the subgroup of $O(2n, \mathbb{R})$ generated by matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

where A and B are n by n block matrices that satisfy

$$\bar{A}^t A + \bar{B}^t B = \mathbb{I}_n, \qquad \overline{(\bar{A}^t B)^t} = \bar{A}^t B.$$

If the manifold is orientable we can take $\det M = b$, so the group reduces to $U(n) \times \mathbb{Z}_2$. Moreover, the subbundle $\mathcal{D}$ is orientable if and only if we can take $\det C = 1$, in which case the group reduces to $U(n) \times 1$. $\square$

Combining Theorem 3.2.3 and Proposition 3.1.10 we have

Corollary 3.2.4: *Let M be a $2n+1$-dimensional contact manifold, then the frame bundle can be reduced to the subgroup $U(n) \times \mathbb{Z}_2$.*

(i) *If n is odd, then M can be reduced to the group $U(n) \times 1$ if and only if M is co-orientable.*
(ii) *If n is even, then M can be reduced to the group $U(n) \times 1$ if and only if M is orientable.*
(iii) *M has a strict contact structure if and only if M can be reduced to the group $U(n) \times 1$.*

We shall now discuss the standard definition of an almost contact structure [**27, 28, 224**], which is more closely related to strict contact structures than contact structures. However, even in this regard the definition includes additional structure, namely a choice of vector field. Also from the point of view of G-structures, it would be more consistent to define almost contact structures using a 'transverse symplectic form' instead of a 'transverse almost complex structure'. Of course, in the presence of a metric the two are equivalent, so we shall stick with what is commonly done. At least for the purposes of this section we shall add the word 'strict' to the definition.

Definition 3.2.5: *A (strict) almost contact structure on a differentiable manifolds M is a triple (ξ, η, Φ), where Φ is a tensor field of type $(1,1)$ (i.e. an endomorphism of TM), ξ is a vector field, and η is a 1-form which satisfy*

$$\eta(\xi) = 1 \text{ and } \Phi \circ \Phi = -\mathbb{I} + \xi \otimes \eta,$$

where $\mathbb{I}$ is the identity endomorphism on TM. A smooth manifold with such a structure is called an **almost contact manifold**.

Such an almost contact structure (ξ, η, Φ) has three other almost contact structures canonically associated with it, namely $(\xi, \eta, -\Phi)$, $(-\xi, -\eta, \Phi)$, and $(-\xi, -\eta, -\Phi)$. Notice that an almost contact manifold is necessarily odd dimensional, and one easily verifies the identities [**27, 28, 224**]

$$\Phi \circ \xi = 0, \qquad \eta \circ \Phi = 0. \tag{3.2.1}$$

Thus, an almost contact manifold M comes canonically equipped with the following structures: Since the vector field ξ is nowhere vanishing, it generates a 1-dimensional subbundle L_ξ of the tangent bundle TM; hence, an almost contact manifold M has associated to it a 1-dimensional foliation $\mathcal{F}_\xi$, called the *characteristic foliation*. The 1-form η is called the *characteristic 1-form* and it plays the role of the characteristic form χ in foliation theory [**210**], even though χ requires a Riemannian metric for its definition. The codimension one subbundle $\mathcal{D} = \ker\,\eta$ has an almost complex structure defined by $J = \Phi|\mathcal{D}$. Thus, there is a canonical splitting of the tangent bundle

$$TM \simeq \mathcal{D} \oplus L_\xi. \tag{3.2.2}$$

and from the point of view of G-structures, a choice of a strict almost contact structure on a $2n+1$ dimensional manifold M corresponds to a reduction of the frame bundle to the subgroup $GL(n,\mathbb{C}) \times 1$ of $GL(2n+1,\mathbb{R})$. Thus, in particular, an almost contact structure is naturally an almost product structure of type $(2n,1)$ with a transverse almost complex structure. Hence, the subbundle $\mathcal{D}$, which we refer to as the *horizontal subbundle* of TM, together with the endomorphism J provides M with an almost CR structure of codimension one. Combining this discussion with Theorem 1.1.5 we have arrived at

Theorem 3.2.6: *Let M be a smooth manifold of dimension $2n+1$. There is a 1-1 correspondence between the strict almost contact structures (ξ,η,Φ) on M and the reductions of the frame bundle $L(M)$ to the group $GL(n,\mathbb{C}) \times 1$.*

Notice that a choice of strict almost contact structure (ξ,η,Φ) on M chooses both an orientation and a transverse orientation on $(M,\mathcal{F}_\xi)$. The almost contact structure $(-\xi,-\eta,\Phi)$ chooses the opposite orientation, and $(\xi,\eta,-\Phi)$ chooses the opposite orientation on $\mathcal{D}$ if and only if n is odd.

The above discussion gives some obvious topological obstructions to the existence of a strict almost contact structure, namely that the first and last Stiefel-Whitney classes $w_1(M)$ and $w_{2n+1}(M)$ vanish, but more is true [**100**]. Since a manifold M has a strict almost contact struture if and only if it has a reduction to the group $U(n) \times 1$, and the universal $U(n)$ bundle $BU(n)$ has vanishing odd Stiefel-Whitney classes. This implies the vanishing of all Stiefel-Whitney numbers, and by a famous theorem of Thom that M is the boundary of a compact manifold. Summarizing, we have

Theorem 3.2.7: [**100**] *Let (M,ξ,η,Φ) be a (strict) almost contact manifold of dimension $2n+1$, then the odd Stiefel-Whitney classes $w_{2i+1}(M)$ vanish. Thus, all Stiefel-Whitney numbers vanish, and M is the boundary of a compact manifold.*

Moreover, Gray [**100**] shows that the primary obstruction to the reduction of $SO(2n+1)$ to the group $U(n) \times 1$ is the third Stiefel-Whitney class $w_3(M)$ and an immediate consequence of this is

Corollary 3.2.8: [**100**] *A 5-dimensional orientable manifold M admits an almost contact structure (ξ,η,Φ) if and only if the integral Stiefel-Whitney class $W_3(M)$ vanishes.*

Equivalence and Automorphisms

Two strict almost contact manifolds (M,ξ,η,Φ) and (M',ξ',η',Φ') are *isomorphic* if there is a diffeomorphism $f : M \longrightarrow M'$ such that

$$\xi' = f_*\xi, \qquad \eta' = (f^{-1})^*\eta, \qquad \Phi' \circ f_* = f_*\Phi. \tag{3.2.3}$$

Such an isomorphism induces an isomorphism of the corresponding symplectic vector bundles $(\mathcal{D}, d\eta)$ and $(\mathcal{D}', d\eta')$. We shall also be interested in the automorphism group $\mathfrak{A}(\xi,\eta,\Phi)$ of an almost contact structure defined by

$$\mathfrak{A}(\xi,\eta,\Phi) = \{f \in \mathrm{Diff}\ M \mid f_*\xi = \xi,\ f^*\eta = \eta,\ f_*\Phi = \Phi f_*\}. \tag{3.2.4}$$

The main theorem on automorphisms of strict almost contact structures is due to Sasaki [**?**].

Theorem 3.2.9: *Let M be a compact manifold with a strict almost contact structure (ξ,η,Φ). Then the group of automorphisms $\mathfrak{A}(\xi,\eta,\Phi)$ is a Lie group.*

Add discussion about groupoids and equivalence of groupoids.

3.3. Almost Contact Metric Structures

In order to investigate further we need a compatible metric structure.

Definition 3.3.1: *Let M be a (strict) almost contact manifold. A Riemannian metric g on M is said to be* **compatible** *with the almost contact structure if for any vector fields X, Y on M we have*

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

An almost contact structure with a compatible metric is called an **almost contact metric structure**.

One immediately sees by setting $Y = \xi$ in the equation above that $g(\xi, X) = \eta(X)$. It is easy to construct compatible Riemannian metrics on an almost contact manifold. Compatible metrics are often called *associated metrics* [**28**].

Exercise 3.6: Let g' be any Riemannian metric on a strict almost contact manifold (M,ξ,η,Φ) and define

$$g = \frac{1}{2}\big(g' \circ (\Phi^2 \otimes \Phi^2) + g' \circ (\Phi \otimes \Phi)\big) + \eta \otimes \eta.$$

Show that g defines a Riemannian metric on M that is compatible with the almost contact structure. Conversely, show that any compatible metric must take the form $g = g_{\mathcal{D}} + \eta \otimes \eta$ where $g_{\mathcal{D}}$ is a metric on the vector bundle $\mathcal{D}$.

Almost contact metric structures correspond to almost product structures of type $(2n, 1)$ with a transverse almost Hermitian structure. As with Theorem 3.2.6 we have

Proposition 3.3.2: *There is a 1-1 correspondence between strict almost contact metric structures (ξ,η,Φ,g) on M and reductions of the frame bundle to the group $U(n) \times 1$.*

If M is a smooth manifold of dimension $2n+1$ with an almost contact metric structure, then there is a natural 2-form Ψ on M of rank $2n$ defined by

$$\Psi(X,Y) = g(X, \Phi Y). \tag{3.3.1}$$

In this case it is easy to see that the $2n+1$ form $\eta \wedge (\Psi)^n \neq 0$ everywhere on M, and that Ψ is skew-symmetric. The following lemma is a reformulation of Lemma 1.5.36 in terms of almost contact manifolds.

Lemma 3.3.3: *Let (M, ξ, η, Φ) be an almost contact manifold. Then the following are equivalent:*
(i) There exists a compatible Riemannian metric on M such that the orbits of ξ are geodesics.
(ii) $\pounds_\xi \eta = 0$.
(iii) $\xi \rfloor d\eta = 0$.

We remark that for 1-dimensional foliations condition (i) in Lemma 3.3.4 is equivalent to *tautness* [**210**]. Thus, we can rephrase Lemma 3.3.4 as:

Lemma 3.3.4: *The characteristic foliation $\mathcal{F}_\xi$ of an almost contact manifold (M, ξ, η, Φ) is taut if and only the characteristic 1-form η is invariant under the flow of ξ.*

Next we wish to characterize those almost contact structures that have Riemannian metrics that are 'compatible' with the characteristic foliation, that is that the characteristic foliation be a Riemannian foliation 1.5.15. As shown in Proposition 1.5.18 this is equivalent to the condition that the compatible metric be bundle-like, cf 1.5.17. We show that this happens precisely when the transverse metric $g_{\mathcal{D}} = g|\mathcal{D}$ is invariant under the flow of ξ, that is when $\pounds_\xi g_{\mathcal{D}} = 0$.

Proposition 3.3.5: *Let (M, ξ, η, Φ) be an almost contact manifold and g be a Riemannian metric that is compatible with the almost contact structure (ξ, η, Φ). Then g is bundle-like and $\mathcal{F}_\xi$ is taut if and only if ξ is a Killing field of g.*

Proof. By Exercise 3.6 any compatible metric g takes the form

$$g = g_{\mathcal{D}} \oplus \eta \otimes \eta, \qquad g_{\mathcal{D}} = \frac{1}{2}\big(g' \circ (\Phi^2 \otimes \Phi^2) + g' \circ (\Phi \otimes \Phi)\big)$$

where g' is an arbitrary Riemannian metric. The computation is local and there is a local frame of foliate vector fields. Computing the Lie derivatives we find for foliate horizontal vector fields X, Y

$$(\pounds_\xi g)(X,Y) = (\pounds_\xi g_{\mathcal{D}})(X,Y), \qquad (\pounds_\xi g)(X,\xi) = d\eta(\xi, X), \qquad (\pounds_\xi g)(\xi,\xi) = 0$$

from which the result easily follows by Lemma 3.3.4. □

It would be interesting to find examples of almost contact structures whose characteristic foliation is not taut, but nevertheless admits a compatible bundle-like metric. Next we give the orbifold version of a Theorem of Hatakeyama [Hat] who shows that the total space of a principal G_1 bundle over an almost complex manifold where G_1 is a 1-dimensional Lie group admits the structure of an almost contact manifold in a natural way. Now let (N, J) be an almost complex orbifold, and $\pi : M \longrightarrow N$ a principal G_1 V-bundle over N. Generally, the total space M is an orbifold with fibres diffeomorphic to G_1. There are two cases (we assume G_1 is connected): (1) $G_1 = \mathbb{R}$ and (2) $G_1 = S^1$. But according to Definition ?? there is group homomorphism from the local uniformizing groups Γ_i to G_1. But for case (1) $G_1 = \mathbb{R}$ the only homomorphism from a finite group to the additive reals is trivial. Hence, $\pi : M \longrightarrow N$ is an absolute V-bundle, i.e. an ordinary real line bundle, and these are classified by homotopy classes of maps $[N, B\mathbb{Z}_2] \approx H^1(N, \mathbb{Z}_2)$. In this case the singularities of N are not disingularized by the V-bundle, so if N is a non-trivial orbifold so is M. Henceforth, for case (1) we restrict ourselves to the smooth category where Hatakeyama's theorem holds. Case (2) with $G_1 = S^1$ is more interesting since M can desingularize N. By Lemma ?? this occurs precisely when the homomorphisms $h_i : \Gamma_i \longrightarrow S^1$ are injective.

In either case Hatakeyama's construction goes through with only slight modification. Let ξ denote the vector field on M generating the G_1 action, and let η be a connection 1-form on the V-bundle $\pi : M \longrightarrow N$. Then with respect to the basis 1 of the Lie algebra $\mathbb{R}$ of G_1, we have $\eta(\xi) = 1$, and the horizontal bundle $\mathcal{D} = \ker \eta$. Now any point $u \in M$ can be written as $u = (\tilde{x}, a) \in \tilde{U} \times S^1$ where $\tilde{U}$ is local uniformizing neighborhood, and there is a vector space isomorphism $\pi_* : \mathcal{D}_u \longrightarrow T_{\tilde{x}}\tilde{U}$. Let $q_u : T_{\tilde{x}}\tilde{U} \longrightarrow \mathcal{D}_u$ denote the inverse map. Then we have a commutative diagram

(3.3.2)
$$\begin{array}{ccc} T_{\tilde{x}}\tilde{U} & \xrightarrow{R_{h(\gamma)*}} & T_{\gamma^{-1}\tilde{x}}\tilde{U} \\ \Big\downarrow{\scriptstyle q_u} & & \Big\downarrow{\scriptstyle q_{uh(\gamma)}} \\ \mathcal{D}_u & \xrightarrow{R_{h(\gamma)*}} & \mathcal{D}_{uh(\gamma)}, \end{array}$$

where $(\tilde{U}, \Gamma, \phi)$ is a local uniformizing system with $\gamma \in \Gamma$, and h is the homomorphism ?. We then define the tensor field Φ by

3.11
$$\Phi_u = q_u \circ J_{\tilde{x}} \circ \pi_* .$$

This tensor field is invariant under the group of the bundle, namely, $\Phi_{ua} = (R_a)_* \Phi_u$.

Theorem 3.3.6: *Let (N, J) be an almost complex orbifold and let $\pi : M \longrightarrow N$ be a principal G_1 V-bundle over N. Suppose that ξ is a generator of the G_1 action that corresponds to 1 in the Lie algebra $\mathfrak{g}_1 \approx \mathbb{R}$, and that η is a connection 1-form on M. Then (ξ, η, Φ) with Φ defined by 3.11 defines an almost contact structure on M. Furthermore, if h is a Hermitian metric on N compatible with J then*

$$g = \pi^* h + \eta \otimes \eta$$

is a bundle-like Riemannian metric on M compatible with the almost contact structure, and ξ is a Killing vector field for g.

Our next task is to formulate a converse to Theorem 3.12. Notice that the ingredients are a bundle-like metric with ξ a Killing vector field, and a ξ invariant tensor field Φ. On an almost contact manifold (M, ξ, η, Φ) these conditions are related mainly through Proposition 3.9; however, the invariance of Φ is not pinned down completely.

Lemma 3.3.7: *On an almost contact manifold (M, ξ, η, Φ) the following relations hold:*

$$\pounds_\xi \Phi(\xi) = 0, \quad \eta \circ (\pounds_\xi \Phi) \circ \Phi = \pounds_\xi \eta, \quad (\pounds_\xi \Phi)^h \circ \Phi + \Phi \circ (\pounds_\xi \Phi)^h = 0$$

where the superscript h means horizontal component. So if the tensor field Φ is invariant under the flow of ξ then $\mathcal{F}_\xi$ is taut.

Proof. The first equation follows from 3.2 To prove the remaining two equations, take the Lie derivative with respect to ξ of the second equation in Definition 3.1 giving

3.14
$$\pounds_\xi \Phi \circ \Phi + \Phi \circ \pounds_\xi \Phi = \xi \otimes \pounds_\xi \eta .$$

Taking η of this implies that the second equation holds on $\mathcal{D}$ while 3.2 implies that it hold also on ξ. Taking the horizontal component equation 3.14 gives the third equation. The last statement follows from the second equation and Lemma 3.7. $\square$

First we shall make the simplifying assumption that the leaves of the characteristic foliation $\mathcal{F}_\xi$ are compact. In this case the leaves are all diffeomorphic to a circle; however, the foliation need not be simple.

Theorem 3.3.8: *Let (M,ξ,η,Φ) be an almost contact manifold such that the leaves of the characteristic foliation are all compact. Suppose also that (M,ξ,η,Φ) admits a compatible Riemannian metric g such that ξ is a Killing vector field which leaves the tensor field Φ invariant. Then the space of leaves $M/\mathcal{F}_\xi$ is an almost complex orbifold such that the canonical projection $\pi : M\longrightarrow M/\mathcal{F}_\xi$ is an orbifold Riemannian submersion, and a principal S^1 V-bundle over $M/\mathcal{F}_\xi$ with connection 1-form η.*

PROOF. Since ξ is a Killing vector field for the compatible Riemannian metric g, Proposition 3.3.5 says that g is bundle-like and the leaves of $\mathcal{F}_\xi$ are geodesics. Thus, $\mathcal{F}_\xi$ is a taut Riemannian foliation. Hence, since the leaves are all compact Molino's theorem [**160**] implies that the space of leaves is an orbifold, and the projection $\pi : M\longrightarrow M/\mathcal{F}_\xi$ is a orbifold Riemannian submersion. Moreover, since the leaves are geodesics a theorem of Wadsley [Wad] implies that the leaves are the orbits of an S^1 action on M. Since the holonomy groups of $\mathcal{F}_\xi$ are finite cyclic, this action is locally free, and thus $\pi : M\longrightarrow M/\mathcal{F}_\xi$ is a principal S^1 V-bundle over $M/\mathcal{F}_\xi$. The horizontal subbundle $\mathcal{D}$ of TM is invariant under the S^1 action and thus defines a connection on $\pi : M\longrightarrow M/\mathcal{F}_\xi$. It is clear from Definition 3.1 that η is a connection 1-form of $\mathcal{D}$. Finally the invariance of the tensor field Φ implies that $J=\pi_*\Phi|\mathcal{D}$ defines an almost complex structure on the orbifold $M/\mathcal{F}_\xi$. $\square$

3.4. Contact Metric Structures

The case of most interest to us is when the almost contact structure is a strict contact structure. We now investigate this case further. First we have

Lemma 3.4.1: *On a strict contact manifold (M,η) there is a unique vector field ξ, called the Reeb vector field, satisfying the two conditons*

$$\xi\rfloor\eta = 1, \qquad \xi\rfloor d\eta = 0.$$

PROOF. The volume element $\eta\wedge(d\eta)^n$ gives an isomorphism between the $C^\infty(M)$-module of vector fields on M and the $C^\infty(M)$-module of $2n$-forms on M. Thus, there is a unique vector field ξ that satisfies the equation $\xi\rfloor(\eta\wedge(d\eta)^n) = (d\eta)^n$. Contracting again with respect to ξ gives $\xi\rfloor(d\eta)^n = 0$, and expanding gives $(d\eta)^n = \eta(\xi)(d\eta)^n - \eta\wedge(\xi\rfloor(d\eta)^n) = \eta(\xi)(d\eta)^n$ which implies $\eta(\xi)=1$. But then since $d\eta$ is a 2-form $0 = \xi\rfloor(d\eta)^n = n(\xi\rfloor d\eta)\wedge(d\eta)^{n-1}$. But since the rank of the 2-form $d\eta$ is n, this implies that $\xi\rfloor d\eta = 0$. $\square$

The Reeb vector field is often called the *characteristic vector field* and it uniquely determines a 1-dimensional foliation $\mathcal{F}_\xi$ on (M,η) called, of course, the *characteristic foliation.*

REMARK 3.4.1: It is clear the the Reeb vector field depends on the 1-form η of the strict contact structure. Thus, in the underlying contact structure different choices of 1-form η produce different characteristic foliations which can have drastically different behavior. We shall see explicit examples of the this shortly.

We now wish to relate strict contact structures to strict almost contact structures. While a strict contact structure uniquely determines the characteristic vector field ξ, we need to add some more structure, namely the endomorphism Φ, in order to obtain an underlying almost contact structure! Recall that on a contact manifold a choice of 1-form η gives the contact bundle the structure of a symplectic vector bundle, namely $(\mathcal{D}, d\eta)$. An almost complex structure J in $\mathcal{D}$ is said to be *compatible* (cf. [**?**]) with the symplectic form $d\eta$ if the two conditions $d\eta(JX, JY) = d\eta(X, Y)$ and $d\eta(X, JY) > 0$ hold. We can extend the endomorphism J to an endomorphism Φ on all of TM by setting $\Phi\xi = 0$. The conditions then become

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y), \qquad d\eta(\Phi X, Y) > 0, \tag{3.4.1}$$

and one easily checks that (ξ, η, Φ) defines an almost contact structure on M. Accordingly, we have

Definition 3.4.2: *Let (M, η) be a contact manifold. Then an almost contact structure (ξ, η', Φ) is said to be* **compatible** *with the contact structure if $\eta' = \eta$, ξ is its Reeb vector field, and there is a smooth section Φ of the endomorphism bundle of TM such that $\Phi^2 = -\mathbb{I} + \xi \otimes \eta$ and conditions 3.4.1 hold. We denote by $\mathcal{AC}(\eta)$ the set of compatible almost contact structures on (M, η).*

Every compatible almost contact structure is determined uniquely by the endomorphism Φ which in turn is determined uniquely by the almost complex structure on $\mathcal{D}$. So we can give the set $\mathcal{AC}(\eta)$ the subspace topology of the space of sections $\Gamma(\mathrm{End}\, TM)$ with the C^∞ compact open topology. Then we have

Proposition 3.4.3: *Let (M, η) be a strict contact manifold. Then there is a 1-1 correspondence between compatible almost complex structures J on the symplectic vector bundle $(\mathcal{D}, d\eta)$ and compatible almost contact structures (ξ, η, Φ). Moreover, the space $\mathcal{AC}(\eta)$ is contractible.*

PROOF. The first statement is clear from what was said above, and the second statement follows from Proposition 2.61 of [**?**]. □

Notice that $d\eta(X, \Phi Y)$ is symmetric in X and Y and is essentially the Levi form L_η of the almost CR-structure defined by $(\mathcal{D}, \Phi|\mathcal{D})$. So the last of conditions 3.4.1 implies that the Levi form L_η is strictly pseudoconvex; hence, $L_\eta = d\eta \circ (\Phi \otimes \mathbb{I})$ defines a Riemannian metric $g_{\mathcal{D}}$ on the vector bundle $\mathcal{D}$. We can then define a metric g on M by

$$g = g_{\mathcal{D}} + \eta \otimes \eta = d\eta \circ (\Phi \otimes \mathbb{I}) + \eta \otimes \eta \tag{3.4.2}$$

which, as one can easily check, is compatible with the almost contact structure (ξ, η, Φ), i.e. (ξ, η, Φ, g) is an almost contact metric structure. Exercise **??** shows that compatible Riemannian metrics always exist on a strict contact manifold. Riemannian metrics of the form 3.4.2 are called *associated Riemannian metrics*, and there is precisely one for every compatible almost contact structure. Thus,

Proposition 3.4.4: *The set of associated Riemannian metrics are in 1-1 correspondence with elements of $\mathcal{AC}(\eta)$.*

Definition 3.4.5: *A (strict) contact manifold (M,η) with a compatible (strict) almost contact metric structure (ξ,η,Φ,g) such that $g(\Phi X,Y)=d\eta(X,Y)$ is called a* **contact metric structure**, *and (M,ξ,η,Φ,g) is called a* **contact metric manifold**.

REMARK 3.4.2: It is possible to have a contact manifold (M,η) with a compatible almost contact structure (ξ,η,Φ) and a Riemannian metric g which is compatible with the almost contact structure, but which is not the associated metric, i.e. $g\neq d\eta\circ(\Phi\otimes\mathbb{I})$. We refer to such a structure as a (strict) *contact structure with a compatible Riemannian metric*. Of course, a contact metric structure is a contact structure with a compatible Riemannian metric. Unless stated to the contrary, the metric of a contact metric structure will mean an associated Riemannian metric.

An immediate corollary of Lemmas 3.3.4 and 3.4.1 is

Corollary 3.4.6: *On a strict contact manifold (M,η) the characteristic foliation $\mathcal{F}_\xi$ is taut with respect any compatible Riemannian metric.*

We are interested in the case when the characteristic foliation $\mathcal{F}_\xi$ of a contact metric structure is a Riemannian foliation. Although a Riemannian flow is not necessarily isometric in general, it turns out that if the characteristic flow $\mathcal{F}_\xi$ of a contact metric structure is Riemannian if and only if it is isometric. First we have

Definition 3.4.7: *A contact metric structure (ξ,η,Φ,g) is called* **K-contact** *if ξ is a Killing vector field of g, i.e. if $\pounds_\xi g=0$.*

Proposition 3.4.8: *On a complete contact metric manifold (M,η,g), the following are equivalent:*

(i) *The characteristic foliation $\mathcal{F}_\xi$ is a Riemannian foliation.*
(ii) *g is bundle-like.*
(iii) *The Reeb flow is an isometry.*
(iv) *The Reeb flow leaves the almost complex structure J on $\mathcal{D}$ invariant.*
(v) *The Reeb flow leaves the $(1,1)$ tensor field Φ invariant.*
(vi) *The contact metric structure (M,η,g) is K-contact.*

PROOF. The equivalence of (i) and (ii) is Proposition 1.5.18. The two conditions of Lemma 3.4.1 that characterize the Reeb vector field imply that both the transverse symplectic form $d\eta$ and the contact form η are invariant under the Reeb flow. From Definition 1.5.17 g is bundle-like if and only if the transverse metric $g_{\mathcal{D}}$ is basic, that is, if and only if the Reeb flow leaves $g_{\mathcal{D}}$ invariant as well. Hence, the equivalence of (ii) and (iii). Since $d\eta$ is invariant under the Reeb flow, $g_{\mathcal{D}}$ is invariant if and only if J is invariant, or equivalently if and only if Φ is invariant. This implies the equivalence of (iii) with both (iv) aand (v). Also (iii) and (vi) are clearly equivalent $\square$

I want to add the following somewhere:

Let (M,η) be a strict contact manifold. An immersed submanifold $\iota:N\longrightarrow M$ is called an *integral submanifold* of the contact structure if $\iota^*\eta=0$. An integral submanifold N is said to be *maximal* if it is not contained as a proper subset of any other integral submanifold. Maximal integral submanifolds of a contact manifold are often called *Legendrian submanifolds.*

Proposition 3.4.9: *On a contact manifold (M,η) of dimension $2n+1$ the Legrendrian submanifolds have dimension n.*

PROOF. It is easy to see that integral submanifolds of dimension n exist. By Darboux's theorem there is a coordinate chart $U, x^1, \cdots, x^n, y^1, \cdots, y^n, z)$ such that η takes the form of equation 3.1.1. Then define the submanifold locally by $x^i = a^i, z = c$ for some constants a^i, c. This shows that submanifolds of dimension n exist. On the other hand suppose that N is an integral submanifold of dimension k, and let X and Y be vector fields that are tangent to N. Then the equation

$$g(\Phi X, Y) = d\eta(X, Y) = 0$$

says that ΦX is normal to N. Since ξ is clearly normal to N there can be at most n linearly independent local vector fields tangent to N. □

3.5. Structures on Cones

We wish to describe certain types of conical geometry. Let M be a smooth manifold, and consider the *cone on* M as $C(M) = M \times \mathbb{R}^+$. We shall identify M with $M \times \{1\}$. We first describe the Riemannian situation.

Riemannian Cones

In this case the natural Riemannian structure on $C(M)$ is not the product metric, but the so-called *warped product*[**?, 169**] $M \times_{r^2} \mathbb{R}^+$, where r denotes the coordinate on $\mathbb{R}^+$. We have

Definition 3.5.1: *For any Riemannian metric g_M on M, the cone metric on $C(M)$ is the Riemannian metric defined by*

$$g = dr^2 + r^2 g_M.$$

It is clear that there is a 1-1 correspondence between Riemannian metrics on M and cone metrics on $C(M)$. The cone metric admits a group of homothety transformations defined by $(x, r) \mapsto (x, \lambda r)$. The infinitesimal generator of the homothety group is the Liouville (or Euler) vector field defined by

$$\Psi = r\frac{\partial}{\partial r}. \tag{3.5.1}$$

It is clear that the cone metric of Definition 3.5.1 is homogeneous of degree 2 and satisfies

$$\pounds_\Psi g = 2g.$$

Almost Complex and Almost Hermitian Cones

We are interested in the geometry on $C(M)$ that corresponds to an almost contact structure on M. Given an almost contact structure (ξ, η, Φ) on M we define a section I of the endomorphism bundle of the tangent bundle $TC(M)$ of the cone by

$$IY = \Phi Y + \eta(Y)\Psi, \qquad I\Psi = -\xi. \tag{3.5.2}$$

It is easy to see that I defines an almost complex structure on $C(M)$ that is homogeneous of degree 0 in r. Alternatively, I is invariant under the flow of Ψ and we have

$$\pounds_\Psi I = 0. \tag{3.5.3}$$

Conversely, given an almost complex structure on $C(M)$ which is invariant under Ψ, we can define a nowhere vanishing vector field ξ on M by setting $\xi = I\Psi$. Letting L_ξ denote the trivial line bundle generated by ξ, we have an exact sequence

$$0 \longrightarrow L_\xi \longrightarrow TM \longrightarrow Q \longrightarrow 0,$$

and there is a 1-1 correspondence between the splittings of this exact sequence and 1-forms η on M that satisfy $\eta(\xi) = 1$. This correspondence is given by $\eta \mapsto \ker \eta$. One can then easily define an endomorphism Φ of TM by

$$\Phi = \begin{cases} I, & \text{on } \ker \eta \\ 0 & \text{on } L_\xi. \end{cases} \tag{3.5.4}$$

It is easy to check that $\Phi^2 = -\mathbb{I} + \xi \otimes \eta$, so (ξ, η, Φ) defines an almost contact structure on M.

The choice of η in the paragraph above can be determined by giving a compatible Riemannian structure. Explicitly,

Proposition 3.5.2: *There is a 1-1 correspondence between almost contact structures (ξ, η, Φ) on M and almost complex structures I on $C(M)$ that are invariant under Ψ. Furthermore, if g is a Riemannian metric on M that is compatible with the almost contact structure, then the cone metric $h = dr^2 + r^2 g$ is Hermitian with respect to the almost complex structure I.*

Proof. As described in the paragraph above an almost contact structure (ξ, η, Φ) on M determines a unique almost complex structure I on $C(M)$ that is invariant under Ψ, and vice-versa. Furthermore, if g is a metric on M satisfying $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$, then one checks (exercise) that the cone metric $h = dr^2 + r^2 g$ is Hermitian, that is that h satisfies $h(IX, IY) = h(X, Y)$ where I is given by equation 3.5.2. This correspondence is clearly injective. Finally, given an almost Hermitian structure (I, h) where h is a cone metric and I is invariant under Ψ, we define η by $\eta(X) = h(I\Psi, X)$ and Φ by equation 3.5.4 and check that g is compatible with the almost contact structure. $\square$

Exercise 3.7: Fill in the details of the proof of Proposition 3.5.2 by showing that the cone metric $h = dr^2 + r^2 g$ is Hermitian if and only if g is compatible with the almost contact structure.

Symplectic Cones

We have

Definition 3.5.3: *A* **symplectic cone** *is a cone $C(M) = M \times \mathbb{R}^+$ with a symplectic structure ω which has a one parameter group ρ_t of homotheties whose infinitesimal generator is a vector field on $\mathbb{R}^+$.*

Definition 3.5.3 is referred to as a *symplectic Liouville structure* in [LM]. Notice that there is a freedom regarding the degree of homogeneity of the homotheties. As we shall want the symplectic structure on $C(M)$ to be compatible with the standard cone metric given in Definition 3.5.1, we ask that the symplectic form ω be homogeneous[1] of degree 2, that is the equation

$$2\omega = \pounds_\Psi \omega = d(\Psi \rfloor \omega) \tag{3.5.5}$$

[1] Note that in [**36**] we took ω to be homogeneous of degree one.

holds. Notice that it follows from this equation that ω is an exact 2-form, so we define the one form $\tilde{\eta}$ on $C(M)$ by $\tilde{\eta} = \Psi \rfloor \omega$ giving $\omega = d\tilde{\eta}$. We can define a 1-form η on M by $\eta = \tilde{\eta}|_{M\times\{1\}}$. Then we see that $\tilde{\eta} = r^2\eta$, and this gives ω as

$$\omega = 2rdr \wedge \eta + r^2 d\eta. \tag{3.5.6}$$

The condition that $\omega^{n+1} \neq 0$ implies that η gives M the structure of a strict contact manifold with a fixed 1-form. Conversely, one can easily reverse this procedure: given a strict contact manifold (M,η), defining ω on $C(M)$ by equation 3.5.6 gives the cone $C(M)$ a symplectic structure with homotheties. In [Eli, MS] $(C(M),\omega)$ is called a *symplectization* of M while it is called a *symplectification* of M in [LM]. The terminology is apparently due to Arnold. We have arrived at the well-known result

Proposition 3.5.4: *Let η be a 1-form on the manifold M. Then η defines a strict contact structure on M if and only if the 2-form $d(r^2\eta)$ defines a symplectic structure on the cone $C(M) = M \times \mathbb{R}^+$.*

Let (η, g) be a contact metric structure on M with g an associated metric. Then by Proposition 3.4.4 there is a unique almost contact structure (ξ,η,Φ) satisfying $g(X,Y) = d\eta(X,\Phi Y)$ and $\xi \rfloor d\eta = 0$. The metric $g = d\eta \circ (\Phi \otimes \mathbb{I}) + \eta \otimes \eta$ on M corresponds to the metric $dr^2 + r^2(d\eta \circ (\Phi \otimes \mathbb{I}) + \eta \otimes \eta)$ on $C(M)$. We have arrived at

Proposition 3.5.5: *There is a 1-1 correspondence between the contact metric structures (ξ,η,Φ,g) on M and almost Kähler structures $(dr^2 + r^2 g, d(r^2\eta), I)$ on $C(M)$.*

So what does the K-contact condition on M correspond to? We have

Proposition 3.5.6: *A contact metric structure (ξ,η,Φ,g) is K-contact if and only if $\Psi - i\xi$ pseudo-holomorphic with respect to the almost Kähler structure $(dr^2 + r^2 g, d(r^2\eta), I)$ on $C(M)$.*

PROOF. By the properties of the Reeb vector field one easily sees that $\Psi - i\xi$ is pseudo-holomorphic, i.e., an infinitesimal automorphism of I if and only if $\pounds_\xi I = \pounds_\xi \Phi = 0$. But by Proposition 3.4.8 this holds if and only if $\pounds_\xi g_{\mathcal{D}} = 0$. $\square$

REMARK 3.5.1: In the case that the complex vector field $\Psi - i\xi$ on $C(M)$ is not pseudo-holomorphic, a quotient formed by dividing by the resulting $\mathbb{C}^*$ action, or equivalently, by the symplectic reduction of the S^1 action will lose both the almost complex structure and the Riemannian structure.

Integrability and Normality

The main thrust of this monograph will deal with the case in which the complex structure on the $C(M)$ is integrable. This leads to [**?**]

Definition 3.5.7: *An almost contact structure (ξ,η,Φ) is said to be* **normal** *if the corresponding almost complex structure I on $C(M)$ is integrable.*

Combining this definition with Proposition 3.5.2 immediately gives

Proposition 3.5.8: *There is a 1-1 correspondence between the normal almost contact structures (ξ,η,Φ) on M and complex structures I on $C(M)$ that are invariant under Ψ. Furthermore, if g is a Riemannian metric on M that is compatible with the almost contact structure, then the cone metric $h = dr^2 + r^2 g$ is Hermitian with respect to the complex structure I.*

As is well known and discussed in Example 1.3.6 the integrability of I is equivalent to the vanishing of its Nijenhius torsion tensor N_I

$$N_I(X,Y) = [IX,IY] - [X,Y] - I[X,IY] - I[IX,Y] \tag{3.5.7}$$

More generally the Nijenhius torsion tensor N_Φ can be defined for any $(1,1)$-tensor field Φ, viz.

$$N_\Phi(X,Y) = [\Phi X,\Phi Y] + \Phi^2[X,Y] - \Phi[X,\Phi Y] - \Phi[\Phi X,Y] \tag{3.5.8}$$

Note that the Nijenhuis torsion tensor of a $(1,1)$-tensor field is a tensor field of type $(1,2)$, and that $N_\Phi(Y,X) = -N_\Phi(X,Y)$. In particular we want to determine the normality condition in terms of the Nijenhuis torsion tensor for the tensor field Φ of an almost contact structure.

Theorem 3.5.9: *An almost contact structure (ξ,η,Φ) on M is normal if and only if $N_\Phi = -2\xi\otimes d\eta$.*

Proof. Since N_I is a tensor field of type $(1,2)$ on $C(M)$, it suffices to compute it on pairs of vector fields of the form X,Y and X,Ψ where X and Y are vector fields on M. Using equation 3.5.2

$$\begin{aligned}
N_I(X,Y) &= [IX,IY] - [X,Y] - I[X,IY] - I[IX,Y]\\
&= [\Phi X + \eta(X)\Psi, \Phi Y + \eta(Y)\Psi] - [X,Y]\\
&\quad - I[X,\Phi Y + \eta(Y)\Psi] - I[\Phi X + \eta(X)\Psi, Y]\\
&= [\Phi X,\Phi Y] + \Phi^2[X,Y] - \Phi[X,\Phi Y] - \Phi[\Phi X,Y] + [\eta(X)\Psi,\Phi Y] + [\Phi X,\eta(Y)\Psi]\\
&\quad - \eta([X,Y])\xi - X\eta(Y)I\Psi + Y\eta(X)I\Psi - \eta([\Phi X,Y] + [X,\Phi Y])\Psi\\
&= N_\Phi(X,Y) - (\Phi Y\eta(X) - \Phi X\eta(Y) - \eta([\Phi X,Y] + [X,\Phi Y]))\Psi\\
&\quad + (X\eta(Y) - Y\eta(X) - \eta([X,Y]))\xi\\
&= N_\Phi(X,Y) + 2d\eta(X,Y)\xi + 2(d\eta(\Phi X,Y) + d\eta(X,\Phi Y))\Psi\\
&= N_\Phi(X,Y) + 2d\eta(X,Y)\xi - 2\big((\pounds_{\Phi Y}\eta)(X) - (\pounds_{\Phi X}\eta)(Y)\big)\Psi.
\end{aligned}$$

Next we compute

$$\begin{aligned}
N_I(X,\Psi) &= [IX,I\Psi] - [X,\Psi] - I[X,I\Psi] - I[IX,\Psi]\\
&= -[\Phi X + \eta(X)\Psi,\xi] + \Phi[X,\xi] + \eta([X,\xi])\Psi\\
&\quad - \Phi[\Phi X + \eta(X)\Psi,\Psi] - \eta([\Phi X + \eta(X)\Psi,\Psi])\\
&= -[\Phi X,\xi] + \Phi[X,\xi] + \big(\eta([X,\xi]) + \xi\eta(X)\big)\Psi\\
&= (\pounds_\xi\Phi)(X) + (\pounds_\xi\eta)(X)\Psi.
\end{aligned}$$

At this stage it is tradicional to define certain tensor fields, and we shall follow suit.

$$N^{(1)}(X,Y) = N_\Phi(X,Y) + 2d\eta(X,Y)\xi, \tag{3.5.9}$$

$$N^{(2)}(X,Y) = (\pounds_{\Phi X}\eta)(Y) - (\pounds_{\Phi Y}\eta)(X), \tag{3.5.10}$$

$$N^{(3)}(X) = (\pounds_\xi\Phi)(X), \qquad N^{(4)}(X) = (\pounds_\xi\eta)(X). \tag{3.5.11}$$

Equating to zero separately the components of $N_I(X,Y)$ that are tangent to M and those that are proportional to Ψ, shows that (ξ,η,Φ) is normal if and only if $N^{(i)}(X,Y) = 0$ for all $i = 1,\cdots,4$. However, the following lemma shows that the vanishing of $N^{(1)}$ is both necessary and sufficient for normality.

Lemma 3.5.10: *If $N^{(1)}$ vanishes so does $N^{(i)}$ for $i = 2,3,4$.*

PROOF. For any vector field X on M, consider

$$\begin{aligned}
N^{(1)}(X,\xi) &= N_\Phi(X,\xi) + 2d\eta(X,\xi)\xi = \Phi^2[X,\xi] - \Phi[\Phi X,\xi] - \xi\eta(X)\xi - \eta([X,\xi])\xi \\
&= -[X,\xi] + \eta([X,\xi]) - \Phi[\Phi X,\xi] - \xi\eta(X)\xi - \eta([X,\xi]) \\
&= -[X,\xi] - \Phi[\Phi X,\xi] - \xi\eta(X)\xi
\end{aligned}$$

Applying η to this equation shows that the vanishing of $N^{(1)}$ implies $\eta([X,\xi]) + \xi\eta(X) = 0$ which is equivalent to $\pounds_\xi\eta = \xi\rfloor d\eta = 0$. This shows that $N^{(4)} = 0$.

Next we notice that replacing X by ΦX in the equation for $N^{(1)}(X,\xi)$, gives

$$N^{(1)}(\Phi X,\xi) = [\xi,\Phi X] - \Phi[\xi,X] = (\pounds_\xi\Phi)(X) = N^{(3)}(X). \tag{3.5.12}$$

So the vanishing of $N^{(1)}$ implies the vanishing of $N^{(3)}$.

Finally, we apply η to $N^{(1)}(\Phi X,Y)$ which gives

$$\begin{aligned}
\eta(N^{(1)}(\Phi X,Y)) &= \eta(N_\Phi(\Phi X,Y)) + 2d\eta(\Phi X,Y) = \eta([\Phi^2 X,\Phi Y]) + 2d\eta(\Phi X,Y) \\
&= -\eta([X,\Phi Y]) + \eta([\eta(X)\xi,\Phi Y]) + 2d\eta(\Phi X,Y) \\
&= -\eta([X,\Phi Y] - \Phi Y\eta(X) + \eta(X)\eta([\xi,\Phi Y]) + 2d\eta(\Phi X,Y) \\
&= 2d\eta(\Phi X,Y) - 2d\eta(\Phi Y,X) - 2\eta(X)d\eta(\xi,\Phi Y) \\
&= 2\big((\pounds_{\Phi X}\eta)(Y) - (\pounds_{\Phi Y}\eta)(X)\big) - 2\eta(X)d\eta(\xi,\Phi Y) \\
&= 2N^{(2)}(X,Y) - 2\eta(X)d\eta(\xi,\Phi Y)
\end{aligned}$$

But we saw in the argument for the vanishing of $N^{(4)}$ that the vanishing of $N^{(1)}$ implies $\xi\rfloor d\eta = 0$; hence, the last term on the right vanishes whenever $N^{(1)}$ vanishes, so the vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$. □

This completes the proof of the theorem. □

Let us compare normality with the integrability of the undelying almost CR-structure. A contact metric structure (ξ,η,Φ,g) defines a pseudoconvex almost CR-structure $(\mathcal{D},\Phi|_{\mathcal{D}})$ with Levi form $L_\eta = d\eta\circ(\Phi|_{\mathcal{D}}\otimes\mathbb{I})$.

Corollary 3.5.11: *An almost contact metric structure (ξ,η,Φ,g) on M is normal if and only if*

(i) *The almost CR-structure $(\mathcal{D},\Phi|_{\mathcal{D}})$ is integrable, and*
(ii) *Φ is invariant under ξ, i.e. $N^{(3)} = 0$.*

PROOF. We set $J = \Phi|_{\mathcal{D}}$ and compute the Nijenhuis tensor for X,Y sections of $\mathcal{D}$.

$$N_J(X,Y) = [JX,JY] - [X,Y] - J([X,JY] + [JX,Y]) = N_\Phi(X,Y) + 2d\eta([X,Y])\xi.$$

Moreover, from equations 3.5.9 and 3.5.12 we have

$$(\pounds_\xi\Phi)(X) = N^{(1)}(\Phi X,\xi) = N_\Phi(\Phi X,\xi) + 2d\eta(\Phi X,\xi)\xi.$$

Using these two equations the result follows from Theorem 3.5.7. □

In the case of a contact metric structure the four tensor fields defined above simplify a bit.

Proposition 3.5.12: *Let (ξ,η,Φ,g) be a contact metric structure. Then $N^{(2)} = N^{(4)} = 0$, and $N^{(3)} = 0$ if and only if (ξ,η,Φ,g) is K-contact.*

PROOF. First we have

$$N^{(4)} = (\pounds_\xi \eta) = d(\eta(\xi)) + \xi \rfloor d\eta = 0$$

when (ξ, η, Φ, g) is contact. Next we have

$$\begin{aligned} N^{(2)}(X,Y) &= (\pounds_{\Phi X}\eta)(Y) - (\pounds_{\Phi Y}\eta)(X) \\ &= \Phi X(\eta)(Y)) - \eta([\Phi X, Y]) - \Phi Y(\eta)(X)) + \eta([\Phi Y, x]) \\ &= 2d\eta(\Phi X, Y) - 2d\eta(\Phi Y, X) = 2g(Y,X) - 2g(X,Y) = 0. \end{aligned}$$

For $N^{(3)}$ we notice first that since (ξ, η, Φ, g) is contact, $\pounds_\xi \eta = \pounds_\xi d\eta = 0$. Since g is the associated metric 3.4.2, we have

$$\pounds_\xi g = d\eta \circ (\mathbb{I} \otimes \pounds_\xi \Phi).$$

So the vanishing of $N^{(3)}$ implies that ξ is Killing. Conversely, if ξ is Killing, the nondegeneracy of the bilinear form $d\eta$ on $\mathcal{D}$ implies that $N^{(3)}(X) = 0$ when X is a section of $\mathcal{D}$. But since $\Phi\xi = 0$, $N^{(3)}$ vanishes identically on ξ. □

Our final definition of this subsection is the major of object of study in this treatise.

Definition 3.5.13: *A normal contact metric structure (ξ, η, Φ, g) is called a* **Sasakian** *structure. A smooth manifold with such a structure is called a* **Sasakian manifold**.

It follows immediately from Corollary 3.5.11 and Proposition 3.5.12 that

Proposition 3.5.14: *A Sasakian structure (ξ, η, Φ, g) is a K-contact structure whose almost CR-structure is integrable.*

Actually as seen in Lemma 3.3.4 $N^{(4)}$ vanishes as soon as there is a compatible metric of the almost contact structure such that integral curves of ξ are geodesics. From the definition of normality 3.5.7 and Proposition 3.5.5, Definition 3.5.13 is equivalent to:

Definition 3.5.15: *A contact metric manifold (M, ξ, η, Φ, g) is* **Sasakian** *if its metric cone $(C(M), dr^2 + r^2 g, d(r^2\eta))$ is Kähler.*

The Einstein Condition

Before leaving this chapter we give an important well known lemma concerning the Einstein condition on a metric cone.

Lemma 3.5.16: *Let (M, g) be a Riemannian manifold of dimension n, and consider $(M \times \mathbb{R}^+, \bar{g})$ the cone on M with metric $\bar{g} = dr^2 + r^2 g$. Then if $\bar{g}$ is Einstein, it is Ricci-flat, and $\bar{g}$ is Ricci-flat if and only if g is Einstein with Einstein constant $n-1$.*

PROOF. Let $\{\theta^i\}_{i=1}^n$ be a local orthonormal coframe for (M, g), then we obtain a local orthonormal coframe $\{\phi^\mu\}$ for the cone metric $\bar{g}$ on $M \times \mathbb{R}^+$ by setting

$$\phi^i = r\theta^i, \qquad \phi^0 = dr,$$

where $0 \leq \mu \leq n$. (We employ the convention that the greek indices range from 0 to n, whereas, the latin indices range from 1 to n). Consider the Cartan structure equations

$$d\phi^\mu + \sum_\nu \omega^\mu_\nu \wedge \phi^\nu \;=\; 0$$

$$d\omega^\mu_\nu + \sum_\lambda \omega^\mu_\lambda \wedge \omega^\lambda_\nu \;=\; \sum_{\lambda<\rho} \bar{R}^\mu_{\nu\lambda\rho}\phi^\lambda \wedge \phi^\rho$$

Here ω^μ_ν denotes the connection 1-forms with respect to the Levi-Civita connection on $M\times\mathbb{R}^+$ which satisfies $\omega^\mu_\nu + \omega^\nu_\mu = 0$, and $\bar{R}^\mu_{\nu\lambda\rho}$ denotes the Riemannian curvature. The first Cartan equations for the cone metric gives $\omega^i_0 = \theta^i$, and that ω^i_j are the connection 1-forms for the Levi-Civita connection with respect to g on M. The second equations then imply

$$\bar{R}^i_{0\mu\nu} = 0 \qquad \bar{R}^i_{jkl} = \frac{1}{r^2}R^i_{jkl} - \delta^{k[i}\delta^{j]l}$$

where δ^{ik} denotes the Kronecker delta symbol, and the brackets mean antisymmetrize over i and j. Forming the Ricci tensor one now easily proves the result. $\square$

flcmlcm

CHAPTER 4

Kähler Manifolds and Orbifolds

In this chapter we will recall some basic properties of Kähler manifolds and Kähler metrics. We will also discuss how such properties generalize to the category of Kähler orbifolds.

4.1. Kähler Metrics and Kähler Potentials

Let us start with a definition of a complex manifold. There are several frequently used approaches to define such a structure. Perhaps the most natural one is the one that imitates the usual definition of a smooth structure on a real manifold.

Definition 4.1.1: *Let M be a real manifold of dimension $2n$. A complex chart on M is a pair (U, ϕ) such that $U \subset M$ is open and $\phi : U \to \mathbb{C}^n$ is diffeomorphism between U and an open set $\phi(U) \subset \mathbb{C}^n$. We say that M is a complex manifold if it admits and atlas $\mathcal{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in \Omega}$ of complex charts whose transition functions are holomorphic, as maps from $\mathbb{C}^n$ to itself.*

Recall that if M is a real manifold than a smooth section J of the bundle of endomorphisms $\mathrm{End}(TM)$ such that $J^2 = -id$ is called an almost complex structure. Note that we must have $\dim(M) = 2n$. We can extend J to act on the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity. Then J induces a splitting $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$ and $T^{0,1}M$ are eigenspaces with eigenvalues $\pm i$, respectively. Note that $T^{1,0}M$ is naturally isomorphic to TM. Namely, consider $X \in TM$. Then X under the above splitting X admits a unique decomposition $X = V + \overline{V}$ into its holomorphic and anti-holomorphic part. We define the map $TM \to T^{0,1}M$ by $X \mapsto V$. This is clearly a vector space isomorphism as V can be uniquely written as $V = \frac{1}{2}(X - iJX)$.

An almost complex structure is said to be integrable if M admits an atlas of complex charts with holomorphic transition functions such that J corresponds to the induced complex multiplication on $TM \otimes_{\mathbb{R}} \mathbb{C}$. Hence, a manifold with an integrable almost complex structure is (by definition) a complex manifold. Conversely, given a complex manifold M we can define J by

$$J \frac{\partial}{\partial z_1} = i \frac{\partial}{\partial z_j},$$

where $(z_1, \ldots, z_n)$ is a local holomorphic coordinate system.

The theorem of Newlander and Nirenberg [?] asserts that integrability of J is equivalent of the vanishing of the Nijenhuis tensor $N_J(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]$. In view of this we have the following

Theorem/Definition 4.1.2: *Let M be a real manifold and J an almost complex structure on M. The Nijenhuis tensor $N_J \equiv 0$ on M if and only if J is integrable in which case we call J a complex structure and the pair (M, J) a complex manifold.*

The second part of this statement is very often taken as an alternative definition of a complex manifold. As we will see it is perhaps the most geometric and indeed we will use it most often.

The third way of describing a complex manifold as a real manifold with torsion-free $GL(n, \mathbb{C})$-structure was described earlier in Example 1.3.6.

Let us consider some example of complex manifolds.

Example 4.1.3: Let $M = \mathbb{R}^{2n}$ and let $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ be the global coordinate chart on M. We can define

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_{j+1}}, \qquad J\frac{\partial}{\partial x_{j+1}} = -\frac{\partial}{\partial x_j}.$$

Clearly, $J^2 = -id$ and $N_J \equiv 0$ as J is a constant matrix on M. We can define $z_1, \ldots, z_n$ by $z_j = x_j + ix_{n+j}$ for $j = 1, \ldots, n$. We have

$$J\frac{\partial}{\partial z_j} = i\frac{\partial}{\partial z_j}, \qquad J\frac{\partial}{\partial \bar{z}_j} = -i\frac{\partial}{\partial \bar{z}_j}$$

so that $(z_1, \ldots, z_n) \in \mathbb{C}^n$ is a (global) holomorphic chart compatible with J.

Example 4.1.4: Let $\mathrm{Gr}_p(\mathbb{C}^{q+p})$ be the set of p-dimensional complex subspaces of the complex vector space $\mathbb{C}^{q+p}$. We will define complex structure on the complex Grassmannian $\mathrm{Gr}_p(\mathbb{C}^{q+p})$ by construction an atlas of holomorphic charts. Let $(z_1, \ldots, z_{p+q})$ be the natural coordinates on $\mathbb{C}^{q+p}$. We can think of z_α as the complex linear mapping $z_\alpha : \mathbb{C}^{q+p} \to \mathbb{C}$. Now, consider the partition of the set $\{1, \ldots, p+q\} = \alpha \cup \alpha^c = \{\alpha_1, \ldots, \alpha_p\} \cup \{\alpha_{p+1}, \ldots, \alpha_p\}$, where where both sets are ordered in the increasing order. Let let $U_\alpha \subset \mathrm{Gr}_p(\mathbb{C}^{q+p})$ be the subset of p-planes W for which $z_{\alpha_1}|_W, \ldots, z_{\alpha_p}|_W$ are linearly independent. Since, for each $W \in U_\alpha$, the mappings $z_{\alpha_1}|_W, \ldots, z_{\alpha_p}|_W$ form a basis for the dual space of W, we can write

$$z_{\alpha_{p+k}}|_W = \sum_{j=1}^{p} w_k^j (z_{\alpha_j}|_W) i, \qquad k = 1, \ldots, q.$$

The above expression uniquely defines a $p \times q$ complex matrix which maybe thought of as an element of $\mathbb{C}^{pq}$. Setting

$$\psi_\alpha(W) = (w_k^j)$$

for each partition α defines maps $\psi_\alpha : U_\alpha \to \mathbb{C}^{pq}$. It is a simple exercise to check that that these maps are injective and onto. Hence, the family $\mathcal{S} = \{U_\alpha, \psi_\alpha\}$ of $\binom{p+1}{p}$ charts form a holomorphic atlas on $\mathrm{Gr}_p(\mathbb{C}^{q+p})$. When $p = 1, q = n$ we get the standard atlas of the $(n+1)$ holomorphic charts on the projective space $\mathrm{Gr}_1(\mathbb{C}^{n+1}) \simeq \mathbb{CP}(n)$. To see it note that, in this special case, choosing the partition $\alpha = \{\alpha_1 = k\}$ is equivalent to choosing one complex coordinate z_k and this the set U_α is clearly the set of complex lines for which $z_k \neq 0$. Let $W = [z_1 : \cdots : z_{n+1}]$ be such a line. Then we can write, for each z_j, $j \neq k$

$$z_\alpha|_W = \frac{z_j}{z_k} z_k.$$

Hence, the maps ψ_k are simply given by

$$\psi_k([z_1 : \cdots : z_{n+1}]) = (z_1/z_k, \ldots, z_{n+1}/z_k), \qquad k = 1, \ldots, n+1.$$

The charts ψ_k are called inhomogeneous coordinate system on $\mathbb{CP}(n)$.

EXAMPLE 4.1.5: Let (M_1, J_1), (M_2, J_2) be two complex manifolds. A map $f : M_1 \to M_2$ is called *holomorphic* if $J_2(df(v)) = df(J_1(v))$ for all $v\Gamma(TM)$. If, in addition, f^{-1} exists and is holomorphic than f is called a *biholomorphism* between M_1 and M_2. (M, J) be a complex manifold and let N be a submanifold of M. Then, in particular at each point $p \in N$, the tangent space $T_pN \subset T_pM$ is a vector subspace. We say that N is a complex submanifold of M if T_pN is a complex subspace for each $p \in N$, i.e., $J(T_pN) = T_pN$. Now, if N is a complex submanifold then the restriction J_N of J to TN is a complex structure on N and the inclusion map $\iota : N \hookrightarrow M$ has the property holomorphic.

$$J(d_p\iota(w)) = (d_p\iota)(J_N(w)), \qquad \forall w \in T_pN,$$

that is ι is a holomorphic map between two complex manifolds (N, J_N) and (M, J).

EXAMPLE 4.1.6: Consider the previous example with $M = \mathbb{CP}(n)$ and let the inclusion map $\iota : N \hookrightarrow \mathbb{CP}(n)$ be defined by setting

$$N = N_{\mathbf{f}} = \{m \in \mathbb{CP}(n) \quad |\mathbf{f}(m) = 0\},$$

where $\mathbf{f} = (f_1, \ldots, f_k)$ and $f_k \in \mathbb{C}[z_0, \ldots, z_n]$ are complex homogeneous polynomials in $(n+1)$ variables for each $k = 0, 1, \ldots, n$. By definition, any such is a *complex projective algebraic variety*. If, in addition N_J is smooth than it is a complex submanifold in the way described by previous example. For instance, let $\mathbf{f} = f$ where f is

$$f([z_0 : \cdots : z_n]) = z_0^p + \cdots + z_{n+1}^p.$$

Then $N_{\mathbf{f}}$ is called a *Fermat hypersurface*. When $p = 2$ we get a *complex quadric*.

Once can extend J to act on the complexified cotangent bundle $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ which then splits as $T^{*(1,0)}M \oplus T^{*(0,1)}M$. This in turn defines the splitting of the bundle of complex p-forms on M

$$\Lambda^p T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{j=0}^{p} \Lambda^j T^{*(1,0)}M \oplus \Lambda^{p-j} T^{*(0,1)}M = \bigoplus_{j=0}^{p} \Lambda^{j,p-j}M,$$

where the last equality defines the space of forms of *type* $(j, p-j)$. Note that such decomposition holds even if J is only almost complex.

Proposition 4.1.7: *Let (M, J) be an almost complex manifold. The following conditions are equivalent:*

(i) $[V, W] \in \Gamma(T^{1,0}M)$ *for all* $V, W \in \Gamma(T^{1,0}M)$,
(ii) $[V, W] \in \Gamma(T^{0,1}M)$ *for all* $V, W \in \Gamma(T^{0,1}M)$,
(iii) $\mathrm{Im}(d)|_{\Gamma(\Lambda^{1,0}M)} \subset \Gamma(\Lambda^{2,0}M) \oplus \Gamma(\Lambda^{1,1}M)$,
$\mathrm{Im}(d)|_{\Gamma(\Lambda^{0,1}M)} \subset \Gamma(\Lambda^{0,2}M) \oplus \Gamma(\Lambda^{1,1}M)$,
(iv) $\mathrm{Im}(d)|_{\Gamma(\Lambda^{p,q}M)} \subset \Gamma(\Lambda^{p+1,q}M) \oplus \Gamma(\Lambda^{p,q+1}M)$,
(v) $N_J \equiv 0$.

PROOF. We will only show the equivalence of (i), (iii), and (v) as the rest is evident. To establish equivalence of (i) and (iii) we use

$$2d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]),$$

which, if ω is a 1-form of type $(1, 0)$ and X, Y are vector fields of type $(0, 1)$, clearly vanishes. Hence, $d\omega$ does not have any $(0, 2)$-part. To establish equivalence of (i) and (v) we consider $B = [X - iJX, Y - iJY]$ for any two real vector fields X, Y on

M. It is easy to see that $B + iJB = -N(X,Y) - iJ(N(X,Y))$. But $B + iJB = 0$ if and only if B is a vector field of type (1,0) if and only if $N_J \equiv 0$. □

Thus, if (M,J) is a complex manifold, composing the exterior derivative acting on (k,l)-forms with projections on $\Gamma(\Lambda^{k+1,l}M)$ and $\Lambda^{k,l+1}M$, defines natural splitting of d into two parts. They are denoted by ∂ and $\bar{\partial}$, where $d = \partial + \bar{\partial}$. Since $d^2 = 0$ we get $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$. We further define the *Dolbeault cohomology groups* of (M,J) as

$$H^{k,l}_{\bar{\partial}}(M) = \frac{\mathrm{Ker}\big(\bar{\partial} : \Gamma(\Lambda^{k,l}) \to \Gamma(\Lambda^{k,l+1})\big)}{\mathrm{Im}\big(\bar{\partial} : \Gamma(\Lambda^{k,l-1}) \to \Gamma(\Lambda^{k,l})\big)}.$$

Definition 4.1.8: *Let (M,J) be an almost complex manifold and let g be a Riemannian metric on M such that*

$$g(JX,JY) = g(X,Y), \qquad X,Y \in \Gamma(TM).$$

Then g is called a Hermitian metric and (M,J,g) is called an almost Hermitian manifold. Furthermore, given a Hermitian metric we define its fundamental 2-form ω_g of g by

$$\omega_g(X,Y) = g(JX,Y), \qquad X,Y \in \Gamma(TM).$$

The triple (J,g,ω_g) is called an almost Hermitian structure on M.

Note that an almost complex manifold always admits a Hermitian metric. An almost Hermitian structure (J,g,ω_g) with an integrable J is called Hermitian. Given a Hermitian metric g we can also define a *Hermitian scalar product* on $T^{1,0}_pM$ by setting $h(V,W) = g(V,\overline{W})$ for any $V,W \in T^{1,0}_pM$ and $p \in M$. Locally, in a complex coordinate chart $(\mathcal{U}; z_1,\ldots,z_n)$ we can write the metric h as

$$h = \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \otimes d\bar{z}_j, \tag{4.1.1}$$

where $h_{i\bar{j}}(p)$ is a Hermitian $n \times n$ matrix. Then the fundamental 2-form can be written as

$$\omega_g = \frac{i}{2}\sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \wedge d\bar{z}_j. \tag{4.1.2}$$

Conversely, suppose M is a complex manifold with a Hermitian metric h on the holomorphic tangent bundle. Locally in a chart $(\mathcal{U}; z_1,\ldots,z_n)$ written as 4.1.1. If we write $z_k = x_k + ix_{n+k}$ and use $(\mathcal{U}; x_1,\ldots,x_{2n})$ as local real smooth chart on M then the real part $g = \mathrm{Re}(h)$ can be written as

$$g = \sum_{a,b=1}^{2n} g_{ab} dx_a \otimes dx_b,$$

with

$$g = (g_{ab}) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad \text{where } \; h = (h_{i\bar{j}}) = A + iB.$$

In particular, we can see that $\det(g) = |det(h)|^2$. Now, $\omega_h = -\frac{1}{2}\mathrm{Im}(h)$ can be then written as 4.1.2. It follows that ω_h is a global real $(1,1)$-form on M. Also, note that

$$\omega_h^n = n!\left(\frac{i}{2}\right)^n \det(h_{i\bar{j}}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

and, hence, $\omega_h^n = n! d\mathrm{vol}_h$, where $d\mathrm{vol}_h$ is the volume form of (M, h).

EXAMPLE 4.1.9: We consider the flat example to carefully to illustrate our conventions. Let $M = \mathbb{R}^{2n}$ and let $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ is the global chart on $\mathbb{R}^{2n}$. Let $\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{2n}}\right)$ be the orthonormal frame in TM. The flat metric

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij}, \qquad i, j = 1, \ldots 2n,$$

and, hence, we can write

$$g = \sum_{i,j=1}^{2n} g_{ij} dx_i \otimes dx_j = \sum_{i=1}^{2n} dx_i \otimes dx_i.$$

Taking the complex structure J described in example 4.1.3 one can easily see that the associated 2-form ω_g is the standard symplectic 2-form

$$\omega_g = \sum_{i=1}^{n} dx_i \wedge dx_{i+1}.$$

Now let us rewrite the metric and the fundamental 2-form in complex coordinates: Let $(z_1, \ldots, z_n)$ be the global holomorphic chart with $z_i = x_i + i x_{n+i}$. Since $dz_i = dx_i + i dx_{n+i}, d\bar{z}_i = dx_i - i dx_i n + i$, $i = 1, \ldots, n$ and the Hermitian metric h reads

$$h = \sum_{i,j=1}^{n} h_{i\bar{j}} dz^i \otimes d\bar{z}^j = \sum_{i=1}^{n} dz_i \otimes d\bar{z}_i.$$

As the fundamental 2-form ω_g is simply minus the imaginary part of h we can write

$$\omega_g = \frac{i}{2} \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \wedge d\bar{z}_j = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i.$$

Definition 4.1.10: *Let (M, J, g) be a Hermitian manifold. We say that (M, J, g) is Kähler if ω_g is a closed 2-form. We call g a Kähler metric and ω_g its Kähler form.*

From the previous discussion we see that Kähler form is a real $(1,1)$-form which is both ∂-closed and $\bar{\partial}$-closed. Hence $[\omega_g]$ defines a Dolbeault cohomology class in $H^{1,1}_{\bar{\partial}}(M)$. Next we state several propositions and lemmas which describes some basic local and global properties of Kähler manifolds.

Proposition 4.1.11: *Let (M, J, g) be an almost Hermitian manifold of real dimension $2n$ and let ω_g be the fundamental 2-form associated to g. Let ∇ be the Levi-Civita connection of g. Then the following conditions are equivalent:*

(i) $\nabla J = 0$,
(ii) $\nabla \omega_g = 0$,
(iii) *The holonomy group* $\mathrm{Hol}(M, g)$ *is contained in* $U(n)$.
(iv) (M, J, g) *is Kähler.*

Proposition 4.1.12: *A Hermitian metric g on a complex manifold (M, J) if and only if for any $p \in \mathcal{U} \subset M$, there exists a l coal holomorphic chart $(\mathcal{U}; z_1, \ldots, z_n)$ such that $h_{i\bar{j}}(p) = \delta_{ij}$, $dh_{i\bar{j}}(p) = 0$. Such a coordinate is said to be normal at p.*

Now, let $f \in C^\infty(M)$. As the $i\partial\bar{\partial} f = \frac{1}{2}(\partial + \bar{\partial})(\bar{\partial} - \partial) f$ is real we conclude that $i\partial\bar{\partial} f$ is a real closed 2-form of type (1,1). We have the following two lemmas

Lemma 4.1.13: [**local $i\partial\bar\partial$-lemma**] *Let $\rho \in \Gamma(\Lambda^{1,1}U)$ be a smooth, closed, real (1,1)-form on a unit disc $U = D^n_{\mathbb{C}}(1) \subset \mathbb{C}^n$. Then there exists $f \in C^\infty(U)$ such that $\rho = i\partial\bar\partial f$.*

Lemma 4.1.14: [**global $i\partial\bar\partial$-lemma**] *Let (M, J) be a compact complex manifold and let g_1, g_2 be two different Kähler metrics on M. Suppose $[\omega_{g_1}] = [\omega_{g_2}] \in H^2(M, \mathbb{R})$. Then there exists $f \in C^\infty(M)$ such that*

$$\omega_{g_2} = \omega_{g_1} + i\partial\bar\partial f,$$

and f is unique up to a constant.

Lemma 4.1.13 says that on a Kähler manifold locally one can always find a function which "generates" the Kähler form (and, hence, the Kähler metric) via the simple formula $\omega_g = i\partial\bar\partial f$. Such a functions is called a *Kähler potential*. However, if M is compact Kähler potential cannot exists globally. This is a simple consequence of the following

Lemma 4.1.15: *Let (M, J, g) be a compact Kähler manifold with the Kähler form ω_g. Then $[\omega_g^k] \in H^{2k}(M, \mathbb{R})$ is non-trivial.*

Proof. This is a simple consequence of the Stokes' theorem and the fact that $\omega_g^n = n! d\mathrm{vol}_g$. For let $\omega_g^k = d\alpha$ be exact. Then

$$0 < n!\mathrm{vol}(M) = \int_M \omega_g^n = \int_M \omega_g^k \wedge \omega_g^{n-k} = \int_M d\alpha \wedge \omega_g^{n-k} = \int_M d(\alpha \wedge \omega_g^{n-k}) = 0,$$

so we get a contradiction. In particular, when $k = 1$ the lemma implies that there is no globally defined function on M such that $\omega_g = i\partial\bar\partial f$. □

4.2. Curvature of Kähler Manifolds

Let (M, J, g, ω_g) be a Kähler manifold and ∇ the Levi-Civita connection. We can extend ∇ in a $\mathbb{C}$-linear way to $\Gamma(TM \otimes_{\mathbb{R}} \mathbb{C})$. In a local chart $(\mathcal{U}; z_1, \ldots, z_n)$ we have $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\}$ and $\{\frac{\partial}{\partial \bar z_1}, \ldots, \frac{\partial}{\partial \bar z_n}\}$ as bases for $T^{1,0}M$ and $T^{0,1}M$, respectively. Define *the Christoffel symbols* Γ^C_{AB} as follows

$$\nabla_{\partial/\partial z_A} \frac{\partial}{\partial z_B} = \sum_{C=1}^n \Gamma^C_{AB} \frac{\partial}{\partial z_C}, \qquad A, B, C \in \{1, \ldots, n, \bar 1, \ldots, \bar n\}.$$

It follows by $\mathbb{C}$-linearity that

$$\overline{\Gamma^C_{AB}} = \Gamma^{\bar C}_{\bar A \bar B}, \qquad \Gamma^C_{AB} = \Gamma^C_{BA}.$$

Lemma 4.2.1: *The only non-vanishing Christoffel symbols are Γ^k_{ij} and $\Gamma^{\bar k}_{\bar i \bar j}$, where*

$$\Gamma^n_{ij} = \sum_{\bar k} g^{n\bar k} \frac{\partial g_{j\bar k}}{\partial z_i}.$$

Proof. We have

$$\nabla_{\partial/\partial z_i} \frac{\partial}{\partial z_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial z_k} + \sum_{\bar k} \Gamma^{\bar k}_{ij} \frac{\partial}{\partial \bar z_k},$$

$$\nabla_{\partial/\partial z_i} \frac{\partial}{\partial \bar z_j} = \sum_k \Gamma^k_{i\bar j} \frac{\partial}{\partial z_k} + \sum_{\bar k} \Gamma^{\bar k}_{i\bar j} \frac{\partial}{\partial \bar z_k},$$

Since, $\nabla J = 0$ and J acts by multiplication by i on the basis of $T^{1,0}M$ and $-i$ on the basis of $T^{0,1}M$, we get

$$\nabla_{\partial/\partial z_i}\left(J\frac{\partial}{\partial z_j}\right) = J\nabla_{\partial/\partial z_i}\frac{\partial}{\partial z_j}.$$

By comparing two sides of this equation we get

$$i\left(\sum_k \Gamma_{ij}^k\frac{\partial}{\partial z_k} + \sum_{\bar{k}}\Gamma_{ij}^{\bar{k}}\frac{\partial}{\partial \bar{z}_k}\right) = i\left(\sum_k \Gamma_{ij}^k\frac{\partial}{\partial z_k} - \sum_{\bar{k}}\Gamma_{ij}^{\bar{k}}\frac{\partial}{\partial \bar{z}_k}\right),$$

which means that $\Gamma_{ij}^{\bar{k}} = 0$. Similar argument applied to the second equation yields $\Gamma_{i\bar{j}}^{\bar{k}} = 0$, while $\Gamma_{\bar{i}j}^{\bar{k}} = 0$ follows by the symmetry two lower indices. Hence, the only non-zero Christoffel symbols are Γ_{ij}^k and their complex conjugate $\Gamma_{\bar{i}\bar{j}}^{\bar{k}}$. Since $X\cdot g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ we obtain

$$\frac{\partial}{\partial z_i}g_{j\bar{k}} = \frac{\partial}{\partial z_i}g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) = g\left(\sum_l \Gamma_{ij}^l\frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar{z}_k}\right) = \sum_l \Gamma^l_{ij}g_{l\bar{k}}.$$

Summing both sides of this equation with the inverse of the metric gives the formula. □

In a local chart $(\mathcal{U}; z_1, \ldots, z_n)$ we can define the following matrix-valued $(1,0)$-form

$$\theta = g^{-1}\partial g = \sum_{i=1} g^{-1}\frac{\partial g}{\partial z_i}dz_i,$$

where $g = (g_{i\bar{j}})$. The above lemma implies that

Proposition 4.2.2: *Let (M, J, g, ω_g) be a Kähler manifold and let ∇ be the Levi-Civita connection of g extended to $TM\otimes_{\mathbb{R}}\mathbb{C} \simeq T^{1,0}M \oplus T^{0,1}M$. Accordingly, ∇ decomposes $\nabla = \nabla^{1,0} + \nabla^{0,1}$, θ is the connection 1-form of $\nabla^{1,0}$.*

Note that we have the following simple expression

$$\sum_j \Gamma_{ij}^j = \sum_{j,\bar{k}} g^{j\bar{k}}\frac{\partial g_{j\bar{k}}}{\partial z_i} = \frac{\partial}{\partial z_i}\ln G,$$

where $G = \det(g) = \det(g_{i\bar{j}})$. The determinant $G \in C^\infty(\mathcal{U})$ is a smooth real function defined locally on $\mathcal{U}$. The last equality follows from the well-known formula $\det(A) = \exp\{\operatorname{Tr}\ln A\}$ as we have

$$\frac{\partial}{\partial z_i}\ln G = \operatorname{Tr}\left(g^{-1}\frac{\partial g}{\partial z_i}\right) = \sum_{\bar{n},k} g^{\bar{n}k}\frac{\partial g_{k\bar{n}}}{\partial z_i}.$$

These simple formulas for the Christoffel symbols simplify the expression for the full Riemann curvature tensor. Recall that the Riemannian curvature tensor R of the metric g one M is defined by $R(X,Y) : \Gamma(TM)\longrightarrow\Gamma(TM)$, where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X, Y, Z \in \Gamma(TM).$$

Point-wise $R(X,Y)$ is a skew-endomorphism of T_pM at any $p \in \mathcal{U}$. Since also $R(X,Y) = -R(Y,X)$ one can think of R as a 2-from on M with values in skew-endomorphisms of TM. It is sometimes convenient to think of R as the *curvature operator*, that is a sections of $\odot^2\Lambda^2 TM$ defined by

$$\mathcal{R}(X\wedge Y, W\wedge Z)) = g(R(X,Y)Z, W), \qquad X, Y, Z, W \in \Gamma(TM).$$

Alternatively, one can introduce R as the section of $\otimes^4(T^*M)$, i.e., the four-linear map $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$, defined by

$$R(X, Y, V, Z) = g(R(X, Y)Z, V), \qquad X, Y, V, Z \in \Gamma(TM).$$

All these different pictures are useful.

If (M, J, g) is complex than one can extend all these curvature tensors by $\mathbb{C}$-linearity to $TM \otimes_{\mathbb{R}} \mathbb{C}$. In addition, on a Kähler manifold we have $\nabla J = 0$ so that

$$R(V, W)JZ = JR(V, W)Z, \qquad V, W, Z \in \Gamma(TM \otimes_{\mathbb{R}} \mathbb{C}).$$

The above property has many important consequences. Since g is Hermitian we get $R(U, V, JW, JZ) = R(U, V, W, Z)$. Hence, $R(U, V, W, Z) = 0$ unless W, Z are of different type. In particular, in the Kähler case one can define the following curvature tensors:

Definition 4.2.3: *Let (M, J, g, ω_g) be a Kähler manifold and let R be the curvature tensor of (M, g). Extending R to $TM \otimes_{\mathbb{R}} \mathbb{C}$ we define the following:*

(i) *Riemann curvature tensor R as section of $\otimes^4(T^{*1,0}M)$, i.e., the map $R : \Gamma(T^{1,0}M) \times \Gamma(T^{0,1}M) \times \Gamma(T^{1,0}M) \times \Gamma(T^{0,1}M) \longrightarrow C^\infty(M, \mathbb{C})$, given by*

$$(X, Y, Z, W) \mapsto R(X, \bar{Y}, Z, \bar{W}),$$

(ii) *Riemann curvature tensor $R_{X,\bar{Y}}$ as a section of $\mathrm{End}(T^{1,0}M)$, i.e., the map $R_{X,\bar{Y}} : (\Gamma(T^{1,0}M)) \longrightarrow \Gamma(T^{0,1}M)$, $X, Y \in \Gamma(T^{1,0}M)$ given by*

$$R_{X,\bar{Y}}(Z) = R(X, \bar{Y})\bar{Z},$$

(iii) *A real $(1,1)$-form Ω with values in $\Gamma(\mathrm{End}(T^{1,0}M))$, i.e., a skew-Hermitian map $\Omega : \Gamma(T^{1,0}M) \times \Gamma(T^{0,1}M) \longrightarrow \Gamma(\mathrm{End}(T^{1,0}M))$*

$$\Omega(X, \bar{Y}) = R_{X,\bar{Y}}, \qquad X, Y \in \Gamma(T^{1,0}M),$$

called the Kähler Riemann curvature form.

(iv) *Riemann curvature operator $\mathcal{R}$ as a section of $\odot^2\Gamma(\Lambda^{1,1}M)$, i.e., the map $\mathcal{R} : \Gamma(\Lambda^{1,1}M) \times \Gamma(\Lambda^{1,1}M) \longrightarrow C^\infty(M, \mathbb{C})$ given by*

$$\mathcal{R}(X \wedge \bar{Y}, V \wedge \bar{W}) = R(X, \bar{Y}, Z, \bar{W}).$$

Proposition 4.2.4: *Let $(\mathcal{U}; z_1, \ldots, z_n)$ be a local coordinate chart on a Kähler manifold (M, J, g, ω_g). Let $R \in \Gamma(\otimes^4(T^{*1,0}M))$ be the Riemann curvature tensor. Then*

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} - \sum_{p,\bar{r},m} g^{p\bar{r}} \frac{\partial g_{k\bar{r}}}{\partial z_i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j}.$$

PROOF. It remains to do the local computations. We have

$$R_{i\bar{j}k\bar{l}} = R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right) = g\left(R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)\frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial z_k}\right).$$

Since

$$R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)\frac{\partial}{\partial \bar{z}_l} = \nabla_{\partial/\partial z_i}\nabla_{\partial/\partial \bar{z}_j}\frac{\partial}{\partial \bar{z}_l} - \nabla_{\partial/\partial \bar{z}_j}\nabla_{\partial/\partial z_i}\frac{\partial}{\partial \bar{z}_l} = \nabla_{\partial/\partial z_i}\left(\sum_{\bar{n}} \Gamma^{\bar{n}}_{\bar{j}\bar{l}}\frac{\partial}{\partial \bar{z}_n}\right).$$

Hence

$$R_{i\bar{j}k\bar{l}} = g\left(\nabla_{\partial/\partial z_i}\left(\sum_{\bar{n}} \Gamma^{\bar{n}}_{\bar{j}\bar{l}}\frac{\partial}{\partial \bar{z}_n}\right), \frac{\partial}{\partial z_k}\right) = \frac{\partial}{\partial z_i} g\left(\sum_{\bar{n}} \Gamma^{\bar{n}}_{\bar{j}\bar{l}}\frac{\partial}{\partial \bar{z}_n}, \frac{\partial}{\partial z_k}\right) -$$

$$g\Big(\sum_{\bar n}\Gamma^{\bar n}_{\bar j\bar l}\frac{\partial}{\partial \bar z_n},\sum_{p}\Gamma^{m}_{ik}\frac{\partial}{\partial z_m}\Big)=\frac{\partial}{\partial z_i}\Big(\sum_{\bar n}\Gamma^{\bar n}_{\bar j\bar l}g_{\bar n k}\Big)-\sum_{\bar n,p}\Gamma^{\bar n}_{\bar j\bar l}\Gamma^{m}_{ik}g_{\bar n m}=$$

$$=\frac{\partial}{\partial z_i}\Big(\sum_{\bar n,p}g^{\bar n p}\frac{\partial g_{\bar l p}}{\partial \bar z_j}g_{\bar n k}\Big)-\sum_{\bar n,p,\bar r,m}g^{\bar n p}\frac{\partial g_{\bar l p}}{\partial \bar z_j}g^{m\bar r}\frac{\partial g_{k\bar r}}{\partial z_i}g_{\bar n m}=\frac{\partial^2 g_{k\bar l}}{\partial z_i\partial \bar z_j}-\sum_{p,\bar r,m}g^{p\bar r}\frac{\partial g_{k\bar r}}{\partial z_i}\frac{\partial g_{p\bar l}}{\partial \bar z_j}.$$

□

Definition 4.2.5: *Let $R_{X,\bar Y}:\Gamma(T^{1,0}M)\longrightarrow\Gamma(T^{1,0}M)$, $X,Y\in\Gamma(T^{1,0}M)$ be the Riemann curvature tensor on a Kähler manifold (M,J,g,ω_g). The Ricci curvature tensor is the map $\mathrm{Ric}_\omega:\Gamma(T^{1,0}M)\otimes\Gamma(T^{1,0})\longrightarrow C^\infty(M,\mathbb{C})$ defined by the trace*

$$\mathrm{Ric}_\omega(X,\bar Y)=\mathrm{Tr}(R_{X,\bar Y}).$$

A priori it is not clear that $\mathrm{Ric}_\omega(X,\bar Y)$ is the Ricci curvature tensor in the usual sense. However, it is clear that $\mathrm{Ric}(X,\bar Y)$ defines a Hermitian symmetric form on $T^{1,0}_pM$. For with respect to any unitary basis $\{e_1,\dots,e_n\}$ of $T^{1,0}_pM$ we have

$$\mathrm{Ric}_\omega(X,\bar Y)=\sum_{i=1}^{n}R(X,\bar Y,e_i,\bar e_i),$$

so that

$$\overline{\mathrm{Ric}_\omega(X,\bar Y)}=\sum_{i=1}^{n}R(\bar X,Y,\bar e_i,e_i)=\sum_{i=1}^{n}R(Y,\bar X,e_i,\bar e_i)=\mathrm{Ric}_\omega(Y,\bar X).$$

Proposition 4.2.6: *The Ricci curvature tensor $\mathrm{Ric}_\omega:T^{1,0}M\otimes T^{1,0}M\to\mathbb{C}$ is a Hermitian symmetric form on $T^{1,0}_pM$ at each $p\in M$. In a local chart $(\mathcal{U};z_1,\dots,z_n)$ this form can be written as*

$$\mathrm{Ric}_\omega=\sum_{i,\bar j=1}^{n}R_{i\bar j}dz_i\otimes d\bar z_j,$$

where

$$R_{i\bar j}=-\frac{\partial^2}{\partial z_i\partial\bar z_j}(\ln G).$$

The real part of this Hermitian symmetric form is the Ricci curvature tensor of the metric g while the imaginary part is a real $(1,1)$-form on M called the Ricci form and denoted by $\rho_\omega\equiv\rho_g$. With respect to the local chart $(\mathcal{U};z_1,\dots,z_n)$ we have

$$\rho_\omega=-\frac{i}{2}\sum_{i,\bar j}R_{i\bar j}dz_i\wedge d\bar z_j=-\partial\bar\partial(\ln G).$$

Proof. By definition we have

$$R_{i\bar j}=-\sum_{k\bar l}g^{k\bar l}R_{i\bar j k\bar l}=-\frac{\partial^2}{\partial z_i\partial\bar z_j}(\ln G).$$

A priori it is not obvious that $\mathrm{Ric}(X,\bar Y)$ coincides with the usual definition of the Ricci curvature. A simple calculation shows that this is indeed the case. □

Note that the Kähler Riemann curvature 2-form $\Omega=d\theta+\theta\wedge\theta$ is simple the curvature 2-form of the $\nabla^{1,0}$ part of the Levi-Civita connection. In local coordinates we can write

$$\Omega^l_k=\sum_{i,\bar j,\bar p}R_{i\bar j k\bar p}g^{l\bar p}dz_i\wedge d\bar z_j$$

and, hence, the Ricci 2-form $\rho_\omega = \mathrm{Tr}(\Omega)$.

There are several more curvatures typically considered in the context of complex and Kähler manifolds. The usual notion of sectional curvature is one of them. Recall that for any $X, Y \in \Gamma(TM)$ we define the sectional curvature of the 2-plane $\sigma \subset T_pM$ spanned at point $p \in M$ by X_p and Y_p as

$$K(\sigma, p) = K(X, Y) = \frac{R(X,Y,Y,X)}{|X \wedge Y|^2},$$

where $|X \wedge Y|^2 = |X|^2|Y|^2 - [g(X,Y)]^2 = g(X,X)g(Y,Y) - [g(X,Y)]^2$ is the area square of the parallelogram spanned by X, Y. Now, if (M, g) is complete, simply connected and of constant sectional curvature then (depending on the sign) (M, g) is isometric to S^n, $\mathbb{R}^n$, or the real hyperbolic ball $B^n_{\mathbb{R}}(1)$. Now, in complex dimension greater than 1 the only Kähler manifolds of constant sectional curvature are flat. To obtain a Kähler analogue of the real space forms we introduce the following

Definition 4.2.7: *Let (M, J, g, ω_g) be a Kähler manifold and let $X, Y \in \Gamma(T^{1,0}M)$. Then*

$$B(X,Y) = \frac{R(X,\bar{X},Y,\bar{Y})}{g(X,\bar{X})g(Y,\bar{Y})},$$

is called the bisectional curvature of g in the direction of X, Y while

$$H(X) = B(X,X) = \frac{R(X,\bar{X},X,\bar{X})}{[g(X,\bar{X})]^2}$$

is called the holomorphic sectional curvature of g in the direction of X.

If we write $X = \frac{1}{\sqrt{2}}(U + iJU)$, $Y = \frac{1}{\sqrt{2}}(V + iJV)$ for some real vector fields $U, V \in \Gamma(TM)$ we have

$$B(X,Y) = \frac{|JU \wedge V|^2}{|U|^2|V|^2}K(JU,V) + \frac{|U \wedge V|^2}{|U|^2|V|^2}K(U,V),$$

and $H(X) = K(U, JU)$, where K the sectional curvature of g. In particular, if g is of positive (negative, nonpositive, nonnegative) sectional curvature then g is of positive (negative, nonpositive, nonnegative) bisectional curvature. Clearly, the holomorphic sectional curvature is the curvature of all J-invariant planes. It is not hard to show that on a Kähler manifold the holomorphic sectional curvature H determines the Riemann curvature tensor completely.

Example 4.2.8: (Fubini-Study Metric on the Complex Projective Space) Let $M = \mathbb{CP}^n$ and $\mathcal{U}_0 = \{[1 : z_1 : \cdots : z_n] \in M \mid z_0 \neq 0\} \simeq \mathbb{C}^n$. Let $|\mathbf{z}|^2 = |z_1|^2 + \cdots + |z_n|^2$ and define

$$g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln(1 + |\mathbf{z}|^2),$$

or, equivalently¡

$$\omega_g = \frac{i}{2}\partial\bar{\partial}\ln(1+|\mathbf{z}|^2) = \frac{i}{2}\left(\frac{\sum_i dz_i \wedge d\bar{z}_i}{1+|\mathbf{z}|^2} - \frac{(\sum_i \bar{z}_i dz_i) \wedge (\sum_j z_j d\bar{z}_j)}{(1+|\mathbf{z}|^2)^2}\right).$$

One can easily check that this formula defines ω_g globally on any other chart U_α. An easy computation gives

$$\omega_g^n = (i/2)^2 \frac{n!}{(1+|\mathbf{z}|^2)^{n+1}} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

which means that $G = \det(g_{i\bar{j}}) = (1 + |\mathbf{z}|^2)^{-(n+1)}$. Hence, The Ricci curvature tensor

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln G = (n+1) g_{i\bar{j}}.$$

It can be shown that the Fubini-Study metric is $U(n+1)$-invariant and that $U(n+1)$ acts on M transitively. It follows that in order to compute the full Riemann curvature tensor $R_{i\bar{j}k\bar{l}}$ it is sufficient to compute it at one point $[1 : 0 : \cdots : 0]$. Using local expressions for $R_{i\bar{j}k\bar{l}}$ and $g_{i\bar{j}}$ one can easily see that at $\mathbf{z} = 0$ we have $R_{i\bar{j}k\bar{l}} = \delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{j}}$. In particular, as $g_{i\bar{j}} \mid_{\mathbf{z}=0} = \delta_{i\bar{j}}$ at any other point

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$$

and M is of constant holomorphic sectional (and bisectional) curvature $c = +1$.

EXAMPLE 4.2.9: (Bergman Metric on the Complex Ball) Consider $M = B^n_{\mathbb{C}}(1) = \{\mathbf{z} \in \mathbb{C}^n \mid |\mathbf{z}|^2 < 1\}$ and let

$$\omega_g = \frac{i}{2}\partial\bar{\partial}\ln(1 - |\mathbf{z}|^2).$$

Likewise, one can show that Bergman metric g is $U(n,1)$-invariant and that $U(n,1)$ acts on $M = B^n_{\mathbb{C}}(1)$ transitively. Repeating the calculations of the previous example we can easily see that $R_{i\bar{j}} = -(n+1)g_{i\bar{j}}$ and (M, g) is of constant holomorphic sectional curvature $c = -1$.

It can be shown that any two simply-connected complete Kähler manifolds with constant sectional c are holomorphically isometric. Hence, the above examples together with the flat metric on $\mathbb{C}^n$ show that

Theorem 4.2.10: *Let (M^{2n}, J, g, ω_g) be a complete Kähler manifold of constant holomorphic sectional curvature c. Depending on the sign of c and up to scaling, the universal cover $\widetilde{M}$ is holomorphically isometric to $\mathbb{CP}^n$, $\mathbb{C}^n$, or $B^n_{\mathbb{C}}(1)$.*

4.3. Curvature and Chern Forms

Let (M, J) be a complex manifold and let $E \longrightarrow M$ be a complex vector bundle of rank r over M. Let ∇ be a complex connection on E with curvature Ω. Consider the space $\mathcal{M}_{r\times r}(\mathbb{C})$ of complex $r \times r$ matrices. For any $\mathbb{A} \in \mathcal{M}_{r\times r}(\mathbb{C})$ we define

$$\det(\mathbb{A} + \lambda\mathbb{I}) = f_r(\mathbb{A}) + \lambda f_{r-1}(\mathbb{A}) + \cdots + \lambda^{r-1} f_1(\mathbb{A}) + \lambda^r.$$

Clearly, for any $i = 1, \ldots, r$ the function $f_i : \mathcal{M}_{r\times r}(\mathbb{C}) \longrightarrow \mathbb{C}$ is a $GL(r, \mathbb{C})$-invariant, complex homogeneous polynomial of $\deg(f_i) = i$. Note that f_i is the i^{th} elementary symmetric function of the eigenvalues of $\mathbb{A}$. In particular, $f_r(\mathbb{A}) = \det(\mathbb{A})$ and $f_1(\mathbb{A}) = \mathrm{Tr}(\mathbb{A})$.

Definition 4.3.1: *Let $E \longrightarrow M$ be rank r complex vector bundle over (M, J) and let ∇ be a complex connection on E with curvature 2-form Ω. For each $i = 1, \ldots, r$ we define the $2i$-form*

$$f_i\left(\frac{i}{2\pi}\Omega\right) = c_i(E, \nabla)$$

and call it the i^{th} Chern from of E.

We have the following

Proposition 4.3.2: *Given (E,∇) and any $1 \le i \le r$, the i^{th} Chern from $c_i(E,\nabla)$ is closed. Furthermore, if $\widetilde{\nabla}$ is another complex connection on E the difference $c_i(E,\nabla) - c_i(E,\widetilde{\nabla})$ is exact, i.e., the cohomology class $[c_i(E,\nabla)] \in H^{2i}_{DR}(M)$ is independent of ∇.*

Now, let us consider special case of a Kähler manifold (M,J,g,ω_g) with $(E,\nabla) = (T^{1,0}M,\nabla^{1,0})$, where the complex connection $\nabla^{1,0}$ on the holomorphic tangent bundle $T^{1,0}M$ is given locally by Christoffel symbols Γ^k_{ij}. The curvature of the connection 1-form of $\nabla^{1,0}$ is the Kähler curvature 2-form Ω. We have

Definition 4.3.3: *Let (M^{2n},J,g,ω_g) be a Kähler manifold with the Kähler curvature 2-form Ω. For each $i = 1,\dots,n$ the cohomology class of the $2i$-form $f_i\big(\frac{i}{2\pi}\Omega\big)$ is called the i^{th} Chern class of M and denoted by $c_i(M)$.*

Note that the top class $c_n(M) = e(M)$ is the Euler class. It follows that Chern classes do not depend on the Kähler (or even simply Hermitian) metric. They depend only on the homotopy class of the complex structure J on M.

In the previous chapter we showed that on a Kähler manifold cohomology class defined by the Ricci form ρ_g does not depend on the Kähler metric. Since $\rho_g = \mathrm{Tr}(\Omega)$ we conclude that

Proposition 4.3.4: *Let (M,J,g,ω_g) be a Kähler manifold and ρ_ω the Ricci form. Then*

$$[\rho_\omega] = 2\pi c_1(M) \in H^2_{DR}(M) \cap H^{1,1}(M,\mathbb{C}).$$

Definition 4.3.5: *Let (M,J) be a complex manifold. We say that $c_1(M) > 0$ $(c_1(M) < 0)$if $c_1(M)$ can be represented be a positive (negative) $(1,1)$-form on M. Furthermore, we say that $c_1(M) = 0$ if the first Chern class is cohomologous to zero.*

4.4. Hodge Theory on Kähler Manifolds

Let (M^{2n},J,g,ω_g) be a compact Hermitian manifold. We consider the following map $\star : \Gamma(\Lambda^{k,l}M) \longrightarrow \Gamma(\Lambda^{n-k,n-l})M$ defined by

$$\phi \wedge \eta = g(\phi,\bar\eta)d\mathrm{vol}_g = g(\phi,\bar\eta)\frac{\omega_g^n}{n!}, \qquad \phi,\eta \in \Gamma(\Lambda^{k,l}M).$$

Note that this map is linear over $\mathbb{R}$ but anti-linear over $\mathbb{C}$. In addition, we have $\star^2\phi = (-1)^{k+1}\phi$. The action of the Hodge star operator on any complex k-form $\phi \in \Gamma(\Lambda^k M \otimes \mathbb{C})$ is defined by the splitting of ϕ as a sum of forms of type $(k-j,j)$. Locally, in a holomorphic chart $(\mathcal{U};z_1,\dots,z_n)$, $n = \dim_{\mathbb{C}}(M)$ one can write

$$\Gamma(\Lambda^{k,l}M) = \Big\{ \sum_{|I|=k,|J|=l} \omega_{I\bar J} dz_I \wedge d\bar z_J, \quad \omega_{I\bar J} \in C^\infty(\mathcal{U};\mathbb{C}) \Big\},$$

where

$$I = (i_1,\dots,i_k), \quad 1 \le i_1 < \cdots < i_k \le n, \quad dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_k},$$

$$J = (j_1,\dots,j_l), \quad 1 \le j_1 < \cdots < j_k \le n, \quad d\bar z_J = d\bar z_{k_1} \wedge \cdots \wedge d\bar z_{j_l}.$$

Let us assume that $(\mathcal{U};z_1,\dots,z_n)$ are normal coordinates, i.e., $\{dz_1,\dots,dz_n\}$ is a unitary frame of $\Gamma(\Lambda^{1,0}M)$. Denote by $\hat I, \hat J$ the likewise ordered complements of

I, J in $\{1, \dots, n\}$ and define $\sigma(I)$ by

$$dz_I \wedge dz_{\hat{I}} = (-1)^{\sigma(I)} dz_1 \wedge \cdots \wedge dz_n.$$

Then we have an explicit formula

$$\star(\omega_{I\bar{J}} dz_I \wedge d\bar{z}_J) = 2^{k+l-m} i^n \epsilon_{IJ} \overline{\omega_{I\bar{J}}} dz_{\hat{I}} \wedge d\bar{z}_{\hat{J}},$$

where the sign $\epsilon_{IJ} = (-1)^{n(n-1)/2+(n-k)l+\sigma(I)+\sigma(J)}$. Since M is compact we can define an L^2-norm on each $\Gamma(\Lambda^{k,l}M)$ by

$$(\phi, \eta) = \int_M \phi \wedge \star\eta, \qquad \phi, \eta \in \Gamma(\Lambda^{k,l}M).$$

Once can check that $(\star\phi, \star\eta) = (\phi, \eta)$. We proceed to define

$$\bar{\partial}^* = - \star \bar{\partial} \star : \Gamma(\Lambda^{k,l}M) \longrightarrow \Gamma(\Lambda^{k,l-1}M).$$

It follows that this is a a formal adjoint of the $\bar{\partial}$ operator as for any $\phi \in \Gamma(\Lambda^{k,l-1}M)$ and any $\eta \in \Gamma(\Lambda^{k,l}M)$ we have

$$\bar{\partial}\phi \wedge \star\eta - \phi \wedge \star\bar{\partial}^* = \bar{\partial}\phi \wedge \star\eta + (-1)^{k+l-1}\phi \wedge \bar{\partial} \star \eta = \bar{\partial}(\phi \wedge \star\eta) = d(\phi \wedge \star\eta),$$

so that, by Stokes' theorem, $(\bar{\partial}\phi, \eta) = (\phi, \bar{\partial}^*\eta)$. Similarly, once can define

$$\partial^* = - \star \partial \star : \Gamma(\Lambda^{k,l}M) \longrightarrow \Gamma(\Lambda^{k-1,l}M)$$

which is an adjoint of the ∂ operator. Hence, we have 6 different operators acting on forms $d, d^*, \partial, \partial^*, \bar{\partial}, \bar{\partial}^*$ and 3 natural Laplacians

$$\Delta_d = dd^* + d^*d, \qquad \Delta_\partial = \partial\partial^* + \partial^*\partial, \qquad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

The Laplacian $\Delta_{\bar{\partial}}$ is a self-adjoint operator and

$$(\Delta_{\bar{\partial}}\phi, \phi) = (\bar{\partial}\phi, \bar{\partial}\phi) + (\bar{\partial}^*\phi, \bar{\partial}^*\phi) \geq 0.$$

Definition 4.4.1: *The space* $\mathcal{H}^{k,l}(M) \subset \Gamma(\Lambda^{k,l}M)$

$$\mathcal{H}^{k,l}(M) = \{\phi \in \Gamma(\Lambda^{p,q}M) \ \mid \ \Delta_{\bar{\partial}}\phi = 0\}.$$

is called the space of harmonic (k,l)-forms. Furthermore, $\dim(\mathcal{H}^{k,l}(M)) = h^{k,l}(M)$ *are called the Hodge numbers of M.*

Theorem 4.4.2: *On a compact Hermitian manifold (M^{2n}, J, g, ω_g) the dimensions* $\dim(\mathcal{H}^{k,l}(M)) < \infty$ *for all $0 \leq k, l \leq n$. Furthermore, we have the decompositions*

$$\Gamma(\Lambda^{k,l}M) = \mathcal{H}^{k,l}(M) \oplus \bar{\partial}(\Gamma(\Lambda^{k,l-1}M)) \oplus \bar{\partial}(\Gamma(\Lambda^{k,l-1}M)).$$

In particular, any class in $H^{k,l}_{\bar{\partial}}(M)$ has a unique harmonic representative, i.e., we have a complex vector space isomorphism

$$\mathcal{H}^{k,l}(M) \simeq H^{k,l}_{\bar{\partial}}(M).$$

Let us introduce the projections

$$\pi^{k,l} : \Gamma(\Lambda^*TM \otimes \mathbb{C}) \longrightarrow \Gamma(\Lambda^{p,q}M.$$

$$\pi^r = \oplus_{k+l=r}\pi^{k,l} : \Gamma(\Lambda^*TM \otimes \mathbb{C}) \longrightarrow \Gamma(\Lambda^r M \otimes \mathbb{C}).$$

Since the metric g is Hermitian we have $\pi_{k,l}\star = \star\pi_{k,l}$. In general, given a complex Δ_d-harmonic r-form ψ the projection $\pi_{k,l}\psi$ need not be $\Delta_{\bar{\partial}}$-harmonic. However, on a Kähler manifold we have some additional structure.

To show this let us introduce the operator

$$L : \Gamma(\Lambda^{k,l}M) \longrightarrow \Gamma(\Lambda^{k+1,l+1}M)$$

defined by wedging with the Kähler form ω_g, i.e., $L(\phi) = \phi \wedge \omega_g$. The adjoint of L in the L^2-norm is given by

$$L^* = (-1)^{p+q} \star L \star : \Gamma(\Lambda^{k,l} M) \longrightarrow \Gamma(\Lambda^{k-1,l-1} M).$$

We have the following

Lemma 4.4.3 (Kähler Identities): *Let (M, J, g, ω_g) be a Kähler manifold. Then*

$$[L, \partial^*] = i\bar{\partial}, \qquad [L, \bar{\partial}^*] = -i\partial.$$

$$[L^*, \partial] = i\bar{\partial}^*, \qquad [L^*, \bar{\partial}] = -i\partial^*.$$

Proof. Since both sides of all these inequalities are first order differential operators it is enough to check the for the Euclidean space $\mathbb{C}^n$ with the standard Hermitian metric. The general statement follows then from the fact that Kähler condition is equivalent to the existence of normal coordinates. The computation follows from the formula (*.*). □

Proposition 4.4.4: *Let (M, J, g, ω_g) be a compact Kähler manifold. The*

(i) $[L^*, \Delta_d] = [L^*, \Delta_d] = 0$,
(ii) $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.
(iii) $[\Delta_d, \pi^{p,q}] = 0$

Proof. Now, the proof of the proposition. We have

$$\Delta_d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) = \Delta_\partial + \Delta_{\bar{\partial}} + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\bar{\partial}\partial^* + \partial^*\bar{\partial}).$$

Using Kähler identities we we can replace $\bar{\partial}^* = -i[L^*, \partial]$ and

$$(\partial\bar{\partial}^* + \bar{\partial}^*\partial) = -i(\partial[L^*, \partial] + [L^*, \partial]\partial) = 0.$$

On the other hand, using $[L^*, \bar{\partial}] = -i\partial^*$, we have

$$-i\Delta_\partial = \partial(-i\partial^*) + (-i\partial^*)\partial = \partial(L^*\bar{\partial} - \bar{\partial}L^*) + (L^*\bar{\partial} - \bar{\partial}L^*)\partial =$$

$$= \partial L^*\bar{\partial} - L^*\partial\bar{\partial} + \bar{\partial}\partial L^* - \bar{\partial}L^*\partial = [\partial, L^*]\bar{\partial} + \bar{\partial}[\partial, L^*] = -i(\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*) = -i\Delta_{\bar{\partial}}.$$

□

There are two important consequences of the above propositions. The first is the so-called Hodge decomposition. The second is the so-called hard Lefschetz theorem. Let us first discuss the Hodge structure. We define

$$H^{k,l}(M, \mathbb{C}) = \frac{Z_d^{k,l}(M)}{d\Gamma(\Lambda^* M \otimes \mathbb{C}) \cap Z_d^{k,l}(M)},$$

to be the space of d-closed forms of type (k, l) modulo d-exact forms and let $\mathcal{H}_d^r(M)$ be the space of Δ_d-harmonic r-forms. Since the usual Laplacian Δ_d is real, commutes with $\pi^{k,l}$ and equals $1/2$ of the $\bar{\partial}$-Laplacian we have

$$\mathcal{H}^r(M) = \oplus_{k+l=r} \mathcal{H}^{k,l}(M),$$

$$H_{\bar{\partial}}^{k,l}(M) = \mathcal{H}^{k,l}(M) = \overline{\mathcal{H}^{k,l}(M)} = H^{k,l}(M, \mathbb{C}).$$

Using the usual Hodge theorem $H_{DR}^*(M) = \mathcal{H}^*(M)$ for the Laplacian Δ_d we get

Theorem 4.4.5: *For a compact Kähler manifold we have*

$$H^r(M, \mathbb{C}) = \oplus_{k+l=r} H^{k,l}(M, \mathbb{C}), \qquad \overline{H^{k,l}(M, \mathbb{C})} = H^{l,k}(M, \mathbb{C}).$$

Since $[\star, \Delta_{\bar\partial}] = 0$, by Kodaira-Serre duality the map

$$\star : \mathcal{H}^{k,l}(M) \longrightarrow \mathcal{H}^{n-k,n-l}(M),$$

is an isomorphism. Summarizing, we have the following properties of the Hodge numbers

Corollary 4.4.6: *Let (M^{2n}, J, g, ω_g) be a compact Kähler manifold, $h^{k,l}(M)$ be the Hodge numbers, and $b_r(M) = \sum_{k+l=r} h^{k,l}(M)$ the Betti numbers of M. Then for all $0 \leq k, l \leq n$ we have*

(i) $h^{k,l}(M) < \infty$,
(ii) $h^{k,k}(M) \geq 1$ *and* $h^{n,n} = h^{0,0} = 1$,
(iii) $h^{k,l}(M) = h^{l,k}(M) = h^{n-k,n-l}(M)$,
(iv) $b_{2r+1}(M)$ *are even and* $h^{1,0}(M) = h^{0,1}(M) = \frac{1}{2}b_1(M)$ *is a topological invariant.*

For any compact Kähler manifold once can consider the Hodge diamond, that is the arrangement of the Hodge numbers in a diamond-shape array. For instance, when $n = 3$ we can have

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array}$$

The above corollary implies that every Hodge diamond has two symmetries: conjugation gives the symmetry through the central vertical axis while Hodge star yields the symmetry through the center of the diamond. There are more relations between Hodge numbers. They are due to the Lefschetz decomposition which we will discuss below. Let us define an operator $h = \sum_{r=0}^{2n}(n-r)\pi^r$. Then we have the following

$$[L^*, L] = h, \qquad [h, L] = -2L, \qquad [h, L^*] = 2L^*.$$

Hence, $\{, L, L^*, h\}$ are generators of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

Definition 4.4.7: *Let (M, J, g, ω_g) be a compact Kähler manifold. We define primitive cohomology groups of M as kernels of the L^*-operator, i.e.,*

$$P^r(M, \mathbb{C}) = \ker\big(L^* : H^r(M, \mathbb{C}) \longrightarrow H^{r-2}(M, \mathbb{C})\big),$$

$$P^{k,l}(M, \mathbb{C}) = \ker\big(L^* : H^{k,l}(M, \mathbb{C}) \longrightarrow H^{k-1,l-1}(M, \mathbb{C})\big).$$

Dimension of $P^r(M, \mathbb{C})$ is called primitive r^{th} Betti number of M.

Since $\{L, L^*, h\}$ all commute with the Kähler Laplacian we have

$$P^r(M, \mathbb{C}) = \oplus_{k+l=r} P^{k,l}(M, \mathbb{C}).$$

The generators $\{L, L^*, h\}$ give a finite-dimensional representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ acting on $H^*(M,\mathbb{C})$, where $H^r(M,\mathbb{C})$ is the eigenspace of the operator h with eigenvalue $(n-r)$. The next theorem follows directly from the theory of finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$:

Theorem 4.4.8: *(Hard Lefschetz) On a compact Kähler manifold M the map $L^k : H^{n-k}(M) \longrightarrow H^{n+k}(M)$ is an isomorphism for $1 \leq k \leq n$. Furthermore,*

$$H^r(M) = \bigoplus_{k\geq 0} L^k P^{r-2k}(M).$$

4.5. Calabi Conjecture and Calabi-Yau Theorem

We have seen that on a Kähler manifold Ricci curvature 2-form ρ_ω of any Kähler metric represents the cohomology class $2\pi c_1(M)$. The well-known Calabi Conjecture is the question whether or not the converse is also true. To be more specific we begin with a couple of definitions

Definition 4.5.1: *Let (M, J, g, ω_g) be a compact Kähler manifold. The Kähler cone of M*

$$K(M) = \{[\omega] \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R}) \ \ | \ [\omega] = [\omega_h] \text{ for some Kaehler metric } h\}$$

is the set of all possible Kähler classes on M.

It is easy to show that $K(M)$ is a convex open set in $H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$.

Definition 4.5.2: *Let (M, J, g, ω_g) be a compact Kähler manifold and $K(M)$ it Kähler cone. For any fixed Kähler class $[\omega] \in K(M)$ we define*

$$\mathcal{K}_{[\omega]} = \{h \in \Gamma(\odot^2(TM)) \ \ | \ h \text{ his a Kaehler metric and } [\omega] = [\omega_h]\}$$

to be the space of all Kähler metrics in a given cohomology class..

Global $i\partial\bar\partial$-lemma provides for a very simple description of the space of Kähler metrics $\mathcal{K}_{[\omega]}$. Suppose we have a Kähler metric g in a with Kähler class $[\omega_g] = [\omega] \in K(M)$. If $h \in \mathcal{K}_{[\omega]}$ is another Kähler metric then, up to a constant, there exists a global function $\phi \in C^\infty(M,\mathbb{R})$ such that $\omega_h - \omega_g = i\partial\bar\partial\phi$. We could fix constant by requiring, for example, that $\int_M \phi\omega_g^n = 0$. Hence, we have

Corollary 4.5.3: *Let (M, J, g, ω_g) be a compact Kähler manifold with $[\omega_g] = [\omega] \in K(M)$. Then, relative to the metric g the space of all Kähler metric is $\mathcal{K}_{[\omega]}$ can be described as*

$$\mathcal{K}_{[\omega]} = \{\phi \in C^\infty(M,\mathbb{R}) \ \ | \ \ \omega_h = \omega_g + i\partial\bar\partial\phi > 0, \ \ \int_M \phi\omega_g^n = 0\},$$

where the 2-form $\omega_h > 0$ means that $\omega_h(X, JY)$ is a Hermitian metric on M.

The following theorem is the statement of the famous conjecture made by Calabi and proved by Yau in 1978.

Theorem 4.5.4: *Let (M, J, g, ω_g) be a compact Kähler manifold, $[\omega_g] = \omega \in K(M)$ the corresponding Kähler class and ρ_g the Ricci form. Consider any positive $(1,1)$-form on M such that $[\Omega] = 2\pi c_1(M)$. Then there exists a unique Kähler metric $h \in \mathcal{K}_{[\omega]}$ such that $\Omega = \rho_h$.*

Let us reformulate the problem using the global $i\partial\bar{\partial}$-lemma. We start with a given Kähler metric g on M in Kähler class $[\omega_g] = [\omega]$. Since ρ_g also represents $2\pi c_1(M)$ there exists a globally defined function $f \in C^\infty(M, \mathbb{R})$ such that

$$\rho_g - \Omega = i\partial\bar{\partial} f.$$

Appropriately, f could be called *a discrepancy potential function* for the Calabi problem and we could fix the constant by asking that $\int_M (e^f - 1)\omega_g = 0$.

Now, supposed the desired solution of the problem is a metric $h \in \mathcal{K}_{[\omega]}$. We know that the Kähler form of h can be written as

$$\omega_h = \omega_g + i\partial\bar{\partial}\phi,$$

for some smooth function $\phi \in C^\infty(M, \mathbb{R})$. We normalize ϕ as in previous corollary. Combining these two equations we see that

$$\rho_h - \rho_g = i\partial\bar{\partial} f.$$

If we define a smooth function $F \in C^\infty(M, \mathbb{R})$ relating the volume forms of the two metrics $\omega_h^n = e^F \omega_g^n$ then the left-hand side of the above equation takes the following form

$$i\partial\bar{\partial} F = \rho_h - \rho_g = i\partial\bar{\partial}\phi,$$

and, hence, simply $i\partial\bar{\partial}(F - f) = 0$. Hence, $F = f + c$. But since we normalized $\int_M (e^f - 1)\omega_g = 0$ we must have $c = 0$. Hence, $F = f$, or $\omega_h^n = e^f \omega_g^n$. We can now give two more equivalent formulations of the Calabi Problem.

Theorem 4.5.5: *Let (M, J, g, ω_g) be a compact Kähler manifold, $[\omega_g] = \omega \in K(M)$ the corresponding Kähler class and ρ_g the Ricci form. Consider any positive $(1,1)$-form on M such that $[\Omega] = 2\pi c_1(M)$. Let $\rho_g - \Omega = i\partial\bar{\partial} f$, with $\int_M (e^f - 1)\omega_g = 0$.*

(i) *There exists a unique Kähler metric $h \in \mathcal{K}_{[\omega]}$ whose volume form ω_h^n equals to $e^f \omega_g^n$.*

(ii) *Let $(\mathcal{U}, z_1, \ldots, z_n)$ be a local complex chart on M with respect to which the metric $g = (g_{i\bar{j}})$. Then, up to a constant, there exists a unique smooth function ϕ in $\mathcal{K}_{[\omega]}$, which satisfies the following equation*

$$\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right)}{\det(g_{i\bar{j}})} = e^f.$$

The equation in (ii) is called the Monge-Ampére equation. Part (i) gives a very simple geometric characterization of the Calabi-Yau theorem. On a compact Kähler manifold one can always find a metric with arbitrarily prescribed volume form. The uniquensess part of this theorem was already proved by Calabi. This part involves only Maximum Principle. The existence proof uses continuity methods and it involves several difficult a priori estimates. These were found by Yau in 1978. For the details we refer the reader to two excelent expositions in Joyce [**?**] and Tian [**?**]. Here we only list some immediate consequences of this theorem:

Corollary 4.5.6: *Let (M, J, g, ω_g) be a compact Kähler manifold with $c_1(M) = 0$. Then M admits a unique Kähler Ricci-flat metric.*

Such a metric has holonomy group inside $SU(n)$. Kähler manifolds with this property are called Calabi-Yau manifolds. We will discuss this geometry in the following section.

Corollary 4.5.7: *Let (M, J, g, ω_g) be a compact Kähler manifold with $c_1(M) > 0$. Then M admits a a Kähler metric of positive Ricci curvature.*

4.6. Kähler-Einstein Metrics

One of the consequences of the Yau's theorem is that a compact Kähler manifold with $c_1(M) = 0$ must admit a Ricci-flat, hence, Einstein metric. More generally, we can consider existence of Kähler-Einstein metrics with arbitrary Einstein constant λ. By scaling we can assume that $\lambda = 0, \pm 1$. Specifically, let (M, J, g, ω_g) be a compact Käher manifold. We would like to know if one can always find a Kähler-Einstein metric $h \in \mathcal{K}_{[\omega_g]}$. Recall that on a Kähler-Einstein manifold $\rho_g = \lambda\omega_g$. This implies that $2\pi c_1(M) = \lambda[\omega_g]$. Now, if $c_1(M) > 0$ we must have $\lambda = +1$ because $[\omega_g]$ is the Kähler class. Similarly, when $c_1(M) < 0$ the only allowable sign of a Kähler-Einstein metric on M is $\lambda = -1$. Clearly, when $c_(M) = 0$ we must have $\lambda = 0$ as $[\omega_g] \neq 0$. As we have already pointed out the $\lambda = 0$ case follows from Yau's solution to the Calabi conjecture. For the reminder of this chapter we shall assume that $\lambda = \pm 1$.a

Remark 4.6.1: Note that when M is not compact one can have Kähler-Einstein metrics on M of any sign. The simplest example here is $M = \mathbb{C}^n$. First we can take the flat metric which is Einstein with $\lambda = 0$. Then we can identify M with a complex hyperbolic ball and put the Bergman metric on it with $\lambda = -1$. Finally, we can take $M = \mathcal{U}_0 \subset \mathbb{CP}(n)$ with the Fubini-Study metric and $\lambda = +1$. The first two metrics are complete, the third is not. One cannot have a complete metric of positive scalar curvature on a non-compact manifold.

Let (M, J, g, ω_g) be a Kähler manifold and $[\omega_g] = [\omega] \in K(M)$ the Kähler class. Let us reformulate the existence problem using the global $i\partial\bar{\partial}$-lemma. Suppose there exists an Einstein metric $h \in \mathcal{K}_{[\omega]}$. Starting with the original Kähler metric g on M we have a globally defined function $f \in C^\infty(M, \mathbb{R})$ such that

$$\rho_g - \lambda\omega_g = i\partial\bar{\partial}f.$$

As before we will call f *a discrepancy potential function.* We also fix the constant by asking that $\int_M (e^f - 1)\omega_g = 0$. Let $h \in \mathcal{K}_{[\omega]}$ be an Einstein metric for which $\rho_h = \lambda\omega_h$. Using global $i\partial\bar{\partial}$-lemma once again we have a globally defined function $\phi \in C^\infty(M, \mathbb{R})$ such that $\omega_h - \omega_g = i\partial\bar{\partial}\phi$. We shall fix the constant in ϕ later. Using these two equations we easily get

$$\rho_g - \rho_h = i\partial\bar{\partial}(f - \lambda\phi).$$

Defining F so that $\omega_h^n = e^F\omega_g^n$ we can write this equation as

$$i\partial\bar{\partial}F = i\partial\bar{\partial}(f - \lambda\phi).$$

This implies that $F = f - \lambda\phi + c$. We have already fixed the constant in f so c depends only on the choice of ϕ. We can make $c = 0$ by choosing ϕ such that $\int_M (e^{f-\lambda\phi} - 1)\omega_g^n = 0$. Hence, we have the following

Proposition 4.6.1: *Let (M, J) be a compact Kähler manifold with $\lambda c_1(M) > 0$, where $\lambda = \pm 1$. Let $[\omega] \in K(M)$ be a Kähler class and g, h two Kähler metrics in $\mathcal{K}_{[\omega]}$ with Ricci forms ρ_g, ρ_h. Let $f, \phi \in C^\infty(M, \mathbb{R})$ be defined by $\rho_g - \lambda\omega_g = i\partial\bar{\partial}f$, $\omega_h - \lambda\omega_g = i\partial\bar{\partial}f$. Fix the relative constant of $f - \lambda\phi$ by setting $\int_M (e^{f-\lambda\phi} - 1)\omega_g = 0$.*

Then the metric h is Einstein with Einstein constant λ if and only if ϕ satisfies the following Monge-Ampére equation

$$\omega_h^n = e^{f-\lambda\phi}\omega_g^n,$$

in a local complex chart $(\mathcal{U}, z_1, \ldots, z_n)$ written as

$$\frac{\det\Big(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\Big)}{\det(g_{i\bar{j}})} = e^{f-\lambda\phi}.$$

Note that by setting $\lambda = 0$ we get the Monge-Ampére equation for the original Calabi problem. The character of the Monge-Ampére equation above very much depends on the choice of λ. The case of $\lambda = -1$ is actually the simplest as the necessary a priori C^0-estimates can be derived using teh Maximum Principle. This was done by Aubin and independently by Yau. We have

Theorem 4.6.2: *Let (M, J, g, ω_g) be a compact Kähler manifold with $c_1(M) < 0$. Then there exists a unique Kähler metric $h \in \mathcal{K}_{[\omega_g]}$ such that $\rho_h = -\omega_h$.*

When $k = +1$ the problem is much harder. It has been known for quite some time that there are actually non-trivial obstructions to the existance. Let $\mathfrak{h}(M)$ be the complex Lie algebra of all holomorphic vector fields on M. Matsushima proved that one a compact Kähler-Einstein maniofld with $c_1(M) > 0$ $\mathfrak{h}(M)$ must be reductive, i.e., $\mathfrak{h}(M) = Z(\mathfrak{h}(M) \oplus [\mathfrak{h}(M), \mathfrak{h}(M)]$. Later Lichnerowicz proved a stronger result

Theorem 4.6.3: *Let (M, J, g, ω_g) be a compact Kähler manifold of constant scalar curvature. Then $\mathfrak{h}(M) = \mathfrak{a}(M) \oplus \mathfrak{h}_0(M)$, where $\mathfrak{a}(M)$ in the subalgebra of $\mathfrak{h}(M)$ consisting of all parallel vector fileds and $\mathfrak{h}_0(M)$ is the complexification of the Lie algebra of Killing vector fields. In particular, $\mathfrak{h}(M)$ is reductive.*

Assuming existance one can still show uniqueness and it was done by Bando and Mabuchi.

4.7. Calabi-Yau Varieties and Mirror Conjecture

4.8. Fano Varieties

4.9. Kähler-Einstein Metrics on del Pezzo Surfaces

Let us begin with the surface case. We have the following classical result concerning smooth Fano varieties in complex dimension 2 (typically called del Pezzo surfaces). The classical classification result in surface theory states that

Theorem 4.9.1: *Let (M, J) be a smooth compact del Pezzo surface. The M is one of the following*

(i) $\mathbb{CP}^2$,
(ii) *a smooth quadric $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$,*
(iii) *a double cover F_1 of a quadric cone $Q'\mathbb{CP}^3$ ramified along a smooth curve of degree 6 not passing through the vertex of the cone,*
(iv) *a double cover F_2 of $\mathbb{CP}^2$ ramified along a smooth curve of degree 4,*
(v) *a surface $F_d \subset \mathbb{CP}^d$, $d = K^2_{F_d}$, where $3 \le d \le 7$,*
(vi) *a geometrically ruled surface $\mathbb{F}_1$ with the exceptional section s, $s^2 = -1$.*

One can show that surfaces F_d, $d = 1, \ldots, 7$ can be obtained by blowing up of $9-p$ points on $\mathbb{CP}^2$ which are in general position, i.e., no two of these points lie on a line nor any three lie on a conic. Also, $\mathbb{F}_1$ can be obtained by blowing $\mathbb{CP}^2$ at 1 point. Hence, we have the following corollary

Corollary 4.9.2: *Let (M, J, g, ω_g) be a smooth compact Kähler manifold of complex dimension 2 and with $c_1(M) > 0$. Then as a smooth manifold M must be diffeomorphic to $\mathbb{CP}^2$, $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$, or $\mathcal{Z}_l \simeq \mathbb{CP}^2 \# l\overline{\mathbb{CP}}^2$, where $1 \leq l \leq 8$.*

One can describe moduli space of complex structures on the del Pezzo surfaces using the fact that, with the exception of the quadric Q, they can obtained by blowing up $\mathbb{CP}^2$ at l distinct points for $0 \leq l \leq 8$. The complex structures on The complex structures on $\mathcal{Z}_l = \mathbb{CP}^2 \# l\overline{\mathbb{CP}}^2$ are then determined by the complex coordinates of the l points modulo the action of the complex automorphism group $\mathfrak{G}(\mathcal{Z}_l)$. Since we can fix precisely four points in general position with the action of $GL(3, \mathbb{C})$, the moduli space of complex structures $\mathcal{M}_l^{\mathbb{C}}$ on $\mathcal{Z}_l$ for $l = 0, \ldots, 4$ is a single point, whereas for $l = 5, 6, 7, 8$ it is an open connected manifold of complex dimension $2(l-4)$.

It is known that when $l > 2$, there exists [**192, ?, ?**] a unique [**14**], up to complex automorphism, Kähler-Einstein structure associated to each complex structure. It follows that in this case ($l > 2$) there is a 1-1 correspondence between the complex structures on $\mathcal{Z}_l$ and the homothety classes of positive Kähler-Einstein metrics modulo complex automorphisms. This gives the identification of moduli spaces

$$\mathcal{M}_l^{\mathbb{C}} \simeq \mathcal{M}_l^{KE}, \quad \text{for } l > 2 \tag{4.9.1}$$

where $\mathcal{M}_l^{KE}$ denotes the moduli space of homothety classes of Kähler-Einstein metrics on $\mathcal{Z}_l$. Moreover, when $l = 1, 2$ the space $\mathcal{M}_l^{KE}$ is well-known to be empty, whereas, $\mathcal{M}_l^{\mathbb{C}}$ is a single point space. Furthermore, for $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ both $\mathcal{M}_l^{\mathbb{C}}$ and $\mathcal{M}_l^{KE}$ are single point spaces. When $4 < l < 9$, it is possible that for two inequivalent complex structures $J_1, J_2 \in \mathcal{M}_l^{\mathbb{C}}$ the Einstein metrics g_1, g_2 solving the corresponding Monge-Ampere equations coincide. However, it follows from a theorem of Pontecorvo [**?**] that up to complex conjugation this can happen only when the metric is anti-self-dual, that is, the anti-self-dual part W_+ of the Weyl conformal tensor must vanish. But it is well-known (cf. [**33**]) that a compact anti-self-dual Kähler surface has zero scalar curvature. So this cannot happen for del Pezzo surfaces. The precise result is given below in Proposition 4.9.3, and the argument works equally well for compact Kähler orbifolds.

In the more general case of log del Pezzo surfaces $\mathcal{Z}$ that is complex compact surfaces with positive first Chern class and at most quotient singularities much less is known. Here we are dealing with Kähler orbifolds and as discussed earlier the existence problem of finding Kähler-Einstein orbifold metrics is still open. However, the Bando-Mabuchi uniqueness theorem carries over to the orbifold case, so when a positive Kähler-Einstein orbifold metric exists it is unique up to homothety and complex automorphisms. Thus, generally, if $\mathcal{M}^{KE}$ denotes the moduli space of homothety classes of Kähler-Einstein metrics on $\mathcal{Z}$, and $\mathcal{M}^{\mathbb{C}}$ denotes the moduli space of complex structures on $\mathcal{Z}$, then there is an injective map

$$\mathcal{M}^{KE} \longrightarrow \mathcal{M}^{\mathbb{C}}. \tag{4.9.2}$$

Next we turn to the equivalence problem for the Einstein metrics. As mentioned previously for the log del Pezzo surfaces with a Y in the last column of Table 1 and the tables of Theorem **??**, there is a unique homothety class of Kähler-Einstein metrics corresponding to each point of $\mathcal{M}^d_{\mathbf{w}}$. But the question remains whether two inequivalent Kähler-Einstein structures can share the same Riemannian metric. This is answered by a Theorem of Pontecorvo [**?**] which can be restated for our purposes as:

Proposition 4.9.3: *Let (J, J') be two distinct log del Pezzo structures on the same underlying orbifold $\mathcal{Z}$ that are both compatible with the same Einstein metric g. Then J and J' are complex conjugates, i.e. $J' = -J$.*

PROOF. By Proposition 3.1 of [**?**] if $J' \neq -J$ then the metric g is anti-self-dual, i.e. $W_+ = 0$. From [**33**] anti-self-dual Kähler metrics must have vanishing scalar curvature, and this cannot happen for log del Pezzo surfaces which are positive. □ □

Conjugate Kähler-Einstein structures correspond to the same point of the moduli space $\mathcal{M}^d_{\mathbf{w}}$, so each point of $\mathcal{M}^d_{\mathbf{w}}$ corresponds to a distinct diffeomorphism class of Kähler-Einstein metrics with scalar curvature $4n(n+1)$.

4.10. Kähler-Einstein Metrics on Fano 3-folds

CHAPTER 5

Sasakian Structures

In this chapter we begin a more detailed study of Sasakian geometry. Since there are many properties that also hold for K-contact structures, we begin with these, that is, the underlying almost CR-structure is not necessarily integrable. The reason for imposing the K-contact condition and not working with the more general contact metric structures comes from Proposition 3.4.8 which says, among other things, that a contact metric structure is K-contact if and only if the characteristic foliation $\mathcal{F}_\xi$ is Riemannian. The later condition is necessary for much of the geometry that we develop. Indeed, early (circa 1960) in the development of global contact geometry the 'nice' situation was that described by the well known Boothby-Wang fibration which describes a 'regular' compact contact manifold as a circle bundle over a compact symplectic manifold. In 1963 Hatakeyama, Ogawa, and Tanno [**?**] introduced the notion of a K-contact manifold, and proved the basic Theorem 5.1.23 found below. There was a systematic development of the curvature properties of both K-contact and Sasakian geometries throughout the sixties and seventies, see the books [**27, 28, 224**].

5.1. K-Contact Structures

We begin with some basic regularity notions of strict contact structures (M, η), or rather of the characteristic foliation $\mathcal{F}_\xi$ of a strict contact structure.

Definition 5.1.1: *The characteristic foliation $\mathcal{F}_\xi$ of a strict contact structure is said to be* **quasi-regular** *if there is a positive integer k such that each point has a foliated coordinate chart (U, x) such that each leaf of $\mathcal{F}_\xi$ passes through U at most k times. If $k = 1$ then the foliation is called* **regular**. *If $\mathcal{F}_\xi$ is not quasi-regular, it is said to be* **irregular**.

This definition is due to Thomas [**207**] who used the terminology *almost regular* instead of quasi-regular. He then proved that the flow of the Reeb vector field for any almost regular contact structure is equivalent to a locally free circle action. However, this follows from the earlier theorem of Wadsley [**?**]. Much later Rukimbira [**?**], based on Carrière's work [**60**] on Riemannian flows, proved that any irregular K-contact structure can be approximated by quasi-regular ones. Moreover, he [**?**] observed that any quasi-regular contact metric structure is necessarily K-contact. Combining these results we have

Theorem 5.1.2 (Structure Theorem)**:** *Let M be compact. Any K-contact structure (ξ, η, Φ, g) on M is either quasi-regular or there is a sequence of quasi-regular K-contact structures $\{(\xi_j, \eta_j, \Phi_j, g_j)\}_j$ converging in the compact-open C^∞-topology to*

(ξ,η,Φ,g). In particular, if M admits a K-contact structure, it admits a locally free circle action.

Conversely, any quasi-regular contact metric structure (ξ,η,Φ,g) on M is K-contact.

Topological Obstructions

We had seen previously in section 3.2 that a necessary condition for a manifold to admit a strict almost contact structure is the vanishing of certain characteristic classes, namely the odd Stiefel-Whitney classes. Here we discuss some additional topological obstructions for a manifold to admit a K-contact structure. The first such obstruction is a recent observation of Itoh [**118**].

Theorem 5.1.3 (Itoh)**:** *Let M be a compact manifold of dimension $2n+1$, and suppose the real cohomology ring $H^*(M,\mathbb{R})$ satisfies $H^{2n+1}(M,\mathbb{R}) = \Lambda^{2n+1}H^1(M,\mathbb{R})$. Then M does not admit a K-contact structure.*

The proof of this theorem is based on the following well known result of Tachabana [**200**].

Lemma 5.1.4 (Tachabana)**:** *Let (M,g) be a compact Riemannian manifold with a Killing vector field X. Then every harmonic 1-form α satisfies $\alpha(X)=0$.*

Proof. Any closed 1-form α can be decomposed into $\alpha = \beta + f\eta$ where η is the 1-form dual to X and β satisfies $\beta(\xi)=0$. By Hodge theory harmonic forms are invariant under isometries, and X is an infinitesimal isometry. Thus,

$$0 = \pounds_X\alpha = d(\alpha(X)) = df,$$

so f is constant. This implies that $d\beta + fd\eta = 0$. So by Stokes theorem

$$0 = \int_M d(\beta\wedge\eta\wedge(d\eta)^{n-1} = -\int_M f\eta\wedge(d\eta)^n,$$

which, since $\eta\wedge(d\eta)^n$ is a volume form, implies that $f=0$. So $\alpha=\beta$ proving the result. □

Proof. (of Itoh's Theorem). By Hodge theory every cohomology class can be represented uniquely by a harmonic form. So by the hypothesis of the theorem, the Riemannian volume $d$vol form can be written uniquely up to order as

$$d\mathrm{vol} = \alpha_1\wedge\cdots\wedge\alpha_{2n+1}$$

where α_i are harmonic 1-forms. If M admits a K-contact structure then the harmonic forms α_i can be given with respect to the K-contact metric g. Moreover, the characteristic vector field ξ is a Killing vector field, and we can extend ξ to a local orhonormal basis $\{\xi,e_1,\cdots,e_{2n}\}$, giving $d\mathrm{vol}(\xi,e_1,\cdots,e_{2n})=0$, which implies the vanishing of the cohomology class $[d\mathrm{vol}]$ giving a contradiction. □

We have an immediate corollary of Theorem 5.1.3:

Corollary 5.1.5: *The manifolds $\Sigma_{g_1}\times\cdots\times\Sigma_{g_l}\times S^1$ do not admit a K-contact structure, where Σ_g denotes a Riemann surface of genus $g\geq 1$. In particular, a torus cannot admit a K-contact structure.*

On the other hand tori do admit contact structures, as evidenced by Example 3.1.20 and Exercise 3.5.

Our next example of an obstruction to K-contact structures involves Gromov's invariants MinVol and $||M||$ discussed at the end of Chapter 1. Since a K-contact structure is in particular a Riemannian flow, we have immediately from Theorem 1.5.41

Corollary 5.1.6: *Let M admit a K-contact structure. Then both* $\mathrm{MinVol}(M)$ *and* $||M||$ *vanish.*

From Theorem 1.5.42, the estimate 1.5.43, and the properties of the simplicial volume discussed in Chapter 1, we have

Corollary 5.1.7: *If M decomposes as $M = M_1\# \cdots \# M_l$ where some M_i is compact and admits a metric whose sectional curvature is bounded from above by $-\delta$ for some $\delta > 0$, then M does not admit a K-contact structure.*

Toral Rank of K-Contact Manifolds

In Proposition 5.1.8 we saw that on a compact manifold the closure of the leaves of a Riemannian foliation has the structure of a singular Riemannian foliation, and from Carrière's Theorem 1.5.32 that the closure of any leaf is a torus. Moreover, the flow is conjugate by the diffeomorphism to a linear flow on the torus. This gave rise to an invariant of the flow called the toral rank.

Here we adapt this to our situation, that is, (ξ, η, Φ, g) is a K-contact structure on a smooth manifold M. We denote the isometry group of (M, g) by $\mathfrak{I}(M, g)$, and the group of automorphisms of the K-contact structure (M, η, g), by $\mathfrak{A}(M, \xi, \eta, \Phi, g)$. We let $\mathfrak{i}(M, g)$ and $\mathfrak{a}(M, \xi, \eta, \Phi, g)$ denote the corresponding Lie algebras. When M is compact the well known theorem of Myers and Steenrod says that $\mathfrak{I}(M, g)$ is a compact Lie group. Moreover, $\mathfrak{A}(M, \xi, \eta, \Phi, g)$ is a closed Lie subgroup of $\mathfrak{I}(M, g)$ [?]. In our case the Reeb flow belongs to the automorphism group $\mathfrak{A}(M, \xi, \eta, \Phi, g)$ which is a compact Lie group. Thus, the closure $\mathfrak{T}$ of the Reeb flow is a compact commutative Lie group which lies in $\mathfrak{A}(M, \xi, \eta, \Phi, g)$. Now the Reeb flow is a strict contact transformation lying in the center of the group of strict contact transformations [LM]; hence, it lies in the center of $\mathfrak{A}(M, \xi, \eta, \Phi, g)$. It follows that $\mathfrak{T}$ also lies in the center of $\mathfrak{A}(M, \xi, \eta, \Phi, g)$. Summarizing we have

Proposition 5.1.8: *Let (ξ, η, Φ, g) be a compact K-contact metric manifold. Then the leaf closures of the Reeb flow are the orbits of a torus $\mathfrak{T}$ lying in the center of the Lie group $\mathfrak{A}(M, \xi, \eta, \Phi, g)$ of automorphisms of (ξ, η, Φ, g) and the Reeb flow is conjugate to a linear flow on $\mathfrak{T}$.*

The dimension of the maximal torus in Proposition 5.1.8 is the toral rank $\mathrm{rk}(M, \mathcal{F}_\xi)$ of (ξ, η, Φ, g), and the contact geometry places a smaller upper bound on the toral rank. We have (see also [Ruk2])

Lemma 5.1.9: *Let (M, ξ, η, Φ, g) be a compact K-contact metric manifold of dimension $2n+1$. Then the rank $rk(M, \mathcal{F}_\xi)$ is independent of the metric g and satisfies $1 \leq rk(M, \mathcal{F}_\xi) \leq n + 1$. Moreover,* $\dim\, \mathfrak{A}(M, \xi, \eta, \Phi, g) \geq rk(M, \mathcal{F}_\xi)$.

Proof. Consider the Lie algebra $\mathfrak{t}$ of $\mathfrak{T}$. It consists of the Reeb vector field ξ together with the infinitesimal generators of the leaf closures. The projections of these generators onto $\mathcal{D}$ are global sections of Molino's commuting sheaf $C(M, \mathcal{F}_\xi)$. Thus, they give integral submanifolds of the subbundle $\mathcal{D}$. But from Proposition 3.4.9 the integral submanifolds of $\mathcal{D}$ of maximal dimension, that is the Legendrian submanifolds of the contact structure, have dimension n. Hence, these together with the Reeb vector field generate a torus of dimension at most $n + 1$. Furthermore, Molino [**160**] shows that the commuting sheaf is independent of the transverse metric, so $\mathrm{rk}(M,\mathcal{F}_\xi)$ is independent of g. □

Rukimbira [**?**] noticed that one can do better than the upper bound given by this lemma. We have

Theorem 5.1.10: *(Rukimbira) Let (M,ξ,η,Φ,g) be a compact K-contact metric manifold of dimension $2n+1$ with first Betti number $b_1=b_1(M)$. Then*

$$1\leq rk(M,\mathcal{F}_\xi)\leq \min(n+1,2n+1-b_1).$$

Proof. By Hodge theory there are precisely b_1 independent harmonic 1-forms $\alpha_1,\cdots,\alpha_{b_1}$. Let $Z_1,\cdots,Z_{b_1}$ denote their dual harmonic vector fields, and let $k=\mathrm{rk}(M,\mathcal{F}_\xi)$. Then by Proposition 5.1.8 the leaf closures of the characteristic foliation $\mathcal{F}_\xi$ are tori $\mathfrak{T}$ whose maximal dimension is k. So on a dense open set the leaf closures are tori $\mathfrak{T}^k$ of dimension precisely k. Furthermore, $\mathfrak{T}^k$ lies in the center of the automorphism group $\mathfrak{A}(\xi,\eta,\Phi,g)$. So if $V\in\mathfrak{t}^k$, the Lie algebra of $\mathfrak{T}^k$, it is a Killing vector field. Thus, by Tachibana's Lemma 5.1.4 we have

$$0=\alpha_i(V)=g(Z_i,V)$$

for all $i=1,\cdots,b_1$ and any $V\in\mathfrak{t}^k$. It follows that $k+b_1\leq 2n+1$ which together with Lemma 5.1.9 proves the result. □

Rukimbira's Theorem has an interesting corollary.

Corollary 5.1.11: *Let (M,ξ,η,Φ,g) be a compact K-contact metric manifold of dimension $2n+1$ whose first Betti number $b_1(M)=2n$. Then M is the total space of a principal S^1 V-bundle over a compact almost Kähler orbifold $\mathcal{Z}$ whose first Betti number $b_1(\mathcal{Z})=2n$.*

We now discuss several other interesting results of Rukimbira. The first is:

Theorem 5.1.12: *(Rukimbira) Let (M,ξ,η,Φ,g) be a compact K-contact manifold, and suppose that ξ has only a finite number of closed integral curves. Then $b_1(M)=0$.*

Add proof later. A corollary of this result is that any toric K-contact manifold M must have $b_1(M)=0$. There is a stronger result for toric K-contact manifolds due to Lerman which says that the fundamental group of any K-contact toric manifold is finite and determined combinatorically by the image of the contact moment map. We will treat toric contact manifolds in detail later.

The case $\mathrm{rk}(M,\mathcal{F}_\xi)=1$ is the quasi-regular case, while the other extreme $\mathrm{rk}(M,\mathcal{F}_\xi)=n+1$ is the toric case studied in [**?, ?, 36**]. Furthermore, Rukimbira [**?**] showed that one can approximate any K-contact structure by a sequence of quasi-regular K-contact structures in the same contact structure. Thus, every K-contact manifold has a structure (ξ,η,Φ,g) of rank 1. Since we shall discuss this approximation in detail in the next section, we only mention here that one chooses a sequence of vector fields ξ_j in $\mathfrak{t}$ with periodic orbits that converges to the Reeb vector field ξ. Then the dual 1-forms η_j are quasi-regular contact forms in the same contact structure.

Here is another theorem of Rukimbira [**?**].

Theorem 5.1.13 (Rukimbira)**:** *Let M be a simply connected manifold of dimension $2n+1$. Suppose M admits a K-contact structure (ξ,η,Φ,g) which has only $n+1$ closed characteristic curves. Then M is homeomorphic to the sphere S^{2n+1}.*

Basic Cohomology of K-Contact Structures

Recall the basic de Rham complex $\Omega_B(\mathcal{F})$ of a foliation and its basic cohomology ring $H^*_B(\mathcal{F})$ from section 1.5. As seen previously a K-contact structure has a taut Riemannian flow, and in this case the spectral sequence of the complex simplifies. Our treatment follows [**210**]. As seen in Proposition 5.1.8 the closure of the leaves of the characteristic foliation $\mathcal{F}_\xi$ is a torus $\mathfrak{T} \subset \mathfrak{A}(\xi,\eta,\Phi,g) \subset \mathfrak{I}(M,g)$. Let $\Omega(M)^{\mathfrak{T}}$ denote the subalgebra of $\Omega(M)$ consisting of $\mathfrak{T}$-invariant forms. Since the orbit of ξ is dense in $\mathfrak{T}$ any basic form is $\mathfrak{T}$-invariant, so we have $\Omega_B(\mathcal{F}) \subset \Omega(M)^{\mathfrak{T}}$. We have

Proposition 5.1.14: *There is an exact sequence of complexes*

$$0 \longrightarrow \Omega^*_B(\mathcal{F}_\xi) \longrightarrow \Omega^*(M)^{\mathfrak{T}} \xrightarrow{\xi\rfloor} \Omega^{*-1}_B(\mathcal{F}_\xi) \longrightarrow 0.$$

Proof. First we show that $\xi\rfloor\alpha$ is basic for a $\mathfrak{T}$-invariant form α. Clearly, $\pounds_\xi(\xi\rfloor\alpha) = 0$, since $\xi \in \mathfrak{T}$. But also, $\xi\rfloor\xi\rfloor\alpha = 0$. Furthermore, $\xi\rfloor$ is a map of complexes since it commutes with $\pounds_\xi$. Next we show that $\xi\rfloor$ is surjective. Let $\alpha \in \Omega^{r-1}_B(\mathcal{F}_\xi)$, then $\xi\rfloor(\eta\wedge\alpha) = \alpha$, and if T is an element of the Lie algebra $\mathfrak{t}$ of $\mathfrak{T}$, we have

$$\pounds_T(\eta\wedge\alpha) = (\pounds_T\eta)\wedge\alpha + \eta\wedge\pounds_T\alpha = 0.$$

The first term vanishes since $T \in \mathfrak{t} \subset \mathfrak{a}(M,\xi,\eta,\Phi,g)$, and the second term vanishes since basic forms are $\mathfrak{T}$-invariant. This proves the surjectivity. Finally, exactness of the middle term follows from the definitions of $\Omega_B(\mathcal{F}_\xi)$ and $\Omega(M)^{\mathfrak{T}}$ and the fact that $\xi \in \mathfrak{t}$. □

Now the cohomology of the complex $\Omega(M)^G$ for any compact connected Lie group is isomorphic to the usual de Rham cohomology [**?**], so we get a long exact cohomology sequence

(5.1.1)
$$\cdots \longrightarrow H^p_B(\mathcal{F}_\xi) \xrightarrow{\iota_*} H^p(M,\mathbb{R}) \xrightarrow{j_p} H^{p-1}_B(\mathcal{F}_\xi) \xrightarrow{\delta} H^{p+1}_B(\mathcal{F}_\xi) \longrightarrow \cdots$$

where δ is the connecting homomorphism given by $\delta[\alpha]_B = [d\eta\wedge\alpha]_B = [d\eta]_B \cup [\alpha]_B$, and j_p is the composition of the map induced by $\xi\rfloor$ with the isomorphism $H^r(\Omega(M)^{\mathfrak{T}}) \approx H^r(M,\mathbb{R})$ mentioned above. On compact K-contact manifolds the basic cohomology groups enjoy some special properties. In particular, there is a transverse Hodge theory [**?, ?, 210**] which we shall briefly review. We begin by defining the transverse Hodge star operator $\bar\star$. It is defined in terms of the usual Hodge star by

$$\bar\star\alpha = \star(\eta\wedge\alpha) = (-1)^r\xi\rfloor\star\alpha. \tag{5.1.2}$$

The adjoint $\delta_B : \Omega^r_B(\mathcal{F}_\xi) \longrightarrow \Omega^{r-1}_B(\mathcal{F}_\xi)$ of d_B is defined by

$$\delta_B = -\bar\star d_B \bar\star. \tag{5.1.3}$$

The *basic Laplacian* Δ_B is defined in terms of d_B and its adjoint δ_B by

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{5.1.4}$$

The space $\mathcal{H}^r_B(\mathcal{F}_\xi)$ of *basic harmonic r-forms* is then defined to be the kernel of $\Delta_B : \Omega^r_B(\mathcal{F}_\xi) \longrightarrow \Omega^r_B(\mathcal{F}_\xi)$, or alternatively $\mathcal{H}^r_B(\mathcal{F}_\xi) = \ker\, d_B \cap \ker\delta_B$. The transverse Hodge theorem [**?**] then says that each basic cohomology class has a unique harmonic representative, i.e. $H^r_B(\mathcal{F}_\xi) = \mathcal{H}^r_B(\mathcal{F}_\xi)$. From transverse Hodge theory we get

Proposition 5.1.15: *Let (M,ξ,η,Φ,g) be a compact K-contact manifold of dimension $2n+1$. Then we have*

(i) *The groups $H^r_B(\mathcal{F}_\xi)$ are finite dimensional and $H^r_B(\mathcal{F}_\xi)=0$ for $r>n$.*
(ii) *$H^{2n}_B(\mathcal{F}_\xi)\approx\mathbb{R}$.*
(iii) *The class $[d\eta]_B\in H^2_B(\mathcal{F}_\xi)$ is nontrivial.*
(iv) *$H^{2r}_B(\mathcal{F}_\xi)>0$ for $0\leq r\leq n$.*
(v) *$H^1(M,\mathbb{R})\approx H^1_B(\mathcal{F}_\xi)$.*
(vi) *(transverse Poincaré duality) There is a nondegenerate pairing*

$$\Psi: H^r(\mathcal{F}_\xi)\otimes H^{2n-r}(\mathcal{F}_\xi)\longrightarrow\mathbb{R}.$$

PROOF. (i) follows from transverse Hodge theory in the usual way. (ii): With $p=2n+1$ the exact sequence 5.1.1 gives an isomorphism

$$H^{2n+1}(M)\overset{j_{2n+1}}{\approx}H^{2n}_B(\mathcal{F}_\xi), \tag{5.1.5}$$

proving (ii). To prove (iii) we notice that the 2-form $d\eta$ satisfies $\eta\wedge(d\eta)^n\neq 0$, and that the isomorphism j_{2n+1} in 5.1.5 which is essentially $\xi\rfloor$ implies that $d\eta$ defines a nontrivial class in $H^2_B(\mathcal{F}_\xi)$. Clearly, (iv) holds since $(d\eta)^n\in H^{2n}_B(\mathcal{F}_\xi)$ is nontrivial. To prove (v) we notice that the beginning of the exact sequence 5.1.1 is

$$0\longrightarrow H^1_B(\mathcal{F}_\xi)\longrightarrow H^1(M,\mathbb{R})\xrightarrow{j_1}\mathbb{R}\xrightarrow{\delta_0}H^2_B(\mathcal{F}_\xi).$$

By (iii) δ_0 is injective, so $\{0\}=\ker\,\delta_0=\mathrm{im}\ j_1$; hence, j_1 is the zero map. (vi) follows by transverse Hodge theory [**?, 210**]. The pairing Ψ is that induced from the pairing

$$\tilde\Psi:\Omega^r_B(\mathcal{F}_\xi)\otimes\Omega^{2n-r}_B(\mathcal{F}_\xi)\longrightarrow\mathbb{R}$$

defined by

$$\tilde\Psi(\alpha,\beta)=\int_M\eta\wedge(\alpha\wedge\beta). \tag{5.1.6}$$

This induces an isomorphism on the transverse harmonic forms, and defines a nondegenerate inner product $<\alpha,\beta>_B=\tilde\Psi(\alpha,\bar\star\beta)$. See [**?, 210**] for details. □

REMARK 5.1.1: The nontrivial class $[d\eta]_B\in H^2_B(\mathcal{F}_\xi)$ is called the *fundamental basic* class of the K-contact structure (ξ,η,Φ,g).

Next we consider some fundamental invariants of K-contact structures [**?**]. We define the *transverse Betti numbers*

$$b^B_r(\mathcal{F}_\xi)=\dim\ H^r_B(\mathcal{F}_\xi), \tag{5.1.7}$$

the *transverse Euler characteristic* of $\mathcal{F}_\xi$ given by

$$\chi(\mathcal{F}_\xi)=\sum_{r=0}^{2n}(-1)^r b^B_r(\mathcal{F}_\xi). \tag{5.1.8}$$

Notice that by (4) of Proposition 5.1.15 $b_1(\mathcal{F}_\xi)=b_1(M)$ a topological invariant of M. Using transverse Poincaré duality we have

Proposition 5.1.16: *On a compact K-contact manifold M^{2n+1} we have*

$$b^B_{2n-r}(\mathcal{F}_\xi)=b^B_r(\mathcal{F}_\xi),\quad \chi(\mathcal{F}_\xi)=2-2b^B_1(\mathcal{F}_\xi)+2b^B_2(\mathcal{F}_\xi)-\cdots+(-1)^n b^B_n.$$

Some Elementary Properties of Contact Metric Structures

Here we develop some useful formulas for K-contact structures. We begin by proving a general expression for the covariant derivative of Φ for an almost contact metric structure.

Lemma 5.1.17: *Let (ξ, η, Φ, g) be an almost contact metric structure on M and define the 2-form Ω by $\Omega(X, Y) = g(X, \Phi Y)$. Then the following formula holds:*

$$\begin{aligned} 2g((\nabla_X \Phi)Y, Z) &= 3d\Omega(X, \Phi Y, \Phi Z) - 3d\Omega(X, Y, Z) + g(N^{(1)}(Y, Z), \Phi X) \\ &\quad + N^{(2)}(Y, Z)\eta(X) + 2d\eta(\Phi Y, X)\eta(Z) - 2d\eta(\Phi Z, X)\eta(Y) \end{aligned}$$

for any vector fields X, Y, Z on M.

Proof. The formula follows by a straightforward albeit tedious computation from the well known formulas for the covariant and exterior derivative, namely

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X), \end{aligned}$$

and

$$\begin{aligned} 3d\Omega(X, Y, Z) &= X\Omega(Y, Z) + Y\Omega(Z, X) + Z\Omega(X, Y) \\ &\quad - \Omega([X, Y], Z) + \Omega([Z, X], Y) - \Omega([Y, Z], X). \end{aligned}$$

The details are left as an exercise. $\square$

Lemma 5.1.17 easily simplifies in the case of a contact metric structure with its associated metric.

Lemma 5.1.18: *Let (ξ, η, Φ, g) be a contact metric structure on M. Then we have*

$$2g((\nabla_X \Phi)Y, Z) = g(N^{(1)}(Y, Z), \Phi X) + 2d\eta(\Phi Y, X)\eta(Z) - 2d\eta(\Phi Z, X)\eta(Y).$$

In particular, $\nabla_\xi \Phi = 0$.

Lemma 5.1.19: *For a contact metric structure we have*

(i) $N^{(3)}$ *is a symmetric endomorphism in the sense that*

$$g(N^{(3)}(X), Y) = g(N^{(3)}(Y), X).$$

(ii) $\nabla \xi = -\Phi - \frac{1}{2}\Phi N^{(3)}$.
(iii) $\Phi \circ N^{(3)} + N^{(3)} \circ \Phi = 0$.
(iv) $N^{(3)}(\xi) = 0, \qquad \eta \circ N^{(3)} = 0$.
(v) $\mathrm{tr}\ N^{(3)} = 0$.

Proof. To prove (i) we first compute

$$\begin{aligned} N^{(3)}(X) &= (\pounds_\xi \Phi)(X) = [\xi, \Phi X] - \Phi[\xi, X] = \nabla_\xi(\Phi X) - \nabla_{\Phi X}\xi - \Phi(\nabla_\xi X - \nabla_X \xi) \\ &= (\nabla_\xi \Phi)(X) + \Phi\nabla_\xi X - \nabla_{\Phi X}\xi - \Phi\nabla_\xi X + \Phi\nabla_X \xi = \Phi\nabla_X \xi - \nabla_{\Phi X}\xi, \end{aligned}$$

where we have used the last statement of Lemma 5.1.18. Thus, $N^{(3)}(\xi) = \Phi\nabla_\xi \xi$ which vanishes by Lemma **??**. Next we claim that $g(N^{(3)}(X), \xi) = \eta(N^{(3)}(X)) = 0$. From above we have $\eta(N^{(3)}(X)) = -\eta(\nabla_{\Phi X}\xi)$ which vanishes since ∇ is a metric connection. This proves (ii) when X or Y is ξ. Now assume that X and Y are orthogonal to ξ. Continuing our computation above

$$\begin{aligned} g(N^{(3)}(X),Y) &= g(\Phi\nabla_X\xi,Y) - g(\nabla_{\Phi X}\xi,Y) = -g(\nabla_X\xi,\Phi Y) + g(\xi,\nabla_{\Phi X}Y) \\ &= \eta(\nabla_X\Phi Y) + \eta(\nabla_{\Phi X}Y), \end{aligned}$$

and hence,

$$g(N^{(3)}(Y),X) = \eta(\nabla_Y\Phi X) + \eta(\nabla_{\Phi Y}X).$$

But it is not difficult to see that the equation $N^{(2)}(X,Y) = 0$ implies that these two expressions are equal. This proves (i).

To prove (ii) we consider the equation of Lemma 5.1.18 with $Y = \xi$, and X and Z orthogonal to ξ. We have

$$\begin{aligned} 2g((\nabla_X\Phi)\xi,Z) &= g(N^{(1)}(\xi,Z),\Phi X) - 2d\eta(\Phi Z,X) \\ &= g(\Phi^2[\xi,Z] - \Phi[\xi,\Phi Z],\Phi X) - 2g(\Phi Z,\Phi X) \\ &= -g(\Phi(\pounds_\xi\Phi)Z,\Phi X) - 2g(Z,X) \\ &= -g((\pounds_\xi\Phi)Z,X) + \eta((\pounds_\xi\Phi)Z)\eta(X) - 2g(Z,X) \\ -2g(\Phi\nabla_X\xi,Z) &= -g(N^{(3)}(Z),X) - 2g(X,Z) = -g(N^{(3)}(X),Z) - 2g(X,Z). \end{aligned}$$

Since this holds for arbitrary Z we have $2\Phi\nabla_X\xi = N^{(3)}(X) + 2X$. Applying Φ to this equation proves (ii).

Both (iii) and (iv) are a result of Lemma 3.3.7. Finally, (v) holds since by (iii) if λ is an eigenvalue of the symmetric operator $N^{(3)}$ so is $-\lambda$. □

Combining Lemma 5.1.19 and Proposition 3.5.12 we have

Proposition 5.1.20: *A contact metric structure (ξ,η,Φ,g) is K-contact if and only if $\nabla\xi = -\Phi$.*

Next we investigate the O'Neill tensors T, A in the case of a K-contact structure (ξ,η,Φ,g). First, one easily sees that $T = 0$, and

$$A_XY = \frac{1}{2}\pi([X,Y]) = \frac{1}{2}\eta([X,Y])\xi = -d\eta(X,Y)\xi. \tag{5.1.9}$$

We now have

Lemma 5.1.21: *For a K-contact manifold the following identities hold:*

(i) $T = 0$,
(ii) $A_XY = -d\eta(X,Y)\xi$,
(iii) $A_X\xi = -\Phi X$.

Proof. It only remains to prove (iii). We have that

$$g(A_X\xi,Y) = -g(\xi,A_XY) = -\eta(A_XY) = d\eta(X,Y) = -g(\Phi X,Y)$$

holds for all horizontal Y. This proves the result since $A_X\xi$ is horizontal. □

We can write A in the succinct form

$$A_X = -\Phi X\otimes\eta - \xi\otimes X\rfloor d\eta. \tag{5.1.10}$$

Combining Lemmas 1.5.24 and 5.1.21 we have

Lemma 5.1.22: *For a K-contact manifold the following identities hold:*

(i) $(\nabla_\xi A)_\xi = 0$,
(ii) $(\nabla_X A)_\xi = A_{\Phi X}$,

(iii) $\eta((\nabla_\xi A)_X\xi) = 0.$

We can obtain more information about the behavior of the covariant derivatives of A from Theorem 1.5.25, but first we need to investigate curvature properties of K-contact manifolds.

Curvature and K-Contact Structures

Here we collect some useful results concerning the Riemannian curvature $R(X,Y)$ of K-contact manifolds. We begin with the fundamental result of [**?**]:

Theorem 5.1.23: *A Riemannian manifold (M,g) is K-contact if and only if (M,g) admits a Killing vector field ξ of unit length such that the sectional curvature of all plane sections containing ξ equals one.*

First we obtain a general expression for $R(X,\xi)Y$ for a metric contact structure.

Lemma 5.1.24: *For a contact metric structure (ξ,η,Φ,g) we have for any X*

$$R(X,\xi)\xi = X - \eta(X)\xi - \frac{1}{4}(N^{(3)})^2(X) + \frac{1}{2}\Phi(\nabla_\xi N^{(3)})(X)).$$

Moreover, if (ξ,η,Φ,g) is K-contact manifold then for any vector field X and for Y is a section of $\mathcal{D}$ the following identities hold:

(i) $R(X,\xi)\xi = X - \eta(X)\xi,$

(ii) $R(X,\xi)Y = -(\nabla_X\Phi)(Y).$

Proof. We compute

$$\begin{aligned} R(X,\xi)Y &= \nabla_X\nabla_\xi Y - \nabla_\xi\nabla_X Y - \nabla_{[X,\xi]}Y \\ &= \nabla_X(\nabla_Y\xi + [\xi,Y]) - \nabla_{\nabla_X Y}\xi - [\xi,\nabla_X Y] - \nabla_{[X,\xi]}Y \\ &= \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi - (\pounds_\xi\nabla)_X Y. \end{aligned}$$

Putting $Y=\xi$ in this equation and using Lemmas 5.1.18 and 5.1.19 we have

$$\begin{aligned} R(X,\xi)\xi &= -\nabla_{\nabla_X\xi}\xi - \big([\xi,\nabla_X\xi] - \nabla_{[\xi,X]}\xi\big) \\ &= -\nabla_{-\Phi X - \frac{1}{2}\Phi N^{(3)}(X)}\xi - [\xi,(-\Phi X - \frac{1}{2}\Phi N^{(3)}(X))] - \Phi[\xi,X] - \frac{1}{2}\Phi N^{(3)}([\xi,X])) \\ &= -\Phi(\Phi X + \frac{1}{2}\Phi N^{(3)}(X)) - \frac{1}{2}\Phi N^{(3)}(\Phi X + \frac{1}{2}\Phi N^{(3)}(X)) + N^{(3)}(X) \\ &+ \frac{1}{2}(N^{(3)})^2(X) + \frac{1}{2}\Phi(\pounds_\xi N^{(3)})(X) \\ &= X - \eta(X)\xi + \frac{1}{4}(N^{(3)})^2(X) + N^{(3)}(X) + \frac{1}{2}\Phi(\pounds_\xi N^{(3)})(X) \end{aligned}$$

Now it is easy to verify the formula

$$\nabla_\xi N^{(3)} = \pounds_\xi N^{(3)} - 2\Phi N^{(3)} - \Phi(N^{(3)})^2 \tag{5.1.11}$$

By applying Φ to this formula and substituting it into the expression for $R(X,\xi)\xi$ above proves the first statement. (i) follows immediately from the first statement by Proposition 3.5.12. To prove (ii) we have from above

$$R(X,\xi)Y = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi - (\pounds_\xi\nabla)_X Y$$

which for a K-contact structure becomes

$$R(X,\xi)Y = -\nabla_X\Phi Y + \Phi\nabla_X Y = -(\nabla_X\Phi)(Y)$$

which proves (ii). □

EXERCISE 5.1: Prove formula 5.1.11 above.

PROOF. (of Theorem 5.1.23) Suppose first that (M,ξ,η,Φ,g) is K-contact. Then ξ is a unit length Killing vector field, and using Lemma 5.1.24 we have for X orthogonal to ξ,

$$K(X,\xi)=\frac{g(R(X,\xi)\xi,X)}{g(X,X)}=\frac{g(X,X)}{g(X,X)}=1.$$

Conversely, suppose that ξ is a unit length Killing vector field. We define η and Φ by

$$\eta(X)=g(\xi,X),\qquad \Phi X=-\nabla_X\xi.$$

Then as in the proof of Lemma **??** we have $\Phi\xi=-\nabla_\xi\xi=0$. Now since ξ is a Killing vector field it leaves ∇ invariant, so we have from above

$$R(X,\xi)\xi=-\nabla_{\nabla_X\xi}\xi=\nabla_{\Phi X}\xi=-\Phi^2X.$$

This gives for any vector field X orthogonal to ξ,

$$1=K(X,\xi)=\frac{g(R(X,\xi)\xi,X)}{g(X,X)}=-\frac{g(\Phi^2X,X)}{g(X,X)}.$$

Since this must hold for all vector fields X orthogonal to ξ, we must have $\Phi^2X=-X$ for all vector fields X orthogonal to ξ. Thus, $\Phi^2X=-X+\eta(X)\xi$ for arbitrary X. Moreover, since ξ is a Killing and ∇ is metric and torsionfree we have

$$\begin{aligned}2d\eta(X,Y)&=X\eta(Y)-Y\eta(X)-\eta([X,Y])=Xg(\xi,Y)-Yg(\xi,X)-g(\xi,[X,Y])\\&=g(\nabla_X\xi,Y)+g(\xi,\nabla_XY)-g(\nabla_Y\xi,X)-g(\xi,\nabla_YX)-g(\xi,[X,Y])\\&=g(\nabla_X\xi,Y)-g(\nabla_Y\xi,X)=2g(\nabla_X\xi,Y)=-2g(\Phi X,Y)=2g(X,\Phi Y).\end{aligned}$$

This proves that the structure (ξ,η,Φ,g) is K-contact. □

Next we consider the Ricci curvature

$$\mathrm{Ric}(X,Y)=\sum_i g(R(X,E_i)E_i,Y)=\sum_i g(R(E_i,X)Y,E_i)\tag{5.1.12}$$

where E_i is a local orthonormal frame. We have

Proposition 5.1.25: *A contact metric structure (ξ,η,Φ,g) on a manifold of dimension $2n+1$ is K-contact if and only if* $\mathrm{Ric}(\xi,\xi)=2n$.

PROOF. Let $(E_0=\xi,E_1,\cdots,E_{2n})$ be a local orthonormal frame, then we have from Lemma 5.1.24

$$\begin{aligned}\mathrm{Ric}(\xi,\xi)&=\sum_{i=1}^{2n}g(R(E_i,\xi)\xi,E_i)\\&=\sum_{i=1}^{2n}g(E_i,E_i)-\sum_{i=1}^{2n}\big(g(\frac14(N^{(3)})^2(E_i),E_i)-\frac12 g(\Phi(\nabla_\xi N^{(3)})(E_i)),E_i)\big)\\&=2n-\frac14\sum_{i=1}^{2n}\big(g(N^{(3)}(E_i),N^{(3)}(E_i))-\frac12 g(\Phi(\nabla_\xi N^{(3)})(E_i)),E_i)\big).\end{aligned}$$

Now since on a contact metric manifold X and ΦX are orthogonal, we can take the local orthonormal frame to be of the form $(\xi, E_1, \cdots, E_n, \Phi E_1, \cdots, \Phi E_n)$. Then we have

$$\begin{aligned}\mathrm{Ric}(\xi,\xi) &= 2n - \frac{1}{4}\sum_{i=1}^{2n}(g(N^{(3)}(E_i), N^{(3)}(E_i)) \\ &- \frac{1}{2}\sum_{i=1}^{n} g(\Phi(\nabla_\xi N^{(3)})(E_i)), E_i) - \frac{1}{2}\sum_{i=1}^{n} g(\Phi(\nabla_\xi N^{(3)})(\Phi E_i)), \Phi E_i)\end{aligned}$$

We claim that the last two terms cancel so that we have

$$\mathrm{Ric}(\xi,\xi) = 2n - \frac{1}{4}\sum_{i=1}^{2n}(g(N^{(3)}(E_i), N^{(3)}(E_i)) = 2n - \frac{1}{4}\mathrm{tr}\ (N^{(3)})^2 \tag{5.1.13}$$

Since each term under the sum is positive, this proves the result. □

Exercise 5.2: Prove the following formula for contact metric structures

$$\Phi R(\Phi X, \xi)\xi = R(X,\xi)\xi - 2X + \frac{1}{2}(N^{(3)})^2(X),$$

and use it to prove the claim above verifying equation 5.1.13.

An immediate corollary of Proposition 5.1.25 and the well-known Myers' Theorem is

Corollary 5.1.26: *If a K-contact structure (ξ,η,Φ,g) on M^{2n+1} is Einstein, then the Einstein constant is positive and equals $2n$. Hence, if M^{2n+1} is complete, then it is compact with finite fundamental group.*

In Section ?? we give a theorem of the authors that proves that if a K-contact structure is Einstein it is Sasakian, that is the underlying almost CR-structure is necessarily integrable. This can be thought of as an odd dimensional version of the well known Goldberg Conjecture. Next we relate the curvature for K-contact manifolds to the O'Neill tensor field A. First we notice that the first two equations and the fourth equation of Theorem 1.5.25 are vacuous in the K-contact case. In what follows X, Y, Z, Z' are always sections of $\mathcal{D}$.

Lemma 5.1.27: *On a K-contact manifold (M,ξ,η,Φ,g) the following identities hold:*

(i) $\eta((\nabla_\xi A)_X Y) = 0$,
(ii) $\eta((\nabla_Z A)_X Y) = -g((\nabla_Z \Phi)(X), Y)$,
(iii) $g(R(X,Y)Z,Z') = g(R^T(X,Y)Z,Z') + 2d\eta(X,Y)d\eta(Z,Z') - d\eta(X,Z)$ $d\eta(Y,Z') - d\eta(X,Z')d\eta(Y,Z)$.

Proof. (i): The third equation of Theorem 1.5.25 gives the identity

$$\begin{aligned}g(R(X,\xi)Y,\xi) &= -g(A_X\xi, A_Y\xi) - g((\nabla_\xi A)_X Y, \xi) \\ &= -g(\Phi X, \Phi X) - g((\nabla_\xi A)_X Y, \xi) = -g(X,Y) - g((\nabla_\xi A)_X Y, \xi).\end{aligned}$$

Using the well known Riemann curvature identities and Lemma 5.1.24 the left hand side of this equation reduces as

$$g(R(X,\xi)Y,\xi) = -g(R(X,\xi)\xi, Y) = -g(X,Y).$$

This proves (i). To prove (ii) we notice that the fifth equation of Theorem 1.5.25 becomes

$$g(R(X,Y)Z,\xi)=g((\nabla_Z A)_X Y,\xi)=\eta((\nabla_Z A)_X Y).$$

Again by the Riemann curvature identities and Lemma 5.1.24, the left hand side is

$$g(R(X,Y)Z,\xi)=g(R(Z,\xi)X,Y)=-g((\nabla_Z\Phi)(X),Y)$$

which proves (ii). Then (iii) follows immediately by (ii) of Lemma 5.1.21. □

From Theorem 1.5.27 one obtains relations between the Ricci curvature of g and the transverse Ricci curvature for K-contact manifolds.

Theorem 5.1.28: *Let (M,ξ,η,Φ,g) be a K-contact manifold and let Ric_T denote the Ricci curvature of the transverse metric g_T. Then the following identities hold:*

(i) $\mathrm{Ric}(X,\xi)=\sum_i g((\nabla_{X_i}\Phi)(X_i),X)$,
(ii) $\mathrm{Ric}(X,Y)=\mathrm{Ric}_T(X,Y)-2g(X,Y)$,

where X,Y are sections of $\mathcal{D}$, and $(\xi,X_1,\cdots,X_{2n})$ is a local orthonormal frame.

Proof. To prove (i) we see that the second equation of Theorem 1.5.27 gives

$$\mathrm{Ric}(X,\xi)=-g((\delta^T A)X,\xi).$$

So choosing a local orthonormal basis $\{X_i\}_i$ of $\mathcal{D}$, we compute using (ii) of Lemma 5.1.27

$$g(\delta^T A)X,\xi)=\sum_i g((\nabla_{X_i}A)_{X_i})(X),\xi)=-\sum_i g((\nabla_{X_i}\Phi)(X_i),X).$$

For (ii) we only need to compute $g(A_X,A_Y)$ using Equation 1.5.30

$$g(A_X,A_Y)=g(A_X\xi,A_Y\xi)=g(\Phi X,\Phi Y)=g(X,Y).$$

□

From Theorem 5.1.28 one gets an important corollary, namely,

Corollary 5.1.29: *A K-contact structure (ξ,η,Φ,g) is Einstein if and only if the two conditions hold*

(i) $\sum_i(\nabla_{X_i}\Phi)(X_i)=2n\xi$,
(ii) *The transverse metric g_T is Einstein with Einstein constant $2n+2$, i.e.* $\mathrm{Ric}_T(X,Y)=(2n+2)g_T(X,Y)$.

For sectional curvatures the only new information from Corollary 1.5.26 for K-contact structures is

$$K(X,Y)=K^T(X,Y)-3(d\eta(X,Y))^2 \tag{5.1.14}$$

which is immediate from the third equation of Corollary 1.5.26 and Lemma 5.1.10. Here X,Y are linearly independent sections of $\mathcal{D}$ of unit length. This becomes more interesting in the case of the so-called Φ*-sectional curvatures* given by $K(X,\Phi X)$. In this case we have

Lemma 5.1.30: *Let X be a section of $\mathcal{D}$ of unit length. Then*

$$K(X,\Phi X)=K^T(X,\Phi X)-3.$$

In the next section we shall see that in the case of Sasakian manifolds, the Φ-sectional curvature determines the full curvature tensor R.

5.2. Curvature Properties of Sasakian Structures

We begin the study of Sasakian structures per se. First we give necessary and sufficient condition on the covariant derivative of Φ for an almost contact structure to be Sasakian.

Theorem 5.2.1: *An almost contact metric manifold (M,ξ,η,Φ,g) is Sasakian if and only if*

$$\nabla_X\Phi = \xi\otimes X\rfloor g - X\otimes\eta.$$

Proof. Let (M,g) be embedded in its cone $(C(M),dr^2+r^2g)$ as the hypersurface $r=1$. Then the almost contact metric structure (ξ,η,Φ,g) is Sasakian if and only if the cone metric $\bar g = dr^2+r^2g$ is Kähler with respect to complex structure given by Equation 3.5.2. Furthermore, the complex structure I is Kähler with respect to the metric $\bar g$ if and only if it is parallel with respect to the Levi-Civita connection $\bar\nabla$ of $\bar g$. Thus, we have, using Gauss' formula $\bar\nabla_XY = \nabla_XY + g(sX,Y)\Psi$ and Weingarten's equation $\bar\nabla_X\Psi = -sX$,

$$\begin{aligned}
0 &= (\bar\nabla_XI)(Y) = \bar\nabla_XIY - I\bar\nabla_XY = \bar\nabla_X(\Phi Y+\eta(Y)\Psi) - (\Phi+\Psi\otimes\eta)\bar\nabla_XY\\
&= \nabla_X\Phi Y + g(sX,\Phi Y)\Psi + X\eta(Y)\Psi - \eta(Y)sX\\
&- \Phi\nabla_XY - g(sX,Y)I\Psi - \eta(\nabla_XY)\Psi\\
&= (\nabla_X\Phi)(Y) - g(X,Y)\xi + \eta(Y)X + \text{normal terms}
\end{aligned}$$

Equating the tangential term to zero gives the result. Notice that we have used the easily verified fact that the 'shape operator' s equals $-\mathbb{I}$ for a hypersurface M embedded metrically in the cone $C(M)$ as $r=1$. □

Next we have a characterization of Sasakian structures in terms of curvature.

Proposition 5.2.2: *A contact metric structure (ξ,η,Φ,g) is Sasakian if and only if*

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$

Proof. We have

$$\begin{aligned}
R(X,Y)\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi\\
&= -\nabla_X(\Phi Y + \frac{1}{2}\Phi N^{(3)}(Y)) + \nabla_Y(\Phi X + \frac{1}{2}\Phi N^{(3)}(X))\\
&+ (\Phi[X,Y] + \frac{1}{2}\Phi N^{(3)}([X,Y]))\\
&= -(\nabla_X\Phi)(Y) + (\nabla_Y\Phi)(X) - \frac{1}{2}\big((\nabla_X\Phi N^{(3)})((Y) - (\nabla_Y\Phi N^{(3)})(X)\big)
\end{aligned}$$

If (ξ,η,Φ,g) is Sasakian then $N^{(3)}=0$, and Theorem 5.2.1 implies the result.

Conversely, if the equation of the Proposition holds we have $R(X,\xi)\xi = X$ for any X orthogonal to ξ. So using Exercise 5.2 we have

$$\frac{1}{2}(N^{(3)})^2(X) = \Phi R(\Phi X,\xi)\xi - R(X,\xi)\xi + 2X = \Phi^2X + X = 0.$$

But since $N^{(3)}$ is symmetric this implies that $N^{(3)}=0$. Thus, (ξ,η,Φ,g) is K-contact. But then as previously we have

$$R(X,\xi)Y = \nabla_X\nabla_Y\xi - \nabla_{\nabla_XY}\xi = \nabla_X\Phi Y - \Phi\nabla_XY = (\nabla_X\Phi)(Y)$$

So using the symmetries of the Riemann curvature tensor we have

$$\begin{aligned} g((\nabla_X\Phi)(Y),Z) &= g(R(X,\xi)Y,Z) = -g(R(Y,Z)\xi,X) = -g(\eta(Y)Z-\eta(Z)Y,X) \\ &= -g(\eta(Y)X-g(X,Y)\xi,Z) = g(g(X,Y)\xi-\eta(Y)X,Z) \end{aligned}$$

Hence, (ξ,η,Φ,g) is Sasakian by Theorem 5.2.1. □

Theorems 5.1.28 and 5.2.1 immediately imply:

Proposition 5.2.3: *On a Sasakian manifold the Ricci curvature satisfies*

$$\mathrm{Ric}(X,\xi) = 2n\eta(X).$$

The proof of Theorem 5.2.1 suggests that we should look for conditions that a real hypersurface in a Kähler manifold be Sasakian. We have

Theorem 5.2.4: *A real hypersurface M in a Kähler manifold N is Sasakian if and only if the shape operator s satisfies $s=-\mathbb{I}+\beta\xi\otimes\eta$ for some smooth function β.*

Proof. Let M be a $2n+1$-dimensional hypersurface in a Kähler manifold $(N,I,\bar g)$ of complex dimensional $n+1$ with complex structure tensor I. Let g denote the restriction of $\bar g$ to M and $\bar\nabla$ the Levi-Civita connection on N. Let ∇ be the Levi-Civita connection on M. Then the Gauss and Weingarten equations are

$$\bar\nabla_XY=\nabla_XY+g(sX,Y)n, \qquad \bar\nabla_Xn=-sX \tag{5.2.1}$$

where X,Y are vector fields on M, and n is the unit normal field to M. Define $\xi=-In$ and Φ by $IX=\Phi X+\eta(X)n$ where η is the dual 1-form to ξ. It is easy to check that (ξ,η,Φ,g) defines an almost contact metric structure on M.

Now since I is parallel with respect to $\bar\nabla$, we have $\bar\nabla_XIY=I\bar\nabla_XY$. The left hand side gives using Equations 5.2.1

$$\bar\nabla_XIY=\bar\nabla_X(\Phi Y+\eta(Y)n)=\nabla_X\Phi Y+g(sX,\Phi Y)n+X\eta(Y)-\eta(Y)sX,$$

while the right hand sign equals

$$I\bar\nabla_XY=I(\nabla_XY+g(sX,Y)n)=\Phi\nabla_XY+\eta(\nabla_XY)n-g(sX,Y)\xi.$$

Equating the components tangent to M gives

$$(\nabla_X\Phi)(Y)=\eta(Y)sX-g(sX,Y)\xi. \tag{5.2.2}$$

By Theorem 5.2.1 M is Sasakian if and only if

$$(\nabla_X\Phi)(Y)=g(X,Y)\xi-\eta(Y)X.$$

Equating this with Equation 5.2.2 gives the solution

$$sX=-X+\beta\eta(X)\xi$$

for some smooth function β, which proves the result. □

Theorem 5.2.4 says that the embedding of a Sasakian hypersurface into a Kähler manifold is umbilic on the contact subbundle. The principal curvatures are all equal there. Another approach to the problem of embedded hypersurfaces in Kähler manifolds is given by

EXERCISE 5.3: Let M be a real hypersurface embedded in a Kähler manifold N. Show that

$$N_\Phi(X,Y) + 2d\eta(X,Y)\xi = \eta(X)[\Phi,s]Y - \eta(Y)[\Phi,s]X.$$

Thus, (M,ξ,η,Φ,g) is Sasakian if $[\Phi,s]=0$.

Add discussion about Φ-sectional curvatures.

Theorem 5.2.5: *On a Sasakian manifold the Φ-sectional curvatures determine the curvature.*

5.3. The Transverse Kähler Geometry of the Characteristic Foliation

Basic Cohomology of Sasakian Structures

The basic cohomology groups $H^{p,q}_B(\mathcal{F}_\xi)$ are fundamental invariants of a Sasakian structure which enjoy many of the same properties as the ordinary Dolbeault cohomology of a Kähler structure. Here we explore some of the elementary properties.

Theorem 5.3.1: *Let (M,ξ,η,Φ,g) be a compact Sasakian manifold of dimension $2n+1$. Then we have*

(i) $H^{n,n}_B(\mathcal{F}_\xi) \approx \mathbb{R}$.
(ii) *The class* $[d\eta]_B \neq 0$ *lies in* $H^{1,1}_B(\mathcal{F}_\xi)$.
(iii) $H^{p,p}_B(\mathcal{F}_\xi) > 0$.
(iv) $H^{2p+1}_B(\mathcal{F}_\xi)$ *has even dimension for* $p < [n/2]$.
(v) *(Transverse Hodge Decomposition)* $H^r_B(\mathcal{F}_\xi) = \bigoplus_{p+q=r} H^{p,q}_B(\mathcal{F}_\xi)$.
(vi) *Complex conjugation induces an anti-linear isomorphism* $H^{p,q}_B(\mathcal{F}_\xi) \approx H^{q,p}_B(\mathcal{F}_\xi)$.
(vii) *(Transverse Serre Duality) There is an isomorphism*

$$H^{p,q}(\mathcal{F}_\xi) \approx H^{n-p,n-q}(\mathcal{F}_\xi).$$

PROOF. The proof is very similar to the standard case [**73**]. As usual we define the operator $L : \Omega^r(\mathcal{F}_\xi) \longrightarrow \Omega^{r+2}(\mathcal{F}_\xi)$ by

$$L\alpha = \alpha \wedge d\eta, \tag{5.3.1}$$

and its adjoint $\Lambda : \Omega^r(\mathcal{F}_\xi) \longrightarrow \Omega^{r-2}(\mathcal{F}_\xi)$ by

$$\Lambda = -\bar{\star} L \bar{\star}. \tag{5.3.2}$$

We also have the adjoint operators

$$\bar\partial^* : \Omega^{p,q}(\mathcal{F}_\xi) \longrightarrow \Omega^{p,q-1}(\mathcal{F}_\xi)$$

and

$$\partial^* : \Omega^{p,q}(\mathcal{F}_\xi) \longrightarrow \Omega^{p-1,q}(\mathcal{F}_\xi)$$

of ∂ and $\bar\partial$, respectively, defined by

$$\bar\partial^* = -\bar{\star}\partial\bar{\star}, \qquad \partial^* = -\bar{\star}\bar\partial\bar{\star} \tag{5.3.3}$$

As usual the following lemma holds:

Lemma 5.3.2: *On a Sasakian manifold the following hold:*

(i) $[\Lambda,\partial] = -i\bar\partial^*$
(ii) $[\Lambda,\bar\partial] = i\partial^*$
(iii) $\partial\bar\partial^* + \bar\partial^*\partial = 0, \quad \partial^*\bar\partial + \bar\partial\partial^* = 0$

(iv) $\partial\partial^* + \partial^*\partial = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$

As usual we now define

$$\Delta^B_\partial = \partial\partial^* + \partial^*\partial, \qquad \Delta^B_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \tag{5.3.4}$$

We then have as usual

Lemma 5.3.3: *On a Sasakian manifold one has*

$$\Delta_B = 2\Delta^B_{\bar{\partial}}.$$

□

REMARK 5.3.1: Item (7) of Theorem 5.3.1 holds without the Kählerian hypothesis. Thus, it holds for normal almost contact structures such that $N^{(3)} = 0$.

Theorem 5.3.1 gives rise to fundamental invariants of the Sasakian structure. Now we have the *basic hodge numbers*

$$h^{p,q}_B(\mathcal{F}_\xi) = \dim\, H^{p,q}_B(\mathcal{F}_\xi) \tag{5.3.5}$$

which are related to the basic Betti numbers by

$$b^B_r(\mathcal{F}_\xi) = \sum_{p+q=r} h^{p,q}(\mathcal{F}_\xi), \tag{5.3.6}$$

and satisfy

$$h^{p,q}_B(\mathcal{F}_\xi) = h^{q,p}_B(\mathcal{F}_\xi), \qquad h^{p,q}_B(\mathcal{F}_\xi) = h^{n-p,n-q}_B(\mathcal{F}_\xi). \tag{5.3.7}$$

In addition to the transverse Euler characteristic on a Sasakian manifold we define the *transverse holomorphic Euler characteristic* by

$$\chi_{hol}(\mathcal{F}_\xi) = \sum_{q=0}^{n}(-1)^q h^{0,q}_B(\mathcal{F}_\xi). \tag{5.3.8}$$

There is also a transverse Lefschetz decomposition [**73**]. Consider the operator $L : \Omega^r_B(\mathcal{F}_\xi)\longrightarrow\Omega^{r+2}_B(\mathcal{F}_\xi)$ and let $P^r(\mathcal{F}_\xi)$ denote the so-called *basic primitive cohomology group*, that is, the kernel of the map $\Lambda : H^r_B(\mathcal{F}_\xi)\longrightarrow H^{r-2}_B(\mathcal{F}_\xi)$. Then we have

Theorem 5.3.4: *(Transverse Hard Lefschetz) On a compact Sasakian manifold M the map $L^k : H^{n-k}_B(\mathcal{F}_\xi)\longrightarrow H^{n+k}_B(\mathcal{F}_\xi)$ is an isomorphism for $1\leq k\leq n$. Furthermore,*

$$H^r_B(\mathcal{F}_\xi) = \bigoplus_{k\geq 0} L^k P^{r-2k}(\mathcal{F}_\xi).$$

If the dimension of M is $4n+1$ we can also define the *basic Hirzebruch signature* $\tau_B(\mathcal{F}_\xi)$ to be the signature of the bilinear form defined by the cup product on the middle basic cohomology group $H^{2n}_B(\mathcal{F}_\xi)$. Using the transverse Lefschetz Theorem 5.3.4 [**73**] one can obtain the usual formula:

$$\tau_B(\mathcal{F}_\xi) = \sum_{p,q}(-1)^p h^{p,q}_B(\mathcal{F}_\xi) = \sum_{p\equiv q(2)}(-1)^p h^{p,q}_B(\mathcal{F}_\xi). \tag{5.3.9}$$

As is usual in complex geometry we introduce the *basic geometric genus* $p_g(\mathcal{F}_\xi) = h^{0,n}_B(\mathcal{F}_\xi)$, the *basic arithmetic genus* $p_a(\mathcal{F}_\xi) = (-1)^n(\chi_{hol}(\mathcal{F}_\xi) - 1)$, and the *basic irregularity* $q = q(\mathcal{S}) = h^{0,1}_B$. We remark that $q = \frac{1}{2}b^B_1(\mathcal{F}_\xi) = \frac{1}{2}b_1(M)$ is actually a topological invariant by (5) of Proposition 1.9 of [**50**]. In the case $n = 2$, things simplify much more. All the basic Hodge numbers are given in terms of the three invariants $q, p_g(\mathcal{F}_\xi)$ and $h_B^{1,1}(\mathcal{F}_\xi)$, and we have the relations

$$\chi_B(\mathcal{F}_\xi) = 2 + 2p_g(\mathcal{F}_\xi) - 4q + h_B^{1,1}(\mathcal{F}_\xi), \qquad \tau_B(\mathcal{F}_\xi) = 2 + 2p_g(\mathcal{F}_\xi) - h_B^{1,1}(\mathcal{F}_\xi)$$

$$\chi_B(\mathcal{F}_\xi) + \tau_B(\mathcal{F}_\xi) = 4\chi_{hol}(\mathcal{F}_\xi).$$

The fact that q is actually a topological invariant has some nice consequences for $n = 2$.

Add statement about the basic cohomology being invariant under diffeomorphisms preserving the foliation.

The Space of Sasakian Stuctures

Here we consider those Sasakian structures that have the same Reeb vector field. Fix a Sasakian structure (ξ, η, Φ, g) and define the set

$$\mathfrak{F}(\xi) = \{\text{Sasakian structures } (\xi', \eta', \Phi', g') \mid \xi' = \xi\} \tag{5.3.10}$$

We give the set $\mathfrak{F}(\xi)$ the C^∞ compact-open topology as sections of vector bundles, and call it *the space of Sasakian structures compatible with* ξ or just the space of Sasakian structures when the Reeb vector field is understood to be ξ. Define the real 1-form $\zeta = \eta' - \eta$. Then ζ is basic, so $[d\eta']_B = [d\eta]_B$. Hence, all Sasakian structures in $\mathfrak{F}(\xi)$ correspond to the same basic cohomology class. We now give an alternative description of $\mathfrak{F}(\xi)$, but first we need the *transverse* $\partial\bar{\partial}$ *lemma* due to El-Kacimi-Alaoui [**73**].

Lemma 5.3.5 (transverse $\partial\bar{\partial}$)**:** *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold, and let ω, ω' be basic real closed $(1,1)$ forms such that $[\omega]_B = [\omega']_B$, then there exists a smooth basic function ϕ such that $\omega' = \omega + i\partial\bar{\partial}\phi$.*

Proof. The proof is similar to the usual case [**73**]. □

A real basic $(1,1)$-form ω on a Sasakian manifold (M, ξ, η, Φ, g) is said to be *positive*, written $\omega > 0$, if $\omega(\Phi X, X) > 0$ for all smooth sections X of $\mathcal{D}$. Now we obtain a characterization of the space of Sasakian metrics on M whose Reeb vector field is ξ.

Proposition 5.3.6: *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold. Then*

$$\mathfrak{F}(\xi) = \{(\phi, \psi) \in C^\infty(M)_B \times C^\infty(M)_B \mid d\eta + i\partial\bar{\partial}\phi > 0, \int_M \phi \,\mathrm{dvol}_g = \int_M \psi \,\mathrm{dvol}_g = 0\}.$$

Proof. Let $(\xi, \eta', \Phi', g') \in \mathfrak{F}(\xi)$, then as mentioned above $d\eta$ and $d\eta'$ represent the same basic cohomology class in $H_B^{1,1}(\mathcal{F}_\xi)$. Thus, by Lemma 5.3.5

$$d\eta' = d\eta + i\partial\bar{\partial}\phi \tag{5.3.11}$$

for some smooth basic function ϕ. Since (ξ, η', Φ', g') is Sasakian we must have $d\eta + i\partial\bar{\partial}\phi > 0$. However, the function ϕ does not uniquely determine the Sasakian structure. Define $d^c = \frac{i}{2}(\bar{\partial} - \partial)$ so that $i\partial\bar{\partial}\phi = dd^c\phi$. Then Equation 5.3.11 holds if and only if there is a smooth basic function ψ such that

$$\eta' = \eta + d^c\phi + d\psi. \tag{5.3.12}$$

Note that ψ must be basic since $\eta'(\xi) = \eta(\xi) = 1$. Since both ϕ and ψ are only defined up to a constant, they can be chosen to satisfy the normalization condition $\int_M f \,\mathrm{dvol}_g = 0$. This then determines the pair (ϕ, ψ) uniquely.

Conversely, let (ξ,η,Φ,g) be a Sasakian structure and (ϕ,ψ) a pair of smooth basic functions satisfying the normalization condition and the positivity condition $d\eta+i\partial\bar{\partial}\phi>0$. Define η' by Equation 5.3.12. Then η' is a contact form on M since if $\eta'\wedge(d\eta+dd^c\phi)^n$ were to vanish at some point, taking the interior product with ξ shows that $(d\eta+dd^c\phi)^n$ would vanish which violates the positivity condition. Now put $\zeta=d^c\phi+d\psi$ and define

$$\Phi'=\Phi-\xi\otimes\zeta\circ\Phi \tag{5.3.13}$$

$$g'=d\eta'\circ(\Phi'\otimes\mathbb{I})+\eta'\otimes\eta'. \tag{5.3.14}$$

We claim that (ξ,η',Φ',g') is a Sasakian structure. Actually, from the point of view of bundle maps we should write Equation 5.3.13 as

$$\Phi'=(\mathbb{I}-\xi\otimes\zeta)\circ\Phi\circ(\mathbb{I}+\xi\otimes\zeta).$$

One easily checks that $\mathbb{I}-\xi\otimes\zeta:TM\longrightarrow TM$ is a vector bundle isomorphism taking the subundle $\mathcal{D}$ to the subundle $\mathcal{D}'=\ker\eta'$, and that $\mathbb{I}+\xi\otimes\zeta$ is the inverse of $\mathbb{I}-\xi\otimes\zeta$. This shows that $(\mathcal{D},\Phi|_{\mathcal{D}})$ and $(\mathcal{D}',\Phi'|_{\mathcal{D}'})$ are isomorphic as complex vector bundles. Thus, the almost CR-structure $\Phi'|_{\mathcal{D}'}$ on $\mathcal{D}'$ is integrable since $\Phi|_{\mathcal{D}}$ is integrable. We need to check the positivity condition $d\eta'\circ(\Phi'\otimes\mathbb{I})>0$, as well as the condition $(\Phi')^2=-\mathbb{I}+\xi\otimes\eta'$. These are straightforward and left as an exercise. Furthermore, under the above identification the topology of $\mathfrak{F}(\xi)$ coincides with the relative topology as a subspace of $C^\infty(M)_B\times C^\infty(M)_B$. □

Remark 5.3.2: It is interesting to note that in the quasi-regular case $\mathfrak{F}(\xi)$ can be viewed as a subspace of the space of connections in a $U(1)$ V-bundle, and indeed even in the irregular case as a subspace of the space of connections in a 1-dimensional foliation. Explicitly, $\mathfrak{F}(\xi)$ is the subspace of connections with strictly positive curvature satifying the normalization condition. The function ψ is then a gauge transformation in $\mathfrak{F}(\xi)$.

It is interesting to note that all of the Sasakian structures in $\mathfrak{F}(\xi)$ have the same volume. Explicitly,

Proposition 5.3.7: *Let M^{2n+1} be a compact Sasakian manifold. Then for $\mathcal{S},\mathcal{S}'\in\mathfrak{F}(\xi)$ arbitrary, we have* $\mathrm{Vol}(M,g)=\mathrm{Vol}(M,g')$.

Proof. The infinitesimal volume element of a Sasakian metric is related to the contact volume element by $d\mathrm{vol}_g=\frac{1}{n!}\eta\wedge(d\eta)^n$. Now by Lemma 5.3.5 there is a smooth basic function ϕ such that

$$d\mathrm{vol}_{g'}=\frac{1}{n!}\eta'\wedge(d\eta')^n=(\eta+d^c\phi)\wedge(d\eta+dd^c\phi)^n=\eta\wedge(d\eta+dd^c\phi)^n,$$

where the last equality holds since $d^c\phi$ is basic. Now expand

$$(d\eta+dd^c\phi)^n=(d\eta)^n+\sum_{i=1}^n\binom{n}{i}(d\eta)^{n-i}\wedge(dd^c\phi)^i,$$

so that

$$\begin{aligned}\mathrm{Vol}(M,g') &= \int_M d\mathrm{vol}_{g'} = \frac{1}{n!}\int_M \eta\wedge(d\eta+dd^c\phi)^n \\ &= \frac{1}{n!}\int_M \eta\wedge\big((d\eta)^n+\sum_{i=1}^n\binom{n}{i}(d\eta)^{n-i}\wedge(dd^c\phi)^i\big) \\ &= \frac{1}{n!}\int_M \eta\wedge(d\eta)^n+\frac{1}{n!}\sum_{i=1}^n\binom{n}{i}\int_M d\big((d\eta)^{n-i}\wedge d^c\phi\wedge(dd^c\phi)^{i-1}\big) \\ &= \int_M d\mathrm{vol}_g = \mathrm{Vol}(M,g),\end{aligned}$$

where the last line follows by Stokes' theorem. □

It is convenient to consider a set of Sasakian structures that is slightly larger than $\mathfrak{F}(\xi)$, namely those Sasakian structures that have the same characteristic foliation. We define $\mathfrak{F}(\mathcal{F}_\xi)$ to be the set of all Sasakian structures whose characteristic foliation is $\mathcal{F}_\xi$. Clearly, we have $\mathfrak{F}(\xi)\subset\mathfrak{F}(\mathcal{F}_\xi)$. For any Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ there is the "conjugate Sasakian structure" defined by $\mathcal{S}^c=(\xi^c,\eta^c,\Phi^c,g)=(-\xi,-\eta,-\Phi,g)\in\mathfrak{F}(\mathcal{F}_\xi)$. Moreover, there is a type of transverse homothety [YK] defined by

$$\xi'=a^{-1}\xi,\quad \eta'=a\eta,\quad \Phi'=\Phi,\quad g'=ag+(a^2-a)\eta\otimes\eta \tag{5.3.15}$$

for $a\in\mathbb{R}^+$. So fixing $\mathcal{S}$ we define

$$\mathfrak{F}^+(\mathcal{F}_\xi)=\bigcup_{a\in\mathbb{R}^+}\mathfrak{F}(a^{-1}\xi), \tag{5.3.16}$$

and $\mathfrak{F}^-(\mathcal{F}_\xi)$ to be the image of $\mathfrak{F}^+(\mathcal{F}_\xi)$ under conjugation. It is then easy to see that we have a decomposition $\mathfrak{F}(\mathcal{F}_\xi)=\mathfrak{F}^+(\mathcal{F}_\xi)\sqcup\mathfrak{F}^-(\mathcal{F}_\xi)$. Notice that for n even conjugation reverses orientation; whereas, for n odd it preserves the orientation. This discussion shows that for each foliation $\mathcal{F}_\xi$ the subset of homology classes represented by compatible Sasakian structures forms a line minus the origin in $H^2_B(\mathcal{F}_\xi)$, and that conjugation interchanges the positive and negative rays. We shall refer to $\mathfrak{F}^\pm(\mathcal{F}_\xi)$ as an *a-homology class* of Sasakian structures, or an *a-deformation class*[1]. We need

Definition 5.3.8: *Two Sasakian structures $\mathcal{S}=(\xi,\eta,\Phi,g)$ and $\mathcal{S}'=(\xi',\eta',\Phi',g')$ in $\mathfrak{F}(\mathcal{F}_\xi)$ on a smooth manifold M are said to be* a-**homologous** *if there is an $a\in\mathbb{R}^+$ such that $\xi'=a^{-1}\xi$ and $[d\eta']_B=a[d\eta]_B$.*

The a-homology classes form a set of two elements that can be identified with positive and negative rays in $H^2_B(\mathcal{F}_\xi)$. So every Sasakian structure in $\mathfrak{F}(\mathcal{F}_\xi)$ is a-homologous to $\mathcal{S}$ or its conjugate $\mathcal{S}^c$.

Here's an example.

[1] The latter appellation will become clear in Section 5.4

EXAMPLE 5.3.9: We begin with the standard contact structure[2] on S^{2n+1}, namely the restriction of $\eta_0 = \sum_{i=0}^{n}(y^i dx^i - x^i dy^i)$ to the unit sphere

$$S^{2n+1} = \{\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} |z^i|^2 = \sum_{i=0}^{n}\big((x^i)^2 + (y^i)^2\big) = 1\}$$

where x^i, y^i denote the real and imaginary components of $z^i \in \mathbb{C}$. In complex coordinates the 1-form η_0 takes the form

$$\eta_0 = -\frac{i}{2}\sum_{j=0}^{n}(z^j d\bar{z}^j - \bar{z}^j dz^j).$$

The Reeb vector field is

$$\xi_0 = \sum_{i=0}^{n}(y^i\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial y^i}) = -i\sum_{j=0}^{n}(z^j\frac{\partial}{\partial z^j} - \bar{z}^j\frac{\partial}{\partial \bar{z}^j})$$

while the $(1,1)$ tensor field Φ_0 is obtained by restricting the standard complex structure

$$I = \sum_{i=0}^{n}(\frac{\partial}{\partial y^i}\otimes dx^i - \frac{\partial}{\partial x^i}\otimes dy^i) = i\sum_{j=0}^{n}(\frac{\partial}{\partial z^j}\otimes dz^j - \frac{\partial}{\partial \bar{z}^j}\otimes d\bar{z}^j)$$

on $\mathbb{R}^{2n+2}$ to $\mathcal{D} \subset TS^{2n+1}$ and extending trivially to the real line bundle generated by ξ_0. Explicitly, we find

$$\Phi_0 = \sum_{i,j}\big((x^ix^j - \delta^{ij})\frac{\partial}{\partial x^i}\otimes dy^j - (y^iy^j - \delta^{ij})\frac{\partial}{\partial y^i}\otimes dx^j + x^jy^i\frac{\partial}{\partial y^i}\otimes dy^j - x^iy^j\frac{\partial}{\partial x^i}\otimes dx^j\big).$$

The canonical metric g_0 on S^{2n+1} is the flat metric restricted to the S^{2n+1}. It has constant sectional curvature 1 and satisfies

$$g_0 = d\eta_0 \circ (\Phi_0 \otimes \mathbb{I}) + \eta_0 \otimes \eta_0.$$

We refer to the Sasakian structure $\mathcal{S}_0 = (\xi_0, \eta_0, \Phi_0, g_0)$ as the **standard Sasakian structure** on S^{2n+1}. We shall also refer to the conjugate structure $\mathcal{S}_0^c = (-\xi_0, -\eta_0, -\Phi_0, g_0)$ as a standard Sasakian structure when no confusion should arise. The reason for this will be seen shortly.

The standard Sasakian structure $\mathcal{S}_0$ is a representative in the *standard a-deformation class* $\mathfrak{F}^+(\mathcal{F}_{\xi_0})$, while $\mathcal{S}_0^c = (-\xi_0, -\eta_0, -\Phi_0, g_0)$ is a representative in $\mathfrak{F}^-(\mathcal{F}_{\xi_0})$. Of course, the other representatives of $\mathfrak{F}(\mathcal{F}_{\xi_0})$ do not, in general, have constant sectional curvature. The diffeomorphism $C : S^{2n+1} \longrightarrow S^{2n+1}$ induced by complex conjugation on $\mathbb{C}^{n+1}$ sends $\mathcal{S}_0$ to $\mathcal{S}_0^c$. Note that C is an element of the contactomorphism group $\mathfrak{C}(S^{2n+1}, \mathcal{D}_0 = \ker\,\eta_0)$ which preserves orientation if n is odd, and reverses orientation for n even. For n odd C reverses the co-orientation.

Quasi-regular Sasakian Stuctures

In the case of a compact quasi-regular Sasakian manifold M we know that the space of leaves is a compact Riemannian orbifold $\mathcal{Z}$. But since the transverse

[2]Our choice of 1-form η_0 here is the negative of the one used in Example 3.1.13. The reason for this is to keep the standard conventions of [**28, 224**] for Sasakian geometry, while at the same time using the standard complex structure on $\mathbb{C}^{n+1}$ and not the complex conjugate. Of course there is no change in the underlying contact structure, but only a change of orientation or co-orientation.

geometry on M is Kähler, the orbifold must be Kähler. But actually in the quasi-regular case more is true. It follows from Theorem 3.3.8 that M is the total space of a V-bundle over $\mathcal{Z}$, and the curvature of the connection form η is precisely the pullback of the Kähler form on $\mathcal{Z}$. Thus, $\mathcal{Z}$ satisfies an orbifold integrality condition which we now elaborate. This integrality condition ties Sasakian geometry on compact manifolds to projective algebraic geometry.

Definition 5.3.10: *A compact Kähler orbifold is called a* **Hodge orbifold** *if the Kähler class* $[\omega]$ *lies in* $H^2_{orb}(\mathcal{Z},\mathbb{Z})$.

With this definition a restatement of the Kodaira-Baily embedding theorem [Ba2] is:

Theorem 5.3.11 (Baily)**:** *A Hodge orbifold is a projective algebraic variety.*

We can now combine the results of this section with Theorem 3.3.8.

Theorem 5.3.12: *Let* $(\mathcal{S},g)$ *be a compact quasi-regular Sasakian manifold of dimension* $2n+1$*, and let* $\mathcal{Z}$ *denote the space of leaves of the characteristic foliation. Then*

(i) *The leaf space* $\mathcal{Z}$ *is a Hodge orbifold with Kähler metric* h *and Kähler form* ω *which defines an integral class* $[\omega]$ *in* $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ *in such a way that* $\pi:(\mathcal{S},g)\longrightarrow(\mathcal{Z},h)$ *is an orbifold Riemannian submersion. The fibers of* π *are totally geodesic submanifolds of* $\mathcal{S}$ *diffeomorphic to* S^1.

(ii) $\mathcal{Z}$ *is also a* $\mathbb{Q}$*-factorial, polarized, normal projective algebraic variety.*

(iii) *The orbifold* $\mathcal{Z}$ *is Fano if and only if* $\mathrm{Ric}_g > -2$. *In this case* $\mathcal{Z}$ *as a topological space is simply connected, and as an algebraic variety is uniruled with Kodaira dimension* $\kappa(\mathcal{Z})=-\infty$.

(iv) $(\mathcal{S},g)$ *is Sasakian-Einstein if and only if* $(\mathcal{Z},h)$ *is Kähler-Einstein with scalar curvature* $4n(n+1)$.

Proof. (i): By Theorem 3.3.8 $\mathcal{Z}$ is a compact orbifold, and the remainder follows exactly as in the regular case except that now the Kähler class $[\omega]$ defines an integral class in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ which is generally only a rational class in the ordinary cohomology $H^2(\mathcal{Z},\mathbb{Q})$. This proves (i). To prove (ii) we notice that as the usual case ω is a positive $(1,1)$-form and thus represents a holomorphic line V-bundle on $\mathcal{Z}$. (ii) now follows from the Kodaira-Baily embedding theorem [Ba2].

In the Sasakian case the O'Neill tensors T and N (See [Bes]) vanish, and the tensor A satisfies $A_XY=-g(\Phi X,Y)\xi$ and $A_X\xi=\Phi X$ for X and Y horizontal. Then one easily sees from [Bes, 9.36] that

$$\mathrm{Ric}_g(X,Y)=\mathrm{Ric}_h(X,Y)-2g(X,Y). \tag{5.3.17}$$

The first statement of (3) now follows from this. Moreover, combining 5.3.17 with the last statement of **??** implies (4). In (3) the simple connectivity of $\mathcal{Z}$ is basically Kobayashi's argument [Kob1] with the usual Riemann-Roch replaced by the singular version of Baum, Fulton, and MacPherson (see [BG1] for details). The uniruledness and Kodaira dimension follow from Miyaoka and Mori [MiMo]. □

Example 5.3.13: We continue the discussion of the standard Sasakian structure on S^{2n+1} of Example 5.3.9. We have the two standard Sasakian structures $\mathcal{S}_0$ and $\mathcal{S}_0^c$. These are both regular Sasakian structures, and are nothing but the Hopf fibrations as principal S^1 bundles over complex projective space $\mathbb{CP}^n$. Now $\mathbb{C}^{n+1}$ with the standard complex structure and the positive oriented $\mathbb{C}^*$-action (i.e. $\mathbf{z}\mapsto re^{it}\mathbf{z}$)

is the total space of the tautological line bundle $\mathcal{O}(-1)$. Since for the Sasakian structure $\mathcal{S}_0$ the circle action has the negative orientation, $\mathcal{S}_0$ is associated with the positive line bundle $\mathcal{O}(1)$, as it must by Theorem 5.3.12. The conjugate Sasakian structure $\mathcal{S}_0^c$ is associated with complex line bundle $\mathcal{O}(-1)$ but with the complex conjugate structure, and as such it is also a positive line bundle.

Notice that a quasi-regular Sasakian structure on a compact manifold can be thought of as a *transverse projective algebraic structure*. For compact Kähler manifolds there is a conjecture attributed to Kodaira that says that any Kähler structure can be approximated or equivalently is a deformation of an algebraic structure. It was shown to be true in case of compact Kähler surfaces by Kodaira [**?**]. There is an analogous result for compact Sasakian manifolds which holds in all dimensions. It is a corollary of the Structure Theorem 5.1.2 and will be stated precisely in the next section.

For quasi-regular Sasakian structures we have:

Definition 5.3.14: *The* **order** *of a quasi-regular Sasakian structure $\mathcal{S}$ is the least common multiple of the orders of the leaf holonomy groups of the characteristic foliation, and is denoted by $\upsilon(\mathcal{S})$.*

Notice that the $\upsilon(\mathcal{S})$ coincides with the order of the orbifold $\mathcal{Z}_{\mathcal{S}}$. The order of a regular Sasakian structure is one.

Next we give an inversion theorem to Theorem 5.3.12. In the regular case this goes back to a construction of Kobayashi [Kob2] together with Hatakeyama [Hat]. There is also a description in the context of Sasakian-Einstein geometry in [FrKat2,BFGK] in the regular case. The non-regular case follows by applying Hatakeyama's results to our more general setup. First we have

Now our inversion theorem is:

Theorem 5.3.15: *Let $(\mathcal{Z},h)$ be a Hodge orbifold. Let $\pi:\mathcal{S}\longrightarrow\mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $[\omega]$, and let η be a connection 1-form in $\mathcal{S}$ whose curvature is $2\pi^*\omega$, then $(\mathcal{S},\eta)$ with the metric $\pi^*h+\eta\otimes\eta$ is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the group of the bundle S^1, the total space $\mathcal{S}$ is a smooth Sasakian manifold.*

Remark 5.3.3: The orbifold structure of $\mathcal{Z}$ is crucial here. Consider the weighted projective space $\mathbb{CP}^2(p_1,p_2,p_3)$ defined by the usual weighted $\mathbb{C}^*$ action on $\mathbb{C}^3-\{0\}$ where p_1,p_2,p_3 are pairwise relatively prime integers. As an algebraic variety $\mathbb{CP}^2(p_1,p_2,p_3)$ is equivalent [Kol] to $\mathbb{CP}^2/\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\mathbb{Z}_{p_3}$. But as orbifolds these are distinct, since the former has $\pi_1^{orb}=0$, whereas the latter has $\pi_1^{orb}=\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\mathbb{Z}_{p_3}$. This is to be contrasted with the smooth case where equivalence as complex manifolds coincides with equivalence as algebraic varieties. Of course, the metrics are also different. In the latter case the metric is just the Fubini-Study metric pushed to the quotient which is Kähler-Einstein. The Sasakian structure on the corresponding V-bundle is just the standard Sasakian-Einstein structure on the lens space $S^5/\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\mathbb{Z}_{p_3}$. In the former case the Sasakian structure is one of the non-standard deformed structures on S^5 described in Example 7.1 of [YKon] which is ellipsoidal and not Einstein. Likewise, the metric h on $\mathbb{CP}^2(p_1,p_2,p_3)$ is not Kähler-Einstein.

Chern Classes and Basic Chern Classes

Consider the complex vector bundle $\mathcal{D}$ on a Sasakian manifold (M,ξ,η,Φ,g). As such $\mathcal{D}$ has Chern classes $c_1(\mathcal{D}),\cdots,c_n(\mathcal{D})$ which can be computed by choosing a connection $\nabla^{\mathcal{D}}$ in $\mathcal{D}$ [**128**]. We choose $\nabla^{\mathcal{D}}$ to be the transverse connection ∇^T of 1.5.14 in which case the curvature is just the transverse curvature R^T given by Equation 1.5.15. Let us choose a local foliate unitary transverse frame $(X_1,\cdots,X_n)$, and denote by Ω^T the transverse curvature 2-form with respect to this frame. We have

Proposition 5.3.16: *Ω_T is a basic $(1,1)$-form.*

PROOF. Since the transverse geometry is Kähler, the 2-form Ω_T has values in the Lie algebra $\mathfrak{u}(n)$. Thus, Ω satisfies $\bar{\Omega}^t_T=-\Omega_T$. is related to the transverse Riemannian curvature R^T by (Finish Later). □

Since the curvature 2-form Ω_T has type $(1,1)$ it follows as in ordinary Chern-Weil theory that

Theorem/Definition 5.3.17: *The kth Chern class $c_k(\mathcal{D})$ of the complex vector bundle $\mathcal{D}$ is represented by the basic (k,k)-form γ_k determined by the formula*

$$\det(\mathbb{I}_n-\frac{1}{2\pi i}\Omega_T)=1+\gamma_1+\cdots+\gamma_k.$$

Since γ_k is a basic (k,k)-form it represents an element in $H^{k,k}_B(\mathcal{F}_\xi)\subset H^{2k}_B(\mathcal{F}_\xi)$ that is called the **basic kth Chern class** *and denoted by $c_k(\mathcal{F}_\xi)$.*

We denote by $c_k(\mathcal{D})_{\mathbb{R}}$ the class $c_k(\mathcal{D})$ viewed as an element of $H^{2k}(M,\mathbb{R})$. From the exact sequence 5.1.1 we have

Lemma 5.3.18: *The basic Chern class $c_k(\mathcal{F}_\xi)$ maps to $c_k(\mathcal{D})_{\mathbb{R}}$ under the natural inclusion ι_*. Moreover, an element $[\alpha]_B\in H^{2k}(\mathcal{F}_\xi)$ maps to $c_k(\mathcal{D})_{\mathbb{R}}$ if and only if there exist a real number a and a closed basic $(2k-2)$-form β such that*

$$[\alpha]_B=c_k(\mathcal{F}_\xi)+a[L\beta]_B.$$

In particular, $c_k(\mathcal{D})_{\mathbb{R}}=0$ if and only if there are a real number a and a closed basic $(2k-2)$-form β such that $c_k(\mathcal{F}_\xi)=a[L\beta]_B$.

Notice that the Chern classes $c_k(\mathcal{D})$ are invariants of the underlying contact structure. We are now interested in the basic Chern classes $c_k(\mathcal{F}_\xi)$. We have

Proposition 5.3.19: *The basic Chern classes $c_k(\mathcal{F}_\xi)$ are independent of the Sasakian structure in $\mathfrak{F}(\mathcal{F}_\xi)$.*

PROOF. Since the transverse Ricci tensor is invariant under a homothety of the transverse metric, the Chern class is invariant under a transverse homothety. □

We now concentrate on the first Chern classes $c_1(\mathcal{D})$ and $c_1(\mathcal{F}_\xi)$. We give an immediate corollary of Lemma 5.3.18, but first a definition.

Definition 5.3.20: *A Sasakian structure (ξ,η,Φ,g) is said to be* **positive (negative)** *if $c_1(\mathcal{F}_\xi)$ is represented by a positive (negative) definite $(1,1)$-form. If either of these two conditions is satisfied (ξ,η,Φ,g) is said to be* **definite**, *and otherwise (ξ,η,Φ,g) is called* **indefinite**. *(ξ,η,Φ,g) is said to be* **null** *if $c_1(\mathcal{F}_\xi)=0$.*

Now we have

Corollary 5.3.21: *Let (ξ,η,Φ,g) be a Sasakian structure with underlying contact bundle $\mathcal{D}$. Then $c_1(\mathcal{D})$ is a torsion class if and only if there exists a real number* *a such that $c_1(\mathcal{F}_\xi) = a[d\eta]_B$. In particular, if $c_1(\mathcal{D})$ is a torsion class then the Sasakian structure is either definite or null.*

In analogy with common terminology of smooth algebraic varieties we see that a positive Sasakian structure is a *transverse Fano structure*, while a null Sasakian structure is a *transverse Calabi-Yau structure*. The negative Sasakian case has no other name. It corresponds to the canonical bundle being ample.

Note that $c_1(\mathcal{D})$ is an invariant of the complex vector bundle $\mathcal{D}$, but it is related to a topological invariant, namely the second Stiefel-Whitney class $w_2(M)$. Indeed, let (M, η) be a strict contact manifold with contact bundle $\mathcal{D}$. Let ξ be the Reeb vector field, then we have a natural splitting

$$TM = \mathcal{D} \oplus L_\xi$$

where L_ξ is the trivial real line bundle generated by ξ. Thus,

$$w_2(M) = w_2(TM) = w_2(\mathcal{D})$$

which is the mod 2 reduction of $c_1(\mathcal{D}) \in H^2(M, \mathbb{Z})$. Since a manifold M admits a spin structure if and only if $w_2(M) = 0$, Corollary 5.3.21 implies

Proposition 5.3.22: *Let (M, ξ, η, Φ, g) be a simply connected Sasakian manifold. Suppose that $c_1(\mathcal{F}_\xi) = a[d\eta]_B$ for some real number a. Then M is spin.*

Recall the *transverse Ricci tensor* Ric_g^T of g_T to be the Ricci tensor of the transverse metric g_T. It is related to the Ricci tensor Ric_g of g by the second equation in Theorem 5.1.28. Now as usual define the *Ricci form* ρ_g and *transverse Ricci form* ρ_g^T by

$$\rho_g(X, Y) = \mathrm{Ric}_g(X, \Phi Y), \qquad \rho_g^T(X, Y) = \mathrm{Ric}_g^T(X, \Phi Y) \tag{5.3.18}$$

for smooth sections X, Y of $\mathcal{D}$. It is easy to check that these are anti-symmetric of type $(1,1)$ and Theorem 5.1.28 implies that they are related by

$$\rho_g^T = \rho_g + 2d\eta. \tag{5.3.19}$$

Thus, as in the usual case we have

Lemma 5.3.23: *The basic class $2\pi c_1(\mathcal{F}_\xi) \in H_B^{1,1}(\mathcal{F}_\xi)$ is represented by the transverse Ricci form ρ_g^T.*

Now suppose that $\mathcal{S}$ is a positive Sasakian structure so that $c_1(\mathcal{F}_\xi) > 0$. Choose a transverse Kähler metric g'_T whose Kähler form $\omega'_T \in c_1(\mathcal{F}_\xi)$. Then both ω'_T and its transverse Ricci form ρ'_T represent $c_1(\mathcal{F}_\xi)$. This transverse Kähler structure defines a positive Sasakian structure $\mathcal{S}' = (\xi, \eta', \Phi', g')$ on M such that $g' = g'_T + \eta' \otimes \eta'$, and ω'_T represents $c_1(\mathcal{F}_\xi)$. The exact sequence 1.11 of [BGN1] becomes

$$\begin{array}{ccccccc} & & & & H^2(M,\mathbb{Z}) & & \\ & & & & \downarrow & & \\ 0 \longrightarrow \mathbb{R} & \xrightarrow{\delta} & H_B^2(\mathcal{F}_\xi) & \xrightarrow{\iota_*} & H^2(M,\mathbb{R}) & \longrightarrow, & \end{array}$$

where $\delta(c) = c[\omega'_T]_B$. But since $c_1(\mathcal{F}_\xi)$ is represented by ω'_T we have $c_1(\mathcal{D}') = \iota_* c_1(\mathcal{F}_\xi) = \iota_*\delta(1) = 0$ in $H^2(M, \mathbb{R})$. But since M is simply connected $c_1(\mathcal{D}') = 0$, so this implies that $w_2(M) = w_2(\mathcal{D}') = 0$.

Actually we have proven more.

Proposition 5.3.24: *For every positive Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on M the first Chern class $c_1(\mathcal{D}) \in H^2(M, \mathbb{Z})$ is torsion.*

5.4. Automorphisms and Deformation Classes of Sasakian Structures

The automorphism group $\mathfrak{A}(\xi, \eta, \Phi, g)$ of a Sasakian structure (ξ, η, Φ, g) is just the automorphism group of the underlying K-contact structure. Recall that we denote the Lie algebra of $\mathfrak{A}(\xi, \eta, \Phi, g)$ by $\mathfrak{a}(\xi, \eta, \Phi, g)$. Before describing the deformations we give a lemma concerning infinitesimal automorphisms of K-contact structures.

Lemma 5.4.1: *Let (ξ, η, Φ, g) be a K-contact structure, and let ρ be a Killing vector field that commutes with ξ. Then ρ is an infinitesimal automorphism of (ξ, η, Φ, g), i.e. $\rho \in \mathfrak{a}(\xi, \eta, \Phi, g)$. Moreover, the following hold:*

$$\xi(\eta(\rho)) = 0, \qquad \rho(\eta(\rho)) = 0.$$

Proof. For the first statement we need to show that $\pounds_\rho \eta$ and $\pounds_\rho \Phi$ both vanish. Since, $\pounds_\rho g = 0$, the vector field ρ is affine, that is, $\pounds_\rho \nabla = 0$, where ∇ is the Levi-Civita connection. Thus,

$$\pounds_\rho \Phi = \pounds_\rho(\nabla \xi) = (\pounds_\rho \nabla)(\xi) + \nabla[\rho, \xi] = 0.$$

To show that $\pounds_\rho \eta$ vanishes, we have

$$\pounds_\rho \eta = \pounds_\rho(\xi \rfloor g) = [\rho, \xi] \rfloor g + \xi \rfloor \pounds_\rho g = 0.$$

To prove the two equations we notice that

$$\xi(\eta(\rho)) = (\pounds_\xi \eta)(\rho) + \eta([\xi, \rho]) = 0,$$

which proves the first equation, while for the second we have

$$\rho(\eta(\rho)) = (\pounds_\rho \eta)(\rho) = 0.$$

□

Since the automorphism group $\mathfrak{A}$ of any Sasakian manifold contains the subgroup generated by ξ, it has dimension at least one. Thus, in the case of quasi-regular structures we can always form nontrivial smooth quotients. Evidently we have

Proposition 5.4.2: *Let (M, ξ, η, Φ, g) be a quasi-regular Sasakian manifold, and let $\mathfrak{G}$ be a finite subgroup of $\mathfrak{A}(\xi, \eta, \Phi, g)$. Then the quotient space $M/\mathfrak{G}$ is an orbifold which inherits the Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ from M. Moreover, if $\mathfrak{G}$ acts freely on M, then $M/\mathfrak{G}$ is a Sasakian manifold. In particular, every Sasakian manifold admits quotients by cyclic subgroups C_m of the circle group generated by ξ. Moreover, if $\gcd(m, \upsilon(\mathcal{S})) = 1$, the quotient M/C_m is a Sasakian manifold.*

Add Tanno's work on automorphism groups.

Deformations of Transverse Holomorphic Structures

Here we briefly sketch the deformation theory of transverse holomorphic structures due to Duchamp and Kalka [**68, 69, 70**], Girbau, Haefliger, and Sundararaman [**94**], and Gomez-Mont [**97**]. In [**68, 69**] an extra assumption was made that is not convenient for our purposes. So we shall mainly follow [**94**], but adapted to our purpose. For the basics of deformation theory we refer to [**?**] and [**?**]. Here M

will be a Sasakian manifold of dimension $2n+1$, but with the Sasakian structure supressed. We are interested for now only in the transverse holomorphic structure. In order to reduce the notation a bit we write $\mathcal{F}$ as the characteristic foliation, dropping the subscript ξ.

Deformations of Sasakian Structures

It is not our purpose here to give a complete theory of deformations of Sasakian structures. We develop only that deformation theory that is needed in the sequel. We consider essentially two types of deformations of Sasakian structures within a fixed transverse holomorphic structure. Type I deformations are those that deform the characteristic foliation $\mathcal{F}_\xi$ by deforming the vector field ξ. These deformations, due to Takahashi [**202**], leave both the underlying contact structure and underlying CR-structure fixed. Type II deformations leave the vector field ξ invariant and deform the contact 1-form.

Deformations of Type I: Here we consider a particular type of deformation due to Takahashi [**202**]. In particular we prove

Theorem 5.4.3 (Takahashi)**:** *Let (ξ,η,Φ,g) be a Sasakian structure on M, and let ρ be a Killing vector field which commutes with ξ. Define the following tensor fields*

$$\tilde{\xi}=\xi+\rho,\qquad \tilde{\eta}=(1+\eta(\rho))^{-1}\eta,\qquad \tilde{\Phi}=\Phi-\Phi\tilde{\xi}\otimes\tilde{\eta} \tag{5.4.1}$$

$$\tilde{g}=\frac{g-\tilde{\eta}\otimes\tilde{\xi}\rfloor g-\tilde{\xi}\rfloor g\otimes\tilde{\eta}+g(\tilde{\xi},\tilde{\xi})\tilde{\eta}\otimes\tilde{\eta}}{1+\eta(\rho))}+\tilde{\eta}\otimes\tilde{\eta}. \tag{5.4.2}$$

Then if $1+\eta(\rho)>0$ holds, $(\tilde{\xi},\tilde{\eta},\tilde{\Phi},\tilde{g})$ is a Sasakian structure on M in the same underlying contact structure. Furthermore, if $\rho=f\xi$ for some $f\in C^\infty(M)$, then $f=a$ is constant.

Proof. It is clear from Equations 5.4.1 that $\tilde{\eta}$ is a contact 1-form that satisfies $\tilde{\eta}(\tilde{\xi})=1$, and that the contact subbundle $\mathcal{D}=\ker\,\eta=\ker\,\tilde{\eta}$ remains unchanged. Furthermore, $\tilde{\Phi}\tilde{\xi}=0$, and if X is a section of $\mathcal{D}$, then

$$\tilde{\Phi}^2X=\tilde{\Phi}(\Phi X-\tilde{\eta}(X)\Phi\rho)=\Phi(\Phi X-\tilde{\eta}(\Phi X)\Phi\rho)=\Phi^2X=-\mathbb{I}.$$

Thus, $\tilde{\Phi}^2=-\mathbb{I}+\tilde{\xi}\otimes\tilde{\eta}$, so $(\tilde{\xi},\tilde{\eta},\tilde{\Phi})$ is an almost contact structure. Next we check that $\tilde{\xi}$ is the Reeb vector field of $\tilde{\eta}$. We have

$$\begin{aligned}\tilde{\xi}\rfloor d\tilde{\eta}&=\tilde{\xi}\rfloor\big((1+\eta(\rho))^{-1}d\eta-(1+\eta(\rho))^{-2}d(\eta(\rho))\wedge\eta\big)\\&=(1+\eta(\rho))^{-1}\rho\rfloor d\eta+(1+\eta(\rho))^{-1}d(\eta(\rho))-(1+\eta(\rho))^{-2}\tilde{\xi}(\eta(\rho))\eta\\&=(1+\eta(\rho))^{-1}\pounds_\rho\eta-(1+\eta(\rho))^{-2}\tilde{\xi}(\eta(\rho))\eta=0\end{aligned}$$

by Lemma 5.4.1.

We need to check that the structure $(\tilde{\xi},\tilde{\eta},\tilde{\Phi})$ is normal and that $\tilde{g}$ is the associated Riemannian metric. According to Corollary 3.5.11 the structure $(\tilde{\xi},\tilde{\eta},\tilde{\Phi})$ is normal if and only if the almost CR-structure $\tilde{\Phi}|_{\mathcal{D}}$ is integrable and $\tilde{\Phi}$ is invariant under $\tilde{\xi}$. First, we note that $\tilde{\xi}\in\mathfrak{a}(\xi,\eta,\Phi,g)$ since both $\xi,\rho\in\mathfrak{a}(\xi,\eta,\Phi,g)$. Moreover, $\tilde{\xi}$ commutes with both ξ and ρ. This is enough to prove that $\pounds_{\tilde{\xi}}\tilde{\Phi}=0$. But clearly $\tilde{\Phi}|_{\mathcal{D}}$ is integrable since $\tilde{\Phi}|_{\mathcal{D}}=\Phi|_{\mathcal{D}}$. Finally, we check the metric $\tilde{g}$. It suffices to

check that
$$\frac{g-\tilde\eta\otimes\tilde\xi\rfloor g-\tilde\xi\rfloor g\otimes\tilde\eta+g(\tilde\xi,\tilde\xi)\tilde\eta\otimes\tilde\eta}{1+\eta(\rho))}=d\tilde\eta\circ(\mathbb{I}\otimes\tilde\Phi).$$
First we see that contracting numerator of the left hand side of the equation with $\tilde\xi$ gives
$$\tilde\xi\rfloor g-\tilde\xi\rfloor g-g(\tilde\xi,\tilde\xi)\tilde\eta+g(\tilde\xi,\tilde\xi)\tilde\eta=0,$$
so it remains to check the equation on sections of $\mathcal{D}$. But for any such section X we have $\tilde\eta(X)=0$. We thus obtain $d\tilde\eta(X,\tilde\Phi Y)=(1+\eta(\rho))^{-1}g(X,Y)$ which proves the result. Hence, $(\tilde\xi,\tilde\eta,\tilde\Phi,\tilde g)$ is a Sasakian structure which clearly belongs to the same underlying contact structure as (ξ,η,Φ,g).

To prove the last statement, we first notice that $0=[\xi,\rho]=[\xi,f\xi]=\xi f$, so f is constant along the leaves of $\mathcal{F}_\xi$. Next we have for any vector fields X,Y
$$\begin{aligned}0=&(\pounds_{f\xi}g)(X,Y)=f\xi g(X,Y)-g([f\xi,X],Y)-g(X,[f\xi,Y])\\ =&f\big(\xi g(X,Y)-g([\xi,X],Y)-g(X,[\xi,Y])\big)+(Xf)g(\xi,Y)+(Yf)g(\xi,X)\\ =&f(\pounds_\xi g)(X,Y)+(Xf)\eta(Y)+(Yf)\eta(X)=(Xf)\eta(Y)+(Yf)\eta(X).\end{aligned}$$
Choosing X to be horizontal and $Y=\xi$ shows that f must be constant. □

This last type of deformation is a special case which always exists, namely when ρ is a constant times ξ. If we put $\rho=(a-1)\xi$, then we have

$$\tilde\xi=a\xi,\qquad \tilde\eta=\frac{1}{a}\eta,\qquad \tilde\Phi=\Phi,\qquad \tilde g=a^{-1}d\eta\circ(\mathbb{I}\otimes\Phi)+a^{-2}\eta\otimes\eta. \tag{5.4.3}$$

Since this transformation rescales the transverse metric, we refer to these deformations as *transverse homotheties.*

An example of a type I deformation is the deformation away from the standard sphere to the weighted sphere as described in Example **??**. These are obtained by considering the maximal torus in the isometry group of the standard round sphere.

EXAMPLE 5.4.4: Consider the standard Sasakian structure $\mathcal{S}_0=(\xi_0,\eta_0,\Phi_0,g_0)$ described in Example 5.3.9. Now the isometry group $\mathfrak{I}(S^{2n+1})$ of g_0 is $O(2n+2,\mathbb{R})$, and the automorphism group $\mathfrak{A}(\xi_0,\eta_0,\Phi_0,g_0)$ of the Sasakian structure is the subgroup of $O(2n+2,\mathbb{R})$ that commutes with ξ_0, namely, $U(n+1)$. The maximal torus $\mathfrak{T}_{n+1}$ of $U(n+1)$ has $n+1$ dimensions. Let $\{e_i\}_{i=0}^n$ denote the standard basis for its Lie algebra $\mathfrak{t}_{n+1}=\mathbb{R}^{n+1}$. Under the action of $\mathfrak{T}_{n+1}$ on S^{2n+1}, the standard basis vectors $\{e_i\}_{i=0}^n$ correspond to the vector fields $\{H_i\}_{i=0}^n$ where
$$H_i=y^i\frac{\partial}{\partial x^i}-x^i\frac{\partial}{\partial y^i},$$
and the Reeb vector field ξ_0 can be written as $\xi_0=\sum_i H_i$. There is a map, called the **moment map**[3] $\mu:S^{2n+1}\longrightarrow\mathfrak{t}^*_{n+1}$ defined by $<\mu,\tau>=\eta_0(X^\tau)$ where X^τ is the vector field on S^{2n+1} corresponding to $\tau\in\mathfrak{t}_{n+1}$, and $\mathfrak{t}^*_{n+1}$ denotes the dual vector space to $\mathfrak{t}_{n+1}$. Let ζ_i denote the coordinates of $\mathfrak{t}^*_{n+1}$ with respect to the dual basis $\{e^*_i\}_{i=0}^n$. Then $\zeta_i=<\mu,e_i>=\eta_0(H_i)=(x^i)^2+(y^i)^2$, and the image of the moment map μ satisfies the two conditions

$$0\leq\zeta_i\leq 1,\qquad \sum_{i=0}^n\zeta_i=\sum_i\big((x^i)^2+(y^i)^2\big)=1. \tag{5.4.4}$$

[3]We will consider moment maps in much more detail in Chapter ?

Thus, the coordinates ζ_i are just the barycentric coordinates for the standard simplex[4] is $\mathbb{R}^{n+1}$.

We now use Theorem 5.4.3 to construct an $n+1$-dimensional family of distinct Sasakian structures on S^{2n+1}. Let $\mathbf{w}=(w_0,\cdots,w_n)$ be any positive vector in $\mathfrak{t}_{n+1}=\mathbb{R}^{n+1}$, that is $w_i>0$ for all $i=0,\cdots,n$. The vector field correponding to $\mathbf{w}$ is denoted by $\xi_{\mathbf{w}}$, and the corresponding 1-form by $\eta_{\mathbf{w}}$. We obtain a deformed Sasakian structure $(\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$ defined by

$$\xi_{\mathbf{w}}=\sum_{i=0}^n w_iH_i, \qquad \eta_{\mathbf{w}}=\frac{\eta}{\sum_{i=0}^n w_i((x^i)^2+(y^i)^2)}, \tag{5.4.5}$$

where $\Phi_{\mathbf{w}}$ and $g_{\mathbf{w}}$ are determined by Equations 5.4.1 and 5.4.2. Notice that

$$\eta(\xi_{\mathbf{w}})=1+\eta(\rho_{\mathbf{w}})=\sum_{i=0}^n w_i((x^i)^2+(y^i)^2)>0.$$

So $(\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$ is indeed a Sasakian structure as long as all the components of $\mathbf{w}$ are positive. We shall refer to $(\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$ as the **weighted Sasakian structure** on S^{2n+1}, and we denote the unit with this Sasakian structure by $S^{2n+1}_{\mathbf{w}}$.

We now have a new moment map corresponding to the deformed strict contact structure, namely $\eta_{\mathbf{w}}(H_i)$, and so we obtain a deformed simplex as well,

$$0\leq\zeta_i\leq\frac{1}{w_i}, \qquad \sum_{i=0}^n w_i\zeta_i=1. \tag{5.4.6}$$

The deformed structures $(\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$, however, are not all distinct. The Weyl group $\mathcal{W}(SU(n+1))=\mathrm{Nor}(\mathfrak{T}_{n+1})/\mathfrak{T}_{n+1}$ acts as outer automorphisms on the maximal torus, and it is well known that $\mathcal{W}(SU(n+1))=\Sigma_{n+1}$, the permutation group on $n+1$ letters. So for any $\sigma\in\Sigma_{n+1}$ the Sasakian structures $(\xi_{\mathbf{w}},\eta_{\mathbf{w}},\Phi_{\mathbf{w}},g_{\mathbf{w}})$ and $(\xi_{\sigma(\mathbf{w})},\eta_{\sigma(\mathbf{w})},\Phi_{\sigma(\mathbf{w})},g_{\sigma(\mathbf{w})})$ are isomorphic. Thus, we shall frequently take the weights to be ordered such that $w_0\leq w_1\leq\cdots\leq w_n$. It is important to note that all the Sasakian structures $\mathcal{S}_{\mathbf{w}}$ belong to the same underlying contact structure on S^{2n+1}, namely the standard one. Later we shall discover Sasakian structures on S^{2n+1} belonging to distinct, nonstandard contact structures.

The Sasakian structures in this example also reside on lens spaces since they are constructed from a cyclic subgroup of the automorphism group $\mathfrak{A}$ of the Sasakian structure.

Example 5.4.5 (Lens spaces): Let $\mathcal{S}_{\mathbf{w}}$ denote the weighted Sasakian structure of weight vector $\mathbf{w}\in(\mathbb{R}^+)^{n+1}$ and $\mathcal{S}_0$ the standard Sasakian structure on S^{2n+1}. Also let C_p denote the cyclic subgroup of the maximal torus $\mathfrak{T}_{n+1}$ generated by the action

$$(z_0,\cdots,z_n)\mapsto(\zeta z_0,\zeta^{q_1}z_1,\cdots,\zeta^{q_n}z_n)$$

where ζ is a primitive pth root of unity, and is relatively prime to the q_i's. The quotient of S^{2n+1} by this action is a smooth manifold known as the lens space $L(p;q_1,\cdots,q_n)$. Since $\mathfrak{T}_{n+1}\subset\mathfrak{A}(\mathcal{S}_{\mathbf{w}})$, the Sasakian structures $\mathcal{S}_{\mathbf{w}}$ as well as $\mathcal{S}_0$ all pass to the quotient manifold $L(p;q_1,\cdots,q_n)$. Notice that if we choose $\mathbf{w}=$

[4] We shall see in Chapter ? that this result generalizes to any compact contact manifold M^{2n+1} which admits the action of an $n+1$ dimensional torus such that the Reeb vector field is an element of the Lie algebra $\mathfrak{t}_{n+1}$ of the torus. The result says that the image of the moment map of a compact toric contact manifold of Reeb type is a convex polytope lying in its characteristic hyperplane.

$(1, q_1, \cdots, q_n)$ the cyclic subgroup $C_p \subset \mathfrak{T}_{n+1}$ is actually a subgroup of the circle group $S^1_{\mathbf{w}}$ generated by the Reeb vector field $\xi_{\mathbf{w}}$. Thus, in the case the Sasakian structure $\mathcal{S}_{\mathbf{w}}$ on $L(p; q_1, \cdots, q_n)$ is the one obtained from Proposition 5.4.2. As in Example 5.4.4 all the Sasakian structures $\mathcal{S}_{\mathbf{w}}$ belong to the same underlying contact structure on $L(p; q_1, \cdots, q_n)$.

Much is known about the topology of lens spaces especially in dimension 3 (i.e. $n = 1$). For example it is well known (cf. [**53**]) that $L(p, q)$ is homotopy equivalent to $L(p, q')$ if and only if either qq' or $-qq'$ is a quadratic residue mod p. There is a theorem of Brody [**55**] (see also [**53**]) that is perhaps less well known. It says that $L(p, q)$ and $L(p, q')$ are homeomorphic if and only if either $q' \equiv \pm q \mod p$ or $qq' \equiv \pm 1 \mod p$.

Type I deformations have an important application, namely in the proof of the Sasakian analogue of Kodaira's conjecture mentioned earlier. The Sasakian case, however, manifests itself somewhat differently than the complex case in that type I deformations do not change the underlying CR-structure.

Theorem 5.4.6: *On a compact manifold every Sasakian structure (ξ, η, Φ, g) is a type I deformation of a quasi-regular (algebraic) Sasakian structure.*

PROOF. If (ξ, η, Φ, g) is quasi-regular there is nothing to do, so let (ξ, η, Φ, g) be a Sasakian structure of rank $k > 1$. By Proposition 5.1.8 the leaf closure $\bar{\mathcal{F}}_\xi$ of the characteristic foliation $\mathcal{F}_\xi$ is a singular foliation whose generic leaves are tori of dimension k. By the approximation theorem **??** there is a sequence of quasi-regular Sasakian structures $\{(\xi_i, \eta_i, \Phi_i, g_i)\}$ converging to (ξ, η, Φ, g) such that for each i the vector field ξ_i is a section of the coherent sheaf of commuting vector fields $\mathfrak{t} \subset \mathfrak{a}(\xi, \eta, \Phi, g)$ generating the singular foliation $\bar{\mathcal{F}}_\xi$. Fix an i. Since $\mathfrak{t}$ lies in the center of $\mathfrak{a}(\xi, \eta, \Phi, g)$, the vector field $\rho = \xi - \xi_i$ is a Killing vector field for g which commutes with ξ. Thus, we can apply Theorem 5.4.3 to the quasi-regular Sasakian structure $(\xi_i, \eta_i, \Phi_i, g_i)$ proving the result. $\square$

The following result is self-evident.

Proposition 5.4.7: *Let (ξ, η, Φ, g) be a Sasakian structure on M. If $\mathfrak{a}(\xi, \eta, \Phi, g)$ is one-dimensional then every type I deformation of (ξ, η, Φ, g) is a transverse homothety.*

Deformations of Type II:

Now fix a Sasakian structure (ξ, η, Φ, g) on M^{2n+1} and consider the G-structure with group $GL(1, 2n; \mathbb{R})$ of Example 1.3.10 which stabilizes the characteristic foliation $\mathcal{F}_\xi$. We wish to deform (ξ, η, Φ, g) through Sasakian structures that have the same fundamental basic cohomology class up to a scale. As in Kähler geometry this deformation space is infinite dimensional. We consider a deformation of the structure (ξ, η, Φ, g) by adding to η a continuous one parameter family of 1-forms ζ_t that are basic with respect to the characteristic foliation. We require that the 1-form $\eta_t = \eta + \zeta_t$ satisfy the conditions

$$\eta_0 = \eta, \qquad \zeta_0 = 0, \qquad \eta_t \wedge (d\eta_t)^n \neq 0 \quad \forall\, t \in [0,1]. \tag{5.4.7}$$

This last nondegeneracy condition implies that η_t is a contact form on M for all $t \in [0, 1]$ which by Gray's Stability Theorem **??** belongs to the same underlying contact structure as η. Moreover, since ζ_t is basic ξ is the Reeb (characteristic) vector field associated to η_t for all t. Now let us define

$$\Phi_t = \Phi - \xi \otimes \zeta_t \circ \Phi \tag{5.4.8}$$

$$g_t = d\eta_t \circ (\Phi \otimes \mathbb{I}_t) + \eta_t \otimes \eta_t. \tag{5.4.9}$$

In [BG3] it was proved that for all $t \in [0,1]$ and every basic 1-form ζ_t such that $d\zeta_t$ is of type $(1,1)$ and such that 5.4.7 holds $(\xi, \eta_t, \Phi_t, g_t)$ defines a continuous 1-parameter family of Sasakian structures on M belonging to the same underlying contact structure as η. Given a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on a manifold M, we define $\mathfrak{F}(\xi)$ to be the family of all Sasakian structures obtained by the deformations above.

Theorem 5.4.8: *Let (M, ξ, η, Φ, g) be a Sasakian manifold. Then for all $t \in [0,1]$ and every basic 1-form ζ_t such that $d\zeta_t$ is of type $(1,1)$ and such that 5.4.7 holds $(\xi, \eta_t, \Phi_t, g_t)$ defines a Sasakian structure on M belonging to the same underlying contact structure as η.*

PROOF. The conditions of 5.4.7 guarantee that $(\xi, \eta_t, \Phi_t, g_t)$ defines a strict contact structure. We need to check that it is a metric contact structure and that it is normal. To see the g_t is a metric it suffices to check that $d\eta_t \circ (\Phi_t \otimes \mathbb{I})$ is symmetric and remains positive for all $t \in [0,1]$. The symmetry follows from the fact $d\eta_t$ is type $(1,1)$ on the contact bundle $\mathcal{D}_t = \ker \eta_t$ while the positivity follows from the fact that $d\eta_t \circ (\Phi_t \otimes \mathbb{I}) = d\eta_t \circ (\Phi \otimes \mathbb{I})$ and that $d\eta_t$ is nondegenerate. One easily checks the identity $\Phi_t^2 = -\mathrm{id} + \xi \otimes \eta_t$. Thus, the compatibility condition

$$g_t(\Phi_t X, \Phi_t Y) = g_t(X, Y) - \eta_t(X)\eta_t(Y)$$

holds. Next we check normality which amounts to checking two conditions, that the almost CR structure defined by Φ_t on the contact bundle $\mathcal{D}_t$ is integrable, and that ξ is a Killing vector field for the metric g_t. The last condition is equivalent to vanishing of the Lie derivative $\mathcal{L}_\xi \Phi_t$ for all t which follows immediately from equation 5.4.8 and the facts that it holds for $t = 0$ and that ζ_t is basic. Integrability follows from the fact that the almost CR structure defined by Φ on $\mathcal{D}$ is integrable, and that equation 5.4.8 is just the projection of the image of Φ onto $\mathcal{D}_t$. □

5.5. New Sasakian Structures from Old

In this section we describe three methods for obtaining new Sasakian structures from old ones. The first is an invariance condition for submanifolds analogous to the situation in Kähler geometry. The second is a join construction which applies to quasi-regular Sasakian structures, while the third is that of symmetry reduction.

Invariant Submanifolds of Sasakian Manifolds

We begin with a definition about submanifolds of Sasakian manifolds.

Definition 5.5.1: *Let $(M^{2n+1}, \xi, \eta, \Phi, g)$ be a Sasakian manifold. An immersed submanifold N of M is said to be an* **invariant submanifold** *if the two conditions hold:*

(i) *ξ is tangent to N at all points of N,*
(ii) *$\Phi T_pN \subset T_pN$ for all points $p \in N$.*

This leads to an important result.

Theorem 5.5.2: *An invariant submanifold N of a Sasakian manifold $(M^{2n+1}, \xi, \eta, \Phi, g)$ is Sasakian with the Sasakian structure given by the restriction of the tensor fields (ξ, η, Φ, g) to N.*

PROOF. The two conditions of Definition 5.5.1 guarentee that ξ and Φ restricted to well-defined tensor fields on N. Of course, the restriction of the tensor fields η and g to N are always well-defined. Hence, the tensor fields (ξ, η, Φ, g) satisfy all the relations on N that they satisfy on M, in particular the normality condition 3.5.9, $N_\Phi = -2\xi \otimes d\eta$, is satisfied. Thus, N with the structure (ξ, η, Φ, g) is Sasakian. □

This theorem will be of great interest to us in Chapter 6.

The Join Construction

In this subsection we apply a construction due to Wang and Ziller [**?**] to define a multiplication on the set of quasi-regular Sasakian orbifolds. This was done originally in [**35**] in the case of Sasakian-Einstein orbifolds which is perhaps of more interest, but there is an easy generalization to the strict Sasakian case. The idea is quite simple and is based on the fact that product of Kähler orbifolds is a Kähler orbifold in a natural way.

Definition 5.5.3: *We denote by $\mathcal{SO}$ the set of compact quasi-regular Sasakian orbifolds, by $\mathcal{SM}$ the subset of $\mathcal{SO}$ that are smooth manifolds, and by $\mathcal{R} \subset \mathcal{SM}$ the subset of compact, simply connected, regular Sasakian manifolds. The set $\mathcal{SO}$ is topologized with the $C^{m,\alpha}$ topology, and the subsets are given the subspace topology.*

The set $\mathcal{SO}$ is graded by dimension, that is,

$$\mathcal{SO} = \bigoplus_{n=0}^{\infty} \mathcal{SO}_{2n+1},$$

and similarly for $\mathcal{SM}$ and $\mathcal{R}$. In the definition of Sasakian structure it is implicitly assumed that $n > 0$. So we want to extend the definition of a Sasakian structure to the case when $n = 0$. This can easily be done since a connected one dimensional orbifold is just an interval with possible boundary, or a circle. So we can just take $\xi = \frac{\partial}{\partial t}, \eta = dt, \Phi = 0$, with the flat metric $g = dt^2$. In this case the space of leaves $\mathcal{Z}$ of the characteristic foliation is just a point. The unit circle S^1 with this structure will play the role of the identity in our monoid. Notice that the identity is smooth.

CHAPTER 6

Links as Sasakian Manifolds

In this chapter we study the Sasakian geometry on the links of isolated hypersurface singularities defined by weighted homogeneous polynomials. We also develop the theory of links of complete intersection singularities. We begin with the main definitions following Milnor's beautiful classical text [**154**]. Let $(z_0, \cdots, z_n)$ denote the standard complex coordinates in $\mathbb{C}^{n+1}$, and let $f \in \mathbb{C}[z_0, \cdots, z_n]$ be a polynomial that is non-constant with respect to each of the variables z_i, i.e. the partial derivative $\frac{\partial f}{\partial z_i}$ is not everywhere zero for any $i = 0, \cdots, n$. We are interested in the zero locus V_f of f which is a complex hypersurface in $\mathbb{C}^{n+1}$. Recall that a point $\mathbf{z} \in V_f$ is called a *singular point* or *critical point* of V_f if its differential df vanishes at $\mathbf{z}$. Otherwise, $\mathbf{z}$ is called a *regular* point. A critical point $\mathbf{z}$ is said to be *isolated* if there is a neighborhood U of $\mathbf{z}$ such that all the points of $U - \{\mathbf{z}\}$ are regular.

Definition 6.0.4: *Let $\mathbf{z}_0$ be any point $\mathbf{z}_0 \in V_f$. We consider a sphere $S^{2n+1}_\epsilon(\mathbf{z}_0)$ of radius ϵ centered about $\mathbf{z}_0$. Then the intersection $L_f(\epsilon) = V_f \cap S^{2n+1}_\epsilon(\mathbf{z}_0)$ is called a* **link** *of V_f.*

If $\mathbf{z}_0$ is a regular point it follows from the well known Morse Lemma [**153**] that for ϵ sufficiently small, L_f is diffeomorphic to the standard sphere S^{2n-1}_ϵ. We are essentially interested in the case when $\mathbf{z}_0$ is singular. It should be clear that if $\mathbf{z}_0$ is an isolated singular point then for ϵ sufficiently small $L_f(\epsilon)$ is independent of ϵ in the sense that $L_f(\epsilon)$ and $L_f(\epsilon')$ are diffeomorphic for $0 < \epsilon' < \epsilon$. Thus, we shall write $L_f(\epsilon)$ as L_f. When $\mathbf{z}_0$ is singular L_f is often referred to as the *link of a hypersurface singularity.* We shall assume that all variables $z_0, \cdots, z_n$ occur in f for if not we would have a polynomial in one less variable. We also assume that there is no constant term so that the origin lies in V_f, and we usually want the origin to be a singular point of f which means that there are no linear terms in f. Consider $df : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$ the differential of f, and let $\theta(f)$ be the Jacobi ideal in $\mathbb{C}[z_0, \cdots, z_n]$ generated by df. The *multiplicity* $\mu_f(\mathbf{z}_0)$ of f at an isolated singular point $\mathbf{z}_0 \in V_f$ is defined to be the degree of the Gauss map $G : S^{2n+1}_\epsilon \longrightarrow S^{2n+1}$ defined by

$$G(\mathbf{z}) = \frac{df(\mathbf{z})}{||df(\mathbf{z})||}. \tag{6.0.1}$$

If $\mathbf{z}_0$ is a regular point then $\mu_f(\mathbf{z}_0)$ is defined to be 0. Let $\mathbf{z}_0$ be the origin in $\mathbb{C}^{n+1}$ and define the *Milnor algebra* as the quotient algebra

$$M(f) = \frac{\mathbb{C}[z_0, z_1, z_2, z_3]}{\theta(f)}. \tag{6.0.2}$$

Then it can be shown [**154**] that

$$\mu_f = \mu_f(0) = \dim_{\mathbb{C}} M(f). \tag{6.0.3}$$

μ_f is usually called the *Milnor number* of the isolated hypersurface singularity. We shall see later how $M(f)$ encodes much of the geometry of the link L_f.

6.1. Weighted Homogeneous Polynomials

In this section we consider the so-called weighted homogeneous polynomials and show that the links of a hypersurface singularity determined by a weighted homogeneous polynomial have a natural Sasakian structure associated with them.

According to the Structure Theorem 5.1.2, in order that a link L_f admits a Sasakian structure it is necessary that it also admits a locally free circle action. Thus, at least locally the hypersurface V_f admits a locally free $\mathbb{C}^*$ action. This leads us to the study of *weighted homogeneous polynomials*, also called *quasi-homogeneous polynomials*. Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action given by

$$(z_0, \ldots, z_n) \mapsto \lambda \cdot \mathbf{z} = (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n), \tag{6.1.1}$$

where $\lambda \in \mathbb{C}^*$, and the *weights* w_j are positive integers[1].

Definition 6.1.1: *A polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ is said to be* **weighted homogeneous** *of* **degree** *d if there are positive integers $w_0, w_1, \cdots, w_n, d$ such that*

$$f(\lambda \cdot \mathbf{z}) = f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n) = \lambda^d f(\mathbf{z}).$$

If $V \subset \mathbb{C}^{n+1}$ is a variety defined by a collection of weighted homogeneous polynomials $f_1, \cdots, f_p$ with weights $(w_0, \cdots, w_n)$. Then V is invariant under the $\mathbb{C}^*$-action 6.1.1. This statement has a converse due to Orlik and Wagreich [**172**].

Proposition 6.1.2 (Orlik-Wagreich)**:** *Let $V \subset \mathbb{C}^{n+1}$ be an irreducible analytic variety with a $\mathbb{C}^*$-action defined by Equation 6.1.1. Then V is an algebraic variety and the ideal of polynomials vanishing on V is generated by weighted homogeneous polynomials of the form 6.1.1.*

Proof. Let $\mathbb{C}\{z_0, \cdots, z_n\}$ denote the ring of convergent power series. Then for $f \in \mathbb{C}\{z_0, \cdots, z_n\}$ there is a unique polynomial f_i for each i such that

$$f(\lambda \cdot \mathbf{z}) = \sum_{i=0}^{\infty} \lambda^i f_i(\mathbf{z}). \tag{6.1.2}$$

This series converges for λ small enough. If f vanishes on V near the origin o in $\mathbb{C}^{n+1}$, then $\mathbf{z} \in V$ implies that $\sum_i \lambda^i f_i(\mathbf{z}) = 0$ for λ small enough. Hence, $f_i(\mathbf{z}) = 0$ for all i and all $\mathbf{z} \in V$ near o. Let $\mathcal{I}(V)$ denote the ideal of functions in $\mathbb{C}\{z_0, \cdots, z_n\}$ vanishing on V, and let $\mathcal{J}$ be the ideal generated by $\{f_i^{(j)}\}$ where the $f^{(j)}$'s are generators of $\mathcal{I}(V)$. Clearly, $\mathcal{J} \subset \mathcal{I}(V)$. Now the locus of zeroes of $\mathcal{J}$ is V, since if $\mathbf{z} \notin V$ and is within the radius of convergence of $f^{(j)}$ for all j, there must be some $f_i^{(j)}$ with $f_i^{(j)}(\mathbf{z}) \neq 0$. Thus, by Hilbert's Nullstellensatz the radical of $\mathcal{J}$ is $\mathcal{I}(V)$. Let $\mathcal{J}'$ be the ideal generated by $\{f_i^{(j)}\}$ in the polynomial ring

[1] Our convention of calling the positive intergers w_j weights is not entirely standard. It is also common to refer to the rational numbers d/w_i as weights (cf. [**155, 172**]), but our convention is more convenient for our purpose.

$\mathbb{C}[z_0, \cdots, z_n]$, and let $\mathcal{I}' = \sqrt{\mathcal{J}'}$. Then $\mathcal{I}'\mathbb{C}[z_0, \cdots, z_n] = \sqrt{\mathcal{J}} = \mathcal{I}(V)$. So $\mathcal{I}(V)$ is generated by polynomials, and the algebraic variety defined by $\mathcal{I}(V)$ is V. Now if f is polynomial in $\mathcal{I}(V)$ then there is only a finite number of i's in Equation 6.1.2 with f_i different from zero. Since the $f^{(j)}$'s are generators of $\mathcal{I}(V)$, the weighted homogeneous polynomials $f_i^{(j)}$ also generate $\mathcal{I}(V)$. □

Notice that for a given $(n+1)$-tuple of weights $(w_0, \cdots, w_n)$ and degree d there may be no such polynomial f. So the existence of a weighted homogeneous polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ puts constraints on the set of positive integers $\{w_0, \cdots, w_n; d\}$. Also as mentioned above, in the hypersurface case we shall assume that when a weighted homogeneous polynomial f does exist that it is *nondegenerate* in the sense that each variable occurs. In this treatise we shall only be interested in the case where either f is regular at the origin o or when o is an isolated singular point. Notice that f will be regular at o if and only if some variable, say z_n, occurs as a linear monomial in f, and this occurs if and only if $w_n = d$. When this happens we can remove the variable z_n and consider the reduced weighted homogeneous polynomial $g \in \mathbb{C}[z_0, \cdots, z_{n-1}]$. Hence, unless otherwise stated we shall always assume that no linear terms occur in f or correspondingly that no w_i equals d. So unless otherwise stated all weighted homogeneous polynomials will have a singular point at the origin. Below we give conditions on the integers $\{w_0, \cdots, w_n; d\}$ such that the hypersurface V_f has an isolated singularity at the origin. First we mention a characterization of weighted homogeneous polynomials in the case of isolated hypersurface singularities due to Saito cf. ([Loo] pg 4).

Proposition 6.1.3 (Saito)**:** *Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a holomorphic map which defines an isolated hypersurface singularity, and let $\theta(f)$ denote the Jacobi ideal generated by the differential df. Then $f \in \theta(f)$ if and only if f is a weighted homogeneous polynomial.*

It will be convenient to view the weights as the components of a vector $\mathbf{w} \in (\mathbb{Z}^+)^{n+1}$, and we shall assume that they are ordered $w_0 \leq w_1 \leq \cdots \leq w_n$ and that $\gcd(w_0, \ldots, w_n) = 1$ unless otherwise stated. The zero set V_f of a nondegenerate weighted homogeneous polynomial f, is just the *weighted affine cone* C_f. If the origin is an isolated singularity in C_f, then it is the only singularity of C_f. Thus, the we can take $\epsilon = 1$ in which case $S^{2n+1}_\epsilon(\mathbf{z}_0)$ becomes the unit sphere S^{2n+1} centered at the origin as in Example 5.3.9. In this case our link

$$L_f = C_f \cap S^{2n+1}. \tag{6.1.3}$$

is a smooth manifold of dimension $2n-1$ which by the Milnor Fibration Theorem 6.2.1 is $(n-2)$-connected. Compact $(n-2)$-connected manifolds M of dimension $2n-1$ have been called *highly connected* by Wall [**212**] since their only non-vanishing homology groups $H_i(M, \mathbb{Z})$ occur for $i = 0, n-1, n, 2n-1$. Wall [**212**] and his students [**15, 213**] have given a rough classification of these highly connected manifolds which will be of much interest to us later. In particular, for $n = 3$ there is a complete classification [**15**].

We wish to find conditions on the weight vector $\mathbf{w}$ that assures us that C_f has an isolated singularity at the origin. The following result is due to Iano-Fletcher

Theorem 6.1.4 (Iano-Fletcher)**:** *Let $f \in \mathbb{C}[z_0, \cdots, z_n]$ be a weighted homogeneous polynomial of degree d and weight vector $\mathbf{w} = (w_0, \cdots, w_n)$. Suppose also* *that no weight w_i equals d, so the origin is a singular point of f. Then the origin is an isolated singular point of f if and only if for every nonempty subset $I = \{i_0, \cdots, i_k\} \subset \{0, \cdots, n\}$ either*

(i) *There is a monomial $z_I^M = z_{i_0}^{m_0} \cdots z_{i_k}^{m_k}$ of degree d, or*
(ii) *for $a = 1, \cdots, k+1$ there are monomials $z_I^{M_a} z_{e_a} = z_{i_0}^{m_{0,a}} \cdots z_{i_k}^{m_{k,a}} z_{e_a}$ of degree d, where $\{e_a\}$ run over $k+1$ distinct elements.*

PROOF. Give later. □

We now come to the important concept of well-formedness. When this condition does not hold certain pathologies exist. –(Maybe this should appear in the orbifold section)–

Definition 6.1.5: *We say that the weight vector $\mathbf{w} = (w_0, \cdots, w_n)$ is* **well-formed** *if for any triple $i, j, k \neq$, we have $\gcd(w_i, w_j, w_k) = 1$. We say that the link L_f is* **well-formed** *if $\mathbf{w}$ is well-formed and the exceptional locus has codimension at least four.*

The circle subgroup $S^1 \subset \mathbb{C}^*$ acts on the affine space by taking $\lambda \in S^1$ in Definition 6.1.1. This gives the weighted circle action on the unit sphere S^{2n+1} generated by the vector field $\xi_{\mathbf{w}}$ of Equation 5.4.5. As discussed in Example 5.3.9 this is part of a Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ on S^{2n+1}. We now have

Proposition 6.1.6: *Let f be a nondegenerate weighted homogeneous polynomial of degree d and weight vector $\mathbf{w}$. The Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ on S^{2n+1} induces by restriction a Sasakian structure, also denoted by $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$, on the link L_f.*

PROOF. According to Theorem 5.5.2 we only need to check that for all points $p \in L_f$ the two conditions hold:

(i) $\xi_{\mathbf{w}}$ is tangent to L_f.
(ii) $\Phi_{\mathbf{w}} T_p L_f \subset T_p L_f$.

To see that (1) holds, we differentiate the defining equation in Definition 6.1.1 with respect to λ,

$$\xi_{\mathbf{w}} f = \lambda \frac{df}{d\lambda} = d\lambda^d f.$$

So $\xi_{\mathbf{w}} f$ vanishes at points of C_f, hence, at points of L_f. This proves (1).

To prove (2) we first notice that since C_f is defined by a holomorphic equation we have $ITC_f \subset TC_f$ where I is the standard almost complex structure on $\mathbb{C}^{n+1}$. Moreover, we have

$$\Phi_{\mathbf{w}}|_{\mathcal{D}_{L_f}} = \Phi|_{\mathcal{D}_{L_f}} = I|_{\mathcal{D}_{L_f}}$$

where $\mathcal{D}_{L_f} = \ker \eta_{L_f}$ and to avoid confusion we let η_{L_f} denote the restriction of $\eta_{\mathbf{w}}$ to L_f. Now at a point $p \in L_f$ we have $(\mathcal{D}_{L_f})_p = \mathcal{D}_p \cap T_p C_f$. Since $\Phi_{\mathbf{w}} \xi_{\mathbf{w}} = 0$, we need only check that $\Phi_{\mathbf{w}} \mathcal{D}_{L_f} \subset \mathcal{D}_{L_f}$. But $\Phi_{\mathbf{w}}$ leaves $\mathcal{D}$ invariant, and equals I on $\mathcal{D}_{L_f}$ which leaves TC_f invariant, so it leaves $\mathcal{D}_{L_f}$ invariant. □

Recall from Example 5.3.9 the unit sphere $S^{2n+1}_{\mathbf{w}}$ with the weighted Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$, but now with $\mathbf{w} \in (\mathbb{Z}^+)^{n+1} \subset (\mathbb{R}^+)^{n+1}$. Thus, Proposition 6.1.6 says that the natural inclusion $\iota : L_f \longrightarrow S^{2n+1}_{\mathbf{w}}$ is a Sasakian embedding. It is also clear since the vector $\mathbf{w}$ lies in the positive integer lattice in $\mathbb{R}^{n+1}$ that the Sasakian structures on both L_f and $S^{2n+1}_{\mathbf{w}}$ are quasi-regular. Thus, by Theorem **??** the spaces of leaves in both cases are compact polarized Hodge orbifolds. We denote the space of leaves of L_f by $\mathcal{Z}_f$, and almost by definition the space of leaves of $S^{2n+1}_{\mathbf{w}}$ is the weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \cdots, w_n)$. Thus, we have a commutative diagram

$$
\begin{array}{ccc}
L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\
\mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}),
\end{array} \tag{6.1.4}
$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 V-bundles and orbifold Riemannian submersions.

6.2. The Milnor Fibration and the Topology of Links

We now consider the well known Fibration Theorem of Milnor [**154**].

Theorem 6.2.1 (Milnor)**:** *Let $\mathbf{z}_0 \in V_f$ be any point of the complex hypersurface V_f, and let $\epsilon > 0$ be sufficiently small. Then the map $\phi : S^{2n+1}_\epsilon(\mathbf{z}_0) - L_f \longrightarrow S^1$ defined by*

$$\phi(\mathbf{z}) = \frac{f(\mathbf{z})}{|f(\mathbf{z})|}$$

is the projection map of a smooth fibre bundle with a smooth parallelizable fibre. If $\mathbf{z}_0$ is an isolated singular point of f, then each fibre F has the homotopy type of a bouquet of n-spheres $S^n \vee \cdots \vee S^n$, and is homotopy equivalent to its closure $\bar{F}$ which is a compact manifold with boundary where the common boundary $\partial \bar{F}$ is precisely L_f. Furthermore, L_f is a smooth $(n-2)$-connected manifold of dimension $2n-1$.

The proof of this well known theorem uses Morse Theory [**153**] and can be found in [**154**]. The number of S^n's in the bouquet described in Theorem 6.2.1 is an important invariant of the link called the *Milnor number* of the link, and is denoted by μ. If the point $\mathbf{z}_0$ is regular, it follows from the well known Morse Lemma [**153**] that $\mu = 0$, and the link L_f is diffeomorphic to the $(2n-1)$-sphere S^{2n-1}.

Using the Wang sequence of the Milnor fibration together with Alexander-Poincare duality gives the exact sequence [**154**]

$$0 \longrightarrow H_n(L_f, \mathbb{Z}) \longrightarrow H_n(F, \mathbb{Z}) \xrightarrow{\mathbb{I}-h_*} H_n(F, \mathbb{Z}) \longrightarrow H_{n-1}(L_f, \mathbb{Z}) \longrightarrow 0, \tag{6.2.1}$$

where h_*, known as the *monodromy* map (or characteristic map), comes from the action of a generator of $\pi_1(S^1)$ on the homology of the fibre. From this we see that $H_n(L_f, \mathbb{Z}) = \ker(\mathbb{I} - h_*)$ is a free Abelian group, and $H_{n-1}(L_f, \mathbb{Z}) = \mathrm{coker}(\mathbb{I} - h_*)$ which in general has torsion, but whose free part equals $\ker(\mathbb{I} - h_*)$. So the topology of L_f is encoded in the monodromy map h_*. The characteristic polynomial $\Delta(t)$ of the monodromy map is another important link invariant. It is a generalization of the Alexander polynomial of knot theory, and has also been called the *Alexander polynomial* [**?, 111**] of L_f. We can get some immediate information from the exact sequence 6.2.1. If $\Delta(1) \neq 0$ then $\mathbb{I} - h_*$ is non-singular which implies $H_n(L_f, \mathbb{Z}) = 0$, that is L_f is a rational homology sphere. Furthermore, if $|\Delta(1)| = 1$, then $\mathbb{I} - h_*$ is an isomorphism, so L_f is a homology sphere, and if $n > 2$ it must be a homotopy sphere by the well known Hurewicz Theorem.

We now briefly consider the case $n = 2$ in more detail. Let us take $\mathbf{z}_0$ to be the origin in $\mathbb{C}^3$. If the origin is at worst an isolated singularity of f, all that we know is that L_f is a smooth connected 3-manifold. But a well known Theorem of Mumford [**?**, **?**] implies more.

Theorem 6.2.2 (Mumford)**:** *If the origin is at worst a normal singularity of f then L_f is simply connected if and only if L_f is diffeomorphic to S^3.*

In fact in the simply connected case the origin is a regular point of f. Thus, hypersurface singularities can say nothing about the Poincaré conjecture. Nevertheless, they play an important role in understanding homology spheres as we shall see in Section ??.

Links of Weighted Homogeneous Polynomials

In the case that f is a weighted homogeneous polynomial (see section 6.1 below), there is a well-known algorithm due to Milnor and Orlik [**155**] for computing the free part of $H_{n-1}(L_f, \mathbb{Z})$ in terms of the characteristic polynomial $\Delta(t) = \det(t\mathbb{I} - h_*)$, namely the Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$.

Now we shall assume that f is a quasi-smooth weighted homogeneous polynomial. In this case the Milnor number μ_f can be calculated explicitly in terms of the degree d and weights $\mathbf{w} = (w_0, \ldots, w_n)$. This result is due to Milnor and Orlik [**155**].

Proposition 6.2.3 (Milnor-Orlik)**:** *Let f be a weighted homogeneous polynomial of degree d and weight vector $\mathbf{w}$ having an isolated singularity at the origin. Then*

$$\mu = \mu(L_f) = \prod_{i=0}^{n} (\frac{d}{w_i} - 1). \tag{6.2.2}$$

Proof. Consider the map $g : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$ defined by

$$g(z_0, \cdots, z_n) = (z_0^{w_0}, \cdots, z_n^{w_n}).$$

Then $f \circ g$ is homogeneous of degree d and the ith component of the differential $df \circ g$ is homogeneous of degree $d - w_i$. So the degree of $df \circ g$ at the origin is $(d - w_0) \cdots (d - w_n)$. But the degree of df at the origin, which by definition is the multiplicity μ_f, is obtained by dividing by the degree of g at the origin, giving the formula. $\square$

Remark 6.2.1: It is remarkable that for isolated hypersurface singularities the product in formula 6.2.2 is always an integer although the ratios d/w_i may not be integers. This is not necessarily the case if the singularity is not isolated.

The closure $\bar{F}$ of F is a manifold with boundary that is homotopy equivalent to F, and whose boundary is precisely the link L_f. The topology of L_f is determined by the *monodromy map* induced by the $S^1_{\mathbf{w}}$ action. In the case of isolated hypersurface singularities determined by weighted homogeneous polynomials this has a particularly nice form.

Proposition 6.2.4 (Milnor)**:** *Let f be a weighted homogeneous polynomial of an isolated hypersurface singularity. Then the Milnor fibre F is diffeomorphic to the* *non-singular hypersurface*

$$\{\mathbf{z} \in \mathbb{C}^{n+1} \mid f(\mathbf{z}) = 1\},$$

and the monodromy map h_* *is induced by the map* $h : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$ *defined by*

$$h(z_0, \cdots, z_n) = (e^{\frac{2\pi w_0 i}{d}} z_0, \cdots, e^{\frac{2\pi w_n i}{d}} z_n).$$

Notice that h is periodic of period d. Thus, the monodromy map h_* is diagonalizable over $\mathbb{C}$ with eigenvalues d roots of unity. Identifying F with $f^{-1}(1)$ we let $h^j : F \longrightarrow F$ denote the j-fold iterate of h, and F^j denote the fixed point subset of F under h^j. The F^j's are determined by

$$F^j = \{\mathbf{z} \in f^{-1}(1) \mid e^{\frac{2\pi i j w_l}{d}} z_l = z_l \text{ for all } l = 0, \cdots, n\} \tag{6.2.3}$$

By computing the Euler number χ_j of the fixed point sets F^j, Milnor [**154**] was able to compute the Alexander polynomial $\Delta(t)$ of the link L_f using the Weil zeta function

$$\zeta(t) = \exp\big(\sum_{j=1}^{\infty} \frac{\chi_j}{j} t^j\big). \tag{6.2.4}$$

The zeta function 6.2.4 and the Euler numbers χ_j are related by integers r_i through

$$\chi_j = \sum_{l|j} l r_l, \qquad \zeta(t) = \prod_{j|d} (1 - t^j)^{-r_j}. \tag{6.2.5}$$

Then we have

Theorem 6.2.5 (Milnor)**:** *The Alexander polynomial* $\Delta(t)$ *of the link* L_f *of an isolated singularity of a weighted homogeneous polynomial* f *is determined by*

$$\Delta(t)^{(-1)^{n+1}} = (t-1) \prod_{j|d} (t^j - 1)^{-r_j}.$$

See [**154**] for the proof of this theorem. Milnor and Orlik [**155**] use these facts to give an algorithm for computing the Alexander polynomial polynomial $\Delta(t)$ of the monodromy map in terms of the degree d and weight vector $\mathbf{w}$. The procedure is this. Associate to any monic polynomial f with roots $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^*$ its divisor

$$\operatorname{div} f = <\alpha_1> + \cdots + <\alpha_k> \tag{6.2.6}$$

as an element of the integral ring $\mathbb{Z}[\mathbb{C}^*]$ and let $\Lambda_n = \operatorname{div}(t^n - 1)$. Then the Λ's satisfy the multiplication rule

$$\Lambda_a \Lambda_b = \gcd(a, b) \Lambda_{\operatorname{lcm}(a,b)}. \tag{6.2.7}$$

Next we write the 'rational weights' $\frac{d}{w_i}$ of [**155**] in irreducible form, viz.

$$\frac{d}{w_i} = \frac{u_i}{v_i}. \tag{6.2.8}$$

We are now ready for

Theorem 6.2.6 (Milnor-Orlik)**:** *The divisor of the Alexander polynomial* $\Delta(t)$ *is given by*

$$\operatorname{div} \Delta = \prod_i \big(\frac{\Lambda_{u_i}}{v_i} - 1\big) = (-1)^{n+1}\big(1 - \sum r_j \Lambda_j\big).$$

PROOF. From Theorem 6.2.5 we have

$$\operatorname{div}\,\Delta = (-1)^{n+1}\big(1-\sum r_j\Lambda_j\big).$$

Since the r_j's are obtained from the Euler numbers χ_j of the fixed point sets F^j, we need to express χ_j in terms of d and the w_i's, or alternatively in terms of the u_i's and v_i's. By Equation 6.2.3 F^j is given by the vanishing of certain coordinates z_i. Then by permuting the variables z_i if necessary we see that if

$$j\equiv 0 \mod u_i \text{ for } i\leq k, \qquad j\not\equiv 0 \mod u_i \text{ for } i>k \tag{6.2.9}$$

then

$$F^j=\{\mathbf{z}\in\mathbb{C}^{n+1}\mid f(z_0,\cdots,z_k,0,\cdots,0)=1\}.$$

The polynomial $f_k(z_0,\cdots,z_k)=f(z_0,\cdots,z_k,0,\cdots,0)$ is weighted homogeneous with an isolated singularity at the origin in $\mathbb{C}^{k+1}$. Thus, F^j is the Milnor fibre of f_k. So its only nonvanishing reduced homology group is $\tilde{H}_k(F^j,\mathbb{Z})\approx Z^{\mu_j}$ where by Proposition 6.2.3

$$\mu_j=\prod_{i=0}^{k}\big(\frac{d}{w_i}-1\big).$$

Thus, the Euler number of F^j is

$$\chi_j=1+(-1)^k\mu_j=1-\prod_{i=0}^{k}\big(1-\frac{d}{w_i}\big)=1-\prod_{i=0}^{k}\big(1-\frac{u_i}{v_i}\big).$$

Notice from Equation 6.2.9 that we can rewrite this equation as

$$1-\chi_j=\prod_{u_i|j}\big(1-\frac{u_i}{v_i}\big). \tag{6.2.10}$$

This determines the χ_j, and hence, the r_j in terms of the w_i's. To prove the second equality we define for any integer m between 0 and n,

$$\delta(w_0,\cdots,w_m)=(-1)^{m+1}\big(1-\sum r_j(w_0,\cdots,w_m)\Lambda_j\big). \tag{6.2.11}$$

Viewing this as an element of the rational group ring $\mathbb{Q}[\mathbb{C}^*]$, we have

Lemma 6.2.7: *The identity holds:*

$$\delta(w_0,\cdots,w_p)=\delta(w_0,\cdots,w_m)\delta(w_{m+1},\cdots,w_p).$$

PROOF. Set $s_l=lr_l$, and let χ_l,s_l be the sequences associated to the weights $w_0,\cdots,w_m$ through Equations 6.2.5 and 6.2.10. Likewise let χ'_l,s'_l be associated to the weights $w_{m+1},\cdots,w_p$, and χ''_l,s''_l be associated to the composite p-tuple $w_0,\cdots,w_p$. Then from Equation 6.2.10 we find

$$\chi''_j=\chi_j+\chi'_j-\chi_j\chi'_j.$$

Substituting $\chi_j=\sum_{i|j}s_i$ and similar expresses for the primed and double primed variables gives

$$\sum_{l|j}s''_l=\sum_{l|j}s_l+\sum_{l|j}s'_l-\sum_{k|j}\sum_{l|j}s_ks'_l.$$

Using this equation an induction argument gives

$$s''_j=s_j+s'_j-\sum_{\mathrm{lcm}(k,l)=j}s_ks'_l. \tag{6.2.12}$$

Now we write the s_j's in terms of the r_j's, and multiply Equation 6.2.12 by Λ_j and sum over j. This gives

$$\sum r''_j \Lambda_j = \sum r_j \Lambda_j + \sum r'_j \Lambda_j - (\sum r_k \Lambda_k)(\sum r'_l \Lambda_l). \tag{6.2.13}$$

Here we have used Equation 6.2.7. The lemma now follows easily from Equation 6.2.13. □

We need one more lemma to finish the proof of the theorem. Namely,

Lemma 6.2.8:

$$\delta(w_0, \cdots, w_n) = \prod_{i=0}^{n} (\frac{\Lambda_{u_i}}{v_i} - 1).$$

Proof. The proof is by induction on n. For $n = 0$ we notice from Equation 6.2.10 that

$$\chi_j(w_0) = \begin{cases} \frac{u_0}{v_0}, & \text{if } u_0 | j; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, s_j satisfies

$$s_j(w_0) = \begin{cases} \frac{u_0}{v_0}, & \text{if } j = u_0; \\ 0, & \text{if } j \neq u_0. \end{cases}$$

This shows that

$$\delta(w_0) = -(1 - r_j(w_0)\Lambda_j) = -(1 - \frac{\Lambda_{u_0}}{v_0}) = \frac{\Lambda_{u_0}}{v_0} - 1,$$

which proves the result for $n = 0$. Now an easy induction using Lemma 6.2.7 proves the result. □

This completes the proof of Theorem 6.2.6. □

The Milnor-Orlik algorythm computes the numbers r_j in terms of $\mathbf{w}$ and d, or alternatively, in terms of u_i's and v_i's. In fact, we have

Corollary 6.2.9:

$$\text{div } \Delta = \sum (-1)^{n+1-s} \frac{u_{i_1} \cdots u_{i_s}}{v_{i_1} \cdots v_{i_s} \text{lcm}(u_{i_1}, \cdots, u_{i_s})} \Lambda_{\text{lcm}(u_{i_1}, \cdots, u_{i_s})},$$

where the sum is taken over all the 2^{n+1} *subsets* $\{i_1, \cdots, i_s\}$ *of* $\{0, \cdots, n\}$.

According to the exact sequence 6.2.1 the $(n-1)$st and nth Betti numbers $b_n(L_f) = b_{n-1}(L_f)$ of the link equals the number of factors of $(t-1)$ in the Alexander polynomial $\Delta(t)$, or equivalently the order of vanishing of $\Delta(t)$ at $t = 1$. Thus, using Corollary 6.2.9 we get

Corollary 6.2.10: *Let* L_f *be the link of an isolated hypersurface singularity of a weighted homogeneous polynomial* f *of degree* d *and weight vector* $\mathbf{w}$, *and let* $\Delta(t)$ *be its Alexander polynomial. Then*

$$b_{n-1}(L_f) = \sum (-1)^{n+1-s} \frac{u_{i_1} \cdots u_{i_s}}{v_{i_1} \cdots v_{i_s} \text{lcm}(u_{i_1}, \cdots, u_{i_s})}, \tag{6.2.14}$$

where the sum is taken over all the 2^{n+1} *subsets* $\{i_1, \cdots, i_s\}$ *of* $\{0, \cdots, n\}$.

This general form for b_{n-1} is suitable for computer computations.

Working with the Alexander polynomial, the characteristic polynomial of the monodromy map, entails working over $\mathbb{C}$, that is studying the map

$$(t\mathbb{I}-h_*):H_n(F,\mathbb{C})\longrightarrow H_n(F,\mathbb{C}).$$

An alternative procedure to the one taken above would be to diagonalize $(t\mathbb{I}-h_*)$, that is find unimodular matrices $U(t)$ and $V(t)$ such that

$$U(t)(t\mathbb{I}-h_*)V(t)=\mathrm{diag}(m_1(t),\cdots,m_\mu(t)). \tag{6.2.15}$$

The minimal polynomial $m_\mu(t)$ contains the same information as the characteristic polynomial $\Delta(t)$. In order to handle torsion, however, one needs to work with the integral polynomial ring $\mathbb{Z}[t]$ which is not a principal ideal domain. This poses several difficulties which have still not been entirely overcome. Nevertheless, there is a conjecture due to Orlik [**171**] saying that the matrices $U(t)$ and $V(t)$ can be taken with values in the ring $\mathbb{Z}[t]$. This implies

Conjecture 6.2.11 (Orlik)**:** *$H_{n-1}(L_f,\mathbb{Z})=\mathbb{Z}_{m_1(1)}\oplus\cdots\oplus\mathbb{Z}_{m_\mu(1)}$ where $\mathbb{Z}_1$ and $\mathbb{Z}_0$ are the trivial and infinite cyclic groups, respectively.*

It is a straightforward though a bit tedious [**171**] to express the integers $m_{j(1)}$ in terms of the u_i's and v_i's. Conjecture 6.2.11 has been shown to be valid in certain special cases [**178, ?**], for example for weighted homogeneous polynomials of the form

$$f=z_0^{a_0}+\cdots+z_n^{a_n}\qquad \text{Brieskorn}-\text{Pham polynomials} \tag{6.2.16}$$

$$f=z_0^{a_0}+z_0z_1^{a_1}+\cdots+z_{n-1}z^{a_n}. \tag{6.2.17}$$

Actually in [**178**] Randell proves that the conjecture holds for certain complete intersections of Brieskorn varieties, that he calls generalized Brieskorn polynomials. We shall develop Randell's method to obtain a condition for Tor $(H_{n-1}(L_f,\mathbb{Z}))$ which will be very useful in the case $n=3$. However, first we mention an important immediate consequence [**154**] of the exact sequence 6.2.1.

Proposition 6.2.12: *The following hold:*

(i) *L_f is a rational homology sphere if and only if $\Delta(1)\neq 0$.*
(ii) *L_f is a homology sphere if and only if $|\Delta(1)|=1$.*
(iii) *If L_f is a rational homology sphere, then the order of $H_{n-1}(L_f,\mathbb{Z})$ equals $|\Delta(1)|$.*

Links of Brieskorn-Pham Type

Here we discuss in more detail links of polynomials of Brieskorn-Pham type 6.2.16 (abbreviated BP type) which are of particular intersest in the case of homology spheres. We also discuss when BP type links admit regular Sasakian structures.

Brieskorn [**54, 66**] considers the links $L(\mathbf{a})$ defined by

$$\sum_{i=0}^{n}|z_i^2|=1,\qquad z_0^{a_0}+\cdots+z_n^{a_n}=0. \tag{6.2.18}$$

To the vector $\mathbf{a}=(a_0,\cdots,a_n)\in\mathbb{Z}_+^{n+1}$ one associates a graph $G(\mathbf{a})$ whose $n+1$ vertices are labeled by $a_0,\cdots,a_n$. Two vertices a_i and a_j are connected if and only if $\gcd(a_i,a_j)>1$. Let C_{ev} denote the connected component of $G(\mathbf{a})$ determined by the even integers. Note that all even vertices belong to C_{ev}, but C_{ev} may contain odd vertices as well. Then we have

Theorem 6.2.13 (Brieskorn)**:** *The following hold:*

(i) *The link $L(\mathbf{a})$ is a rational homology sphere if and only if either $G(\mathbf{a})$ contains at least one isolated point, or C_{ev} has an odd number of vertices and for any distinct $a_i, a_j \in C_{ev}$, $\gcd(a_i, a_j) = 2$.*

(ii) *The link $L(\mathbf{a})$ is an integral homology sphere if and only if either $G(\mathbf{a})$ contains at least two isolated points, or $G(\mathbf{a})$ contains one isolated point and C_{ev} has an odd number of vertices and for any distinct $a_i, a_j \in C_{ev}$, $\gcd(a_i, a_j) = 2$.*

PROOF. The proof of (1) is given in detail in [**66**], and that of (2) in [**54**]. □

Another case of interest for Brieskorn-Pham links is when the weights are all relatively prime. We have

Proposition 6.2.14: *Let $L(\mathbf{a})$ be a BP link with relatively prime weights $\mathbf{w} = (w_0, \cdots, w_n)$. Then $L(\mathbf{a})$ is the total space of a circle bundle over a smooth projective algebraic variety $\mathcal{Z}_{\mathbf{a}}$ and the induced Sasakian structure is regular.*

To develop Randell's method we consider the exceptional locus of the $S^1_{\mathbf{w}}$-action on L_f. Let Γ_x denote the isotropy subgroup of $S^1_{\mathbf{w}}$-action at $x \in L_f$, and let $|\Gamma_x|$ denote its order. For each prime p define the *p-exceptional set*

$$S_p = \{x \in L_f \mid p \text{ divides } |\Gamma_x|\}. \tag{6.2.19}$$

Lemma 6.2.15: *If $\mathbf{w}$ is well-formed then S_p has codimension at least four.*

PROOF. The isotropy subgroup Γ_x for the $S^1_{\mathbf{w}}$-action on L_f is the isotropy subgroup of the $S^1_{\mathbf{w}}$-action on $S^{2n+1}_{\mathbf{w}}$. From this action we see that if p does not divide w_i we must have $z_i = 0$. Since $\mathbf{w}$ is well-formed there must be at least two w_i's that are not divisible by p. Thus, there are at least two of the complex coordinates z_i vanishing at any point of S_p. So S_p has real codimension at least four. □

Consider $\mathbb{Z}(p) = \{\frac{a}{b} \mid \gcd(b,p) = 1\}$, the ring of integers localized at the prime p. Then the p-torsion of $H_*(X, \mathbb{Z})$ is just the torsion subgroup of $H_*(X, \mathbb{Z}(p))$.

Proposition 6.2.16: *If S_p has codimension four in L_f and is $(n-4)$-connected, then*

$$\mathrm{Tor}\ (H_{n-1}(L_f, \mathbb{Z}(p))) \approx \mathrm{Tor}\ (H_{n-3}(S_p, \mathbb{Z}(p))).$$

PROOF. Let $\mathcal{N}$ be an $S^1_{\mathbf{w}}$-invariant tubular neighborhood of S_p in L_f, and set $M = L_f - \mathcal{N}$ which is also $S^1_{\mathbf{w}}$-invariant, and a compact $(2n-1)$-manifold with boundary $\partial M = \partial\mathcal{N}$. Now we have

Claim 1: $H_q(M, \mathbb{Z}) = 0$ *for* $q \neq 0, 3, n-1, n, n+1, 2n-1$.

To prove this write the long exact homology sequence of the pair (L_f, M).

$$\cdots \longrightarrow H_{q+1}(L_f, \mathbb{Z}) \longrightarrow H_{q+1}(L_f, M, \mathbb{Z}) \longrightarrow H_q(M, \mathbb{Z}) \longrightarrow H_q(L_f, \mathbb{Z}) \longrightarrow \cdots. \tag{6.2.20}$$

By excision, Poincaré duality, and deformation retraction we have

$$H_{q+1}(L_f, M, \mathbb{Z}) \approx H_{q+1}(\bar{\mathcal{N}}, \partial\mathcal{N}, \mathbb{Z}) \approx H^{2n-q-2}(\bar{\mathcal{N}}, \mathbb{Z}) \approx H^{2n-q-2}(S_p, \mathbb{Z}). \tag{6.2.21}$$

Combining these two results with the fact that L_f is $(n-2)$-connected and S_p is $(n-4)$-connected proves the claim. Notice also that this shows that $H_{n+1}(M,\mathbb{Z})$ is free since $H^{n-1}(L_f,\mathbb{Z})$ is free.

Now by definition the space M has an $S^1_{\mathbf{w}}$-action whose isotropy subgroups have orders relatively prime to p. Hence, there is a $\mathbb{Z}(p)$ Gysin sequence. First to simplify notation set $\widetilde{M}=M/S^1_{\mathbf{w}}$. Then we have

(6.2.22)
$$\cdots\longrightarrow H_{q-1}(\widetilde{M},\mathbb{Z}(p))\longrightarrow H_q(M,\mathbb{Z}(p))\longrightarrow H_q(\widetilde{M},\mathbb{Z}(p))\longrightarrow H_{q-2}(\widetilde{M},\mathbb{Z}(p))\longrightarrow\cdots.$$

Then using Claim 1 and Universal Coefficients we obtain

$$H_q(\widetilde{M},\mathbb{Z}(p))=0 \text{ for } q=1,3,\cdots,n-2., \qquad \mathrm{Tor}\ (H_n(\widetilde{M},\mathbb{Z}(p)))=0. \tag{6.2.23}$$

Claim 2: Tor $(H_{n-3}(S_p,\mathbb{Z}))\approx$ Tor $(H_n(\partial M,\mathbb{Z}))\approx$ Tor $(H_n(M,\mathbb{Z}))$.

To establish the first isomorphism we notice that since S_p has trivial normal bundle we have Tor $(H_{n-3}(S_p,\mathbb{Z}))\approx$ Tor $(H_n(\partial\mathcal{N},\mathbb{Z}))\approx$ Tor $(H_n(\partial M,\mathbb{Z}))$ by Kunneth. For the second isomorphism consider the exact sequence

(6.2.24)
$$\cdots\longrightarrow H_{n+1}(M,\partial M,\mathbb{Z})\longrightarrow H_n(\partial M,\mathbb{Z})\longrightarrow H_n(M,\mathbb{Z})\longrightarrow H_n(M,\partial M,\mathbb{Z})\longrightarrow\cdots.$$

By duality and Universal Coefficients

$$H_{n+1}(M,\partial M,\mathbb{Z})\approx H^{n-2}(M,\mathbb{Z})\approx FH_{n-2}(M,\mathbb{Z})\oplus \mathrm{Tor}\ (H_{n-3}(M,\mathbb{Z})),$$

where FH_n denotes the free part of H_n. The right hand side of this equation vanishes by Claim 1 for $n\neq 2,5$. (A separate argument must be made for $n=2,5$ which we leave for the reader to fill in). Now Claim 2 follows from the exact sequence 6.2.24 since by duality $H_n(M,\partial M,\mathbb{Z})\approx H^{n-1}(M,\mathbb{Z})$ which is free.

Claim 3: Tor $(H_n(M,\mathbb{Z}))\approx$ Tor $(H_{n-1}(\widetilde{M},\mathbb{Z}))\approx$ Tor $(H_{n-1}(M,\mathbb{Z}))$.

To prove this use the $\mathbb{Z}(p)$ Gysin sequence 6.2.22 and Claim 1 to give

$$0\longrightarrow H_{n-1}(\widetilde{M},\mathbb{Z}(p))\longrightarrow H_n(M,\mathbb{Z}(p))\longrightarrow H_n(\widetilde{M},\mathbb{Z}(p))\longrightarrow 0.$$

The first isomorphism then follows by the second equation in 6.2.23. Again the $\mathbb{Z}(p)$ Gysin sequence 6.2.22 with $q=n-1$ and Equation 6.2.23 proves the claim. (Again separate arguments are needed for $n=2,5$.) Finally, we have

Claim 4: $H_{n-1}(M,\mathbb{Z})\approx H_{n-1}(L_f,\mathbb{Z})$.

To prove this consider the long exact sequence 6.2.20 with $q=n-1$, viz.

(6.2.25)
$$\cdots\longrightarrow H_n(L_f,M,\mathbb{Z})\longrightarrow H_{n-1}(M,\mathbb{Z})\longrightarrow H_{n-1}(L_f,\mathbb{Z})\longrightarrow H_{n-1}(L_f,M,\mathbb{Z})\longrightarrow\cdots.$$

By the isomorphisms 6.2.21 and duality, we have with $q=n-1$

$$H_n(L_f,M,\mathbb{Z})\approx H^{n-1}(S_p,\mathbb{Z})\approx H_{n-4}(S_p,\mathbb{Z})=0,$$

while for $q=n-2$ we get

$$H_{n-1}(L_f,M,\mathbb{Z})\approx H^{n}(S_p,\mathbb{Z})\approx H_{n-5}(S_p,\mathbb{Z})=0.$$

This proves Claim 4. Combining Claims 2, 3 and 4 proves the result. $\square$

An immediate corollary of this result that is of much importance to us is:

Theorem 6.2.17 ([**36**])**:** *Let L_f be the link of an isolated hypersurface singularity defined by a weighted homogeneous polynomial f in four complex variables. Suppose further that L_f is well-formed, then Tor $(H_2(L_f,\mathbb{Z}))=0$.*

CHAPTER 7

Sasakian Geometry on Homotopy Spheres

In this chapter we describe the types of Sasakian geometry that can occur on homotopy spheres. Actually we first give some results concerning rational homology spheres.

7.1. Sasakian Structures on Rational Homology Spheres

The type of Sasakian structures that can occur on rational homology spheres are quite restrictive. Our first result gives a characterization of Sasakian geometry on rational homology spheres in terms of the basic cohomology ring $H_B^*(\mathcal{F}_\xi)$.

Proposition 7.1.1: *Let M^{2n+1} be a manifold with a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$. Then M^{2n+1} is a rational homology sphere if and only if the basic cohomology ring $H_B^*(\mathcal{F}_\xi)$ is isomorphic to the real cohomology ring of complex projective space $\mathbb{CP}^n$. Moreover, any Sasakian structure $\mathcal{S}$ on a rational homology sphere is definite, that is either $c_1(\mathcal{F}_\xi) > 0$ or $c_1(\mathcal{F}_\xi) < 0$.*

Proof. To prove the first statement we analyze the basic long exact sequence 5.1.1. If M is a rational homology sphere this implies the isomorphisms

$$H_B^{p-1}(\mathcal{F}_\xi) \approx H_B^{p+1}(\mathcal{F}_\xi)$$

for $p = 1, \cdots, 2n-1$, and since $H_B^1(\mathcal{F}_\xi) = 0$ and $H_B^2(\mathcal{F}_\xi)$ is generated by $[d\eta]_B$, this implies the ring isomorphism

$$H_B^*(\mathcal{F}_\xi) \approx \mathbb{R}\big[[d\eta]_B\big]/([d\eta]_B)^{n+1}. \tag{7.1.1}$$

Conversely, the ring isomorphism 7.1.1 implies that the connecting homomorphism δ in 5.1.1 is an isomorphism. Putting $p = 2r+1$ then shows that $H^{2r+1}(M, \mathbb{R}) = H^{2r+2}(M, \mathbb{R}) = 0$ for $r = 0, \cdots, n-1$ which implies that M^{2n+1} is a rational homology sphere. This proves the first statement.

For the second statement we first notice that the ring isomorphism 7.1.1 implies that $c_1(\mathcal{F}_\xi) = a[d\eta]_B$ for some $a \in \mathbb{R}$. Since $d\eta$ is the transverse Kähler form it is positive definite, and a cannot vanish since this would imply that the basic geometric genus $p_g(\mathcal{F}_\xi) \neq 0$ implying $b_n(M) \neq 0$ by the transverse Hodge decomposition 5.3.1 which would give a contradiction. Thus, either $a > 0$ or $a < 0$ implying either $c_1(\mathcal{F}_\xi) > 0$ or $c_1(\mathcal{F}_\xi) < 0$. □

We will present many examples in dimensions 5 and 7 of rational homology spheres with positive Sasakian structures. This leads to the intriguing question as to whether there are any rational homology spheres with a negative Sasakian structure. The answer is yes. It is a circle bundle over Mumford's fake projective plane [**?**], and will be described in Chapter 8.

7.2. A Review of Exotic Structures on Homotopy Spheres

Let us briefly recall a construction of exotic differential structures on odd dimensional spheres. In 1956 Milnor [?] created the field of differential topology by constructing exotic differential structures on S^7. Later the work of Milnor and Kervaire [**124**] (see also [**109**]) showed that associated with each sphere S^n with $n \geq 5$ there is an Abelian group Θ_n consisting of equivalence classes of homotopy spheres S^n that are equivalent under oriented h-cobordism. By Smale's h-cobordism theorem this implies equivalence under oriented diffeomorphism. The group operation on Θ_n is connected sum. Θ_n has a subgroup bP_{n+1} consisting of equivalence classes of those homotopy n-spheres which bound parallelizable manifolds V_{n+1}. Kervaire and Milnor [**124**] proved that $bP_{2k+1} = 0$ for $k \geq 1$. Moreover, for $m \geq 2$, bP_{4m} is cyclic of order

$$|bP_{4m}| = 2^{2m-2}(2^{2m-1} - 1) \text{ numerator } \big(\frac{4B_m}{m}\big),$$

where B_m is the m-th Bernoulli number. For bP_{4m+2} the situation is still not entirely understood. It entails computing the Kervaire invariant, which is hard. It is known (see the recent review paper [La] and references therein) that $bP_{4m+2} = 0$, or $\mathbb{Z}_2$ and is $\mathbb{Z}_2$ if $4m + 2 \neq 2^i - 2$ for any $i \geq 3$. Furthermore, bP_{4m+2} vanishes for $m = 1, 3, 7,$ and 15.

Ten years after Milnor's landmark paper, Brieskorn [**54**] showed how exotic spheres can be obtained as links of isolated hypersurface singularities. In particular if g_m denotes the Milnor generator in bP_{4m} (see [**109**]) then Brieskorn showed that for each $k \in \mathbb{Z}^+$ the link defined as the locus of points in $\mathbb{C}^{2m+1}$ that satisfies

$$\sum_{i=0}^{2m} |z_i^2| = 1, \qquad z_0^{6k-1} + z_1^3 + z_2^2 + \cdots + z_{2m}^2 = 0, \qquad k \geq 1, \tag{7.2.1}$$

represents an element in bP_{4m}. We shall denote these homotopy $(4m-1)$-spheres by Σ_k^{4m-1}. The diffeomorphism type of $\Sigma \in bP_{4m}$ is determined [KeMi] by the signature τ of V_{4m} which is necessarily divisible by 8. Two $\Sigma, \Sigma' \in bP_{4m}$ are diffeomorphic if and only if $\tau(V'_{4m}) \equiv \tau(V_{4m})(\mod 8|bP_{4m}|)$, and Brieskorn shows that $\tau(V_{4m}^{(k)}) = (-1)^m 8k$, so that Σ_k^{4m-1} and $\Sigma_{k'}^{4m-1}$ are diffeomorphic if and only if

$$k' \equiv k(\mod |bP_{4m}|).$$

The group operation in bP_{4m} is connected sum and since the signature is additive under connected sum, the element $(-1)^m\Sigma_1^{4m-1}$ is a generator $g_m \in bP_{4m}$, and $(-1)^m|bP_{4m}|g_m$ is diffeomorphic to the standard sphere S^{4m-1}.
Hence, if $n \geq 3$ such an $L(\mathbf{a})$ is homeomorphic to the sphere S^{2n-1}, and by Milnor's fibration theorem [Mil2] describes an element of bP_{2n}.

Clearly, the links in bP_{4m} described by 7.2.1 are of type 1 above, whereas, examples of type 2 links in bP_{4m+2} are given by

$$\sum_{i=0}^{2m} |z_i^2| = 1, \qquad z_0^p + z_1^2 + z_2^2 + \cdots + z_{2m+1}^2 = 0, \qquad p \geq 1. \tag{7.2.2}$$

We shall denote these spheres by Σ_p^{4m+1}. If $p \equiv \pm 1 \mod 8$ then Σ_p^{4m+1} is diffeomorphic to the standard sphere S^{4m+1}, whereas assuming that $4m+2 \neq 2^i - 2$ if

$p \equiv \pm 3 \mod 8$ then Σ_p^{4m+1} is the exotic Kervaire sphere in bP_{4m+2}. This follows from Levine's [Lev] computation of the Arf invariant [**54**].

The polynomials given in 7.2.1 and 7.2.2 are by no means unique. Any other Brieskorn-Pham polynomial 6.2.18 satisfying either of the two conditions on $G(\mathbf{a})$ above will do as well. For example, in lieu of 7.2.1 we can consider

$$\sum_{i=0}^{2m} |z_i^2| = 1, \qquad z_0^{10k-1} + z_1^5 + z_2^2 + \cdots + z_{2m}^2 = 0, \qquad k \geq 1. \tag{7.2.3}$$

In this case we find that Σ_k^{4m-1} and $\Sigma_{k'}^{4m-1}$ are diffeomorphic if and only if $3k' \equiv 3k($ mod $|bP_{4m}|)$.

CHAPTER 8

Sasakian Geometry in Dimensions Three and Five

The topology and geometry of compact three dimensional manifolds is extremely rich and varied. There has been enormous progress in the last twenty years mainly due to Thurston [**?**], but one of the most important conjectures in mathematics, the Poincaré conjecture, is still unsolved. Also there is no smoothing problem in dimension three. Oddly there is a classification of closed simply connected five manifolds due to Barden [**16**] up to diffeomorphism. So dimension five is large enough to have 'enough room' for Smale's proof of the generalized Poincaré's conjecture [**193**] to work, but small enough to obtain a reasonable classification. We should note that allowing a nontrivial fundamental group makes the classification problem undecidable.

8.1. Sasakian Geometry in Dimension Three

As for contact geometry in dimension three, as was mentioned previously there is Martinet's Theorem 3.1.19 that every orientable 3-manifold admits a contact structure. Mention here about tight and overstwisted contact structures. Sasakian structures are also quite managable as shown by a theorem of Belgun [**19**] which is described below.

Discuss Belgun's results, homology spheres, Orlik's classification of links, Neuman's link invariant and the Casson invariant. Also lens spaces, homotopy equiv, homeomorphism a la Bredon.

It is easy to classify weighted homogeneous polynomials in 3 variables [**172**].

Proposition 8.1.1: *Let $h(z_0,z_1,z_2)$ be a weighted homogeneous polynomial in three complex variables having an isolated singularity at the origin. Then there are a permutation σ on the set $\{0,1,2\}$, non-zero complex numbers $\alpha_0,\alpha_1\alpha_2$, and weighted homogeneous polynomials f and g with no monomials in common such that h can be written as $h=f+g$ where $f(\alpha_0z_{\sigma(0)},\alpha_1z_{\sigma(1)},\alpha_2z_{\sigma(2)})$ is equal to*[1]*:*
(I) $z_0^{a_0}+z_1^{a_1}+z_2^{a_2}$,
(II) $z_0^{a_0}+z_1^{a_1}+z_2^{a_2}z_1$ *with* $a_1>1$,
(III) $z_0^{a_0}+z_1^{a_1}z_0+z_2^{a_2}z_1$ *with* $a_1>1, a_2>1$,
(IV) $z_0^{a_0}+z_1^{a_1}z_0+z_2^{a_2}z_1$ *with* $a_0>1$,
(V) $z_0^{a_0}z_1+z_1^{a_1}z_2+z_2^{a_2}z_0$.

We shall use Orlik's notation $K(a_0,a_1,a_2;I),\cdots,K(a_0,a_1,a_2;V)$ when referring to the link associated to these classes of polynomials. The following conjecture of Milnor [**154**] was proven by Orlik [**170**]

[1]Actually Orlik and Wagreich give six classes, but as they mention class VI is equivalent to a special case of class I

Theorem 8.1.2 (Orlik): *Let L be the link of a quasi-smooth weighted homogeneous polynomial in three complex variables of degree d and weight vector $\mathbf{w} = (w_0, w_1, w_2)$. Then*

(i) *The fundamental group $\pi_1(L)$ is infinite and the universal cover of L is an open 3-cell if and only if $d \geq |\mathbf{w}|$.*
(ii) *$\pi_1(L)$ is infinite nilpotent if and only if $d = |\mathbf{w}|$.*

Proof. To the prove the first statement Orlik shows that $\pi_1(L)$ is finite if and only if $|\mathbf{w}| > d$. If $\pi_1(L)$ is finite, then the geometry of L is elliptic and L is covered by S^3. In this case it is well known that $\pi_1(L)$ is one of the finite subgroups Γ of $SU(2)$, namely:

(i) $\Gamma = \mathbb{Z}_m$ the cyclic group of order m,
(ii) $\Gamma = \mathbb{D}^*_m$ a binary dihedral group where m is an integer greater than 2,
(iii) $\Gamma = \mathbb{T}^*$ the binary tetrahedral group,
(iv) $\Gamma = \mathbb{O}^*$ the binary octahedral group,
(v) $\Gamma = \mathbb{I}^*$ the binary icosahedral group.

Orlik shows that all of these geometries are realized by the links occuring in Proposition 8.1.1 and that this exhausts all cases with $|\mathbf{w}| > d$. These are:

(i) $\Gamma = \mathbb{Z}_p$ gives the lens spaces $L(p, 1) = K(2, 2, p; I)$.
(ii) The binary dihedral group $\Gamma = \mathbb{D}^*_m$ gives $K(2, m, 2; II)$.
(iii) The binary tetrahedral group $\Gamma = \mathbb{T}^*$ gives $K(2, 3, 4; I)$.
(iv) The binary octahedral group $\Gamma = \mathbb{O}^*$ gives $K(2, 3, 3; II)$.
(v) The binary icosahedral group $\Gamma = \mathbb{I}^*$ gives $K(2, 3, 5; I)$.

It is easy to check that these exhaust all possible cases when $|\mathbf{w}| > d$ (Recall that $w_i = \frac{d}{a_i}$). This proves the first part of (1), and the second statement of (1) follows from the work of Waldhausen [**170, ?**]. □

Here is the genus formula:

$$g(C_{\mathbf{w}}) = \frac{1}{2}\Big(\frac{d^2}{w_1 w_2 w_3} - d\sum_{i<j}\frac{\gcd(w_i, w_j)}{w_i w_j} + \sum_i \frac{\gcd(d, w_i)}{w_i} - 1\Big).$$

Here I discuss some integral homology 3-spheres. There is a paper (O. Saeki, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), 43-50.) which says that the link L of any isolated hypersurface singularity in 3 complex variables with an isolated singularity coming from a weighted homogeneous polynomial is an integral homology sphere only if it is equivalent to the link $L(\mathbf{a})$ of a Brieskorn-Pham polynomial of the form

$$z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0.$$

The problem of finding which Brieskorn manifolds are homology spheres is solved by Breiskorn's graph Theorem 6.2.13; however, here we give a different proof using the Alexander polynomial.

Proposition 8.1.3: *The link $L(\mathbf{a})$ is a homology sphere if and only if* $\gcd(a_i, a_j) = 1$ *for all $i \neq j = 0, 1, 2$.*

PROOF. First we assume that (a_0, a_1, a_2) are coprime which implies that $d = a_0a_1a_2$ and, hence that $\mathbf{w} = (a_1a_2, a_0a_2, a_0a_1)$. We compute the Alexander polynomial $\Delta(t)$. First the divisor of $\Delta(t)$ is

$$\begin{aligned}\text{Div } \Delta =&(\Lambda_{a_0} - 1)(\Lambda_{a_1} - 1)(\Lambda_{a_2} - 1) = (\Lambda_{a_0a_1} - \Lambda_{a_0} - \Lambda_{a_1} + 1)(\Lambda_{a_2} - 1)\\ =&\Lambda_{a_0a_1a_2} - \Lambda_{a_0a_1} - \Lambda_{a_1a_2} - \Lambda_{a_0a_2} + \Lambda_{a_0} + \Lambda_{a_1} - 1.\end{aligned}$$

It follows immediately that $b_1(L) = 0$. Then from Theorem 6.2.5 we find

$$\begin{aligned}\Delta(t) =&\frac{(t^{a_0a_1a_2} - 1)(t^{a_0} - 1)(t^{a_1} - 1)(t^{a_2} - 1)}{(t^{a_0a_1} - 1)(t^{a_0a_2} - 1)(t^{a_1a_2} - 1)(t - 1)}\\ =&\frac{(t^{(a_2-1)a_0a_1} + \cdots t^{a_0a_1} + 1)(t^{(a_2-1)} + \cdots t + 1)}{(t^{(a_2-1)a_1} + \cdots t^{a_1} + 1)(t^{(a_2-1)a_0} + \cdots t^{a_0} + 1)}.\end{aligned}$$

This implies that $\Delta(1) = 1$ which implies that L is an integral homology sphere by Proposition 6.2.12.

Next we consider the case when two exactly two of the a_i's have gcd greater than one. Without loss of generality we assume that $\gcd(a_0, a_1) = r > 1$. As before we compute the Alexander polynomial obtaining

$$\Delta(t) = \frac{(t^{(a_2-1)s} + \cdots t^s + 1)^r(t^{(a_0-1)} + \cdots t + 1)}{(t^{(a_0-1)a_2} + \cdots t^{a_2} + 1)(t^{(a_2-1)a_1} + \cdots t^{a_1} + 1)}$$

where $s = \operatorname{lcm}(a_0, a_1)$. Thus, $\Delta(1) = a_2^{r-1}$, so L is not a homology sphere in this case. The cases when $\gcd(a_1, a_2) > 1$ and/or $\gcd(a_0, a_2) > 1$ are treated similarly. □

8.2. Topology of Compact 5-Manifolds

Sasakian structures in dimension five are fairly manageable because much is know about projective algebraic surfaces. We begin by reviewing what is known about compact 5-manifolds.

Here we mention Markov's undecidability result. Barden-Smale classification of simply connected compact 5-manifolds. Geiges' work on contact structures.

In 1965 Barden [**16**] proved the following remarkable theorem:

Theorem 8.2.1 (Barden)**:** *The class of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism. Furthermore, any such M is diffeomorphic to one of the spaces $M_{j;k_1,\ldots,k_s} = X_j\#M_{k_1}\#\cdots\#M_{k_s}$, where $-1 \leq j \leq \infty$, $s \geq 0$, $1 < k_1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$. A complete set of invariants is provided by $H_2(M, \mathbb{Z})$ and $i(M)$.*

It is understood that $s = 0$ means that no M_{k_i} occurs. The diffeomorphism invariant $i(M)$ depends only on the second Stiefel-Whitney class $w_2(M)$. When M is spin $i(M) = 0$ as is $w_2(M) = 0$. Otherwise $i(M) \neq 0$. Barden's result is the extension of the similar theorem of Smale for spin manifolds. In fact, in Barden's notation $M_{j;k_1,\ldots,k_s}$ is spin if and only if $j = 0$ in which case $X_0 = S^5$ and $M_{0;k_1,\ldots,k_s} = M_{k_1}\#\cdots\#M_{k_s}$ which is precisely Smale's list [**194**]. Recall that here, $X_{-1} = SU(3)/SO(3)$ is the well-known symmetric space, X_∞ is the non-trivial S^3 bundle over S^2. For all other values of j, X_j is a rational homology sphere determined by its 2-torsion which is $H_2(X_j, \mathbb{Z}) = \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$ and the value of $i(X_j) = j$. All X_j other than X_0 are non-spin, as their second Stiefel-Whitney class $w_2(X_j) \neq 0$. Now, M_k, $1 < k < \infty$ is a rational homology sphere with $H_2(M_k, \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_k$ and vanishing $i(M_k) = 0$. Finally, $M_\infty = S^2 \times S^3$ is the trivial S^3 bundle over S^2. All of the pieces in the above theorem are indecomposable with the exception of $X_1 = X_{-1} \# X_{-1}$. Note that for $1 < j < \infty$, M_{2^j} and X_j have the same $H_2(M, \mathbb{Z})$ but $i(M_{2^j}) = 0$ while $i(X_j) = j$.

We are interested in those simply connected 5-manifolds that admit Sasakian structures. First we describe a result of Geiges [**91**] concerning contact structures on simply connected 5-manifolds.

Theorem 8.2.2 (Geiges)**:** *Let M^5 be a simply connected 5-manifold. Then M^5 admits a contact structure in every homotopy class of almost contact structures.*

By Corollary 3.2.8 M^5 admits an almost contact structure if and only if its 3rd integral Stiefel-Whitney class $W_3(M^5)$ vanishes. From [**16**] this eliminates all the X_j's except for $j = 0, \infty$. So from Geiges' Theorem 8.2.2 we have

Theorem 8.2.3: *A simply connected 5-manifold M^5 admits a contact structure if and only if $M^5 = X_j \# M_{k_1} \# \cdots \# M_{k_s}$, where $j = 0, \infty$, $s \geq 0$, $1 < k_1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$.*

8.3. Regular Sasakian Structures on 5-Manifolds

We begin with a theorem of Geiges [**91**] about regular contact manifolds.

Theorem 8.3.1 (Geiges)**:** *A simply connected 5-manifold M^5 admits a regular contact structure if and only if M^5 has torsionfree homology.*

Proof. If M^5 admits a regular contact structure then by the Boothby-Wang Theorem **??** it is the total space of a principal S^1 bundle over a symplectic base B^4. By the long exact homotopy sequence B^4 is simply connected, and by Poincaré duality $H^3(B^4, \mathbb{Z}) = 0$. Then the Gysin sequence implies that $H^3(M^5, \mathbb{Z})$ is torsionfree, and this is the only possible torsion in a simply connected 5-manifold. The converse will be proved below by proving that the manifolds of Theorem 8.2.3 without torsion admit Sasakian structures. $\square$

Since the classification of compact complex surfaces uses the concept of minimal models we do the same for regular Sasakian structures on compact 5-manifolds. Let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a regular Sasakian structure on a compact 5-manifold M^5. Then by Theorem 5.3.12 M^5 is the total space of an S^1 bundle over a smooth compact polarized algebraic surface X^2. The polarization is determined by a choice of integral Kähler class. The regular Sasakian structures on compact 5-manifolds are classified by picking out the algebraic surfaces in the Enriques-Kodaira classification [**?**] in complex surface theory. We have

Theorem 8.3.2: *Let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a regular Sasakian structure on a compact 5-manifold M^5. Then M^5 is the total space of an S^1-bundle over a projective algebraic surface X^2 which is obtained by blowing-up $k = 0, 1, \cdots$-times one of the following minimal surfaces:*

(i) *a minimal rational surface: $\mathbb{CP}^2$ or a Hirzebruch surface F_n with $n \geq 0, n \neq 1$.*
(ii) *a ruled surface of genus $g \geq 1$.*
(iii) *an algebraic K3 surface.*

(iv) *an Enriques surface.*
(v) *an algebraic 2-torus (abelian variety of dimension 2).*
(vi) *a hyperelliptic surface.*
(vii) *an algebraic minimal properly elliptic surface.*
(viii) *a minimal surface of general type.*

Recall that a compact complex surface is *minimal* if it does not contain a (-1)-curve. Thus, we can say that a regular Sasakian structure $\mathcal{S}$ on M^5 is *minimal* if X^2 is minimal.

EXAMPLE 8.3.3 (Circle Bundle over Mumford's Fake $\mathbb{CP}^2$): In 1979 Mumford [**?**] gave an example, using p-adic analysis, of a compact complex surface X^2 of general type having $b_1 = 0, b_2 = 1$ and $c_1^2 = 9$. Thus, $c_2 = 3$. So the equality $c_1^2 = 3c_2$ holds, and a result of Yau [**?**] says that X^2 is covered by the disc $D^2 \subset \mathbb{C}^2$ and $X^2 = D^2/\Gamma$ where Γ is a discrete co-compact subgroup of $SU(2,1)$. Hence, $\pi_1(X^2)$ is infinite, and $c_1(X^2) < 0$. An explicit description of Γ is still missing. Choose the Kähler class of X^2 to be the positive generator in $H^2(X^2,\mathbb{Z}) \approx \mathbb{Z}$, and let $\pi : M^5 \longrightarrow X^2$ denote the circle bundle whose first Chern class is this class. By the long exact homotopy sequence, $\pi_1(M^5)$ is also infinite, and by the rational Gysin sequence M^5 is a rational homology sphere. Since $c_1(X^2)$ pulls back to $c_1(\mathcal{F}_\xi)$, M^5 has a negative Sasakian structure.

Generally, little is known about the existence of negative Sasakian structures on rational homology 5-spheres, but if one assumes that the Sasakian structure is also regular much more can be said.

Proposition 8.3.4: *Let M^5 be a rational homology 5-sphere that admits a regular negative Sasakian structure, then $\pi_1(M^5)$ is infinite.*

PROOF. M^5 is the total space of an S^1 bundle over a compact complex surface X^2 with $b_1 = 0$ and $b_2 = 1$. By Theorem V.1.1 of [**?**] X^2 is either $\mathbb{CP}^2$ or $\pi_1(X^2)$ is infinite. Since $c_1(\mathcal{F}_\xi) < 0$ and is the pullback of $c_1(X^2)$, the later must hold. But then $\pi_1(M^5)$ is also infinite by the long exact homotopy sequence. $\square$

An immediate corollary of Proposition 8.3.4 is

Corollary 8.3.5: *Every regular Sasakian structure on S^5 is positive.*

CHAPTER 9

Kähler-Einstein Metrics

In this chapter we will recall some basic results concerning existance of Einstein metrics on Kähler manifolds. If (M, J, g) is Kähler and Einstein than it follows that $c_1(M)$ must be aither negative, zero, or positive. We will be mostly interested in the last case but we shall start the discussion with the first two.

9.1. The $c_1(M) < 0$ Case

9.2. The $c_1(M) = 0$ Case – Calabi-Yau Spaces

9.3. The $c_1(M) > 0$ Case – Fano Varieties

CHAPTER 10

Sasakian-Einstein Geometry

10.1. Sasakian η-Einstein Metrics

10.2. General Properties of Sasakian-Einstein Geometry

10.3. Sasakian-Einstein Metrics in Dimensions Five and Seven

10.4. The Sasakian-Einstein Monoid

In this section we apply a construction due to Wang and Ziller to define a multiplication on the set of quasi-regular Sasakian-Einstein orbifolds.

Definition 10.4.1: *We denote by $\mathcal{SE}$ the set of compact quasi-regular Sasakian-Einstein orbifolds with $\pi_1^{orb}=0$, by $\mathcal{SE}^s$ the subset of $\mathcal{SE}$ that are smooth manifolds, and by $\mathcal{R}\subset\mathcal{SE}^s$ the subset of compact, simply connected, regular Sasakian-Einstein manifolds. The set $\mathcal{SE}$ is topologized with the $C^{m,\alpha}$ topology, and the subsets are given the subspace topology.*

The condition $\pi_1^{orb}=0$ is made to avoid complications, and since every Sasakian-Einstein orbifold is covered by one with $\pi_1^{orb}=0$, there is no loss of generality. The set $\mathcal{SE}$ is graded by dimension, that is,

$$\mathcal{SE}=\bigoplus_{n=0}^{\infty}\mathcal{SE}_{2n+1},$$

and similarly for $\mathcal{SE}^s$ and $\mathcal{R}$. In the definition of Sasakian structure it is implicitly assumed that $n>0$. So we want to extend the definition of a Sasakian-Einstein structure to the case when $n=0$. This can easily be done since a connected one dimensional orbifold is just an interval with possible boundary, or a circle. So we can just take $\xi=\frac{\partial}{\partial t},\eta=dt,\Phi=0$, with the flat metric $g=dt^2$. In this case the space of leaves $\mathcal{Z}$ of the characteristic foliation is just a point. The unit circle S^1 with this structure will play the role of the identity in our monoid. We define the *index* of any element of $\mathcal{SE}_1$ to be 0 and understand that $\gcd(0,m)=m$ for any integer m. Then we see that $\mathcal{SE}_1^s=\mathcal{R}_1$ is a single point, namely the unit circle with its flat metric. Also it follows from a theorem of Hamilton [**22**] that $\mathcal{SE}_3^s=\mathcal{R}_3=\{pt\}$, namely S^3 with its standard Sasakian-Einstein structure. Moreover, we know of no examples of elements of $\mathcal{SE}_5^s$ that are not in $\mathcal{R}_5$.

We now define a graded multiplication

$$\mathcal{SE}_{2n_1+1}\times\mathcal{SE}_{2n_2+1}\longrightarrow\mathcal{SE}_{2(n_1+n_2)+1}$$

as follows: Let $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{SE}$ of dimension $2n_1+1$ and $2n_2+1$ respectively. Their respective space of leaves $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are Kähler-Einstein Fano orbifolds of complex dimension n_1, n_2, and metrics h_1, h_2, respectively. Moreover, the scalar curvature of $(\mathcal{Z}_i, h_i)$ is $4n_i(n_i+1)$. Now the product orbifold $(\mathcal{Z}_1 \times \mathcal{Z}_2, h_1+h_2)$ is Kähler, but in general not Kähler-Einstein. However, by rescaling we see that the metric

$$h' = \frac{(n_1+1)h_1 + (n_2+1)h_2}{n_1+n_2+1}$$

is Kähler-Einstein and Fano with scalar curvature $4(n_1+n_2)(n_1+n_2+1)$. Furthermore, if $\mathrm{Ind}(\mathcal{Z}_i)$ denotes the indices of $\mathcal{Z}_i$, then the index of $\mathcal{Z}_1 \times \mathcal{Z}_2$ is just $\gcd(\mathrm{Ind}(\mathcal{Z}_1), \mathrm{Ind}(\mathcal{Z}_2))$. Thus, by Corollary **??** the S^1 V-bundle on $\mathcal{Z}_1 \times \mathcal{Z}_2$ whose first Chern class is

$$\frac{c_1(\mathcal{Z}_1 \times \mathcal{Z}_2)}{\mathrm{Ind}(\mathcal{Z}_1 \times \mathcal{Z}_2)} = \frac{c_1(\mathcal{Z}_1) + c_1(\mathcal{Z}_2)}{\gcd(\mathrm{Ind}(\mathcal{Z}_1), \mathrm{Ind}(\mathcal{Z}_2))} \tag{10.4.1}$$

is simply connected in the orbifold sense with a Sasakian-Einstein structure determined by Theorem 5.3.15. We denote this multiplication by $\star$, and refer to it as the *join*. It is clear from the construction that if $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{SE}$ then also $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathcal{SE}$ with the Sasakian-Einstein structure as defined above, and that this procedure can be iterated in such a way that $\star$ is associative. It is also commutative up to isomorphism. Furthermore, $\star$ is continuous in both factors. We have arrived at:

Theorem 10.4.2: *The operation $\star$ defined above gives $\mathcal{SE}$ the structure of a commutative associative topological monoid.*

Actually we are interested in the subset $\mathcal{SE}^s$ of smooth Sasakian-Einstein manifolds. However, it is certainly not true that $\mathcal{SE}^s$ is a submonoid. We want conditions that guarantee that if $\mathcal{S}_i \in \mathcal{SE}^s$ for $i=1,2$ then $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathcal{SE}^s$. For $i=1,2$ we define the *relative indices* of the pair $(\mathcal{S}_1, \mathcal{S}_2)$ by

$$l_i = \frac{\mathrm{Ind}(\mathcal{S}_i)}{\gcd(\mathrm{Ind}(\mathcal{S}_1), \mathrm{Ind}(\mathcal{S}_2))}. \tag{10.4.2}$$

Then, $\gcd(l_1, l_2) = 1$, and we have,

Proposition 10.4.3: *Let $\mathcal{S}_i \in \mathcal{SE}^s$ with orders m_i, and relative indices l_i, respectively. Then*

- (i) *$Ind(\mathcal{S}_1 \star \mathcal{S}_2) = \gcd(Ind(\mathcal{S}_1), Ind(\mathcal{S}_2))$.*
- (ii) *$\mathcal{S}_1 \star \mathcal{S}_2 \in \mathcal{SE}^s$ if and only if $\gcd(m_1 l_2, m_2 l_1) = 1$.*
- (iii) *$\mathcal{R}$ is a submonoid.*
- (iv) *If $\mathcal{S}_1$ is regular (i.e., $m_1 = 1$) and $Ind(\mathcal{S}_2)$ divides $Ind(\mathcal{S}_1)$, then $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathcal{SE}^s$ independently of m_2.*

PROOF. (i) and (iii) are clear from the discussion above, and (iv) is a special case of (ii), which we now prove. Since $\mathcal{S}_1 \times \mathcal{S}_2$ is a 2-torus V-bundle over $\mathcal{Z}_1 \times \mathcal{Z}_2$, the S^1 V-bundle $\mathcal{S}_1 \star \mathcal{S}_2$ can be realized as a quotient $(\mathcal{S}_1 \times \mathcal{S}_2)/S^1$ by choosing a certain homomorphism $S^1 \longrightarrow S^1 \times S^1$. For the V-bundle with Chern class given by 10.4.1 we have the action on $\mathcal{S}_1 \times \mathcal{S}_2$ given by $(x,y) \mapsto (\tau^{l_2}x, \tau^{-l_1}y)$. The condition that $\mathcal{S}_1 \star \mathcal{S}_2$ is in $\mathcal{SE}^s$ is that there are no fixed points (x,y) under the above action. If m_x, m_y denote the orders of the local uniformizing groups (leaf holonomy groups) of x, y, respectively, then the condition that (x,y) be a fixed point is that $\gcd(m_x l_2, m_y l_1) = g > 1$. That this condition never hold for all pairs (x,y) is precisely the condition $\gcd(m_1 l_2, m_2 l_1) = 1$. ∎

This proposition will allow us to construct smooth examples of quasi-regular Sasakian-Einstein manifolds in all odd dimensions greater than or equal to 9. Before discussing some important examples, we introduce some more terminology.

Definition 10.4.4: *We say that $\mathcal{S} \in \mathcal{SE}$ is $\mathcal{SE}$-irreducible if writing $\mathcal{S} = \mathcal{S}_1 \star \mathcal{S}_2$ implies that either $\mathcal{S}_1$ or $\mathcal{S}_2$ is S^1 with its flat structure. $\mathcal{S}$ is $\mathcal{SE}$-reducible if it is not $\mathcal{SE}$-irreducible.*

It is clear that $\mathcal{SE}$-irreducibility corresponds to Riemannian irreducibility on the space of leaves. Reducibility first occurs in dimension 5, and up to diffeomorphism there is precisely one regular $\mathcal{SE}$-reducible 5-manifold, namely the homogeneous Stiefel manifold $V_2(\mathbb{R}^4) \approx S^2 \times S^3 \approx S^3 \star S^3$. In terms of the homogeneous Sasakian-Einstein manifolds, reducibility has a group theoretic interpretation. A well-known theorem of Borel [**32**] states a simply connected homogeneous Kähler manifold corresponding to a complex semi-simple Lie group G is Riemannian irreducible if and only if G is simple. It is easy to see that in terms of Sasakian-Einstein geometry this implies

Proposition 10.4.5: *A simply connected Sasakian-Einstein homogeneous manifold corresponding to a semi-simple Lie group G is $\mathcal{SE}$-irreducible if and only if G is simple.*

Moreover, our construction also implies

Proposition 10.4.6: *Let $\mathcal{S}_1, \mathcal{S}_2$ be simply connected homogeneous Sasakian-Einstein manifolds, then $\mathcal{S}_1 \star \mathcal{S}_2$ is a simply connected homogeneous Sasakian-Einstein manifold.*

This proposition states that the subset $\mathcal{H}$ of simply connected homogeneous Sasakian-Einstein manifolds forms a submonoid of $\mathcal{R}$.

As noted by Theorem **??**, the regular Sasakian-Einstein 5-manifolds have been classified. This immediately gives a classification of the regular $\mathcal{SE}$-reducible 7-manifolds.

Proposition 10.4.7: *Any simply connected regular $\mathcal{SE}$-reducible 7-manifold is one of the following:*

$$S^3 \star S^3 \star S^3, \quad S^3 \star S^5, \quad S^3 \star \mathcal{S}_k$$

*for $3 \leq k \leq 8$ where $\mathcal{S}_k$ is one of the Sasakian-Einstein circle bundles over the del Pezzo surface discussed in Theorem **??**.*

These examples have already been noted in [**17**]. Non simply connected examples are obtained by quotienting by a cyclic subgroup of the circle generated by the characteristic vector field. Notice also that for $5 \leq k \leq 8$ the 7-manifolds $S^3 \star S_k$ have continuous families of Sasakian-Einstein structures on them. We shall discuss these shortly, but first we mention that in the list given in Proposition 10.4.7 only the first two are homogeneous. In fact as already noted in [**35**] we have

Proposition 10.4.8: *The simply connected homogeneous Sasakian-Einstein 7-manifolds are precisely one of the following:*

(i) *(i) An $\mathcal{SE}$-reducible manifold $S^3 \star S^3 \star S^3$ or $S^3 \star S^5$.*
(ii) *(ii) The real Stiefel manifold $V_{5,2}$.*
(iii) *(iii) A homogeneous 3-Sasakian 7-manifold S^7 or $\mathcal{S}(1,1,1)$.*

Of course, there are other regular $\mathcal{SE}$-irreducible 7-manifolds as noted in Examples **??** and **??**, but so far a complete classification of regular Sasakian-Einstein 7-manifolds is lacking. Actually the results of Mori and Mukai [**164**] on the classification of smooth Fano 3-folds with $b_2 \geq 2$ go quite far toward a classification of regular Sasakian-Einstein 7-manifolds, but this is beyond the scope of the present paper, and is currently under investigation. Nevertheless, there are some immediate consequences of their work worth mentioning here, namely

Proposition 10.4.9: *Let $\mathcal{S}$ be a compact regular Sasakian-Einstein 7-manifold. Then*

(i) *(i) $b_2(\mathcal{S}) \leq 9$.*
(ii) *(ii) If $b_2(\mathcal{S}) \geq 5$ then $\mathcal{S}$ is reducible. Explicitly $\mathcal{S} \approx S^3 \star \mathcal{S}_k$, where $4 \leq k \leq 8$.*

Notice that the upper bound on the second Betti number for regular Sasakian-Einstein 7-manifolds is realized by $S^3 \star S_8$, and there is a continuous 4-dimensional family of Sasakian-Einstein structures on these manifolds. It is interesting to contemplate whether it is generally true that the bound on b_2 is realized by a $\mathcal{SE}$-reducible element in $\mathcal{R}$. If this were true then the regular Sasakian-Einstein 9-manifold with the largest second Betti number would be $S_8 \star S_8$ with $b_2 = 17$. The answer probably lies in Mori theory. For general odd dimension greater than 3 our join construction gives

Proposition 10.4.10: *For each positive integer $n \geq 2$ there exists smooth manifolds of dimension $2n+1$ that admit continuous families $\mathcal{F}$ of regular Sasakian-Einstein structures. Examples of such the manifolds are*

$$S_k, \quad \overbrace{S^3 \star \cdots \star S^3}^{n-2 \text{ times}} \star S_k$$

for $5 \leq k \leq 8$, as well joins with other regular Sasakian-Einstein manifolds. Moreover, the dimension of the family $\mathcal{F}$ of Sasakian-Einstein structures on S_k is $\geq k-4$.

This result should be contrasted with the 3-Sasakian case which is infinitesimally rigid [**175**]. Manifolds that admit continuous families of non-regular Sasakian-Einstein structures also exist as will be discussed below. However, they begin in dimension 11.

Next we turn to the more lucrative non-regular case. Propositions **??** and 10.4.3 imply the following:

Corollary 10.4.11: *Let $\mathcal{S}$ be any compact simply connected 3-Sasakian manifold of dimension $4n+3$ that is not a sphere. Then for any positive integer r the join $S^{2(n+1)r-1} \star \mathcal{S}$ is a smooth Sasakian-Einstein manifold of dimension $4n+2nr+2r+1$. In particular, if $n=1$, so that dim $\mathcal{S}=7$, then $S^3 \star \mathcal{S}$ is a smooth Sasakian-Einstein 9-manifold.*

Corollary 10.4.12: **??** *Let $\mathcal{S}$ be any compact simply connected Sasakian-Einstein 7-manifold of index 2, for example, a 3-Sasakian 7-manifold that is not S^7. Then $S^{2m+1} \star \mathcal{S}$ is smooth if m is odd or if m is even and the order of $\mathcal{S}$ is odd.*

The last case in Corollary 10.4.11 will prove to be of much interest to us. Notice that $\mathrm{Ind}(S^3 \star \mathcal{S}) = \gcd(\mathrm{Ind}(S^3), \mathrm{Ind}(\mathcal{S})) = 2$, so that the procedure iterates arriving at:

Corollary 10.4.13: *Let $\mathcal{S}$ be any compact simply connected 3-Sasakian manifold of dimension 7 that is not a 7-sphere. Then for any positive integer r the r-fold join $S^3\star\cdots\star S^3\star\mathcal{S}$ is a smooth Sasakian-Einstein $2r+7$-manifold of index 2.*

The examples of a 3-Sasakian 7-manifold that we have in mind are the toric 3-Sasakian manifolds $\mathcal{S}(\Omega_k)$ of [**48**] where Ω_k is a k by $k+2$ matrix of integers which satisfy certain gcd conditions (see [**48**] for details), and $b_2(\mathcal{S}(\Omega_k))=k$. In the case of the join of two non-regular Sasakian-Einstein manifolds and/or when the relative indices are different, the gcd conditions in Proposition 10.4.3 are generally fairly restrictive. For example, let $\mathcal{S}_1,\mathcal{S}_2\in\mathcal{SE}^s_{4n+3}$ be two 3-Sasakian manifolds neither of which are spheres. Then $l_1=l_2=1$, and the smoothness conditions become $\gcd(m_1,m_2)=1$. Even this is restrictive since orders tend to be large and have many divisors. However, consider the 3-Sasakian manifolds $\mathcal{S}(1,\cdots,1,2p_i+1)$ discussed in [**41**]. In these cases $m_i=p_i+1$, so if we choose $\gcd(p_1+1,p_2+1)=1$, which is easy to satisfy, we get a smooth join. Notice, however, that $\mathcal{S}\star\mathcal{S}$ is never smooth for $\mathcal{S}$ non-regular.

CHAPTER 11

Quaternionic Kähler Manifolds

In this chapter we will recall some basic results concerning various metric structures on quaternionic manifolds.

11.1. Quaternionic Structures

11.2. Quternionic Kähler Metrics

CHAPTER 12

Hyperkähler Manifolds

In this chapter we will recall some basic results concerning hyperkähler manifolds.

12.1. Definition and Basic Properties

12.2. Hyperkähler Quotients

12.3. Other Constructions and Examples

12.4. Compact Hyperkähler Manifolds

CHAPTER 13

3-Sasakian Manifolds

In this chapter we consider a special kind of Sasakian-Einstein manifolds which we shall call manifolds (several different names have been introduced and used in the earlier literature on this subject). In 1970 Kuo [**141**] refined the notion of a Sasakian structure and introduced manifolds with Sasakian 3-structures (see also [**142**], [**201**]). Independently, the same concept was invented by Udrişte [**211**]. Between 1970 and 1975 this new kind of geometry was investigated almost exclusively by a group of Japanese geometers, including Ishihara, Kashiwada, Konishi, Kuo, Tachibana, Tanno, and Yu. Already in [**141**] we learn that the 3-Sasakian geometry has some interesting topological implications. Using earlier results of Tachibana about the harmonic forms on compact Sasakian spaces [**200**], Kuo showed that odd Betti numbers up to the middle dimension must be divisible by 4. In 1971 Kashiwada observed that every 3-Sasakian manifold is Einstein with a positive Einstein constant [**120**]. In the same year Tanno proved an interesting theorem about the structure of the isometry group of every 3-Sasakian space [**203**]. In a related paper he studied a natural 3-dimensional foliation on such spaces showing that, if the foliation is regular, then the space of leaves is an Einstein manifold of positive scalar curvature [**204**]. Tanno clearly points to the importance of the analogy with the quaternionic Hopf fibration $S^3 \to S^7 \to S^4$, but does not go any further. In fact, Kashiwada's paper mentions a conjecture speculating that every 3-Sasakian manifold is of constant curvature [**120**]. She attributed this conjecture to Tanno and, at the time, these were the only known examples.

Very soon after, however, it became clear that such a conjecture could not possibly be true. This is due to a couple of papers by Ishihara and Konishi [**?**], [**?**]. They made a fundamental observation that the space of leaves of the natural 3-dimensional foliations mentioned above has a "quaternionic structure", part of which is the Einstein metric discovered by Tanno. This led Ishihara to an independent study of this "sister geometry": quaternionic Kähler manifolds [**?**]. His paper is very well-known and is almost always cited as the source of the explicit coordinate description of quaternionic Kähler geometry. Among other results Ishihara showed that his definition implies that the holonomy group of the metric is a subgroup of $Sp(n)\cdot Sp(1)$, thus providing an important connection with the earlier studies of such manifolds by Alekseevsky [**3**], Bonan [**31**], Gray [**98**], Kraines [**134**], and Wolf [**222**]. In 1975 Konishi [**133**] proved the existence of a Sasakian 3-structure on a natural principal $SO(3)$-bundle over any quaternionic Kähler manifold of positive scalar curvature. This, with the symmetric examples of Wolf, gives precisely all of the homogeneous 3-Sasakian spaces. Yet, at the time they did not appear explicitly and escaped any systematic study until much later.

In fact, 1975 seems to be the year when 3-Sasakian manifolds are relegated to an almost complete obscurity which lasted for about 15 years. From that point on the two "sisters" fair very differently. The extent of this can be best illustrated by the famous book on Einstein manifolds by Besse [**22**]. The book appeared in 1987 and provided the reader with an excellent, up-to-date, and very complete account of what was known about Einstein manifolds 10 years ago. But one is left in the dark when trying to find references to any of the papers on 3-Sasakian manifolds we have cited; 3-Sasakian manifolds are never mentioned in Besse. The other "sister", on the contrary, received a lot of space in a separate chapter. Actually Einstein metrics on Konishi's bundle do appear in Besse (see [**22**] 14.85, 14.86) precisely in the context of the $SO(3)$-bundles over positive quaternionic Kähler manifolds as a consequence of a theorem of Bérard-Bergery ([**22**], 9.73). Obviously, the absence of 3-Sasakian spaces in Besse's book was the result rather than the cause of this obscurity. One could even say it was justified by the lack of any interesting examples. The authors have puzzled over this phenomenon without any sound explanation. One can only speculate that it is the holonomy reduction that made quaternionic Kähler manifolds so much more attractive an object. Significantly, the holonomy group of a 3-Sasakian manifold never reduces to a proper subgroup of the special orthogonal group. And when in 1981 Salamon [**?, ?**], independently with Bérard-Bergery [**20**], generalized Penrose's twistor construction for self-dual 4-manifolds introducing the twistor space over an arbitrary quaternionic Kähler manifold, the research on quaternionic Kähler geometry flourished, fueled by powerful tools from complex algebraic geometry.

Finally, in the early nineties, 3-Sasakian manifolds start a comeback mostly due to the work of the authors and B. Mann.

13.1. Definitions and Basic Properties

Using all the definitions of the previous section we now describe a more specialized situation. Again, this can be done by an additional holonomy reduction requirement.

Definition 13.1.1: *Let $(\mathcal{S},g)$ be a Riemannian manifold of real dimension m. We say that $(\mathcal{S},g)$ is 3-Sasakian if the holonomy group of the metric cone on $\mathcal{S}$ $(C(\mathcal{S}),\bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$ reduces to a subgroup of $Sp(\frac{m+1}{4})$. In particular, $m = 4n+3, n \geq 1$ and $(C(\mathcal{S}),\bar{g})$ is hyperkähler.*

Since $C(\mathcal{S})$ is hyperkähler it has a hypercomplex structure $\{I^1, I^2, I^3\}$. We can define $\xi^a = I^a(\partial_r)$ for each $a = 1,2,3$. Then using the well-known properties of a hypercomplex structure together with Proposition 2.1.2 gives:

Proposition 13.1.2: *Let $(\mathcal{S},g)$ be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. Then $\mathcal{S}$ is 3-Sasakian if and only if it admits three characteristic vector fields $\{\xi^1,\xi^2,\xi^3\}$ (that is, satisfying any of the corresponding conditions in Proposition 2.1.2) such that $g(\xi^a,\xi^b) = \delta_{ab}$ and $[\xi^a,\xi^b] = 2\epsilon_{abc}\xi^c$.*

Remark 13.1.1: By using Proposition 2.2.2 we can easily generalize the definition of a 3-Sasakian structure to orbifolds. A Riemannian orbifold $\mathcal{S}$ is a *3-Sasakian orbifold* if it admits three characteristic vector fields satisfying the conditions of Proposition 2.2.2, and if the action of the local uniformizing groups leaves the characteristic vector fields $\{\xi^1,\xi^2,\xi^3\}$ invariant.

The triple $\{\xi^1, \xi^2, \xi^3\}$ defines $\eta^a(Y) = g(\xi^a, Y)$ and $\Phi^a(Y) = \nabla_Y \xi^a$ for each $a = 1, 2, 3$. We call $\{\xi^a, \eta^a, \Phi^a\}_{a=1,2,3}$ the *3-Sasakian structure* on $(\mathcal{S}, g)$. The hyperkähler geometry of the cone $C(\mathcal{S})$ gives $\mathcal{S}$ a "quaternionic structure" reflected by the composition laws of the (1,1) tensors Φ^a. The following proposition describes additional properties of $\{\xi^a, \eta^a, \Phi^a\}$ not listed in Proposition 2.1.3(i-iv).

Proposition 13.1.3: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold and let $\{\xi^a, \eta^a, \Phi^a\}_{a=1,2,3}$ be its 3-Sasakian structure. Then*

$$\begin{aligned} \eta^a(\xi^b) &= \delta^{ab}, \\ \Phi^a \xi^b &= -\epsilon^{abc}\xi^c, \\ \Phi^a \circ \Phi^b - \xi^a \otimes \eta^b &= -\epsilon^{abc}\Phi^c - \delta^{ab}\mathrm{id}. \end{aligned}$$

Remark 13.1.2: For any $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ such that $\tau_1^2 + \tau_3^2 + \tau_3^2 = 1$ the vector field $\xi(\tau) = \tau_1\xi^1 + \tau_2\xi^2 + \tau_3\xi^3$ has the Sasakian property. Therefore a 3-Sasakian manifold has not just 3 but an S^2 worth of Sasakian structures. This is in complete analogy with the hyperkähler case, and perhaps the name *hypersasakian* would have been more consistent. However, most of the existing literature uses the name *Sasakian 3-structure* or, as we do, *3-Sasakian structure*. Thus we have decided to stay with the latter.

Since a hyperkähler manifold is Ricci-flat, Proposition *.*.* and its corollary immediately imply:

Corollary 13.1.4: *Every 3-Sasakian manifold $(\mathcal{S}, g)$ of dimension $4n+3$ is Einstein with Einstein constant $\lambda = 2(2n+1)$. Moreover, if $\mathcal{S}$ is complete it is compact with finite fundamental group.*

The important result that every 3-Sasakian manifold is Einstein was first obtained by Kashiwada [**120**] using tensorial methods. One can also easily verify the structure group of any 3-Sasakian manifold is reducible to $Sp(n) \times \mathbb{I}_3$, where $\mathbb{I}_3$ denotes the three by three identity matrix [**141**]. It follows [**?**] that

Corollary 13.1.5: *Every 3-Sasakian manifold $(\mathcal{S}, g)$ is spin.*

If $(\mathcal{S}, g)$ is compact the characteristic vector fields $\{\xi^1, \xi^2, \xi^3\}$ are complete and define a 3-dimensional foliation $\mathcal{F}_3$ on $\mathcal{S}$. The leaves of this foliation are necessarily compact as $\{\xi^1, \xi^2, \xi^3\}$ defines a locally free $Sp(1)$ action on $\mathcal{S}$. Hence, the foliation $\mathcal{F}_3$ is automatically quasi-regular and the space of leaves is a compact orbifold. We shall denote it by $\mathcal{O}$.

Definition 13.1.6: *Let $(\mathcal{S}, g)$ be be a compact 3-Sasakian manifold of dimension $4n+3$, $n \geq 1$, and let $\mathcal{F}_3$ be the 3-dimensional foliation defined by $\{\xi^1, \xi^2, \xi^3\}$. We say that $\mathcal{S}$ is regular if $\mathcal{F}_3$ is regular.*

Remark 13.1.3: When $\dim(\mathcal{S}) = 3$ the leaf space of the foliation $\mathcal{F}_3$ is a single point so it makes no sense to talk about the regularity of $\mathcal{F}_3$. In this case we will say that $\mathcal{S}$ is *regular* if the foliation $\mathcal{F}_1$ defined by the characteristic vector field ξ^1 is regular.

For any $\tau \in S^2$ we can consider again the characteristic vector field $\xi(\tau)$ associated with the direction τ. This vector field defines a 1-dimensional foliation $\mathcal{F}_\tau \subset \mathcal{F}_3 \subset \mathcal{S}$. This foliation has compact leaves and defines a locally free circle action $U(1)_\tau \subset Sp(1)$ on $\mathcal{S}$. In the next section we will describe the geometry of these foliations. Here, we simply conclude by the following observation concerning regularity properties of the foliations $\mathcal{F}_\tau \subset \mathcal{F}_3$ [**204**]:

Proposition 13.1.7: *Let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold. If $\mathcal{F}_3$ is regular then $\mathcal{F}_\tau$ is regular for all $\tau \in S^2$. Conversely, if $\mathcal{F}_\tau$ is regular for some $\tau = \tau_0 \in S^2$ then it is regular for all τ and, hence, $\mathcal{F}_3$ is regular. Furthermore, if $\mathcal{F}_3$ is regular then either all the leaves are diffeomorphic to $SO(3)$ or all the leaves are diffeomorphic to S^3.*

Actually in the regular case it follows from a deeper result of Simon Salamon [**?**] that all leaves are diffeomorphic to S^3 in precisely one case, namely when $\mathcal{S} = S^{4n+3}$. (See the next section for further discussion.)

Remark 13.1.4: Note that every Sasakian-Einstein 3-manifold must also have a 3-Sasakian structure. This is because in dimension four Ricci-flat and Kähler is equivalent to hyperkähler. Every compact 3-Sasakian 3-manifold, by Proposition 2.1.2(iii), must be a space of constant curvature 1. Hence, $\mathcal{S}$ is covered by a unit round 3-sphere and , in fact, it is always the homogeneous spherical space form S^3/Γ, where Γ is a discrete subgroup of $Sp(1)$ [**187**]. The homogeneous spherical space forms in dimension 3 are well-known. They are $Sp(1)/\Gamma$ where Γ is one of the finite subgroups of $Sp(1)$, namely: $\Gamma = \mathbb{Z}_m$ the cyclic group of order m, $\Gamma = \mathbb{D}^*_m$ a binary dihedral group with m is an integer greater than 2, $\Gamma = \mathbb{T}^*$ the binary tetrahedral group, $\Gamma = \mathbb{O}^*$ the binary octahedral group, $\Gamma = \mathbb{I}^*$ the binary icosahedral group. The only regular 3-Sasakian manifolds in dimension 3 are S^3 and $SO(3)$. More generally, the diffeomorphism classification of compact Sasakian 3-manifolds was recently completed by Geiges [**?**]. In addition to S^3/Γ one gets compact quotients of the double cover of $PSL_2(\mathbb{R})$ and the 3-dimensional Heisenberg group.

Remark 13.1.5: A Sasakian-Einstein structure on a 3-Sasakian manifold does not have to be a part of the 3-Sasakian structure. The simplest example when this is the case is the lens space $\mathbb{Z}_k\backslash S^3$. Consider the unit 3-sphere $S^3 \simeq Sp(1)$ as the unit quaternion $\sigma \in \mathbb{H}$. Such a sphere has two 3-Sasakian structures generated by the left and the right multiplication. Consider the homogeneous space $\mathbb{Z}_k\backslash S^3$, where the $\mathbb{Z}_k$-action is given by the multiplication from the left by $\rho \in Sp(1)$, $\rho^k = 1$. The quotient still has the "right" 3-Sasakian structure. But it also has a "left" Sasakian structure (the centralizer of $\mathbb{Z}_k$ in $Sp(1)$ is an S^1 and it acts on the coset from the left). This left Sasakian structure is actually regular while none of the Sasakian structures of the right 3-Sasakian structure can be regular unless $k = 1, 2$ [**205**].

13.2. The Fundamental Foliations

In this section we discuss the foliations associated 3-Sasakian manifolds and describe their consequences.

13.2.1. The One Dimensional 3-Sasakian Foliation. Fixing a Sasakian structure, say (ξ^1, Φ^1, η^1) in the 3-Sasakian structure, we notice that subbundle $\mathcal{H} = \ker \eta^1$ of $T\mathcal{S}$ together with $I = -\Phi^1|\mathcal{H}$ define the CR structure on $\mathcal{S}$. Actually a 3-Sasakian structure gives a special kind of CR structure, namely, a CR structure with a compatible holomorphic contact structure. Notice that the complex valued one form on $\mathcal{S}$ defined by $\eta^+ = \eta^2 + i\eta^3$ is type $(1, 0)$ on $\mathcal{S}$. Moreover, one checks that η^+ is holomorphic with respect to the CR structure I. Although the 1-form η^+ is not invariant under the circle action generated by ξ^1, the trivial complex line bundle L^+ generated by η^+ is invariant. Thus, the complex line bundle L^+ pushes down to a nontrivial complex V-line bundle $\mathcal{L}$ on $\mathcal{Z}$. Let V denote the one dimensional complex vector space generated by L^+. Writing the circle action as $\exp\,(i\phi\xi^1)$ shows that V is the representation with character $e^{-2i\phi}$, and since $\mathcal{S}$ is a principal S^1 V-bundle over $\mathcal{Z}$, the twisted product $\mathcal{L}\simeq\mathcal{S}\times_{S^1}V$ is a complex line V-bundle on $\mathcal{Z}$. Now we can define a map of V-bundles $\theta : T^{(1,0)}\mathcal{Z}\longrightarrow\mathcal{L}$ by

$$\theta(X)=\eta^+(\hat{X}),$$

where $\hat{X}$ is the horizontal lift of the vector field X on $\mathcal{Z}$. Notice that $\theta(X)$ is not a function on $\mathcal{Z}$ but a section of $\mathcal{L}$. Now a straightforward computation shows that $\eta^+\wedge(d\eta^+)^n$ is a nowhere vanishing section of $\Lambda^{(2n+1,0)}\mathcal{H}$ on $\mathcal{S}$, and thus $\theta\wedge(d\theta)^n$ is a nowhere vanishing section of $K\otimes\mathcal{L}^{n+1}$, where K is the canonical V-line bundle (see the Appendix) on $\mathcal{Z}$. Hence, in $\mathrm{Pic}^{orb}(\mathcal{Z})$ we have the relation $\mathcal{L}^{n+1}\otimes K=1$. So the contact line V-bundle is $\mathcal{L}\simeq K^{-\frac{1}{n+1}}$ in $\mathrm{Pic}^{orb}(\mathcal{Z})$. Alternatively, the subbundle $\ker\theta$ is a holomorphic subbundle of $T^{(1,0)}\mathcal{Z}$ which is maximally non-integrable. This defines the complex contact structure on $\mathcal{Z}$. Of course, this construction depends on a choice of direction $\tau\in S^2$ in the 2-sphere of complex structures. However, the transitive action of $Sp(1)$ on S^2 guarantees that this structure is unique up to isomorphism as complex contact manifolds. We have [**40, ?**]:

Theorem 13.2.1: *Let $\mathcal{S}$ be a complete 3-Sasakian manifold, choose a direction $\tau\in S^2$, and let $\mathcal{Z}_\tau$ denote the space of leaves of the corresponding foliation $\mathcal{F}_\tau$. Then $\mathcal{Z}_\tau$ is a compact $\mathbb{Q}$-factorial contact Fano variety with a Kähler-Einstein metric h of scalar curvature $8(2n+1)(n+1)$ such that the natural projection $\pi:\mathcal{S}\longrightarrow\mathcal{Z}_\tau$ is an orbifold Riemannian submersion with respect to the Riemannian metrics g on $\mathcal{S}$ and h on $\mathcal{Z}_\tau$.*

We call the space $\mathcal{Z}_\tau$, usually just written $\mathcal{Z}$, the *twistor space* associated to $\mathcal{S}$. Actually there is another object that could merit the name the twistor space of $\mathcal{S}$, namely the trivial 2-sphere bundle $S^2\times\mathcal{S}$ with the structure induced from the twistor space $S^2\times C(\mathcal{S})$ of the hyperkähler cone.

An important property of the twistor space in the case of quaternionic Kähler manifolds is that it is ruled by rational curves. The same is true in our case as long as one allows for singularities. We have

Proposition 13.2.2: *$\mathcal{Z}$ is ruled by a real family of rational curves C with possible singularities on the singular locus of $\mathcal{Z}$. All the curves C are simply-connected, but $\pi_1^{orb}(C)$ can be a non-trivial cyclic group.*

For any line V-bundle $\mathcal{L}$ we let $\hat{\mathcal{L}}$ denote $\mathcal{L}$ minus its zero section.

Proposition 13.2.3: *Let $\mathcal{Z}$ be the twistor space of a 3-Sasakian manifold $\mathcal{S}$ of dimension $4n+3$, and assume that $\pi_1^{orb}(\mathcal{Z})=0$. If the contact line V-bundle $\mathcal{L}$ (or equivalently its dual $\mathcal{L}^{-1}$) has a root in Pic$^{orb}(\mathcal{Z})$, then it must be a square root, namely $\mathcal{L}^{\frac{1}{2}}$. Moreover, in this case if both $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}^{\frac{1}{2}}$ are proper in the sense of Kawasaki, then we must have $\mathcal{Z}=\mathbb{P}^{2n+1}$. In particular, this holds if the total space of $\hat{\mathcal{L}}$ is smooth.*

Proof. By Proposition 3.1.3 $\mathrm{Pic}^{orb}(\mathcal{Z})$ is torsion free. So the proof in [**?**] now goes through. By Proposition 3.2.2 $\mathcal{Z}$ is ruled by rational curves C which on the singular locus take the form $\Gamma\backslash\mathbb{P}^1$. Now the restriction $\mathcal{L}^{-1}|C$ is $\mathcal{O}(-2)$ which is a V-bundle if C is singular. In either case it has only a square root namely the tautological V-bundle $\mathcal{O}(-1)$. Since these curves C cover $\mathcal{Z}$ this proves the first statement. The second statement follows from a modification of an argument due to Kobayashi and Ochiai [**130**] and used by Salamon [**?**]. The main point is that since $\hat{\mathcal{L}}, \hat{\mathcal{L}}^{\frac{1}{2}}$ are proper and it follows that we can apply Kawasaki's Riemann-Roch Theorem [**121**] together with the Kodaira-Baily Vanishing Theorem [**?**] to arbitrary powers of the line V-bundle $\mathcal{L}^{\frac{1}{2}}$ to give $(n+1)(2n+3)$ infinitesimal automorphisms of the complex contact structure on $\mathcal{Z}$. Since $\pi_1^{orb}(\mathcal{Z}) = 0$, these integrate to global automorphisms on $\mathcal{Z}$ and the result follows. See the Appendix and [**?**] for details. □

Remark 13.2.1: : There is an error in the statement of Proposition 4.3 of [**?**]. The error is in leaving out the assumptions that $\pi_1^{orb}(\mathcal{Z})$ is trivial and that the contact line bundle is proper. Example 3.1 below shows that the conclusion in Proposition 3.2.3 does not necessarily hold if the hypothesis $\pi_1^{orb}(\mathcal{Z}) = 0$ is omitted. Likewise, Example 3.2 below gives a counterexample when the condition that $\mathcal{L}$ be proper is omitted.

Example 13.2.4: Consider the 3-Sasakian lens space $L(p;q) = \mathbb{Z}_p\backslash S^7$ constructed as follows: S^7 is the unit sphere in the quaternionic vector space $\mathbb{H}^2$ with quaternionic coordinates u_1, u_2. The action of $\mathbb{Z}_p$ is the left action defined by $(u_1, u_2) \mapsto (\tau u_1, \tau^q u_2)$, where $\tau^p = 1$ and p and q are relatively prime positive integers. If $p = 2m$ for some integer m then $-\mathrm{id}$ is an element of $\mathbb{Z}_{2m}$, so the 3-Sasakian manifolds $L(2m;q)$ and $L(m;q)$ both have the same twistor space, namely $\mathcal{Z} = \mathbb{Z}_m\backslash\mathbb{P}^3$, and $\pi_1^{orb}(\mathcal{Z}) \simeq \mathbb{Z}_m$. There are clearly many similar examples in all dimensions equal to 3 mod 4.

Example 13.2.5: Consider the 3-Sasakian 7 manifolds $\mathcal{S}(p_1,p_2,p_3)$ described in Section 8.4 below, where the p_i's are pairwise relatively prime, and precisely one of the p_i's is even, say p_1. $\mathcal{S}(p_1,p_2,p_3)$ is simply-connected and its twistor space $\mathcal{Z}(p_1,p_2,p_3)$ has $\pi_1^{orb}(\mathcal{Z}(p_1,p_2,p_3)) = 0$. Now there is a $\mathbb{Z}_2$ acting on $\mathcal{S}(p_1,p_2,p_3)$, but not freely, which acts as the identity on $\mathcal{Z}(p_1,p_2,p_3)$. Thus, $\mathbb{Z}_2\backslash\mathcal{S}(p_1,p_2,p_3)$ has $\mathcal{Z}(p_1,p_2,p_3)$ as its twistor space, and as a V-bundle $\mathbb{Z}_2\backslash\mathcal{S}(p_1,p_2,p_3) \longrightarrow \mathcal{Z}(p_1,p_2,p_3)$ is not proper in the sense of Kawasaki [**122**]. Thus, the V-bundle $\hat{\mathcal{L}}$ is not proper, and Kawasaki's Riemann-Roch theorem [**121**] cannot be applied.

We now wish to formulate a converse to Theorem 3.2.1.

Definition 13.2.6: *A complete $\mathbb{Q}$-factorial Fano contact variety $\mathcal{Z}$ is said to be good if the total space of the principal circle bundle $\mathcal{S}$ associated with the contact V-line bundle $\mathcal{L}$ is a smooth compact manifold.*

Thus, for good $\mathbb{Q}$-factorial Fano contact varieties, $\mathcal{S}$ desingularizes $\mathcal{Z}$. As discussed in the Appendix this happens precisely when all the leaf holonomy groups inject into the group S^1 of the bundle. Notice also that in this case $\mathcal{S}$ is necessarily compact. We now are ready for:

Theorem 13.2.7: *A good $\mathbb{Q}$-factorial Fano contact variety $\mathcal{Z}$ is the twistor space associated to a compact 3-Sasakian manifold if and only if it admits a compatible Kähler-Einstein metric h.*

Proof. Let $\mathcal{Z}$ be a good $\mathbb{Q}$-factorial Fano contact variety with a compatible Kähler-Einstein metric h. Choose the scale of h so that the scalar curvature is

$8(2n+1)(n+1)$. Let $\pi : \mathcal{S} \to \mathcal{Z}$ denote the principal orbifold circle bundle associated to $\mathcal{L}$. It is a smooth compact submanifold embedded in the dual of the contact V-line bundle $\mathcal{L}^{-1}$. The Kähler-Einstein metric h has Ricci form $\rho = 4(n+1)\omega$, where ω is the Kähler form on $\mathcal{Z}$, and ρ represents the first Chern class of K^{-1}. Let η^1 be the connection in $\pi : \mathcal{S} \to \mathcal{Z}$ with curvature form $2\pi^*\omega$. Then the Riemannian metric $g_{\mathcal{S}}$ on $\mathcal{S}$ can be defined by $g_{\mathcal{S}} = \pi^* h + (\eta^1)^2$. It is standard (see the proof in Example 1 of Section 4.2 in [**?**]) that $g_{\mathcal{S}}$ is Sasakian-Einstein. As in Proposition 2.2.4 of [**199**] the V-bundle $\mathcal{L} \otimes \Lambda^{(1,0)} \mathcal{Z}$ has a section θ such that the Kähler-Einstein metric h decomposes as $h = |\theta|^2 + h_D$, where h_D is a metric in the V-bundle D. Let us write $\pi^*\theta = \eta^+$. Since $\mathcal{S}$ is a circle bundle in $\mathcal{L}^{-1}$, the contact bundle $\mathcal{L}$ trivializes when pulled back to $\mathcal{S}$. This together with the condition that $\theta \wedge (d\theta)^n$ is nowhere vanishing on $\mathcal{Z}$ implies that η^+ is a nowhere vanishing complex valued 1-form on $\mathcal{S}$. So the metric $g_{\mathcal{S}}$ on $\mathcal{S}$ can be written as

$$g_{\mathcal{S}} = (\eta^1)^2 + |\eta^+|^2 + \pi^* h_D.$$

We claim that this metric is 3-Sasakian. To see this consider the total space M of the dual of the contact V-line bundle minus its 0 section which is $\mathcal{S} \times \mathbb{R}^+$. Put the cone metric $dr^2 + r^2 g$ on M. The natural $\mathbb{C}^*$ action on M induces homotheties of this metric. Now using a standard Weitzenböck argument, LeBrun [**145**] shows that M has a parallel holomorphic symplectic structure and his argument works just as well in our case. Let ϑ denote the pullback of the contact form θ to M which is a holomorphic 1-form on M that is homogeneous of degree 1 with respect to the $\mathbb{C}^*$ action. Thus $\Upsilon = d\vartheta$ is a holomorphic symplectic form on M which is parallel with respect to the Levi-Civita connection of the cone metric. Hence, $(M, dr^2 + r^2 g)$ is hyperkähler. Furthermore, if $\{I^a\}_{a=1}^3$ denote hyperkähler endomorphisms on M, ϑ^2, ϑ^3 are the real and imaginary parts of ϑ, and ϑ^1 is the pullback of η^1 to M, then LeBrun shows that

$$\vartheta^1 I^1 = \vartheta^2 I^2 = \vartheta^3 I^3.$$

It then follows from our previous work [**40**] that g is 3-Sasakian. But by construction $\mathcal{Z}$ is the space of leaves of the foliation generated by ξ^1, so $\mathcal{Z}$ must be the twistor space of the compact 3-Sasakian manifold $\mathcal{S}$. □

13.2.2. The Three Dimensional 3-Sasakian Foliation. Next we consider the three dimensional foliation $\mathcal{F}_3$ discussed in Section 2.2.

Proposition 13.2.8: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold such that the characteristic vector fields ξ^a are complete. Let $\mathcal{F}_3$ denote the the canonical three dimensional foliation on $\mathcal{S}$. Then*
(i) The metric g is bundle-like.
(ii) The leaves of $\mathcal{F}_3$ are totally geodesic spherical space forms $\Gamma\backslash S^3$ of constant curvature one, where $\Gamma \subset Sp(1) = SU(2)$ is a finite subgroup.
(iii) The 3-Sasakian structure on $\mathcal{S}$ restricts to a 3-Sasakian structure on each leaf.
(iv) The generic leaves are either $SU(2)$ or $SO(3)$.

PROOF. The proof of (i), (ii), and (iii) follow from the basic relations for 3-Sasakian manifolds as in Proposition 2.1.1. To prove (iv) we notice that the foliation $\mathcal{F}_3$ is regular restricted to the generic stratum $\mathcal{S}_0$. By (ii) and regularity there is a finite subgroup $\Gamma \subset SU(2)$ such that the leaves of this restricted foliation are all diffeomorphic to $\Gamma\backslash S^3$, which is 3-Sasakian by (iii). Now the regularity of $\mathcal{F}_3$ on $\mathcal{S}_0$

implies that its leaves must all be regular with respect to the foliation generated by ξ^1. But a result of Tanno [**203**] says that the only regular 3-Sasakian 3-manifolds have $\Gamma = \mathrm{id}$ or $\mathbb{Z}_2$, in which case (iv) follows. $\square$

EXAMPLE 13.2.9: Consider the 3-Sasakian lens space $L(p;q) = \mathbb{Z}_p\backslash S^7$ of Example 2.1. If p is odd then $-\mathrm{id}$ is not an element of $\mathbb{Z}_p$ so the generic leaf of the foliation $\mathcal{F}_3$ is S^3. The singular stratum consists of two leaves both of the form $\mathbb{Z}_p\backslash S^3$ with leaf holonomy group $\mathbb{Z}_p$. These two leaves are described by $u_2 = 0$ and $u_1 = 0$, respectively. If p is even then $-\mathrm{id}$ is an element of $\mathbb{Z}_p$, so the generic leaf is $SU(2)/\mathbb{Z}_2 = SO(3)$, and the leaf holonomy of the two singular leaves is $\mathbb{Z}_{\frac{p}{2}}$.

The next theorem was first proved by Ishihara [**?**] in the regular case using slightly different methods. First we need to describe our structures in the orbifold category. Recall that a quaternionic Kähler structure on a Riemannian manifold M is defined by

Definition 13.2.10: *A Riemannian orbifold $\mathcal{O}$ is called a quaternionic Kähler orbifold if there is a rank 3 V-subbundle $\mathcal{G}$ of the endomorphism V-bundle End TM of TM which is preserved by the Levi-Civita connection and is locally generated by almost complex structures I, J, K that satisfy the algebra of the quaternions, and the action of the local uniformizing groups preserves the bundle $\mathcal{G}$. An alternative definition which works only in dimension greater than 4 is that $\mathcal{O}$ is a Riemannian orbifold whose holonomy group is a subgroup of $Sp(n)\cdot Sp(1)$.*

It is well-known that the strata of a quaternionic Kähler orbifold are not necessarily quaternionic Kähler [**88**]. The strata will be quaternionic Kähler if the local uniformizing groups act trivially on the fibres of $\mathcal{G}$ [**63**]. The group of the bundle $\mathcal{G}$ is $SO(3)$ with the adjoint representation. Thus, for each local uniformizing system on $\mathcal{O}$ there is a group homomorphism $\psi_i : \Gamma_i \longrightarrow SO(3)$.

Theorem 13.2.11: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold of dimension $4n+3$ such that the characteristic vector fields ξ^a are complete. Then the space of leaves $\mathcal{S}/\mathcal{F}_3$ has the structure of a quaternionic Kähler orbifold $(\mathcal{O}, g_{\mathcal{O}})$ of dimension $4n$ such that the natural projection $\pi : \mathcal{S} \longrightarrow \mathcal{O}$ is a principal V-bundle with group $SU(2)$ or $SO(3)$ and a Riemannian orbifold submersion such that the scalar curvature of $g_{\mathcal{O}}$ is $16n(n+2)$.*

PROOF. We can split $T\mathcal{S} = \mathcal{V}_3 \oplus \mathcal{H}$, where $\mathcal{V}_3$ is the subbundle spanned by the characteristic vector fields $\{\xi^1, \xi^2, \xi^3\}$ and the "horizontal" bundle is the orthogonal complement $\mathcal{H} = \mathcal{V}_3^{\perp}$. Let $h\Phi^a = \Phi^a\mid_{\mathcal{H}}$ be the restriction of characteristic endomorphisms. One can easily see that

$$h\Phi^a \circ h\Phi^b = -\delta^{ab}\mathbf{1} + \sum_c \epsilon^{abc} h\Phi^c.$$

It follows that $\mathcal{H}$ is pointwise a quaternionic vector space and $\mathcal{O}$ is a compact quaternionic orbifold. We must show that the metric $g_{\mathcal{O}}$ obtained from g by the orbifold Riemannian submersion $\pi : \mathcal{S} \to \mathcal{O}$ has its holonomy group reduced to a subgroup of $Sp(n)\cdot Sp(1)$. This can be done by constructing a parallel 4-form on $\mathcal{O}$. Consider $\phi^a = d\eta^a$ and

$$\bar{\Phi}^a = \phi^a + \sum_{b,c} \epsilon^{abc} \eta^b \wedge \eta^c.$$

It is easy to see that the 4-form $\Omega = \sum_a \bar{\Phi}^a \wedge \bar{\Phi}^a$ is horizontal and $Sp(1)$-invariant. It follows that there is a unique 4-form $\hat{\Omega}$ on the orbifold $\mathcal{O}$ invariant under the action of the local uniformizing groups such that $\pi^*\hat{\Omega} = \Omega$. One can show that $\hat{\Omega}$ is parallel on $\mathcal{O}$ using standard tensor computation with O'Neill formulas (see [**40, 88**] for details). In the case $n = 1$ the parallelism of the 4-form does not further restrict Riemannian geometry of $\mathcal{O}$. However, one can show that $(\mathcal{O}, g_{\mathcal{O}})$ is a compact self-dual Einstein orbifold. Self-duality follows easily from the fact that $\mathcal{O}$ is quaternionic. The fact that the metric is Einstein is a simple computation and in the regular case can be found in [**204**]. $\square$

There is an important inversion theorem of Theorem 3.3.3 originally in the regular case due to Konishi [**133**]. By now there are several proofs of this, all of them related. Given a quaternionic Kähler orbifold $\mathcal{O}$ one can construct the Salamon twistor space $\mathcal{Z}$ and then get $\mathcal{S}$ from the inversion theorem of [**?**]. Another approach would be to construct the orbifold version of Swann's bundle [**199**] on $\mathcal{O}$ and then use the results of [**40**] to obtain $\mathcal{S}$. Here our proof is essentially that of Konishi's, only slightly modified to handle the orbifold situation.

Theorem 13.2.12: *Let $(\mathcal{O}, g_{\mathcal{O}})$ be a quaternionic Kähler orbifold of dimension $4n$ with positive scalar curvature $16n(n+2)$. Then there is a principal $SO(3)$ V-bundle over $\mathcal{O}$ whose total space $\mathcal{S}$ admits a 3-Sasakian structure with scalar curvature $2(2n+1)(4n+3)$.*

PROOF. Let $\mathcal{G}$ denote the V-subbundle of End $T\mathcal{O}$ describing the quaternionic structure. Let $\{\tilde{U}_i\}$ be local uniformizing neighborhoods that cover $\mathcal{O}$ and $\mathcal{I}_i^a$ a local framing of $\mathcal{G}$ on $\tilde{U}_i$ that satisfies

$$\mathcal{I}_i^a \circ \mathcal{I}_i^b = -\delta^{ab}\mathrm{id} + \epsilon^{abc}\mathcal{I}_i^c.$$

Since $\mathcal{O}$ is quaternionic Kähler there are 1-forms τ_i^a on each $\tilde{U}_i$ such that

$$\nabla \mathcal{I}_i^a = \epsilon^{abc}\tau_i^b \otimes \mathcal{I}_i^c.$$

Now the structure group of the V-bundle End $T\mathcal{O}$ is $Sp(n)\cdot Sp(1)$, and that of the V-subbundle $\mathcal{G}$ is $SO(3)$. Let $\pi : \mathcal{S} \longrightarrow \mathcal{O}$ denote the principal $SO(3)$ V-bundle associated to $\mathcal{G}$. The local 1-forms τ_i^a are the components of an $\mathfrak{so}(3)$ connection $\tau_i = \sum_{a=1}^3 \tau_i^a e_a$, where $\{e_a\}$ denotes the standard basis of $\mathfrak{so}(3)$ which satisfies the Lie bracket relations $[e_a, e_b] = 2\epsilon^{abc}e_c$. The local connection forms satisfy the well-known relations

$$\tau_i = \mathrm{ad}_{g_{ij}}\tau_j + g_{ij}^{-1}dg_{ij}$$

in $\tilde{U}_i \cap \tilde{U}_j$ for some smooth map $g_{ij} : \tilde{U}_i \cap \tilde{U}_j \longrightarrow SO(3)$. Furthermore, from [**?**] one checks that the curvature forms

$$\omega_i^a = d\tau_i^a + \epsilon^{abc}\tau_i^b \wedge \tau_i^c \tag{13.2.1}$$

satisfy the relation $2g_{\mathcal{O}}(\mathcal{I}_i^a X, Y) = \omega_i^a(X, Y)$. Now on each $\tilde{U}_i$ there exists a smooth local section $\sigma_i : \tilde{U}_i \longrightarrow \mathcal{S}$ and on $\mathcal{S}$ there is a global 1-form η^a such that $\tau_i^a = \sigma_i^*\eta^a$. On $\mathcal{S}$ we define a Riemannian metric by

$$g = \pi^* g_{\mathcal{O}} + \sum_{a=1}^3 \eta^a \otimes \eta^a.$$

By construction the vector fields ξ^a generating the $SO(3)$ action on $\mathcal{S}$ are dual to the forms η^a with respect to this metric, viz $g_{\mathcal{S}}(X, \xi^a) = \eta^a(X)$ for any vector field

X on $\mathcal{S}$. Also by construction the vector fields ξ^a are Killing fields with respect to the metric g. Now define the $(1,1)$ tensor field $\Phi^a = \nabla\xi^a$. Since the ξ^a are mutually orthogonal vector fields of unit length on $\mathcal{S}$, one easily checks that $\Phi^a\xi^b = -\epsilon^{abc}\xi^c$. Thus Φ^a splits as

$$\Phi^a = h\Phi^a + \epsilon^{abc}\xi^b \otimes \eta^c.$$

One then checks using 13.2.1 that on each open set $\pi^{-1}(U_i)$, $h\Phi^a$ equals the horizontal component of $(\sigma_i)_*\mathcal{I}_i^a$. From this one then shows that

$$\Phi^a \circ \Phi^b - \xi^a \otimes \eta^b \;=\; -\epsilon^{abc}\Phi^c - \delta^{ab}\mathrm{id}.$$

The result then follows by Propositions 2.2.2 and 2.2.3. □

We mention that from the discussion in the Appendix it follows that if the homomorphisms $\psi_i : \Gamma_i \longrightarrow SO(3)$ are injective the total space $\mathcal{S}$ will be a smooth 3-Sasakian manifold.

Konishi's construction gives an $SO(3)$ bundle over $\mathcal{O}$. In the case that $\mathcal{O}$ is a smooth manifold there is a well-known obstruction [**?**] to lifting this bundle to an $Sp(1)$ bundle, the Marchiafava-Romani class ε. Actually ε is the obstruction to lifting the principal $Sp(n){\cdot}Sp(1)$ frame bundle to an $Sp(n){\times}Sp(1)$ bundle [**150, ?**]. This obstruction also occurs when $\mathcal{O}$ is an orbifold as long as one uses Haefliger's orbifold cohomology (see Appendix). The class ε is the image of the connecting homomorphism

$$\delta : H^1_{orb}(\mathcal{O},\mathcal{G}) \longrightarrow H^2_{orb}(\mathcal{O},\mathbb{Z}_2), \tag{13.2.2}$$

where $\mathcal{G}$ is the sheaf of germs of smooth orbifold maps from open sets of $\mathcal{O}$ to the group $Sp(n){\cdot}Sp(1)$. If following Salamon [**?**] we write $T\mathcal{O}\otimes\mathbb{C} \simeq \mathbf{E}\otimes\mathbf{H}$, then ε is the second Stiefel-Whitney class w_2 of the bundle $S^2(\mathbf{H})$ over $\mathcal{O}$. We have

Proposition 13.2.13: *The principal $SO(3)$ V-bundle constructed in Theorem 3.3.4 lifts to a principal $Sp(1)$ V-bundle if and only if $\varepsilon \in H^2_{orb}(\mathcal{O},\mathbb{Z}_2)$ vanishes. Moreover, when $\varepsilon = 0$ the 3-Sasakian structure on the total space $\mathcal{S}$ of the $SO(3)$ V-bundle lifts to the total space $\mathcal{S}'$ of the $Sp(1)$ V-bundle.*

Thus, in the case that $\varepsilon = 0$ there are precisely two 3-Sasakian orbifolds $\mathcal{S},\mathcal{S}'$ corresponding to the quaternionic Kähler orbifold $\mathcal{O}$. Let $\mathcal{Z}$ denote the twistor space of the orbifold $\mathcal{O}$. Then likewise, since $S^2 \simeq SO(3)/S^1 \simeq Sp(1)/S^1$ the two 3-Sasakian orbifolds $\mathcal{S}$ and $\mathcal{S}'$ have the same twistor space $\mathcal{Z}$. When $\mathcal{O}$ is a smooth manifold a result of Salamon [**?**] says that $\varepsilon = 0$ if and only if the quaternionic Kähler manifold $\mathcal{O}$ is quaternionic projective space. If we impose the condition that the orbifolds $\mathcal{S}$ and $\mathcal{S}'$ are smooth manifolds, there is a similar result.

Theorem 13.2.14: *If two 3-Sasakian manifolds $\mathcal{S}$ and $\mathcal{S}'$ are associated to the same quaternionic Kähler orbifold $\mathcal{O}$ or equivalently the same twistor space $\mathcal{Z}$, then both $\mathcal{S}$ and $\mathcal{S}'$ have the same universal covering space $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ is a standard 3-Sasakian sphere.*

Proof. We work with the twistor space $\mathcal{Z}$. Now $\mathcal{S}$ and $\mathcal{S}'$ are unit circle bundles in the line V-bundles $\mathcal{L}^{-1}$ and $(\mathcal{L}')^{-1}$ respectively. Moreover, since $\mathcal{S}'$ is a double cover of $\mathcal{S}$, it follows that $\mathcal{L} = (\mathcal{L}')^2$. Consider the universal orbifold cover $\tilde{\mathcal{Z}}$ of $\mathcal{Z}$ with $\pi_1^{orb}(\tilde{\mathcal{Z}}) = 0$. Pull back the V-bundles $\mathcal{L}$ and $\mathcal{L}'$ to V-bundles $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}'$ on $\tilde{\mathcal{Z}}$ respectively. These bundles satisfy $\tilde{\mathcal{L}}' = \widetilde{\mathcal{L}^{\frac{1}{2}}}$. By construction and naturality of the covering maps $\tilde{\mathcal{L}}$ is the contact line bundle on $\tilde{\mathcal{Z}}$. Moreover, $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}'$ which are the total spaces of the pullbacks of $\mathcal{S}$ and $\mathcal{S}'$ to $\tilde{\mathcal{Z}}$ are both smooth manifolds since they cover smooth manifolds. Thus, by Proposition 3.2.3 $\tilde{\mathcal{Z}} \simeq \mathbb{P}^{2n+1}$. It follows that $\tilde{\mathcal{S}}' \simeq S^{4n+3}$. $\square$

Remark 13.2.2: Konishi also considers the case when the quaternionic Kähler manifold has negative scalar curvature. This gives a Sasakian 3-structure on $\mathcal{S}$ with indefinite signature $(3, 4n)$.

Finally, we give some general results concerning the curvature of any 3-Sasakian manifold. Since the curvature of any Riemannian manifold is completely determined by its sectional curvature and the sectional curvature of any Sasakian manifold [**?, ?**] is completely determined by the Φ-sectional curvature, we essentially give the latter. This shows that the local geometry of any 3-Sasakian manifold determines and is determined by that of its associated quaternionic Kähler orbifold.

Proposition 13.2.15: *Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian manifold and let K and $\check{K}$ denote the sectional curvatures of g and its transverse component $\check{g}$, respectively. Then if X is any horizontal vector field of unit length on $\mathcal{S}$, we have $K(X, \Phi^a X) = \check{K}(X, \Phi^a X) - 3$.*

Proof. We first notice that [**22**], 9.29c gives $K(X, \Phi^a X) = \check{K}(X, \Phi^a X) - 3|A_X \Phi^a X|^2$, where A is O'Neill's tensor [**22**] which is essentially the curvature of the $\mathfrak{sp}(1)$-valued orbifold connection [**40**]. One then shows that for any horizontal vector fields X, Y on $\mathcal{S}$ we have

$$A_X Y = \sum_{a=1}^{3} g(\Phi^a X, Y)\xi^a, \tag{13.2.3}$$

and the identity follows. $\square$

13.2.3. The Second Einstein Metric.

Of course, by an Einstein metric we actually mean a homothety class of Einstein metrics. In this section we shall show by using a theorem of Berard-Bergery [**22**] that every 3-Sasakian manifold has at least two distinct homothety classes of Einstein metrics. The method involves the canonical variation [**22**] associated with Riemannian submersions. Due to the local nature of the calculations involved this construction holds equally well for orbifold Riemannian submersions. The canonical variation is constructed as follows [**22**]: Let $\pi : M \longrightarrow B$ be an orbifold Riemannian submersion with g the Riemannian metric on M. Let $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal subbundles of the tangent bundle TM. For each real number $t > 0$ we construct a one parameter family g_t of Riemannian metrics on M by defining

$$g_t|\mathcal{V} = tg|\mathcal{V}, \qquad g_t|\mathcal{H} = g|\mathcal{H}, \qquad g_t(\mathcal{V}, \mathcal{H}) = 0. \tag{13.2.4}$$

So for each $t > 0$ we have an orbifold Riemannian submersion with the same base space. Furthermore, if the fibers of g are totally geodesic, so are the fibers of g_t. We apply the canonical variation to the orbifold Riemannian submersion $\pi : \mathcal{S} \longrightarrow \mathcal{O}$. The metric as well as other objects on $\mathcal{O}$ will be denoted with a check such as $\check{g}$.

Theorem 13.2.16: *Every 3-Sasakian manifold admits a second Einstein metric of positive scalar curvature.*

Proof. We apply the canonical variation to the orbifold Riemannian submersion $\pi : \mathcal{S} \longrightarrow \mathcal{O}$. According to the Bérard-Bergery Theorem [**22**], 9.73 there are several conditions to check. First, the connection $\mathcal{H}$ must be a Yang-Mills connection. The condition for this is [**22**]:

$$\sum_i g((\nabla_{X_i}A)_{X_i}X, \xi^a) = 0$$

for each $a = 1, 2, 3$ and where X_i is a local orthonormal frame of $\mathcal{H}$, X is any horizontal vector field, and A is O'Neill's tensor. Actually we can use standard computations together with Proposition 2.1.2 and Equation 13.2.3 to prove the stronger condition $(\nabla_{X_i}A)_{X_i}X = 0$. Second $|A|^2$ must be constant. To compute this notice that using [**22**] and Equation 13.2.3 we find

$$g(A_{X_i}, A_{X_j}) = 3\delta_{ij}, \qquad g(A\xi^a, A\xi^b) = 4n\delta^{ab}. \tag{13.2.5}$$

This gives $|A|^2 = 12n$. The final condition to be satisfied is $(\check{\lambda})^2 - \hat{\lambda}(12n + 18) > 0$, where $\check{\lambda}$ and $\hat{\lambda}$ are the Einstein constants for $\mathcal{O}$ and the fibers, respectively, and we have made use of 13.2.5. Since in our case $\check{\lambda} = 4(n + 2)$ and $\hat{\lambda} = 2$, we see that the inequality is satisfied. □

The scalar curvature of any metric g_t in the canonical variation of the metric g is given by the formula $s_t = 16n(n + 2) + 6/t - 12nt$ [**22**]. Moreover, the value of t that gives the second Einstein metric is $t_0 = \frac{\hat{\lambda}}{\check{\lambda}-\hat{\lambda}} = \frac{1}{2n+3}$. The ratio of the two metrics depends only on the homothety class and is given by

$$\frac{s_{\frac{1}{2n+3}}}{s_1} = 1 + \frac{6(n + 1)}{(2n + 3)(2n + 1)}.$$

In the special case $n = 1$ that is dim $\mathcal{S} = 7$, both the 3-Sasakian metric and the second Einstein metric have weak G_2 holonomy [**83, 90**]. See Theorem 5.2.9 below.

13.2.4. Invariants and the Classification of 3-Sasakian Structures. We consider the question of equivalence of 3-Sasakian manifolds. A 3-Sasakian structure $\{\xi^a, \eta^a, \Phi^a\}_{a=1}^3$ on a manifold $(\mathcal{S}, g)$ is determined completely by the metric g and the characteristic vector fields ξ^a.

Definition 13.2.17: *Two 3-Sasakian manifolds $(\mathcal{S}, g)$ and $(\mathcal{S}', g')$ are said to be isomorphic if there exist a diffeomorphism $F : \mathcal{S} \longrightarrow \mathcal{S}'$ and an $\phi \in Sp(1)$ such that $F^*g' = g$ and $\acute{\xi}^a = (Ad_\phi)_*F_*\xi^a$, where Ad denotes the adjoint action of $Sp(1)$ on its Lie algebra $\mathfrak{sp}(1)$.*

In practice we shall always choose a basis $\acute{\xi}^a$ of the 3-Sasakian structure on $\mathcal{S}'$ so that $F_*\xi^a = \acute{\xi}^a$. Now given such a diffeomorphism $F : \mathcal{S} \longrightarrow \mathcal{S}'$ it is clear that the corresponding foliations are F-related, that is that $F_*\mathcal{F}_1 = \mathcal{F}_1'$ and $F_*\mathcal{F}_3 = \mathcal{F}_3'$. This induces a commutative diagram of orbifold diffeomorphisms

$$
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{F} & \mathcal{S}' \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{Z} & \xrightarrow{F_1} & \mathcal{Z}' \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{O} & \xrightarrow{F_3} & \mathcal{O}'.
\end{array}
\tag{13.2.6}
$$

This implies that if L_x is the leaf of $\mathcal{F}_3$ at $x \in \mathcal{O}$, then $F(L_x)$ is the leaf at $F_3(x) \in \mathcal{O}'$, that is, $L'_{F_3(x)} = F(L_x)$. Let $G(L)$ denote the leaf holonomy group of the leaf L. Then we have $G(L'_{F_3(x)}) \approx G(L_x)$. More generally let $\mathcal{G}(\mathcal{S})$ denote the holonomy groupoid [**?**] of the foliation $\mathcal{F}_3$, that is the set of triples $(x, y, [\alpha])$ where $x, y \in \mathcal{S}$ lie on the same leaf L_x of $\mathcal{F}_3$ and $[\alpha]$ is the holonomy equivalence class of all piecewise smooth paths from x to y lying in L_x. Multiplication in the groupoid $\mathcal{G}(\mathcal{S})$ is defined on pairs of triples $(x, y, [\alpha]), (x', y', [\alpha'])$ precisely when $y = x'$ and then $(x, y, [\alpha]) \cdot (x', y', [\alpha']) = (x, y', [\alpha'\alpha])$. Furthermore, the subgroup of triples $(x, x, [\alpha])$ with x fixed is identified with the holonomy group $G(L_x)$. With this structure, $\mathcal{G}(\mathcal{S})$ is a locally compact topological groupoid [**?**]. (Actually $\mathcal{G}(\mathcal{S})$ is a smooth manifold of dimension $4n + 6$ but we do not use this here). We have

Proposition 13.2.18: *Let $F : \mathcal{S} \longrightarrow \mathcal{S}'$ be an isomorphism of 3-Sasakian manifolds. Then F induces an isomorphism $F_* : \mathcal{G}(\mathcal{S}) \longrightarrow \mathcal{G}(\mathcal{S}')$ of topological groupoids.*

The groupoid $\mathcal{G}(\mathcal{S})$ will be studied in a forthcoming work. For now we are interested in the unordered list $(\Gamma_1, \Gamma_2, \cdots)$ of holonomy groups in $\mathcal{G}(\mathcal{S})$ up to abstract isomorphism. This list is finite if $\mathcal{S}$ is complete and it provides important invariants of a 3-Sasakian manifold. Since the leaves of the foliation $\mathcal{F}_3$ are all spherical space forms, the groups Γ_i are all either subgroups of $Sp(1)$ or all subgroups of $SO(3)$, depending on whether the Marchiafava-Romani class ε of the quaternionic Kähler orbifold $\mathcal{O}$ is 0 or 1, respectively. Notice that it follows from its definition and 13.2.6 above that the class ε is an invariant of the 3-Sasakian structure on the manifold $\mathcal{S}$. Indeed, ε can be identified with a certain secondary characteristic class of the foliation $\mathcal{F}_3$. Thus, the Marchiafava-Romani class splits the isomorphism classes $\mathbb{S}$ of 3-Sasakian manifolds into the disjoint union $\mathbb{S}_0 + \mathbb{S}_1$ depending on whether ε is 0 of 1. A further rough classification scheme is given by

Definition 13.2.19: *$\mathcal{S}$ is said to be:*
(i) regular if all the Γ_i are the identity.
(ii) of cyclic type if all the Γ_i are cyclic.
(iii) of dihedral type if all the Γ_i are either cyclic or dihedral or binary dihedral with at least one Γ_i non-Abelian.
(iv) of polyhedral type if at least one of the Γ_i is one of the polyhedral groups, tetrahedral, octahedral, or icosahedral (or the corresponding binary double covers) groups.

The definition of regular here coincides with that of Definition 2.2.6. In general 3-Sasakian dimension $4n + 3$ the only known examples of 3-Sasakian manifolds of polyhedral or dihedral type are the spherical space forms $\Gamma\backslash S^{4n+3}$ and $\Gamma\backslash\mathbb{RP}^{4n+3}$, where Γ is a binary polyhedral or a binary dihedral group in the first case and a polyhedral or a dihedral group in the second. The action is that induced by the diagonal action of Γ on the quaternionic vector space $\mathbb{H}^{n+1}$. However, in dimension 7 there exist 3-Sasakian manifolds of dihedral or polyhedral type [**?, 89**] which are not spherical space forms. All other known non-regular 3-Sasakian manifolds are of cyclic type and are discussed in detail in Section 8 The following is essentially due to Salamon:

Theorem 13.2.20: *Let $\mathcal{S}$ be a complete regular 3-Sasakian manifold with $\varepsilon = 0$. Then $\mathcal{S} \simeq S^{4n+3}$ or $\mathbb{RP}^{4n+3}$.*

For more results about regular 3-Sasakian manifolds see Section 5.4 below. Next we consider an important infinitesimal rigidity result. In the regular case this rigidity is a simple consequence of the results of LeBrun [**144**] and Nagatomo [**?**] (see [**90**]). In the general case it was recently proved by Pedersen and Poon.

Theorem 13.2.21: [**?**] *Complete 3-Sasakian manifolds are infinitesimally rigid.*

Proof. The deformation theory of 3-Sasakian manifolds is tied to the deformation theory of hypercomplex manifolds studied previously in [**174**]. Let $\mathcal{S}$ be a complete 3-Sasakian manifold. Then the compact manifold $S^1 \times \mathcal{S}$ has a natural hypercomplex structure [**?**]. Thus, its twistor space W is compact and fibers holomorphically over $\mathbb{CP}^1$. Moreover, there is a holomorphic foliation on W whose leaves are elliptic Hopf surfaces, and whose space of leaves is the twistor space $\mathcal{Z}$ associated to $\mathcal{S}$. The geometry of the corresponding deformation theory is as follows. Deformations $(\mathcal{S}_t, g_t)$ of the 3-Sasakian structure $(\mathcal{S}_0, g_0)$ on $\mathcal{S}$ correspond to deformations of the hypercomplex structure on $S^1 \times \mathcal{S}$ of the form $S^1 \times \mathcal{S}_t$. In turn these deformations correspond to deformations of the holomorphic fibration $p : W \longrightarrow \mathbb{CP}^1$. Thus, there are natural projections:

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow \Phi \\ \mathbb{CP}^1 & & \mathcal{Z}, \end{array} \tag{13.2.7}$$

where each fiber of p is a divisor in W diffeomorphic to $S^1 \times \mathcal{S}$ and Φ is an orbifold submersion whose leaves are elliptic Hopf surfaces. Now the product map $p \times \Phi : W \longrightarrow \mathbb{CP}^1 \times \mathcal{Z}$ is an orbifold submersion whose leaves are elliptic curves. The differential of $p \times \Phi$ induces the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \Theta_W \longrightarrow \Phi^*\Theta_{\mathcal{Z}} \oplus p^*\Theta_{\mathbb{CP}^1} \longrightarrow 0,$$

where $\mathcal{O}_W$ denotes the structure sheaf of W and Θ denotes the holomorphic tangent sheaf. Then using standard techniques together with the Kodaira-Baily vanishing theorem and the orbifold version of the Akizuki-Nagano vanishing theorem, Pedersen and Poon show that the virtual parameter space for 3-Sasakian deformations lies in

$$\begin{aligned} &H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \otimes H^1(F, \mathcal{O}_F) \oplus H^0(\mathcal{Z}, \Theta_{\mathcal{Z}}) \otimes H^1(F, \mathcal{O}_F) \\ &\oplus H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}) \otimes H^0(F, \mathcal{O}_F) \oplus H^1(W, p^*\Theta_{\mathbb{CP}^1}), \end{aligned} \tag{13.2.8}$$

where F is the generic elliptic Hopf surface $S^1 \times Sp(1)$. One then analyzes each summand of 3.8 to show that there are no 3-Sasakian deformations. For example, possible deformations lying in the last summand vanish by results of Horikawa, while 3-Sasakian deformations lying in the second and third summands must preserve the complex contact structure on $\mathcal{Z}$. There are no such deformations in the third summand by the Kodaira-Baily vanishing theorem. Elements in the second summand correspond to complex contact transformations that are invariant under the $U(1) \times U(1)$ action coming from a discrete quotient of the $\mathbb{C}^*$ principal action on $\mathcal{L}$, and there are no such elements. Finally, elements of the first summand correspond to scale changes in the S^1 factor of $S^1 \times Sp(1)$ and these hypercomplex deformations do not come from 3-Sasakian ones. □

While this theorem says that there is no "infinitesimal moduli", there may well be discrete moduli of 3-Sasakian structures. Indeed, we believe that the work of Kruggel [**139**] can be used with the aid of a computer to construct distinct 3-Sasakian structures on the same manifold. See the last paragraph of secion 8.4 below.

13.3. Homogeneous 3-Sasakian Manifolds

Every 3-Sasakian manifold $(\mathcal{S}, g)$ has a nontrivial isometry group $I(\mathcal{S}, g)$ of dimension at least three. We first recall some of the results about $I(\mathcal{S}, g)$.

Definition 13.3.1: *Let $I_0(\mathcal{S}, g) \subset I(\mathcal{S}, g)$ be the subgroup of the isometry group which preserves the 3-Sasakian structure, that is if $\phi^k \in$ Diff $\mathcal{S}$ corresponds to $k \in I_0(\mathcal{S}, g)$ then $\phi^k_* \xi^a = \xi^a$, for all $a = 1, 2, 3$. Then $I_0(\mathcal{S}, g)$ is called the group of 3-Sasakian isometries and when it acts transitively on $(\mathcal{S}, g)$ the space $\mathcal{S}$ is said to be a 3-Sasakian homogeneous space.*

Lemma 13.3.2: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold and $X \in \mathfrak{i}$ be a Killing vector field on $\mathcal{S}$. Let $\mathcal{L}_X$ denote the Lie derivative with respect to X. Then the following conditions are equivalent:*
(i) $\mathcal{L}_X \Phi^a = 0$, for $a = 1, 2, 3$.
(ii) $\mathcal{L}_X \eta^a = 0$, for $a = 1, 2, 3$.
(iii) $\mathcal{L}_X \xi^a = 0$, for $a = 1, 2, 3$.
Furthermore, if any (hence, all) of the conditions above is satisfied, then for any vector field Y on $\mathcal{S}$ we have $X\eta^a(Y) = \eta^a([X, Y])$.

The above lemma gives alternative characterizations of the Lie algebra of $I_0(\mathcal{S}, g)$ and it easily follows from the definition and properties of the 3-Sasakian structure. As its immediate consequence we get the following theorem [**203**]:

Theorem 13.3.3: *Let $(\mathcal{S}, g)$ be a complete 3-Sasakian manifold which is not of constant curvature. $\mathfrak{i}$ and $\mathfrak{i}_0$ denote the Lie algebras of $I(\mathcal{S}, g)$ and $I_0(\mathcal{S}, g)$, respectively. Then as Lie algebras $\mathfrak{i} = \mathfrak{i}_0 \oplus \mathfrak{sp}(1)$, where $\mathfrak{sp}(1)$ is the Lie algebra generated by $\{\xi^1, \xi^2, \xi^3\}$.*

Notice that any of the first three conditions in Lemma 4.2.2 can be used to describe the Lie subalgebra $\mathfrak{i}_0 \subset \mathfrak{i}$. Moreover, the equivalence of conditions (iii) and (i) says that the Lie algebra $\mathfrak{c}(\mathfrak{sp}(1))$ of the centralizer of $Sp(1)$ in $I(\mathcal{S}, g)$ is precisely $\mathfrak{i}_0$. Globally, on the group level we obtain:

Proposition 13.3.4: *Let $(\mathcal{S}, g)$ be a complete 3-Sasakian manifold. Then both the isometry groups $I(\mathcal{S}, g)$ and $I_0(\mathcal{S}, g)$ are compact. Furthermore, if $(\mathcal{S}, g)$ is not of constant curvature then either $I(\mathcal{S}, g) = I_0(\mathcal{S}, g) \times Sp(1)$ or $I(\mathcal{S}, g) = I_0(\mathcal{S}, g) \times SO(3)$. Finally, if $(\mathcal{S}, g)$ does have constant curvature then $I(\mathcal{S}, g)$ strictly contains* *either* $I_0(\mathcal{S},g)\times Sp(1)$ *or* $I_0(\mathcal{S},g)\times SO(3)$ *as a proper subgroup and* $I_0(\mathcal{S},g)$ *is the centralizer of* $Sp(1)$ *or* $SO(3)$.

PROOF. The first assertion follows from Corollary 2.2.5 and a standard result of Myers and Steenrod (cf. [**22**]). Next, since $I_0(\mathcal{S},g)$, $Sp(1)$, and $SO(3)$ are all compact, the direct sum on the Lie algebra level given in Theorem 4.2.3 also gives a direct product of Lie groups. The last assertion follows immediately from lemma 4.2.2. □

Proposition 13.3.5: *Let* $(\mathcal{S},g)$ *be a* 3*-Sasakian homogeneous space. Then all leaves are diffeomorphic and* $\mathcal{S}/\mathcal{F}_3$ *is a quaternionic Kähler manifold where the natural projection* $\pi:\mathcal{S}{\longrightarrow}\mathcal{S}/\mathcal{F}_3$ *is a locally trivial Riemannian fibration. Furthermore,* $I_0(\mathcal{S},g)$ *acts transitively on the space of leaves* $\mathcal{S}/\mathcal{F}_3$.

PROOF. Let $\psi: I_0(\mathcal{S},g)\times\mathcal{S}{\longrightarrow}\mathcal{S}$ denote the action map so that, for each $a\in I_0(\mathcal{S},g)$, $\psi_a=\psi(a,\cdot)$ is a diffeomorphism of $\mathcal{S}$ to itself. Proposition 4.2.4 implies that the isometry group $I(\mathcal{S},g)$ contains $I_0(\mathcal{S},g)\times Sp(1)$ where either $Sp(1)$ acts effectively or its $\mathbb{Z}_2$ quotient $SO(3)\simeq Sp(1)/\mathbb{Z}_2$ acts effectively. Since the Killing vector fields ξ^a for $a=1,2,3$ are both the infinitesimal generators of the group $Sp(1)$ and a basis for the vertical distribution $\mathcal{V}$, it follows that $Sp(1)$ acts transitively on each leaf with isotropy subgroup of a point some finite subgroup $\Gamma\subset Sp(1)$. Now let p_1 and p_2 be any two points of $\mathcal{S}$ and let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the corresponding leaves through p_1 and p_2, respectively. Since $I_0(\mathcal{S},g)$ acts transitively on $\mathcal{S}$, there exists an $a\in I_0(\mathcal{S},g)$ such that $\psi_a(p_1)=p_2$. Now ψ_a restricted to $\mathcal{L}_1$ maps $\mathcal{L}_1$ diffeomorphically onto its image, and, since the $Sp(1)$ factor acts transitively on each leaf and commutes with $I_0(\mathcal{S},g)$, the image of ψ_a lies in $\mathcal{L}_2$. But the same holds for the inverse map $\psi_{a^{-1}}$ with $\mathcal{L}_1$ and $\mathcal{L}_2$ interchanged, so the leaves must be diffeomorphic. Thus, the leaf holonomy is trivial and $\pi:\mathcal{S}{\longrightarrow}\mathcal{S}/\mathcal{F}_3=\mathcal{O}$ is a locally trivial Riemannian fibration. The fact that the space of leaves $\mathcal{O}$ is a quaternionic Kähler manifold now follows from Ishihara's Theorem 3.3.4. Finally, the constructions above shows directly that $I_0(\mathcal{S},g)$ acts transitively on $\mathcal{O}$. □

The following classification theorem is now immediate from Proposition 4.2.5, the result of Alekseevsky which states that all homogeneous quaternionic Kähler manifolds of positive scalar curvature are symmetric [**4**], and Proposition 2.2.7:

Theorem 13.3.6: *Let* $\mathcal{S}$ *be a* 3*-Sasakian homogeneous space. Then* $\mathcal{S}=G/H$ *is precisely one of the following:*

$$\frac{Sp(n+1)}{Sp(n)},\ \frac{Sp(n+1)}{Sp(n)\times\mathbb{Z}_2},\ \frac{SU(m)}{S\big(U(m-2)\times U(1)\big)},\ \frac{SO(k)}{SO(k-4)\times Sp(1)},$$

$$\frac{G_2}{Sp(1)},\qquad \frac{F_4}{Sp(3)},\qquad \frac{E_6}{SU(6)},\qquad \frac{E_7}{\mathrm{Spin}(12)},\qquad \frac{E_8}{E_7}.$$

Here $n\geq 0$, $Sp(0)$ *denotes the trivial group,* $m\geq 3$, *and* $k\geq 7$. *Hence, there is one-to-one correspondence between the simple Lie algebras and the simply-connected* 3*-Sasakian homogeneous manifolds.*

Below we give the fundamental diagram $\diamondsuit(G/H)$ for each 3-Sasakian homogeneous space of Theorem 4.2.6:

$$
\begin{tikzcd}
& \mathbb{R}_+ \times G/H \arrow[dl] \arrow[dr] \arrow[dd] & \\
G/H\cdot U(1) \arrow[dr] & & G/H, \arrow[ll] \arrow[dl] \\
& G/H\cdot Sp(1) &
\end{tikzcd}
\tag{13.3.1}
$$

where $G/H\cdot Sp(1)$ are precisely the Wolf spaces [**222**].

REMARK 13.3.1: Note that a homogeneous 3-Sasakian manifold is necessarily simply-connected with the exception of the real projective space. This is in sharp contrast with the Sasakian-Einstein case. Also notice that a 3-Sasakian manifold can be Riemannian homogeneous (i.e., the full isometry group acts transitively) but not 3-Sasakian homogeneous. This is true for the lens spaces $\Gamma\backslash S^3$, with $|\Gamma| > 2$. Observe that $\mathbb{Z}_k\backslash S^3$, $k > 2$, is a homogeneous Sasakian-Einstein manifold but not 3-Sasakian homogeneous.

Theorem 4.2.6 does not specify what is the 3-Sasakian metric on the coset G/H. In Section 7.2 we will describe a quotient construction of the 3-Sasakian homogeneous spaces with $G = SU(n+1)$ and $G = SO(n+1)$. Here we quote a theorem of Bielawski [**23**], which gives an explicit description of these metrics in all cases.

Theorem 13.3.7: *Let $\mathcal{S} = G/H$ be one of the spaces in Theorem 4.2.6. and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the corresponding decomposition of the Lie algebras. Then there is a natural decomposition $\mathfrak{m} = \mathfrak{sp}(1) \oplus \mathfrak{m}'$ and the metric g on $\mathcal{S}$ is given in terms of the scalar product on $\mathfrak{m}$*

$$\|m\|^2 = - < \sigma, \sigma > - \frac{1}{2} < m', m' >,$$

where $\sigma \in \mathfrak{sp}(1)$, $m' \in \mathfrak{m}'$, and $< \cdot, \cdot >$ is the Killing form on $\mathfrak{g}$. In particular, the metric g is not naturally reductive with respect to the homogeneous structure on $\mathcal{S}$.

REMARK 13.3.2: In the case when $\mathcal{S}$ is of constant curvature the canonical metric on S^{4n+3} (or $\mathbb{RP}^{4n+3}$) is *not* the standard homogeneous metric on the homogeneous space $Sp(n+1)/Sp(n)$ (or $Sp(n+1)/Sp(n) \times \mathbb{Z}_2$) with respect to the reductive decomposition $\mathfrak{sp}(n+1) \simeq \mathfrak{sp}(n) + \mathfrak{m}$. It is, of course, the standard homogeneous metric with respect to the naturally reductive decomposition $\mathfrak{o}(4n+4) \simeq \mathfrak{o}(4n+3) + \mathfrak{m}$. This is quite special to the sphere and orthogonal group. In general the 3-Sasakian homogeneous metrics are not naturally reductive with respect to any reductive decomposition.

13.4. 3-Sasakian Cohomology

In this section we will describe some cohomological properties of 3-Sasakian manifolds $\mathcal{S}$. We prove a vanishing theorem and then derive a relation between the Betti numbers of $\mathcal{S}$ and the Betti numbers of the associated orbifolds $\mathcal{Z}$ and $\mathcal{O}$. We conclude with various implications of these relations in the case $\mathcal{S}$ is regular.

13.4.1. Sasakian Manifolds and Harmonic Theory. We start by recalling some old results about harmonic forms on Sasakian manifolds due to Tachibana [**200**]. Let $(\mathcal{S}, g)$ be a compact Sasakian manifold of dimension $2m+1$ with Sasakian structure $\{\xi, \eta, \Phi\}$ and let $\Omega^p(\mathcal{S})$ be the space of smooth p-forms on $\mathcal{S}$. Furthermore, let $\mathcal{H}^p(\mathcal{S}) = \{u \in \Omega^p(\mathcal{S}) : du = 0 = d * u\}$ denote the finite-dimensional space of harmonic p-forms. By Hodge theory any harmonic form u is necessarily invariant under the isometry group $I(\mathcal{S}, g)$ The tensor Φ extends to an endomorphism of $\Omega^p(\mathcal{S})$ by setting

$$(\mathbf{\Phi}u)(X_1, X_2, \ldots, X_p) = \sum_{i=1}^{p} u(X_1, \ldots, \Phi X_i, \ldots, X_p).$$

With this notation we have [**200**]

Theorem 13.4.1: *: Let $u \in \mathcal{H}^p(\mathcal{S})$, $p \leq m$. Then $\xi \lfloor u = 0$, and $\mathbf{\Phi}(u) \in \mathcal{H}^p(\mathcal{S})$.*

Proof. The first statement is easy to prove in the case $p = 1$. Indeed, let $u = \alpha + f\eta$ be a closed invariant 1-form, where $\alpha(\xi) = 0$, and f is a function. The vanishing of the Lie derivative of u along ξ implies that $0 = d(\xi \lfloor u) = df$, so that f is a constant and $0 = d\alpha + f d\eta$. Then

$$0 = \int_{\mathcal{S}} d(\alpha \wedge (d\eta)^{m-1} \wedge \eta) = -\int_{\mathcal{S}} f(d\eta)^m \wedge \eta.$$

Since $(d\eta)^m \wedge \eta$ is a non-zero multiple of the volume form of $\mathcal{S}$, we obtain $f = 0$ and $\xi \lfloor u = 0$. The general case of the original proof uses an explicit computation in local coordinates and we omit it here. The second statement follows immediately from 5.1.1 and the fact that Φ preserves horizontal subspaces. □

Let us define the following $\mathcal{I} : \mathcal{H}^p(\mathcal{S}) \to \mathcal{H}^p(\mathcal{S})$ endomorphism for $p \leq m$:

$$(\mathcal{I}u)(X_1, \ldots, X_p) = u(\Phi X_1, \ldots, \Phi X_p)$$

The basic identity 2.1.3(i) together with Theorem 5.1.2 shows that $\mathcal{I}u$ is a linear combination of $(\mathbf{\Phi})^k u$ for $0 \leq k \leq p$. Thus $\mathcal{I}$ also maps $\mathcal{H}^p(\mathcal{S})$ into itself. The following proposition is now a simple consequence of the definition 5.1.3 and Theorem 5.1.2 [**?**, **?**]:

Proposition 13.4.2: *Let $\mathcal{I} : \mathcal{H}^p(\mathcal{S}) \to \mathcal{H}^p(\mathcal{S})$ and $p \leq m$. Then $\mathcal{I} \circ \mathcal{I} = (-1)^p$. In particular, when p is odd, $\mathcal{I}$ defines an almost complex structure on the vector space $\mathcal{H}^p(\mathcal{S})$.*

Corollary 13.4.3: *Let $(\mathcal{S}, g)$ be a compact Sasakian manifold of dimension $2m+1$. Then the Betti numbers b_p for p odd and $p \leq m$ are even.*

In the case of compact Sasakian-Einstein manifolds this and the fact that the fundamental group is finite are the only known general topological restrictions on $\mathcal{S}$. Under some additional curvature conditions we can get further restrictions. For example, it is known [**?**] that a compact simply-connected Sasakian manifold of positive sectional curvature is isometric to a sphere. For other similar results see [**?**, **?**] and references therein.

13.4.2. A Vanishing Theorem. Now, let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold of dimension $4n + 3$ and 3-Sasakian structure $\{\xi^a, \eta^a, \Phi^a\}$. Throughout this section we shall suppose that $p \leq 2n + 1$. Referring to the splitting of the tangent bundle of $\mathcal{S}$ into $T\mathcal{S} = \mathcal{V}_3 \oplus \mathcal{H}$, we shall say that a p-form $u \in \Omega^p(\mathcal{S})$ has *bidegree* $(i, p-i)$ if it is a section of the subbundle of $\bigwedge^p T^*\mathcal{S}$ isomorphic to the dual of $\bigwedge^i \mathcal{V}_3 \otimes \bigwedge^{p-i} \mathcal{H}$. In particular, u is called *3-horizontal* if it has bidegree $(0, p)$, or equivalently if $\xi^a \lfloor u = 0$ for $a = 1, 2, 3$. An element $\omega \in \Omega^p(\mathcal{S})$ is called *invariant* if $h^*\omega = \omega$ for all $h \in Sp(1)$. In the regular case, there is a principal $Sp(1)$-bundle $\pi : \mathcal{S} \to \mathcal{O}$, and ω is both 3-horizontal and invariant if and only if it is the pullback $\pi^*\hat{\omega}$ of a form $\hat{\omega}$ on the quaternionic Kähler base $\mathcal{O}$. Now the curvature forms $\bar{\Phi}^a$ defined in the proof of Theorem 3.3.3 are horizontal with respect to the foliation $\mathcal{F}_3$. The Killing fields ξ^a transform according to the adjoint representation of $Sp(1)$, and the same is true of the associated triples η^a, $d\eta^a$, and Φ^a. For example, if $h \in Sp(1)$, we may write

$$h_*\Phi^a = \sum_b h^{ab}\Phi^b, \quad a = 1, 2, 3,$$

where h^{ab} are components of the image of h in $Sp(1)/\mathbb{Z}_2 \cong SO(3)$. The 3-forms

$$\Upsilon = \eta^1 \wedge \eta^2 \wedge \eta^3, \qquad \Theta = \sum_a \eta^a \wedge \bar{\Phi}^a = \sum_a \eta^a \wedge d\eta^a + 6\Upsilon \tag{13.4.1}$$

have respective bidegrees $(3, 0), (1, 2)$, and are clearly invariant. Their exterior derivatives are

$$d\Upsilon = \eta^1 \wedge \eta^2 \wedge \bar{\Phi}^3 + \eta^2 \wedge \eta^3 \wedge \bar{\Phi}^1 + \eta^3 \wedge \eta^1 \wedge \bar{\Phi}^2, \qquad d\Theta = \Omega + 2d\Upsilon, \tag{13.4.2}$$

where the 4-form Ω is defined in section 3.3. In fact, Ω is the canonical 4-form determined by the quaternionic structure of Proposition 2.2.3 of the subbundle $\mathcal{H}$, and is the pullback of the fundamental 4-form $\hat{\Omega}$ on the quaternionic Kähler orbifold $\mathcal{O}$ (see section 3.2).

Theorem 5.1.2(i) implies that any harmonic p-form with $p \leq 2n + 1$ on the compact 3-Sasakian manifold $\mathcal{S}$ is 3-horizontal. Apply 5.1.1 so as to obtain $\mathbf{\Phi}^a$: $\mathcal{H}^p(\mathcal{S}) \to \mathcal{H}^p(\mathcal{S})$, $\quad a = 1, 2, 3, \quad p \leq 2n + 1$, and 5.1.3 to get

$$(\mathcal{I}^a u)(X_1, X_2, \ldots, X_p) = u(\Phi^a X_1, \Phi^a X_2, \ldots, \Phi^a X_p).$$

Now, using the basic identities of Proposition 2.2.3 we can generalize Proposition 5.1.4 to get the following result due to Kuo [**141**]:

Proposition 13.4.4: *Let $\mathcal{I}^a : \mathcal{H}^p(\mathcal{S}) \to \mathcal{H}^p(\mathcal{S})$, $a = 1, 2, 3$, and $p \leq 2n + 1$. Then*

$$\mathcal{I}^b \circ \mathcal{I}^a = (-\delta^{ab})^p \mathbf{I} + \sum_c (\epsilon^{abc})^p \mathcal{I}^c.$$

In particular, when p is odd, $\{\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3\}$ defines an almost quaternionic structure on the vector space $\mathcal{H}^p(\mathcal{S})$.

We are now ready to prove the main theorem of this section:

Theorem 13.4.5: *Let $u \in \mathcal{H}^p(\mathcal{S})$, $p \leq 2n + 1$.*
(i) If p is odd then $u \equiv 0$.
(ii) If p is even then $\mathcal{I}^a u = u$ for $a = 1, 2, 3$.

Proof. Let $u \in \mathcal{H}^p(\mathcal{S})$. We shall in fact show that $\mathcal{I}^1 u = \mathcal{I}^2 u$ irrespective of whether p is even or odd; the result then follows from the identities in Proposition 5.2.1 and symmetry between the indices 1,2,3. By 13.4.2, we may choose an isometry $h \in Sp(1)$ so that $h_*\Phi^1 = \Phi^2$. Both u and $\mathcal{I}^1 u$ are harmonic, so $h^*u = u$ and

$$(\mathcal{I}^1 u)(X_1, \ldots, X_p) = (h_*(\mathcal{I}^1 u))(X_1, \ldots, X_p) = u((h_*\Phi^1)(X_1), \ldots, (h_*\Phi^1)(X_p)) =$$

$$= u(\Phi^2 X_1, \ldots, \Phi^2 X_p) = (\mathcal{I}^2 u)(X_1, \ldots, X_p).$$

□

Corollary 13.4.6: *Let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold of dimension $4n+3$. Then the odd Betti numbers b_{2k+1} of $\mathcal{S}$ are all zero for $0 \leq k \leq n$.*

We should point out that Corollary 5.2.3 does not apply to compact Sasakian or even Sasakian-Einstein manifolds. In [?] the authors construct examples of Sasakian-Einstein manifolds with certain non-vanishing odd Betti numbers within the range given in Corollary 5.2.3. For example, in dimension 7 there are circle bundles over Fermat hypersurfaces in $\mathbb{CP}^3$, as well as circle bundles over certain complete intersections that admit Sasakian-Einstein structures and have $b_3 \neq 0$. These are the only known examples of Sasakian-Einstein manifolds which cannot admit any 3-Sasakian structure.

13.4.3. 3-Sasakian Cohomology As Primitive Cohomology. We are going to consider connection between the cohomology of $\mathcal{S}$ and that of $\mathcal{Z}$ and $\mathcal{O}$. We will use the vanishing theorem and orbifold Gysin sequence arguments for the diagram of orbifold bundles of $\diamondsuit(\mathcal{S})$:

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] & \mathcal{Z}. \arrow[dl] \\ \mathcal{O} & \end{tikzcd} \tag{13.4.3}$$

Proposition 13.4.7: *Let $\mathcal{S}$ be a compact 3-Sasakian manifold of dimension $4n+3$ and $\mathcal{Z} = \mathcal{S}/S^1$ be the twistor space. Then $b_p(\mathcal{S}) = b_p(\mathcal{Z}) - b_{p-2}(\mathcal{Z})$, for $p \leq 2n+1$. In particular, all odd Betti numbers of $\mathcal{Z}$ vanish.*

Proof. The result follows form the rational Gysin sequence applied to the orbifold fibration $S^1 \to \mathcal{S} \to \mathcal{Z}$. First, note that the bundle $S^1 \longrightarrow \mathcal{S} \longrightarrow \mathcal{Z}$ is a circle V-bundle over a compact Kähler-Einstein orbifold $\mathcal{Z}$. As explained in Section 3, up to a possible $\mathbb{Z}_2$ cover, $\mathcal{S}$ is the total space of the unit circle bundle in the dual of the contact line V-bundle on $\mathcal{Z}$, and the Kähler-Einstein metric of $\mathcal{Z}$ arises in accordance with the orbifold version of the Kobayashi's theorem [**126, ?**] . It follows that the connecting homomorphism δ is given by wedging with a non-zero multiple of the Kähler form of $\mathcal{Z}$. When $\mathcal{Z}$ is smooth this is well-known to be injective so long as $p \leq 2n+2$. However, the Lefschetz decomposition is equally true for compact orbifolds and the result still holds in this more general situation [?]. The Gysin sequence therefore reduces to a series of short exact sequences up to and including $H^{2n+1}(\mathcal{S})$, and the proposition follows. □

Proposition 13.4.8: *Let $\mathcal{S}$ be a compact 3-Sasakian manifold of dimension $4n+3$ and let $\mathcal{O} = \mathcal{S}/\mathcal{F}_3$. Then $b_{2p}(\mathcal{S}) = b_{2p}(\mathcal{O}) - b_{2p-4}(\mathcal{O})$, for $p \leq 2n+1$.*

Proof. The result follows form the Gysin sequence applied to the orbifold fibration $\mathcal{L} \to \mathcal{S} \to \mathcal{O}$. Since the principal orbit of the $Sp(1)$ action (or generic leaf

$\mathcal{L}$) is either S^3 or $SO(3)$ the usual Gysin sequence argument applies as well in this situation (see the Appendix). We have

$$\cdots \to H^i(\mathcal{S},\mathbb{Q}) \to H^{i-3}(\mathcal{O},\mathbb{Q}) \xrightarrow{\delta} H^{i+1}(\mathcal{O},\mathbb{Q}) \to H^{i+1}(\mathcal{S},\mathbb{Q}) \to H^{i-2}(\mathcal{S},\mathbb{Q}) \to \cdots$$

and the statement of the proposition follows easily from the vanishing of the odd Betti numbers of $\mathcal{S}$. □

Recall that the vector space of primitive harmonic p-forms $\mathcal{H}_0^p(\mathcal{Z},\mathbb{Q})$ of the orbifold $\mathcal{Z}$ is isomorphic to the cokernel of the injective mapping $L_{\mathcal{Z}} : \mathcal{H}^{p-2}(\mathcal{Z}) \hookrightarrow \mathcal{H}^p(\mathcal{Z}), \quad p \leq 2n$ defined by wedging with the Kähler 2-form. We define the primitive Betti numbers $b_p^0(\mathcal{Z})$ of $\mathcal{Z}$ as the dimension of $\mathcal{H}_0^p(\mathcal{Z})$. Proposition 5.3.1 says that the primitive Betti numbers of $\mathcal{Z}$ are the usual Betti numbers of $\mathcal{S}$ and it follows from the fact that for, $0 \leq r \leq 2n+1$, an r-form on $\mathcal{S}$ is harmonic if and only if it is the lift of a primitive harmonic form on $\mathcal{Z}$ [**?**]. Similarly, the vector space of primitive harmonic p-forms $\mathcal{H}_0^p(\mathcal{O},\mathbb{Q})$ of the orbifold $\mathcal{O}$ is isomorphic to the cokernel of the injective mapping $L_{\mathcal{O}} : \mathcal{H}^{p-4}(\mathcal{O}) \hookrightarrow \mathcal{H}^p(\mathcal{O}), \quad p \leq 2n+2$ defined by wedging with the quaternionic Kähler 4-form Ω. The injectivity of this mapping is well-known in the smooth case [**31, 85, 134**] and it extends to the orbifold case. We define the primitive Betti numbers $b_p^0(\mathcal{O})$ of $\mathcal{O}$ as the dimension of $\mathcal{H}_0^p(\mathcal{O})$. Proposition 5.3.2 says that the primitive Betti numbers of $\mathcal{O}$ are the usual Betti numbers of $\mathcal{S}$. Again, Proposition 5.3.3 is a consequence of the fact that an r-form on $\mathcal{S}$ is harmonic if and only if it is the lift of a primitive harmonic form on $\mathcal{O}$, $0 \leq r \leq 2n+1$.

13.4.4. Regular 3-Sasakian Cohomology, Finiteness, and Rigidity. In this Section we shall assume that $\mathcal{S}$ is regular and, hence, both $\mathcal{Z}$ and $\mathcal{O}$ are smooth. In this instance, using the results of the previous section, one can easily translate all the results about strong rigidity of positive quaternion Kähler manifolds [**143, 146, 184**] (see the chapter in this volume on Quaternionic Kähler Manifolds by Salamon) to compact regular 3-Sasakian manifolds. In particular, we get

Proposition 13.4.9: *Let $\mathcal{S}$ be a compact regular 3-Sasakian manifold of dimension $4n+3$. Then $\pi_1(\mathcal{S}) = 0$ unless $\mathcal{S} = \mathbb{RP}^{4n+3}$ and*

$$\pi_2(\mathcal{S}) = \begin{cases} \mathbb{Z} & \text{iff } \mathcal{S} = SU(n+2)/S(U(n)\times U(1)), \\ \text{finite} & \text{otherwise.} \end{cases}$$

Furthermore, up to isometries, for each $n \geq 1$ there are only finitely many regular 3-Sasakian manifolds $\mathcal{S}$.

Proof. Using the long exact homotopy sequence for the vertical map in diagram 5.1, this follows from the strong rigidity theorem of LeBrun and Salamon [**146, 143**] for positive quaternionic Kähler manifolds, and Salamon's theorem that a positive quaternionic Kähler manifold with vanishing Marchiafava-Romani class must be $\mathbb{HP}^n$. □

n	relation on Betti numbers or coefficients thereof
2	$b_2 = b_4$
3	$b_2 = b_6$
4	$2b_2 + b_4 = b_6 + 2b_8$
5	$5b_2 + 4b_4 = 4b_8 + 5b_{10}$
6	$5b_2 + 5b_4 + 2b_6 = 2b_8 + 5b_{10} + 5b_{12}$
7	$7b_2 + 8b_4 + 5b_6 = 5b_{10} + 8b_{12} + 7b_{14}$
8	$28b_2 + 35b_4 + 27b_6 + 10b_8 = 10b_{10} + 27b_{12} + 35b_{14} + 28b_{16}$
9	$12b_2 + 16b_4 + 14b_6 + 8b_8 = 8b_{12} + 14b_{14} + 16b_{16} + 12b_{18}$
10	$15, 21, 20, 14, 5$
16	$40, 65, 77, 78, 70, 55, 35, 12$
28	$126, 225, 299, 350, 380, 391, 385, 364, 330, 285, 231, 170, 104, 35$

Table 1

Proposition 13.4.10: *The Betti numbers of a regular compact 3-Sasakian manifold $\mathcal{S}$ of dimension $4n+3$ satisfy*
(i) $b_2 \leq 1$, with equality iff $\mathcal{S} = SU(l+2)/S(U(l)\times U(1))$,
(ii) $\displaystyle\sum_{k=1}^{n} k(n+1-k)(n+1-2k)b_{2k} = 0.$

Proof. (i) follows from Proposition 5.4.1 and (ii) for Salamon's relation on Betti numbers of $\mathcal{O}$ via Theorem 5.3.2. □

The following is a 3-Sasakian version of a theorem of Salamon [**90**]:

Proposition 13.4.11: *Let $\mathcal{S}$ be a regular compact 3-Sasakian manifold of dimension $4n+3$. If $n = 3, 4$ and $b_4 = 0$, then $\mathcal{S}$ is either a sphere S^{4n+3} or a real projective space $\mathbb{RP}^{4n+3}$.*

The linear Betti number relations in Proposition 5.4.2(ii) exhibit an interesting symmetry of the coefficients which, for lower values of n, are listed in Table 1.

One can compute the Poincaré polynomials of all known regular 3-Sasakian manifolds, that is 3-Sasakian homogeneous space of Theorem 4.2.6. We get [**90**]

Proposition 13.4.12: *The Poincaré polynomials of the homogeneous 3-Sasakian manifolds are as given in Table 2.*

G	H	$P(G/H,t)$
$SU(n+2)$	$SU(n)\times_{\mathbb{Z}_n} T^1$	$\sum_{i=0}^{n}(t^{2i} + t^{4n+3-2i}$
$SO(2k+3)$	$SO(2k-1)\times SU(2)$	$\sum_{i=0}^{k-1}(t^{4i} + t^{8k-1-4i})$
$Sp(n+1)$	$Sp(n)$	$1 + t^{4n+3}$
$SO(2l+4)$	$SO(2l)\times SU(2)$	$t^{2l} + t^{6l+3} + \sum_{i=0}^{l}(t^{4i} + t^{8l+3-4i})$
E_6	$SU(6)$	$1 + t^6 + t^8 + t^{12} + t^{14} + t^{20} + \cdots$
E_7	$\mathrm{Spin}(12)$	$1 + t^8 + t^{12} + t^{16} + t^{20} + t^{24} + t^{32} + \cdots$
E_8	E_7	$1 + t^{12} + t^{20} + t^{24} + t^{32} + t^{36} + t^{44} + t^{56} + \cdots$
F_4	$Sp(3)$	$1 + t^8 + t^{23} + t^{31}$
G_2	$SU(2)$	$1 + t^{11}$

Table 2

We conclude this section with a translation of two well-known classification results for positive quaternionic Kähler manifolds.

Theorem 13.4.13: *Let $(\mathcal{S}, g)$ be a compact regular 3-Sasakian manifold of dimension $4n+3$. If $n < 3$ then then $\mathcal{S} = G/H$ is homogeneous, and hence one of the spaces listed in Theorem 4.2.6.*

The $n = 0$ case is trivial and it was an observation made by Tanno [**204**]. The $n = 1$ case is based on [**112, 84**] and it was first observed in [**82, ?**]. The $n = 2$ case is based on [**177**] and was stated in [**40**].

13.5. Symmetry Reduction

In this section we give a general 3-Sasakian reduction procedure which constructs new 3-Sasakian manifolds from a given 3-Sasakian manifold $\mathcal{S}$ with a nontrivial 3-Sasakian isometry group $I_0(\mathcal{S}, g)$ [**?**]. Actually, this is a reduction that is associated with a quadruple of spaces of the fundamental diagram $\diamondsuit(\mathcal{S})$. At the level of the hyperkähler cone $C(\mathcal{S})$ the reduction was discovered by Lindström and Roček [**147**] in the context of supersymmetric σ-model and later rigorously described by Hitchin et al. in [**114**]. In the case of the quaternionic Kähler base $\mathcal{O}$ the reduction was discovered by the second author and Lawson [**87, 88**]. The lift of the quaternionic Kähler quotient to the twistor space $\mathcal{Z}$ was described by Hitchin [**113**]. In this section we restrict ourselves to describing the general procedure of reduction together with the homogeneous case arising from reduction by a circle group, as well as a brief discription of the singular case. The large class of 3-Sasakian toric manifolds obtained by reduction is relagated to a separate section, namely Section 8. It should also be understood that every 3-Sasakian reduction gives as well a reduction procedure for each of the spaces of the fundamental diagram $\diamondsuit(\mathcal{S})$.

13.5.1. The 3-Sasakian Moment Map. Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a nontrivial group $I_0(\mathcal{S}, g)$ of 3-Sasakian isometries. By the Definition 2.2.1, $C(\mathcal{S}) = \mathcal{S} \times \mathbb{R}^+$ is a hyperkähler manifold with respect to the cone metric $\bar{g}$. The isometry group $I_0(\mathcal{S}, g)$ extends to a group $I_0(C(\mathcal{S}), \bar{g}) \cong I_0(\mathcal{S}, g)$ of isometries on $C(\mathcal{S})$ by defining each element to act trivially on $\mathbb{R}^+$. Furthermore, it follows easily from the definition of the complex structures I^a that these isometries $I_0(C(\mathcal{S}), \bar{g})$ are hyperkähler; that is, they preserve the hyperkähler structure on $C(\mathcal{S})$. Recall [**114**] shows that any subgroup $G \subset I_0(M, \bar{g})$ gives rise to a hyperkähler moment map $\mu : M \rightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$, where $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^*$ is its dual. Thus, we can define a 3-Sasakian moment map

$$\mu_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3 \tag{13.5.1}$$

by restriction $\mu_{\mathcal{S}} = \mu \mid \mathcal{S}$. We denote the components of $\mu_{\mathcal{S}}$ with respect to the standard basis of $\mathbb{R}^3$, which we have identified with the imaginary quaternions, by $\mu^a_{\mathcal{S}}$. Recall that ordinarily moment maps determined by Abelian group actions (in particular, those associated to 1-parameter groups) are only specified up to an arbitrary constant. This is not the case for 3-Sasakian moment maps since we require that the group $Sp(1)$ generated by the Sasakian vector fields ξ^a acts on the level sets of $\mu_{\mathcal{S}}$. However, we shall see that 3-Sasakian moment maps are given by a particularly simple expression.

Proposition 13.5.1: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries. Let τ be an element of the Lie algebra $\mathfrak{g}$ of G and let X^τ denote the corresponding infinitesimal isometry. Then there is a unique 3-Sasakian moment map $\mu_\mathcal{S}$ such that the zero set $\mu_\mathcal{S}^{-1}(0)$ is invariant under the group $Sp(1)$ generated by the vector fields ξ^a. This moment map is given by*

$$< \mu_\mathcal{S}^a, \tau > = \frac{1}{2}\eta^a(X^\tau).$$

Furthermore, the zero set $\mu_\mathcal{S}^{-1}(0)$ is G invariant.

Proof. Using the Definition 2.2.1 we can define the 2-forms $\omega_\mathcal{S}^a$ on $\mathcal{S}$ as the restriction of the hyperkähler 2-forms ω^a. Then any 3-Sasakian moment map $\mu_\mathcal{S}^a(\tau)$ determined by $\tau \in \mathfrak{g}$ satisfies $2d\mu_\mathcal{S}^a(\tau) = 2X^\tau \rfloor \omega_\mathcal{S}^a = -X^\tau \rfloor d\eta^a$. As X^τ is a 3-Sasakian infinitesimal isometry, Lemma 4.2.2 implies that $2 < \mu_\mathcal{S}^a, \tau >$ differs from $\eta^a(X^\tau)$ by a constant depending on a and τ. One then uses the invariance of the zero set $\mu_\mathcal{S}^{-1}(0)$ to show that these constants must vanish. See [**?**] for details. $\square$

Henceforth by the 3-Sasakian moment map, we shall mean the moment map $\mu_\mathcal{S}$ determined in Proposition 7.1.1. Hence, the Definition 2.2.1 and Proposition 7.1.1 imply

Theorem 13.5.2: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ smoothly and properly by 3-Sasakian isometries. Let $\mu_\mathcal{S}$ be the corresponding 3-Sasakian moment map and assume both that 0 is a regular value of $\mu_\mathcal{S}$ and that G acts freely on the submanifold $\mu_\mathcal{S}^{-1}(0)$. Furthermore, let $\iota : \mu_\mathcal{S}^{-1}(0) \longrightarrow \mathcal{S}$ and $\pi : \mu_\mathcal{S}^{-1}(0) \longrightarrow \mu_\mathcal{S}^{-1}(0)/G$ denote the corresponding embedding and submersion. Then $(\mathcal{S}/\!/\!/G = \mu_\mathcal{S}^{-1}(0)/G, \check{g})$ is a smooth 3-Sasakian manifold of dimension $4(n - dim\ \mathfrak{g}) + 3$ with metric $\check{g}$ and characteristic vector fields $\check{\xi}^a$ determined uniquely by the two conditions $\iota^* g = \pi^* \check{g}$ and $\pi_*(\xi^a \,|\, \mu_\mathcal{S}^{-1}(0)) = \check{\xi}^a$.*

We conclude this part with the following fact concerning 3-Sasakian isometries whose proof can be found in [**?**].

Proposition 13.5.3: *Assume that the hypothesis of Theorem 7.1.2 holds. In addition assume that $(\mathcal{S}, g)$ is complete and hence compact. Let $C(G) \subset I_0(\mathcal{S}, g)$ denote the centralizer of G in $I_0(\mathcal{S}, g)$ and let $C_0(G)$ denote the subgroup of $C(G)$ given by the connected component of the identity. Then $C_0(G)$ acts on the submanifold $\mu_\mathcal{S}^{-1}(0)$ as isometries with respect to the restricted metric $\iota^* g$ and the 3-Sasakian isometry group $I_0(\mathcal{S}/\!/\!/G, \check{g})$ of the quotient $(\mathcal{S}/\!/\!/G, \check{g})$ determined in Theorem 7.1.2 contains an isomorphic copy of $C_0(G)$. Furthermore, if $C_0(G)$ acts transitively on $\mathcal{S}/\!/\!/G$, then $\mathcal{S}/\!/\!/G$ is a 3-Sasakian homogeneous space.*

It should be mentioned that it is not required that the isometry group $I_0(\mathcal{S}/\!/\!/G, \check{g})$ acts effectively.

13.5.2. Regular Quotients And Classical Homogeneous Metrics. We now apply the reduction procedure given in Theorem 7.1.2 to the round unit sphere S^{4n+3} to explicitly construct the Riemannian metrics for the 3-Sasakian homogeneous manifolds arising from the simple classical Lie algebras. These metrics are precisely the ones associated to the three infinite families appearing in Theorem 4.2.6. The quotient construction applied to $\diamondsuit(\mathcal{S}^{4n+3})$ explicitly describes all metrics in the fundamental diagrams $\diamondsuit(G/H)$, where G is either the special unitary

$SU(n+1)$ or the orthogonal group $SO(n+1)$. To carry out this reduction we must set some conventions. We describe the unit sphere S^{4n+3} by its embedding in flat space and we represent an element $\mathbf{u} = (u_1, \cdots, u_{n+1}) \in \mathbb{H}^{n+1}$ as a column vector. The quaternionic components of this vector are denoted by $\mathbf{u}^0$ for the real component and by $\mathbf{u}^a$ for the three imaginary components so that we can write $\mathbf{u} = \mathbf{u}^0 + i\mathbf{u}^1 + j\mathbf{u}^2 + k\mathbf{u}^3$ using the quaternionic units $\{i, j, k\}$. We also define quaternionic conjugate $\bar{\mathbf{u}} = \mathbf{u}^0 - i\mathbf{u}^1 - j\mathbf{u}^2 - k\mathbf{u}^3$.

Now, the infinitesimal generators of the subgroup $Sp(1) \subset \mathbb{H}^*$ acting by the right multiplication on $\mathbf{u}$ are the defining vector fields ξ^a for the Sasakian 3-structure. These vector fields are given by

$$\xi^a_r \;=\; \mathbf{u}^0 \cdot \frac{\partial}{\partial \mathbf{u}^a} - \mathbf{u}^a \cdot \frac{\partial}{\partial \mathbf{u}^0} - \epsilon^{abc}\mathbf{u}^b \cdot \frac{\partial}{\partial \mathbf{u}^c},$$

where the dot indicates sum over the vector components u_i and the subscript r means that these vector fields are the generators of the right action.

We will first consider $G = U(1)$ acting on the sphere S^{4n+3} as follows

$$\varphi_t(\mathbf{u}) = \tau\mathbf{u}, \qquad \tau = e^{2\pi i t}, \qquad \mathbf{u} \in S^{4n+3}.$$

Note that this action is actually free on S^{4n+3} and hence it will be automatically free on the level set of the moment map. To compute the moment map we identify the imaginary quaternions $\mathbb{R}^3$ with the Lie algebra $\mathfrak{sp}(1)$ in equation 13.5.1 and the Lie algebra of $U(1)$ with $\mathbb{R}$, so the moment map is $\mu_{\mathcal{S}} : S^{4n+3} \longrightarrow \mathbb{R} \otimes \mathfrak{sp}(1)$ and it can easily be computed

$$\mu_{\mathcal{S}}(\mathbf{u}) = \sum_{\alpha=1}^{n+1} \bar{u}_\alpha i u_\alpha.$$

One can easily identify the zero-level set of the moment map with the Stiefel manifold of complex 2-frames in $\mathbb{C}^{n+1}$, and the following proposition is then an immediate consequence of Theorem 7.1.2.

Proposition 13.5.4: *Let $N = \mu_{\mathcal{S}}^{-1}(0)$ and $\iota : N \hookrightarrow S^{4n+3}$ be the inclusion. Then ι is an embedding and $(N, \iota^* g_{can})$ is the complex homogeneous Stiefel manifold $V^{\mathbb{C}}_{2,n+1} = SU(n+1)/SU(n-1)$ of 2-frames in $\mathbb{C}^{n+1}$. Hence, the 3-Sasakian quotient $S^{4n+3}/\!\!/\!\!/U(1) = V^{\mathbb{C}}_{2,n+1}/U(1) = SU(n+1)/S(U(n-1) \times U(1))$ with the 3-Sasakian metric $\breve{g}$ given by inclusion ι and submersion $\pi : N \longrightarrow N/U(1)$, i.e., $\iota^* g_{can} = \pi^* \breve{g}$.*

Remark 13.5.1: A similar construction can be carried out for the $Sp(1)$-action on S^{4n+3} defined by the left multiplication of $\mathbf{u}$ by a unit quaternion σ, i.e.,

$$\varphi_\sigma(\mathbf{u}) = \sigma\mathbf{u}, \qquad \sigma\bar{\sigma} = 1, \qquad \mathbf{u} \in S^{4n+3}.$$

This action is free on S^{4n+3} and the zero-level set of the corresponding moment map can be identified with the real Stiefel manifold $V^{\mathbb{R}}_{4,n+1} \simeq SO(n+1)/SO(n-3)$ of 4-frames in $\mathbb{R}^{n+1}$ with $n \geq 4$. Hence, the reduced space $S^{4n+3}/\!\!/\!\!/Sp(1) = \frac{SO(n+1)}{SO(n-3) \times Sp(1)}$. For the more detailed and uniform description of the geometry of these two quotients see [?].

13.5.3. The Structure of Singular Quotients. In this section we will describe a more general situation, when the zero-level set of the 3-Sasakian moment map 13.5.1 is not necessarily smooth and the group action on the level set is not necessarily locally free.

Let G be a Lie group acting smoothly and properly on a manifold $\mathcal{S}$ and let $H \subset G$ be a subgroup. Using standard notation we will denote by $\mathcal{S}_H \subset \mathcal{S}$ the set of points in $\mathcal{S}$ where the stability group is exactly equal to H and by $\mathcal{S}_{(H)} \subset \mathcal{S}$ the set of points with stabilizer conjugate to H in G. It follows than that the normalizer $N(H)$ of H in G acts freely on $\mathcal{S}_{(H)}$. Then we have the following theorem due to Dancer and Swann [**63**]:

Theorem 13.5.5: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ smoothly and properly by 3-Sasakian isometries. Let $\mu_{\mathcal{S}}$ be the corresponding 3-Sasakian moment map. Then the quotient $\mu_{\mathcal{S}}^{-1}(0)/G$ is a union of the smooth, 3-Sasakian manifolds $(\mathcal{S}_{(H)} \cap \mu_{\mathcal{S}}^{-1}(0))/G$, where (H) runs over the conjugacy classes of stabilizers of points in $\mathcal{S}$.*

Quite often $\mathcal{S}_{(H)}$ does not meet the zero locus of the moment map. Then the stratum $(\mathcal{S}_{(H)} \cap \mu_{\mathcal{S}}^{-1}(0))/G$ is empty.

Example 13.5.6: We start with the 3-Sasakian sphere S^{4n+3} in the notation of the previous section. But now we consider a different circle action $U(1)$, namely

$$\varphi_t^{p,q;m}(\mathbf{u}) = (\tau^p u_1, \dots, \tau^p u_m, \tau^q u_{m+1}, \dots, \tau^q u_{n+1}), \qquad \tau = e^{2\pi i t}, \qquad \mathbf{u} \in S^{4n+3},$$

where $0 \leq m \leq n+1$ and $p, q \in \mathbb{Z}$. Let $\mathcal{S}(p,q;m) = S^{4n+3} /\!\!/\!\!/ U(1)$ be the quotient.
(i) First, let p, q be relatively prime positive integers bigger than 1 and $2 \leq m \leq n-1$. Then, the stratified manifold $\mathcal{S}(p,q;m)$ consists of 3 strata. The stratum of the highest dimension corresponding to $H = \{\mathrm{id}\}$ is an open incomplete 3-Sasakian manifold. The two strata of lower dimension are easily seen to be the homogeneous spaces: one with $H = \mathbb{Z}_q$ is the homogeneous 3-Sasakian space of $SU(n+1-m)$ and the one with $H = \mathbb{Z}_p$ is the homogeneous 3-Sasakian space of $SU(m)$. In this case, $\mathcal{S}(p,q;m)$ is actually a compact 3-Sasakian orbifold and the stratification of Theorem 7.3.1 coincides with the orbifold stratification.
(ii) Consider $\mathcal{S}(0,p;m)$, where $p > 1$ and $2 \leq m \leq n-1$. There are two strata now: the stratum of the highest dimension corresponds to $H = \mathbb{Z}_p$ and the second stratum is just the sphere S^{4m-1} with $H = U(1)$. The space $\mathcal{S}(0,p;m)$ is not an orbifold but, as pointed out in [**63**], it does have a length space structure.
(iii) Consider $\mathcal{S}(0,1;n)$. Here H is either $U(1)$ or trivial but the set S^{4n+3}_{id} does not meet the zero locus of the moment map. Hence, there is only one stratum and $\mathcal{S}(0,1;n) = S^{4n-1}$.
(iv) Finally, consider $\mathcal{S}(0,1;n-1)$. The stability group H is either $U(1)$ or trivial. The stratum corresponding to $H = U(1)$ is the sphere S^{4n-5}. We leave it as an exercise to the reader to show that $\mathcal{S}(0,1;n-1)$ is an orbifold and that it can be identified with $S^{4n-1}/\mathbb{Z}_2$, where $(w_1, ..., w_n) \in S^{4n-1}$, where $\mathbb{Z}_2$ acts on the last quaternionic coordinate by multiplication by ± 1.

13.6. Toric 3-Sasakian Manifolds

In this section we shall describe the quotient construction of large families of 3-Sasakian manifolds $\mathcal{S}(\Omega)$. They all have the property that $I_0(\mathcal{S}(\Omega), g(\Omega)) \supset T^m$, where $\dim(\mathcal{S}(\Omega)) = 4m-1$, and following the ideas of [**26**] we shall call such 3-Sasakian manifolds *toric* (See 8.6.1 for a precise definition). We also describe some interesting geometric and topological properties of such spaces. Up until now all known examples of 3-Sasakian manifolds are either homogeneous or toric or discrete quotients of them.

13.6.1. Toral Reductions of Spheres. Using the notation of the previous section we start with the unit $(4n+3)$-dimensional sphere embedded in the quaternionic vector space $\mathbb{H}^{n+1}$. The subgroup of the full isometry group $O(4n+3)$ that preserves the quaternionic structure is $Sp(n+1)\cdot Sp(1)$ acting by

$$\varphi_{\mathbb{A},\sigma}(\mathbf{u}) = \mathbb{A}\mathbf{u}\sigma^{-1},$$

where $\mathbb{A} \in Sp(n+1)$ is the quaternionic $(n+1)\times(n+1)$ matrix of the quaternionic representation of $Sp(n+1)$, and $\sigma \in Sp(1)$ is a unit quaternion. As the diagonal $\mathbb{Z}_2$ acts trivially this is indeed an $Sp(n+1)\cdot Sp(1)$ action. The group $Sp(n+1)$ is the subgroup of $Sp(n+1)\cdot Sp(1)$ which preserves the 3-Sasakian structure on S^{4n+3}, so we have $I_0(S^{4n+3}, g_{can}) = Sp(n+1)$. We shall consider the maximal torus $T^{n+1} \subset Sp(n+1)$ and its subgroups. Every quaternionic representation of a k-torus T^k on $\mathbb{H}^{n+1}$ can be described by an exact sequence $0 \longrightarrow T^k \xrightarrow{f_\Omega} T^{n+1} \longrightarrow T^{n+1-k} \longrightarrow 0$. The monomorphism f_Ω can be represented by the matrix

$$f_\Omega(\tau_1,\ldots,\tau_k) = \begin{pmatrix} \prod_{i=1}^{k} \tau_i^{a_1^i} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \prod_{i=1}^{k} \tau_i^{a_{n+1}^i} \end{pmatrix},$$

where $(\tau_1,..,\tau_k) \in S^1\times\cdots\times S^1 = T^k$ are the complex coordinates on T^k, and $a_j^i \in \mathbb{Z}$ are the coefficients of a $k\times(n+1)$ integral *weight* matrix $\Omega = (a_\alpha^l)_{\alpha=1,\ldots,n+1}^{l=1,\ldots,k} \in \mathcal{M}_{k,n+1}(\mathbb{Z})$.

Let $\{e_l\}_{l=1}^k$ denote the standard basis for $\mathfrak{t}_k^* \simeq \mathbb{R}^k$. Then the 3-Sasakian moment map $\mu_\Omega : S^{4n+3} \longrightarrow \mathfrak{t}_k^* \otimes \mathbb{R}^3$ of the k-torus action defined by $\varphi_{(\tau_1,\ldots,\tau_k)}(\mathbf{u}) = f_\Omega(\tau_1,\ldots,\tau_k)\mathbf{u}$, is given by $\mu_\Omega = \sum_l \mu_\Omega^l e_l$ with

$$\mu_\Omega^l(\mathbf{u}) = \sum_\alpha \bar{u}_\alpha i a_\alpha^l u_\alpha.$$

Let us further denote the triple $(T^k, f_\Omega, \varphi_{(\tau_1,\ldots,\tau_k)})$ by $T^k(\Omega)$.

Definition 13.6.1: $N(\Omega) = \mu_\Omega^{-1}(0)$ *and* $\mathcal{S}(\Omega) = S^{4n+3}/\!\!/\!\!/T^k(\Omega) = N(\Omega)/T^k(\Omega)$.

Let S_H^{4n+3} denote all the points on the sphere where the stability group $H \subset T^k$ is exactly H. Because T^k is Abelian $S_H^{4n+3} = S_{(H)}^{4n+3}$. Furthermore, let $K_H = T^k/H$ and denote by $\mathcal{S}(\Omega; H) = S_H^{4n+3} \cap N(\Omega)/K_H$. Following Theorem 7.3.1 we have

Proposition 13.6.2: *The quotient* $\mathcal{S}(\Omega) = \bigcup_H \mathcal{S}(\Omega; H)$ *is a disjoint union of 3-Sasakian manifolds, where each stratum* $\mathcal{S}(\Omega; H)$ *is smooth.*

We will be interested in the case when $\mathcal{S}(\Omega)$ is a compact orbifold (all stability groups H for which $S_H^{4n+3} \cap N(\Omega)$ are non-empty are discrete) or a compact smooth manifold (there is only one stratum). Necessary and sufficient conditions for this to happen can be expressed in terms of properties of the matrix Ω. First observe that, without loss of generality, we can assume that the rank of Ω equals k. Otherwise, one simply has an action of a torus of lower dimension and the whole problem reduces to considering another weight matrix Ω with fewer rows.

We introduce the following terminology: Consider the $\binom{n}{k}$ minor determinants

$$\Delta_{\alpha_1\ldots\alpha_k} = \det\begin{pmatrix} a^1_{\alpha_1} & \cdots & a^1_{\alpha_k} \\ \vdots & & \vdots \\ a^k_{\alpha_1} & \cdots & a^k_{\alpha_k}\end{pmatrix} \tag{13.6.1}$$

obtained by deleting $n+1-k$ columns of Ω.

Definition 13.6.3: *Let $\Omega \in \mathcal{M}_{k,n+1}(\mathbb{Z})$ be the weight matrix.*
(i) If $\Delta_{\alpha_1\cdots\cdots\alpha_k} \neq 0, \forall \quad 1 \leq \alpha_1 < \cdots < \alpha_k \leq n+1$, then we say that Ω is non-degenerate.

Suppose Ω is non-degenerate and let g be the kth determinantal divisor, i.e., the gcd of all the k by k minor determinants $\Delta_{\alpha_1\cdots\cdots\alpha_k}$. Then Ω is said to be admissible if in addition we have
(ii) $\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\alpha_k}) = g$ *for all sequences of length (k+1) such that $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n+1$.*

13.6.2. Equivalence Problem and Admissibility. Before we show how these properties of the matrix Ω impact on the geometry of the quotient $\mathcal{S}(\Omega)$ we need to discuss the notion of the equivalence of T^k-actions on S^{4n+3} and obtain a normal form for admissible weight matrices. We are free to change bases of the Lie algebra $\mathfrak{t}_k$. This can be done by the group of unimodular matrices $GL(k,\mathbb{Z})$. Moreover, if we fix a maximal torus T^{n+1} of $Sp(n+1)$, its normalizer, the Weyl group $\mathcal{W}(Sp(n+1)) \simeq \Sigma_{n+1} \bowtie (\mathbb{Z}_2)^{n+1}$, preserves the 3-Sasakian structure on S^{4n+3} and intertwines the T^k actions. Thus, there is an induced action of $GL(k,\mathbb{Z}) \times \mathcal{W}(Sp(n+1))$ on the set of weight matrices $\mathcal{M}_{k,n+1}(\mathbb{Z})$. The group $GL(k,\mathbb{Z})$ acts on $\mathcal{M}_{k,n+1}(\mathbb{Z})$ by matrix multiplication from the left, and the Weyl group $\mathcal{W}(Sp(n+1))$ acts by permutation and overall sign changes of the columns. Actually we want a slightly stronger notion of equivalence than that described above. If the ith row of Ω has a gcd d_i greater than one, then by reparameterizing the one-parameter subgroup $\tau'_i = \tau_i^{d_i}$ we obtain $\tau_i^{a^j_\alpha} = (\tau'_i)^{b^i_\alpha}$, where $\gcd\{b^i_\alpha\}_\alpha = 1$. So the action obtained by using the matrix whose ith row is divided by its gcd d_i is the same as the original action. The integers d_i all divide the kth determinantal divisor g. We say that a non-degenerate matrix Ω is in *reduced* form (or simply *reduced*) if $g=1$. The following easy lemma says that among non-degenerate matrices it is sufficient to consider matrices in a reduced form.

Lemma 13.6.4: *Every non-degenerate weight matrix Ω is equivalent to a matrix in a reduced form.*

Henceforth, we shall only consider matrices in a reduced form.

Definition 13.6.5: *Let $\mathcal{A}_{k,n+1}(\mathbb{Z}) \subset \mathcal{M}_{k,n+1}(\mathbb{Z})$ denote the subset of reduced admissible matrices. This subset is invariant under the action of $GL(k,\mathbb{Z})\times\mathcal{W}(Sp(n+1))$, so the set $\mathcal{A}_{k,n+1}(\mathbb{Z})/GL(k,\mathbb{Z})\times\mathcal{W}(Sp(n+1))$ of equivalence classes $[\Omega]$ is well defined. We let $\mathcal{E}_{k,n+1}(\mathbb{Z}) \subset \mathcal{A}_{k,n+1}(\mathbb{Z})$ denote a fundamental domain for the action.*

Our interest in $\mathcal{A}_{k,n+1}(\mathbb{Z})$ is the following:

Theorem 13.6.6: *Let $\mathcal{S}(\Omega)$ be the quotient space of definition 13.6.1. Then*

(i) if Ω is non-degenerate, $\mathcal{S}(\Omega)$ is an orbifold.
(ii) If Ω is degenerate, then either $\mathcal{S}(\Omega)$ is a singular stratified space which is not an orbifold or it is an orbifold obtained by reduction of a lower dimensional sphere $S^{4n-4r-1}$ by a torus $T^{k-r}(\Omega')$ or a finite quotient of such, where $1 \leq r \leq k$ and Ω' is non-degenerate. (When $r = k$ the quotient is the sphere $S^{4n-4k-1}$).
(iii) Assuming that Ω is non-degenerate $\mathcal{S}(\Omega)$ is a smooth manifold if and only if Ω is admissible.

One can easily see that the non-degeneracy of Ω is not necessary for the quotient space $\mathcal{S}(\Omega)$ to be smooth or a compact orbifold (see Example 13.5.6(iv)). However, Theorem 8.2.3(ii) shows that then we can reformulate the whole problem in terms of another quotient and a new non-degenerate weight matrix Ω' and can be found in [**46**]. Theorem 8.2.3(iii) shows then the importance of admissible matrices in the construction and it easily follows from the fact that non-degeneracy implies that at most $n-k$ quaternionic coordinates u_j can simultaneously vanish on $N(\Omega)$ [**?**].

Remark 13.6.1: Our discussion shows clearly that, if $\Omega, \Omega' \in \mathcal{A}_{k,n+1}(\mathbb{Z})$ such that $[\Omega] = [\Omega']$ then the quotients $\mathcal{S}(\Omega) \simeq \mathcal{S}(\Omega')$ are equivalent as 3-Sasakian manifolds. We believe that the converse of this is also true, though we will establish it later only in certain cases.

13.6.3. Combinatorics and Admissibility. In general Theorem 8.2.3 is not yet an existence theorem, since $\mathcal{A}_{k,n+1}(\mathbb{Z})$ could be empty. Indeed, for many pairs (k,n) this is the case and we shall demonstrate this next.

Let $\Omega \in \mathcal{A}_{k,n+1}(\mathbb{Z})$. Since Ω is reduced there is a k by k minor determinant that is odd. By permuting columns if necessary this minor can be taken to be the first k columns. Now consider the mod 2 reduction $\mathcal{M}_{k,n+1}(\mathbb{Z}) \longrightarrow \mathcal{M}_{k,n+1}(\mathbb{Z}_2)$. We have the following commutative diagram

$$\begin{array}{ccc} GL(k,\mathbb{Z}) \times \mathcal{M}_{k,n+1}(\mathbb{Z}) & \longrightarrow & \mathcal{M}_{k,n+1}(\mathbb{Z}) \\ \Big\downarrow & & \Big\downarrow \\ GL(k,\mathbb{Z}_2) \times \mathcal{M}_{k,n+1}(\mathbb{Z}_2) & \longrightarrow & \mathcal{M}_{k,n+1}(\mathbb{Z}_2). \end{array} \tag{13.6.2}$$

Let $\tilde{\Omega} \in \mathcal{A}_{k,n+1}(\mathbb{Z}_2)$ denote the mod 2 reduction of $\Omega \in \mathcal{A}_{k,n+1}(\mathbb{Z})$. Since the first k by k minor determinant of Ω is odd, the mod 2 reduction of this minor in $\tilde{\Omega}$ is invertible. Thus, we can use the $GL(k,\mathbb{Z}_2)$ action to put $\tilde{\Omega}$ in the form

$$\tilde{\Omega} = \begin{pmatrix} 1 & 0 & \dots & 0 & a^1_{k+1} & \dots & a^1_{n+1} \\ 0 & 1 & \dots & 0 & a^2_{k+1} & \dots & a^2_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 1 & a^k_{k+1} & \dots & a^k_{n+1} \end{pmatrix} \tag{13.6.3}$$

with $a^i_j \in \mathbb{Z}_2$.

Lemma 13.6.7: *The set $\mathcal{A}_{k,n+1}(\mathbb{Z})$ is empty for $n > k+1$ and $k > 4$.*

Proof. The second admissibility condition is equivalent to the condition that every k by $k+1$ submatrix of $\tilde{\Omega}$ has rank k. By considering $k-1$ of the first k columns and 2 of last $n+1-k$ columns, this condition implies $(a^j_l, a^j_m) \neq (0,0)$ for all $j = 1, \cdots, k$, and $k+1 \le l < m \le n+1$. Similarly, by considering $k-2$ of the first k columns and 3 of last $n+1-k$ columns 13.6.3 implies

$$\begin{pmatrix} a_l^i & a_m^i & a_r^i \\ a_l^j & a_m^j & a_r^j \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} a_l^i & a_m^i & a_r^i \\ a_l^j & a_m^j & a_r^j \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \tag{13.6.4}$$

where the last inequality is understood to be up to column permutation. Hence, it follows that, up to column and row permutations, that any four triples of the last $n-k$ columns of an admissible $\tilde{\Omega}$ must have the form

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{13.6.5}$$

So we see that we cannot add another row without violating the above conditions. It follows that $k \le 4$. □

Similar analysis shows that

Lemma 13.6.8: *The set $\mathcal{A}_{k,n+1}(\mathbb{Z})$ is empty if $k > 1$ and $n-k \ge 4$.*

Remark 13.6.2: In view of the above lemmas and the fact that in the remainder of this section we will be interested only in the smooth and compact quotients we are left with the following possibilities:

(i) Trivial case of $n = k$. Then there are many admissible matrices Ω but $\dim(\mathcal{S}(\Omega)) = 3$ and it follows that $\mathcal{S}(\Omega) = S^3/\mathbb{Z}_p$, where $p = p(\Omega)$ depends on Ω. This case is of little interest.

(ii) Bi-quotient geometry with $k = 1$ and $n > 1$ arbitrary. Here Ω is just a row vector $\mathbf{p}$. The admissibility condition means that the entries are non-zero and pairwise relatively prime. The quotient $\mathcal{S}(\mathbf{p})$ turns out to be a bi-quotient of the unitary group $U(n+1)$ and we shall discuss its geometry and topology in the next subsection.

(iii) The most interesting, 7-dimensional case of $k = n-1$. Here one easily sees that there are many admissible matrices and we analyze the geometry and topology of the quotients in a separate subsection.

(iv) "Special" quotients: $(k,n) = \{(2,4),(2,5),(3,5),(3,6),(4,6),(4,7)\}$. These quotients are 11- or 15-dimensional and we give examples of admissible weight matrices in each case. We shall show also that they provide counterexamples to certain Betti number relations that are satisfied in the regular case [**90**].

13.6.4. 3-Sasakian Structures on Bi-Quotients.

When $k = 1$ we have $\Omega = \mathbf{p} = (p_1, ..., p_{n+1})$ and we shall write $\mathcal{S}(\Omega) = \mathcal{S}(\mathbf{p})$, $N(\Omega) = N(\mathbf{p})$, $f_\Omega = f_{\mathbf{p}}$, and $\tau_1 = \tau$. The quotients $\mathcal{S}(\mathbf{p})$ are generalizations of the homogeneous examples discussed in Section 7.2. We get

$$\mathcal{A}_{1,n+1}(\mathbb{Z}) = \{\mathbf{p} \in (\mathbb{Z})^{n+1} \mid p_i \neq 0 \ \forall i = 1, ..., n+1 \text{ and } \gcd(p_i, p_j) = 1 \qquad \forall i \neq j\},$$

$$\mathcal{E}_{1,n+1}(\mathbb{Z}) = \{\mathbf{p} \in \mathbb{Z}^{n+1} \mid 0 < p_1 \le \cdots \le p_{n+1} \text{ and } \gcd(p_i, p_j) = 1 \qquad \forall i \neq j\}.$$

Note that $\mathcal{E}_{1,n+1}(\mathbb{Z})$ can be identified with a certain integral lattice in the positive Weyl chamber in $\mathfrak{t}^*_{n+1}$.

First, by studying the geometry of the foliations in the diagram $\diamondsuit(\mathcal{S}(\mathbf{p}))$ [**45**] one can solve the equivalence problem in this case. We get [**42**]:

Proposition 13.6.9: *Let $n \geq 2$ and $\mathbf{p}, \mathbf{q} \in \mathcal{A}_{1,n+1}(\mathbb{Z})$ so the quotients $\mathcal{S}(\mathbf{p})$ and $\mathcal{S}(\mathbf{q})$ are smooth manifolds. Then $\mathcal{S}(\mathbf{p}) \simeq \mathcal{S}(\mathbf{q})$ are 3-Sasakian equivalent if and only if $[\mathbf{p}] = [\mathbf{q}]$.*

It is easy to see that for $\mathbf{p} \in \mathcal{A}_{1,n+1}(\mathbb{Z})$ the zero locus of the moment map $N(\mathbf{p})$ is always diffeomorphic to the Stiefel manifold $V^{\mathbb{C}}_{2,n+1}$ of complex 2-frames in $\mathbb{C}^{n+1}$. Hence, the quotient $\mathcal{S}(\mathbf{p}) = V^{\mathbb{C}}_{2,n+1}/S^1$. We first observe that one can identify $V^{\mathbb{C}}_{2,n+1}$ with the homogeneous space $U(n+1)/U(n-1)$. Using this identification we have

Proposition 13.6.10: *For each $\mathbf{p} \in \mathcal{E}_{1,n+1}(\mathbb{Z})$, there is an equivalence $\mathcal{S}(\mathbf{p}) \simeq U(1)_{\mathbf{p}} \backslash U(n+1)/U(n-1)$ as smooth $U(1)_{\mathbf{p}} \times U(n-1)$-spaces, where the action of $U(1)_{\mathbf{p}} \times U(n-1) \ \subset \ U(n+1)_L \times U(n+1)_R$ is given by the formula*

$$\varphi^{\mathbf{p}}_{\tau,\mathbb{B}}(\mathbb{W}) = f_{\mathbf{p}}(\tau)\mathbb{W}\begin{pmatrix} \mathbb{I}_2 & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix}.$$

Here $\mathbb{W} \in U(n+1)$ and $(\tau, \mathbb{B}) \in S^1 \times U(n-1)$.

Note that the identification $\mathcal{S}(\mathbf{p}) \simeq U(1)_{\mathbf{p}} \backslash U(n+1)/U(n-1)$ is only true after assuming that all the weights are positive, as the right-hand side is not invariant under such sign changes. Proposition 8.4.2 shows that, in a way, the quotients $\mathcal{S}(\mathbf{p})$ can be though of as a discrete "bi-quotient deformation" of the homogeneous model $\mathcal{S}(\mathbf{1})$. Now let $\iota_{\mathbf{p}} : N(\mathbf{p}) \hookrightarrow S^{4n+3}$ be the inclusion and $\pi_{\mathbf{p}} : N(\mathbf{p}) \to \mathcal{S}(\mathbf{p})$ be the Riemannian submersion of the moment map. Then the metric $g(\mathbf{p})$ is the unique metric on $\mathcal{S}(\mathbf{p})$ that satisfies $\iota^*_{\mathbf{p}} g_{can} = \pi^*_{\mathbf{p}} g(\mathbf{p})$. Using the geometry of the inclusion $\iota_{\mathbf{p}}$ one can show the following [**42, 45**]

Theorem 13.6.11: *Let $I_0(\mathcal{S}(\mathbf{p}), g(\mathbf{p}))$ be the group of 3-Sasakian isometries of $\big(\mathcal{S}(\mathbf{p}), g(\mathbf{p})\big)$ and let k be the number of 1's in $\mathbf{p}$. Then the connected component of I_0 is $S(U(k) \times U(1)^{n+1-k})$, where we define $U(0) = \{e\}$. Thus, the connected component of the isometry group is the product $S(U(k) \times U(1)^{n+1-k}) \times SO(3)$ if the sums $p_i + p_j$ are even for all $1 \leq i, j \leq n+1$, and $S(U(k) \times U(1)^{n+1-k}) \times Sp(1)$ otherwise.*

In the case that $\mathbf{p}$ has no repeated 1's, the cohomogeneity can easily be determined, viz. [**42**]

Corollary 13.6.12: *If the number of 1's in $\mathbf{p}$ is 0 or 1 then the dimension of the principal orbit in $\mathcal{S}(\mathbf{p})$ equals $n+3$ and the cohomogeneity of $g(\mathbf{p})$ is $3n-4$. In particular, the 7-dimensional $\mathcal{S}(\mathbf{p})$ the family $(\mathcal{S}(\mathbf{p}), g(\mathbf{p}))$ contains metrics of cohomogeneity 0,1, and 2.*

Combining Proposition 8.4.2 with techniques developed by Eschenburg [**74, 75**] in the study of certain 7-dimensional bi-quotients of $SU(3)$ one can compute the integral cohomology ring of $\mathcal{S}(\mathbf{p})$ [**?**]:

Theorem 13.6.13: *Let $n \geq 2$ $\mathbf{p} \in \mathcal{E}_{1,n+1}(\mathbb{Z})$. Then, as rings,*

$$H^*\big(\mathcal{S}(\mathbf{p}), \mathbb{Z}\big) \ \cong \ \left(\frac{\mathbb{Z}[b_2]}{[b_2^{n+1} = 0]} \otimes E[f_{2n+1}]\right)/\mathcal{R}(\mathbf{p}).$$

Here the subscripts on b_2 and f_{2n+1} denote the cohomological dimension of each generator. Furthermore, the relations $\mathcal{R}(\mathbf{p})$ are generated by $\sigma_n(\mathbf{p})b_2^n = 0$ and $f_{2n+1}b_2^n = 0$, where $\sigma_n(\mathbf{p}) = \sum_{j=1}^{n+1} p_1 \cdots \hat{p_j} \cdots p_{n+1}$ is the n^{th} elementary symmetric polynomial in the entries of $\mathbf{p}$.

Notice that Theorem 8.4.5 shows that $H^{2n}(\mathcal{S}(\mathbf{p});\mathbb{Z}) = \mathbb{Z}_{\sigma_n(\mathbf{p})}$ and hence has the following corollary.

Corollary 13.6.14: *The quotients $(\mathcal{S}(\mathbf{p}), g(\mathbf{p}))$ give infinitely many homotopy inequivalent simply-connected compact inhomogeneous 3-Sasakian manifolds in dimension $4n-1$ for every $n \geq 2$. In fact, there are infinite families that are not homotopy equivalent to any homogeneous space.*

Remark 13.6.3: Corollary 8.4.6 shows that the finiteness results for regular 3-Sasakian manifolds discussed in Section 5 fail for non-regular 3-Sasakian manifolds. Moreover, combining our results with a well-known finiteness theorem of Anderson [**6**] we have

Corollary 13.6.15: *For each $n \geq 2$ there are infinitely many 3-Sasakian $4n-1$-manifolds with arbitrarily small injectivity radii.*

When $n = 2$ the spaces $\mathcal{S}(\mathbf{p}) = \mathcal{S}(p_1, p_2, p_3)$ give a subfamily of the more general bi-quotients of $U(3)$ studied by Eschenburg [**74, 75**]. This large collection of spaces contains not only our 3-Sasakian subfamily, but also the well-known Aloff-Wallach spaces [**5**] which are of much interest since they admit Einstein metrics of positive sectional curvature [**216**]. These two subfamilies intersect at the homogeneous 3-Sasakian manifold $\mathcal{S}(1,1,1)$, that is $\mathcal{S}(1,1,1)$ is diffeomorphic to the Aloff-Wallach space $N_{1,1}$ mentioned in 6.1. Then following Eschenburg [**74**] we can make use of the Cheeger ρ^*-topology on the space of Riemannian manifolds to show the existence of an infinite number of 3-Sasakian manifolds that admit metrics of positive sectional curvature. More precisely [**?**],

Corollary 13.6.16: *For all sufficiently large odd positive integers c, the 3-Sasakian manifolds $\mathcal{S}(c, c+1, c+2)$ admits a metric of positive sectional curvature.*

In the next subsection we give a result in the opposite direction. We shall exhibit an infinite family of 3-Sasakian manifolds that cannot admit any metric whose sectional curvature is bounded from below by a fixed arbitrary negative number.

We end this subsection with a discussion of topological and differential invariants of the 7-manifolds $\mathcal{S}(p_1, p_2, p_3)$. Homotopy invariants for Eschenburg space have been worked out independently by Kruggel [**137, 138**] and Milgram [**152**]. The homeomorphism and diffeomorphism classification was first done for a certain subclass of Eschenburg spaces which include some of the $\mathcal{S}(p_1, p_2, p_3)$ by Astey, Micha, and Pastor [**11**]. Later Kruggel [**139**] obtained the diffeomorphism and homeomorphism classification of all Eschenburg's bi-quotients by computing the Kreck-Stolz invariants [**135**]. This, in principle, gives a complete differential topological description of the 7-dimensional family $\mathcal{S}(p_1, p_2, p_3)$. Using this classification together with the help of a computer program, one would expect to find examples $\mathcal{S}(\mathbf{p})$ and $\mathcal{S}(\mathbf{q})$ with $[\mathbf{p}] \neq [\mathbf{q}]$ such that the quotients are homeomorphic, but not diffeomorphic, as well as examples that are diffeomorphic, but not 3-Sasakian equivalent. The later would show that a smooth 7-manifold can admit more than one inequivalent 3-Sasakian structure. In the case of the former, such exotic structures are known to exist for the family of Aloff-Wallach spaces [**136**], but the examples involve large integers and were obtained with help of a computer program. The analysis of the above mentioned invariants for our family $\mathcal{S}(p_1, p_2, p_3)$ proves even harder due to the positivity of the weights. For a fixed $\sigma_2 = p_1p_2 + p_2p_3 + p_3p_1$ there are only finitely many positive integer solution $\mathbf{p} \in \mathcal{E}_{1,3}(\mathbb{Z})$.

13.6.5. 7-dimensional Toric 3-Sasakian Manifolds. In this case we can easily see that there are many examples of admissible weight matrices Ω. The simplest family of examples is given by matrices of the form

$$\Omega = \begin{pmatrix} 1 & 0 & \dots & 0 & a_1 & b_1 \\ 0 & 1 & \dots & 0 & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_k & b_k \end{pmatrix},$$

for which we have

Proposition 13.6.17: *Let k be a positive integer, and let $\Omega \in \mathcal{M}_{k,k+2}(\mathbb{Z})$ be as in 13.6.5. Then $\Omega \in \mathcal{A}_{k,k+2}(\mathbb{Z})$ if and only if $(\mathbf{a}, \mathbf{b}) \in (\mathbb{Z}^*)^k \oplus (\mathbb{Z}^*)^k$ and the components (a^i, b^i) are pairs of relatively prime integers for $i = 1, \cdots, k$ such that if for some pair i, j $a^i = \pm a^j$ or $b^i = \pm b^j$ then we must have $b^i \neq \pm b^j$ or $a^i \neq \pm a^j$, respectively.*

Proposition 8.5.1 shows that $\mathcal{A}_{k,k+2}(\mathbb{Z})$ is never empty and we have many examples of compact smooth 7-dimensional quotients $\mathcal{S}(\Omega)$ for arbitrary $k > 1$. Some of these examples were first mentioned in [**40**] and the idea of the quotient is based on the result of [**89**]. As we shall not present here the complete solution to the equivalence problem, we shall further assume that $\Omega \in \mathcal{A}_{k,k+2}(\mathbb{Z})$ is arbitrary and shall determine some important topological properties of the quotients $\mathcal{S}(\Omega)$. More explicitly,

Theorem 13.6.18: *Let $\Omega \in \mathcal{A}_{k,k+2}(\mathbb{Z})$. Then $\pi_1(\mathcal{S}(\Omega)) = 0$ and $\pi_2(\mathcal{S}(\Omega)) = \mathbb{Z}^k$.*

Because of Corollary 5.2.3 and Poincare duality, Theorem 8.5.2 completely determines the rational homology of the 3-Sasakian 7-manifolds $\mathcal{S}(\Omega)$. The proof given below is a compilation with some simplifications of the proofs in [**?, 47**], while some of the more tedious details are left to those references.

Proof. First note that the groups $T^{k+2} \times Sp(1)$ and $T^2 \times Sp(1)$ act as isometry groups on $N(\Omega)$ and $\mathcal{S}(\Omega)$, respectively. Let us define the following quotient spaces:

$$Q(\Omega) = N(\Omega)/T^{k+2} \times Sp(1), \qquad B(\Omega) = N(\Omega)/Sp(1).$$

We have the following commutative diagram

$$\begin{array}{ccc} N(\Omega) & \longrightarrow & B(\Omega) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{S}(\Omega) & \longrightarrow & Q(\Omega). \end{array} \tag{13.6.6}$$

The top horizontal arrow and the left vertical arrow are principal bundles with fibers $Sp(1)$ and T^k, respectively. The remaining arrows are not fibrations. The right vertical arrow has generic fibers T^{k+2}, while the lower horizontal arrow has generic fibers $T^2 {\cdot} Sp(1)$ homeomorphic either to $T^2 \times \mathbb{RP}^3$ or $T^2 \times S^3$ depending on Ω. The dimension of the orbit space $Q(\Omega)$ is 2.

The difficulty is in proving that both $N(\Omega)$ and $B(\Omega)$ are 2-connected. Once this is accomplished the result follows by applying the long exact homotopy sequence to the left vertical arrow in diagram 13.6.6.

Lemma 13.6.19: *Both $N(\Omega)$ and $B(\Omega)$ are 2-connected.*

To prove this lemma we construct a stratification giving a Leray spectral sequence whose differentials can be analyzed. Let us define the following subsets of $N(\Omega)$: (Recall that, in this case, at most one quaternionic coordinate can vanish.)

$$
\begin{aligned}
N_0(\Omega) &= \{\mathbf{u} \in N(\Omega) |\ u_\alpha = 0 \text{ for some } \alpha = 1, \cdots, k+2\},\\
N_1(\Omega) &= \{\mathbf{u} \in N(\Omega) |\ \text{for all } \alpha = 1, \cdots, k+2,\ u_\alpha \neq 0 \text{ and there is a pair } (u_\alpha, u_\beta)\\
&\qquad \text{that lies on the same complex line in } \mathbb{H}\},\\
N_2(\Omega) &= \{\mathbf{u} \in N(\Omega) |\ \text{for all } \alpha = 1, \cdots, k+2,\ u_\alpha \neq 0 \text{ and no pair } (u_\alpha, u_\beta)\\
&\qquad \text{lies on the same complex line in } \mathbb{H}\}.
\end{aligned}
$$

Clearly, $N(\Omega) = N_0(\Omega) \sqcup N_1(\Omega) \sqcup N_2(\Omega)$ and $N_2(\Omega)$ is a dense open submanifold of $N(\Omega)$. This stratification is compatible with the diagram 13.6.6 and induces corresponding stratifications

$$B(\Omega) = B_0(\Omega) \sqcup B_1(\Omega) \sqcup B_2(\Omega), \quad Q(\Omega) = Q_0(\Omega) \sqcup Q_1(\Omega) \sqcup Q_2(\Omega). \tag{13.6.7}$$

The $B_i(\Omega)$ fiber over the $Q_i(\Omega)$ whose fibers are tori T^{k+i}. The strata are labeled by the dimension of the cells in the resulting CW decomposition of $Q(\Omega)$. Using known results about cohomogeneity 2 actions [**52**] one can easily prove:

Lemma 13.6.20:
(i) The orbit space $Q(\Omega)$ is homeomorphic to the closed disc $\bar{D}^2$, and the subset of singular orbits $Q_1(\Omega) \sqcup Q_0(\Omega)$ is homeomorphic to the boundary $\partial\bar{D}^2 \simeq S^1$.
(ii) $Q_2(\Omega)$ is homeomorphic to the open disc D^2.
(iii) $Q_1(\Omega)$ is homeomorphic to the disjoint union of $k+2$ copies of the open unit interval.
(iv) $Q_0(\Omega)$ is a set of $k+2$ points.

Next one can easily show that $\pi_1(B(\Omega))$ is Abelian; hence, $\pi_1(B(\Omega)) = H_1(B(\Omega))$. Now we claim that $H_1(B(\Omega)) = H_2(B(\Omega)) = 0$, and since the fibers of the top horizontal arrow of 13.6.6 are S^3's this together with Hurewicz will prove Lemma 8.5.3. To prove this claim we define $Y_0 = Q_0(\Omega),\ Y_1 = Q_0(\Omega) \cup Q_1(\Omega)$, and $Y_2 = Q(\Omega)$. Then, we filter $B(\Omega)$ by $X_i = \pi^{-1}(Y_i)$ to obtain the increasing filtration

$$X_0 = B_0(\Omega),\ X_1 = B_0(\Omega) \cup B_1(\Omega), \text{ and } X_2 = B(\Omega).$$

The Leray spectral sequence associated to this filtration has E^1 term given by

$$E^1_{s,t} \ \cong\ H_{s+t}(X_t, X_{t-1}; \mathbb{Z})$$

with differential $d_1 : H_{s+t}(X_t, X_{t-1}; \mathbb{Z}) \longrightarrow H_{s+t-1}(X_{t-1}, X_{t-2}; \mathbb{Z})$, where we use the convention that $X_{-1} = \emptyset$.

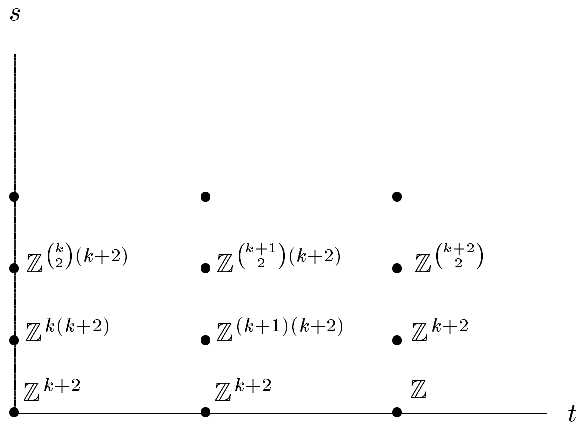
To compute these E^1 terms notice that all the pairs (X_t, X_{t-1}) are relative manifolds so that one can apply the Alexander-Poincaré duality theorem. Hence, by 13.6.7

$$H_s(X_0; \mathbb{Z}) \ \cong\ H_s(\sqcup_{k+2} T^k; \mathbb{Z}); \tag{13.6.8}$$

$$H_s(X_1, X_0; \mathbb{Z}) \ \cong\ H^{k+2-s}(\sqcup_{k+2} T^{k+1}; \mathbb{Z}); \tag{13.6.9}$$

$$H_s(X_2, X_1; \mathbb{Z}) \ \cong\ H^{k+4-s}(T^{k+2}; \mathbb{Z}), \tag{13.6.10}$$

where $\sqcup_j T^l$ means the disjoint union of j copies of T^l. Hence, the $E^1_{s,t}$ term of the spectral sequence is described by the diagram



$E^1_{s,t}$

The computation of the differentials is fairly tedious and we refer the reader to [**47**] for details. Suffice it to say here that after making certain choices Lemma 8.5.4 can be used to represent $Q(\Omega)$ topologically as a polygon

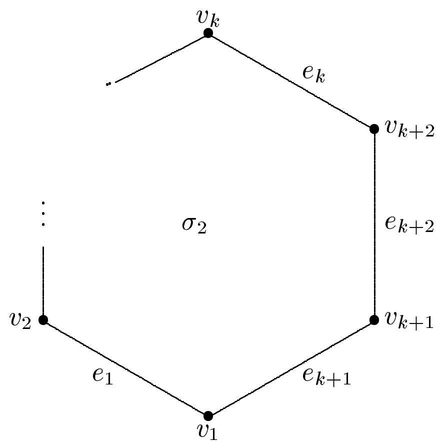


Diagram 8.5.12

The d_1 differential can then be computed and the result is that the $E^2_{s,t}$ term has zeros for $(s,t) = (1,0), (2,0), (1,1), (1,2), (0,1), (0,2)$. Then $E^2_{s,t} = E^\infty_{s,t}$ which converges to $H_{s+t}(B(\Omega), \mathbb{Z})$, so this proves Lemma 8.5.3 and hence, Theorem 8.5.2. $\square$

By further analysis of the differentials it should be possible to determine the torsion in $H_3(\mathcal{S}(\Omega), \mathbb{Z})$. This should be given in terms of symmetric functions of the invariants $|\Delta_{\alpha_1,\cdots,\alpha_k}|$.

REMARK 13.6.4: It was pointed out to the authors by Karsten Grove that if one takes the metric geometry into account, the internal angles in Diagram 8.5.12 are all less than 90 degrees. This indicates the presence of hyperbolic geometry.

Now, using Theorem 8.5.2 with Propositions 5.3.1, 5.3.2 gives:

Proposition 13.6.21: *Let $\Omega \in \mathcal{A}_{k,k+2}(\mathbb{Z})$ so that $\mathcal{S}(\Omega)$ is a smooth manifold. Let $\mathcal{Z}(\Omega)$ and $\mathcal{O}(\Omega)$ be the associated twistor space and quaternionic Kähler orbifold, respectively. Then we have*

$$b_2(\mathcal{S}(\Omega)) = b_2(\mathcal{O}(\Omega)) = b_2(\mathcal{Z}(\Omega)) - 1 = k.$$

This shows that inequality $b_2 \leq 1$ in Proposition 5.4.2 does not hold for non-regular 3-Sasakian manifolds. Finally we give several interesting corollaries of our work.

Corollary 13.6.22: *There exists a simply-connected 3-Sasakian 7-manifold for every rational homology type allowed by Corollary 5.2.3.*

Our next corollary follows from the results of this section and remarkable theorem of Gromov [**?**]:

Corollary 13.6.23: *For any non-positive real number κ there are infinitely many 3-Sasakian 7-manifolds which do not admit metrics whose sectional curvatures are all greater than or equal to κ.*

For such an infinite family of 3-Sasakian 7-manifolds, the appearance of negative curvature is foretold by Remark 8.4. Corollary 8.5.7 can be contrasted with Corollary 8.4.8.

Corollary 13.6.24: *There exist 7-manifolds with arbitrary second Betti number having metrics of weak holonomy G_2.*

Of course, Corollary 8.5.7 also applies to these weak holonomy G_2 metrics.

Corollary 13.6.25: *There exist $\mathbb{Q}$-factorial contact Fano 3-folds X with $b_2(X) = l$ for any positive integer l.*

This corollary should be contrasted to the smooth case, where Mori and Mukai have proven that $b_2 \leq 10$ [**?**]:

Corollary 13.6.26: *If the second Betti number $b_2(\mathcal{S}(\Omega)) = k > 3$, the 3-Sasakian manifolds $\mathcal{S}(\Omega)$ are not homotopy equivalent to any homogeneous space.*

This corollary can be compared to Corollary 8.4.6. Finally we have

Corollary 13.6.27: *There exist compact, T^2-symmetric, self dual Einstein orbifolds of positive scalar curvature with arbitrary second Betti number.*

Again this should be contrasted to the smooth case where we must have $b_2 \leq 1$.

13.6.6. Higher Dimensional Toric 3-Sasakian Manifolds. We begin with the definition of a toric 3-Sasakian manifolds which is motivated by the hyperkähler case [**26**].

Definition 13.6.28: *A 3-Sasakian manifold (orbifold) of dimension $4m-1$ is said to be a toric 3-Sasakian manifold (orbifold) if it admits an effective action of a m-torus T^m that preserves the 3-Sasakian structure.*

The importance of toric 3-Sasakian manifolds is underlined by the following recent Delzant-type theorem of Bielawski:

Theorem 13.6.29: *Let $\mathcal{S}$ be a toric 3-Sasakian manifold of dimension $4n-1$. Then $\mathcal{S}$ is isomorphic as a 3-Sasakian T^n-manifold to a 3-Sasakian quotient of a sphere by a torus, that is to a $\mathcal{S}(\Omega)$ for some Ω.*

This theorem includes the degenerate case when the quotient is a sphere or a discrete quotient of such. The Betti numbers of a 3-Sasakian orbifold obtained by a toral quotient of a sphere were computed by Bielawski [**24**] using different techniques than the ones employed in Section 8.5:

Theorem 13.6.30: *Let $\Omega \in \mathcal{M}_{k,n+1}(\mathbb{Z})$ be non-degenerate so that $\mathcal{S}(\Omega)$ is a compact 3-Sasakian orbifold of dimension $4(n-k)+3$. Then we have*

$$b_{2i} = \binom{k+i-1}{k}, \qquad i < n+1-k. \tag{13.6.11}$$

Furthermore, the Betti number constraints of Proposition 5.4.2(ii) can hold for $\mathcal{S}(\Omega)$ if and only if $k=1$.

Combining Theorems 8.6.2 and 8.6.3 with Lemmas 8.3.1 and 8.3.2 which give obstructions to smoothness gives the somewhat surprising result [**46**],

Theorem 13.6.31: *Let $\mathcal{S}$ be a toric 3-Sasakian manifold.*
(i) If the dimension of $\mathcal{S}$ is 19 or greater, then $b_2(\mathcal{S}) \leq 1$.
(ii) If the dimension of $\mathcal{S}$ is 11 or 15, then $b_2(\mathcal{S}) \leq 4$.
(iii) If $b_2(\mathcal{S}) > 4$, then the dimension of $\mathcal{S}$ is 7.

A corollary due to Bielawski [**25**] is:

Corollary 13.6.32: *Let $\mathcal{S}$ be a regular toric 3-Sasakian manifold. Then $\mathcal{S}$ is one of the 3-Sasakian homogeneous spaces $S^{4n-1}, \mathbb{RP}^{4n-1}$ or $\frac{SU(n)}{S\big(U(n-2)\times U(1)\big)}$.*

Next we give an explicit construction of toric 3-Sasakian manifolds not eliminated by Theorem 8.6.4. It is enough to show that $\mathcal{A}_{4,8}$ and $\mathcal{A}_{4,7}$ are not empty as the rest follow by deletion of rows of the corresponding $\Omega \in \mathcal{A}_{4,*}$. We shall present two three parameter families of solutions, namely

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1+2^l \\ 1 & 16 & 1+2^m \\ -1 & 3 & 2c \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1+2^{l'} & -1 \\ 1 & 16 & 1+2^{2n} & 3 \\ -1 & 3 & 2c' & -1 \end{pmatrix},$$

where $l, l', m, n \in \mathbb{Z}^+$, and $c, c' \in \mathbb{Z}$.

With the aid of MAPLE symbolic manipulation program, we find

Lemma 13.6.33: [**46**] *Let $\Delta = 2(31c + 6 + 19\cdot 2^{l-1} - 7\cdot 2^{m-1})$.*
(i) $\Omega_1 = \big(\mathbb{I}_4\ A_1\big)$ is admissible if and only if $c \neq 0$ and is not divisible by 3, and $\Delta \neq 0$ and is not divisible by 7, 19 nor 31.
(ii) $\Omega_2 = \big(\mathbb{I}_4\ A_2\big)$ is admissible if and only if c and all minor determinants of A_2 are non-vanishing, and $c' \not\equiv 0 \pmod 3$, $l' \not\equiv 0 \pmod 4$, $c' \not\equiv 5 \pmod 7$, and 11, 19, 37, 71 do not divide $\det A = 19\cdot 2^{2n} - 63 - 148c' - 11\cdot 2^{l'}$, and the following conditions hold:

$$\begin{array}{lcl} \gcd(3, 4c' + 2^{l'} + 1, 2c' - 2^{l'} - 1) & = & 1 \\ \gcd(7, 2^{2n+1} - 2^{l'} + 1, 3\cdot 2^{l'} + 2^{2n} + 4) & = & 1 \\ \gcd(19, 2^{2n} - 2^{l'+4} - 15, 3\cdot 2^{l'} + 2^{2n} + 4) & = & 1 \\ \gcd(25, 32c' - 3\cdot 2^{2n} - 3, 6c' + 2^{2n} + 1) & = & 1. \end{array} \tag{13.6.12}$$

The conditions in this proposition guarantee that the quotient spaces denoted by $\mathcal{S}(c, l, m)$ and $\mathcal{S}(c', l', n)$ are smooth manifolds of dimension 11 and 15, respectively.

It is routine to verify that the three parameter infinite family given by

$$c \equiv 14 \pmod{21}, \qquad l \not\equiv 1 \pmod{5}, \qquad m \not\equiv \alpha(c) \pmod{18},$$

where $2^{\alpha(c)} = 22(31c+6) \pmod{18}$ satisfies the conditions in (i) of Lemma 8.6.6. This gives examples in dimension 11. (Notice that as 2 is a primitive root of 19 the equation defining $\alpha(c)$ has a unique solution $\pmod{18}$ for each value of c.) Similarly, it is straightforward to verify that the infinite family given by

$$c' = 2, \qquad l' = 1, \qquad n \equiv 21 \pmod{90},$$

satisfies the conditions (ii) of Lemma 8.6.6. We have arrived at:

Theorem 13.6.34: [**46**] *There exist toric 3-Sasakian manifolds $\mathcal{S}$ of dimensions* 11 *and* 15 *with* $b_2(\mathcal{S}) = 2, 3, 4$. *Consequently, the Betti number relations of Proposition 5.4.2 do not hold generally. More explicitly there are compact 11-dimensional 3-Sasakian manifolds for which $b_2 \neq b_4$, and compact 15-dimensional 3-Sasakian manifolds for which $b_2 \neq b_6$.*

13.7. Non-Toric 3-sasakian Manifolds

CHAPTER 14

Killing Spinors

In this section we discuss some additional properties of Sasakian and Sasakian-Einstein manifolds which are connected with spin structure and eigenvalues of the Dirac operator.

14.1. Parallel Spinors and Holonomy

14.2. Real Killing Spinors and Bär's Theorem

Definition 14.2.1: *Let (M,g) be a complete n-dimensional Riemannian spin manifold, and let $S(M)$ be the spin bundle of M and ψ a smooth section of $S(M)$. We say that ψ is a Killing spinor if*

$$\nabla_X\psi = \alpha \cdot X\cdot\psi, \qquad \forall X\in\Gamma(TM),$$

where ∇ is the Levi-Civita connection of g and $X\cdot\psi$ denotes the Clifford product of X and ψ. We say that ψ is imaginary when $\alpha\in\mathrm{Im}(\mathbb{C}^*)$, ψ is parallel if $\alpha=0$ and ψ is real if $\alpha\in\mathrm{Re}(\mathbb{C}^*)$.

From the point of view of Einstein geometry the importance of Killing spinors is an immediate consequence of the following theorem of Friedrich [**80**]:

Theorem 14.2.2: *Let (M,g) be an n-dimensional; complete Riemannian spin manifold with a Killing spinor. Then M is Einstein with Einstein constant $\lambda = 4(n-1)\alpha^2$. In particular, when $\alpha\in\mathrm{Re}(\mathbb{C}^*)$, M is compact of positive scalar curvature.*

On the other hand, Friedrich showed that if M is a compact spin manifold of positive scalar curvature and R_0 is the minimum of the scalar curvature, then for all eigenvalues β of the Dirac operator D one has $\beta^2\geq \frac{1}{4}\frac{nR_0}{n-1}$ [**80**]. If the equality holds than it follows that the corresponding eigenspinor must be a Killing spinor with $\alpha=\pm\frac{1}{2}[\frac{R_0}{n(n-1)}]^{1/2}$. We have the following important property of manifolds with Killing spinors [**?**]:

Theorem 14.2.3: *Let (M^n,g) be a connected Riemannian spin manifold admitting a non-trivial Killing spinor with $\alpha\neq 0$. Then (M,g) is locally irreducible. Furthermore, if M is locally symmetric, or $n\leq 4$, then M is a space of constant sectional curvature equal $4\alpha^2$.*

From now on we will be interested only in the case of real Killing spinors. It was Friedrich and Kath [**81, 82**] who first noticed that in some low odd dimensions the existence of real Killing spinors leads naturally to the existence of Sasakian-Einstein or 3-Sasakian structures. Later, the problem found a simple classification in terms of the holonomy of the associated metric cone $C(M)$ [**15**]. First, we have the following definition:

Definition 14.2.4: *We say that M is of type (p,q) if it carries exactly p linearly independent real Killing spinors with $\alpha > 0$ and exactly q linearly independent real Killing spinors with $\alpha < 0$, or vice versa.*

For, example, the standard sphere S^n is of type $(2^{[n/2]}, 2^{[n/2]})$. Bär shows that when M admits a real Killing spinor then the cone $(C(M), \bar{g})$ has a parallel spinor. In particular, $C(M)$ is always Ricci-flat and, when M is simply-connected, then only a few holonomy groups $\mathrm{Hol}(\bar{g})$ are possible [**218**]:

Theorem 14.2.5: [**15**] *Let (M,g) be a simply-connected Riemannian spin manifold admitting a non-trivial Killing spinor and let $\mathrm{Hol}(\bar{g})$ be the holonomy group of the metric cone $(C(M), \bar{g})$. Then there are only the following 6 possibilities for the triple $\big(\dim M, \mathrm{Hol}(\bar{g}), (p,q)\big)$:*

$\dim M$	$\mathrm{Hol}(\bar{g})$	(p,q)
n	id	$(2^{[n/2]}, 2^{[n/2]})$
$4m+1$	$SU(2m+1)$	$(1,1)$
$4m+3$	$SU(2m+2)$	$(2,0)$
$4m+3$	$Sp(m+1)$	$(m+2,0)$
7	$\mathrm{Spin}(7)$	$(1,0)$
6	G_2	$(1,1)$

Here $m \geq 1$, and $n > 1$.

The first case is special as M is the n-dimensional round sphere. Since M is assumed to be simply-connected, in the next two cases, by Proposition 2.1.4, M must be Sasakian-Einstein. In the fourth case above it follows from Definition 2.2.1 that M is 3-Sasakian. Specifically, we get the following theorem [**15**]:

Theorem 14.2.6: [**15**] *Let (M^n,g) be a complete simply-connected Riemannian spin manifold admitting a non-trivial Killing spinor with $\alpha > 0$ or $\alpha < 0$. If $n = 4m+1$ with $m \geq 1$, then there are two possibilities:*
(i) $(M,g) = (S^n, g_{can})$,
(ii) (M,g) *is of type* $(1,1)$ *and it is a Sasakian-Einstein manifold.*

Conversely, if (M,g) is a complete simply-connected Sasakian-Einstein manifold of dimension $4m+1$, then M carries Killing spinors with $\alpha > 0$ and $\alpha < 0$.

Remark 14.2.1: Note that in the converse statement we do not need to assume that M is spin. When $\pi_1(M) = 0$ this is automatic by Corollary 2.1.6. When $\pi_1(M) \neq 0$ then the 'if' part of Theorem 6.1.6 can be generalized and we still get two possibilities: (i) either M is a spin spherical space form, or (ii) it is of type $(1,1)$ with a Sasakian-Einstein structure and $\mathrm{Hol}(\bar{g}) = SU(m+1)$ [**218**].

Theorem 14.2.7: [**15**] *Let (M^n,g) be a complete simply-connected Riemannian spin manifold admitting a non-trivial Killing spinor with $\alpha > 0$ or $\alpha < 0$. If $n = 4m+3$, $m \geq 2$, then there are three possibilities:*
(i) $(M,g) = (S^n, g_{can})$,
(ii) (M,g) *is a Sasakian-Einstein manifolds of type* $(2,0)$*, but* (M,g) *is not 3-Sasakian,*
(iii) (M,g) *is of type* $(m+2,0)$ *and it is 3-Sasakian.*

Conversely, if (M,g) is a complete simply-connected 3-Sasakian manifold, of dimension $4m+3$ which is not of constant curvature, then M carries $(m+2)$ linearly independent Killing spinors with $\alpha > 0$. If (M,g) is a complete simply-connected

Sasakian-Einstein manifold of dimension $4m+3$ *which is not 3-Sasakian then* M *carries* 2 *linearly independent Killing spinors with* $\alpha > 0$.

REMARK 14.2.2: Note that in Theorem 6.1.7(ii) we are not excluding the possibility of M having another 3-Sasakian structure with a different metric g'. We are only saying that the holonomy group $\mathrm{Hol}(\bar{g}) = SU(2m+2)$ rather than $Sp(m+1) \subset SU(2m+2)$, which, by definition, means that g cannot be 3-Sasakian. However, we are not aware of any such example. We have excluded $\dim(M) = 7$ because in this case we have one more possibility due to Theorem 6.1.5 and we want to discuss the associated geometry in more detail later. Again, one can generalize Theorem 6.1.7 to $\pi_1(M) \neq 0$. For the full list of possible holonomy groups $\mathrm{Hol}(\bar{g})$ see [**217**]. The coresponding M are then only locally Sasakian-Einstein or locally 3-Sasakian [**173, 176**]. The problem of the existence of Killing spinors on a Sasakian-Einstein or 3-Sasakian manifold with $\pi_1(M) \neq 0$ is, however, more subtle.

Corollary 14.2.8: *Let* $(\mathcal{S}, g)$ *be a compact Sasakian-Einstein manifold of dimension* $2m+1$. *Then* $\mathcal{S}$ *is locally symmetric if and only if* $\mathcal{S}$ *is of constant curvature. Moreover,* $(\mathcal{S}, g)$ *is locally irreducible as a Riemannian manifold.*

PROOF. If necessary, go to the universal cover $\tilde{\mathcal{S}}$. This is a compact simply-connected Sasakian-Einstein manifold; hence, it admits a non-trivial Killing spinor by Theorems 6.1.6 and 6.1.7. The statement then follows from the Theorem 6.1.3. □

Corollary 14.2.9: *Let* $(\mathcal{S}, g)$ *be a compact Sasakian-Einstein manifold of dimension* $2m+1$. *Then* $\mathrm{Hol}(g) = SO(2m+1)$.

PROOF. Let us consider universal cover $\tilde{\mathcal{S}}$. This is a compact simply-connected Sasakian-Einstein manifold; hence, it admits a non-trivial Killing spinor. By the previous corollary, it can be symmetric if only if it is isomorphic to a space of constant curvature, that is, a sphere. Then $\mathcal{S}$ is a spherical space form and $\mathrm{Hol}(g) = SO(2m+1)$. Assume $\mathcal{S}$ is not locally symmetric. By Corollary 6.1.8 $\mathcal{S}$ is locally Riemannian irreducible, so for dimensional reasons and Berger's famous classification theorem [**21**], the only possibilities for the restricted holonomy group $\mathrm{Hol}^0(g)$ are $SO(2m+1)$ and G_2 in dimension 7. But G_2 holonomy implies Ricci-flat and, hence, not Sasakian-Einstein. Hence, the restricted holonomy group $\mathrm{Hol}^0(g) = SO(2m+1)$. Since $\mathcal{S}$ is orientable this coincides with the holonomy group $\mathrm{Hol}(g)$. □

14.3. Geometric Structures Associated with 3-Sasakian 7-manifolds

Let us assume that $\dim(\mathcal{S}) = 7$. Remarlably, one can "squash" the whole diamond

$$\begin{tikzcd}
& C(\mathcal{S}) \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathcal{Z} \arrow[dr] & & \mathcal{S}. \arrow[ll] \arrow[dl] \\
& \mathcal{O} &
\end{tikzcd}$$

to obtain 4 more Einstein geometries naturally associated with the geometry of $\mathcal{S}$. Actually, in this case $\mathcal{O}$ is more fundamental as the self-dual Einstein metric on it remains the same. In this section we will discuss the four remaing geometric structures.

14.3.1. $Spin(7)$-Structure.

14.3.2. Weak G_2-Structure. Recall, that geometrically G_2 is defined to be the Lie group acting on $\mathbb{R}^7$ and preserving the 3-form

$$\begin{aligned}\varphi &= \alpha_1\wedge\alpha_2\wedge\alpha_3+\alpha_1\wedge(\alpha_4\wedge\alpha_5-\alpha_6\wedge\alpha_7)\\ &+\alpha_2\wedge(\alpha_4\wedge\alpha_6-\alpha_7\wedge\alpha_5)+\alpha_3\wedge(\alpha_4\wedge\alpha_7-\alpha_5\wedge\alpha_6),\end{aligned}\tag{14.3.1}$$

where $\{\alpha_i\}_{i=1}^7$ is a fixed orthonormal basis of the dual of $\mathbb{R}^7$. A G_2 structure on a 7-manifold M is, by definition, a reduction of the structure group of the tangent bundle to G_2. This is equivalent to the existence of a global 3-form $\varphi\in\Omega^3(M)$ which may be written locally as 14.3.1. Such a 3-form defines an associated Riemannian metric, an orientation class, and a spinor field of constant length. The following terminology is due to Gray [**99**]:

Definition 14.3.1: *Let (M,g) be a complete 7-dimensional Riemannian manifold. We say that that (M,g) has weak holonomy G_2 if there exist a global 3-form $\varphi\in\Omega^3(M)$ which locally can be written in terms of a local orthonormal basis as in 14.3.1, and $d\varphi=c\star\varphi$, where $\star$ is the Hodge star operator associated to g and c is a constant whose sign is fixed by an orientation convention.*

The equation $d\varphi=c\star\varphi$ implies that φ is 'nearly parallel' in the sense that only a 1-dimensional component of $\nabla\varphi$ is different from zero [**76**]. Thus, a weak holonomy G_2 structure is sometimes called a *nearly parallel G_2 structure.* The case of $c=0$ is somewhat special. In particular, it is known [**?**] that the condition $d\varphi=0=d\star\varphi$ is equivalent to the condition that φ be parallel, i.e., $\nabla\varphi=0$ which is equivalent to the condition that the metric g has holonomy group contained in G_2. For a discussion of this very interesting and very difficult case, see the article by D. Joyce in this volume. The following theorem provides the connection with the previous discussion on Killing spinors [**15**]

Theorem 14.3.2: *Let (M,g) be a complete 7-dimensional Riemannian manifold with weak holonomy G_2. Then the holonomy group $\mathrm{Hol}(\bar{g})$ of the metric cone $(C(M),\bar{g})$ is contained in $\mathrm{Spin}(7)$. In particular, $C(M)$ is Ricci-flat and M is Einstein with positive Einstein constant $\lambda=6$.*

Remark 14.3.1: The sphere S^7 with its constant curvature metric is isometric to the isotropy irreducible space $Spin(7)/G_2$. The fact that G_2 leaves invariant (up to constants) a unique 3-form and a unique 4-form on $\mathbb{R}^7$ implies immediately that this space has weak holonomy G_2.

Definition 14.3.3: *Let (M,g) be a complete 7-dimensional Riemannian manifold. We say that g is a proper G_2-metric if $\mathrm{Hol}(\bar{g})=\mathrm{Spin}(7)$.*

Theorem 14.3.4: *Let (M^7,g) be a complete simply-connected Riemannian spin manifold of dimension 7 admitting a non-trivial Killing spinor with $\alpha>0$ or $\alpha<0$. Then there are four possibilities:*
(i) (M,g) is of type $(1,0)$ and it is a proper G_2-manifold,
(ii) (M,g) is of type $(2,0)$ and it is a Sasakian-Einstein manifold, but (M,g) is not 3-Sasakian,
(iii) (M,g) is of type $(3,0)$ and it is 3-Sasakian,

(iv) $(M,g)=(S^7,g_{can})$ *and is of type* $(8,8)$.

Conversely, if (M,g) *is a compact simply-connected proper* G_2*-manifold then it carries a Killing spinor with* $\alpha>0$. *If* (M,g) *is a compact simply-connected Sasakian-Einstein 7-manifold which is not 3-Sasakian then* M *carries* 2 *linearly independent Killing spinors with* $\alpha>0$. *Finally, if* (M,g) *is a 3-Sasakian 7-manifold, which is not of constant curvature, then* M *carries* 3 *linearly independent Killing spinors with* $\alpha>0$.

REMARK 14.3.2: The four possibilities of the Theorem 6.2.4 correspond to the sequence of inclusions

$$\mathrm{Spin}(7)\supset SU(4)\supset Sp(2)\supset\{\mathrm{id}\}.$$

All of the corresponding cases are examples of weak holonomy G_2 metrics. If we exclude the trivial case when the associated cone is flat, we have three types of the weak holonomy G_2 geometries. Following [**83**] we use the number of linearly independent Killing spinors to classify the types of weak holonomy G_2 geometries. We call these type I, II, and III corresponding to cases (i), (ii), and (iii) of Theorem 6.2.4, respectively.

REMARK 14.3.3: In the case $\pi_1(M)\neq 0$, then M is either a spin spherical space form or $\mathrm{Hol}(\bar g)$ equals to $SU(4)$, $SU(4)\bowtie\mathbb{Z}_2$, $Sp(2)$, or $\mathrm{Spin}(7)$ of type $(2,0),(1,0),(3,0),(1,0)$, respectively. Hence, we have just one more possible geometry for M [**218**]. Note that in the case $\mathrm{Hol}(\bar g)=SU(4)\bowtie\mathbb{Z}_2$, the cone $C(M)$ is not Kähler so that M cannot be Sasakian, but it is locally so.

Recall that S^7, regarded as the space $Sp(2)/Sp(1)$ and fibering over S^4, admits a 'squashed' Einstein metric which does not have constant curvature. This metric also has weak holonomy G_2 since the associated cone metric has holonomy equal to $Spin(7)$ and therefore S^7 with this metric is a proper G_2-manifold. We can generalize this example to get [**90, 83**]:

Theorem 14.3.5: *Let* $(\mathcal{S},g)$ *be a 7-dimensional 3-Sasakian manifold. Then the metric* g *has weak holonomy* G_2. *Moreover, the second Einstein metric* g' *given by Theorem 3.4.1 has weak holonomy* G_2. *In fact* g' *is a proper* G_2 *metric.*

PROOF. For the second Einstein metric g' we have three mutually orthonormal 1-forms $\alpha^1=\sqrt{t}\eta^1,\quad \alpha^2=\sqrt{t}\eta^2,\quad \alpha^3=\sqrt{t}\eta^3$, where t is the parameter of the canonical variation discussed in section 3.4. Let $\{\alpha^4,\alpha^5,\alpha^6,\alpha^7\}$ be local 1-forms spanning the annihilator of $\mathcal{V}_3$ in $T^*\mathcal{S}$ such that

$$\begin{aligned}\bar\Phi^1&=2(\alpha^4\wedge\alpha^5-\alpha^6\wedge\alpha^7),\\ \bar\Phi^2&=2(\alpha^4\wedge\alpha^6-\alpha^7\wedge\alpha^5),\\ \bar\Phi^3&=2(\alpha^4\wedge\alpha^7-\alpha^5\wedge\alpha^6).\end{aligned}$$

Then the set $\{\alpha^1,\cdots\alpha^7\}$ forms a local orthonormal coframe for the metric g'. In terms of the 3-forms Υ and Θ of 13.4.1 we have $\varphi=\frac{1}{2}\sqrt{t}\Theta+\sqrt{t}^3\Upsilon$. One easily sees that this is of the type of Equation 14.3.1 and, therefore, defines a compatible G_2-structure. Moreover, a straightforward computation gives

$$d\varphi=\frac{1}{2}\sqrt{t}\Omega+\sqrt{t}(t+1)d\Upsilon,\qquad \star\varphi=-\frac{1}{2}td\Upsilon-\frac{1}{24}\Omega.$$

Thus, $d\varphi=c\star\varphi$ is solved with $\sqrt{t}=1/\sqrt{5}$, and $c=-12/\sqrt{5}$. So g' has weak holonomy G_2. That g' is a proper G_2 metric is due to [**83**]. The idea is to use

Theorem 6.2.4. Looking at the four possibilities given in that theorem, we see that it suffices to show that g' is not Sasakian-Einstein. The details are in [**83**]. □

Example 14.3.6: 3-Sasakian 7-manifolds are plentiful and examples will be discussed in next section. These give, by Theorem 6.2.5, many examples of type I and type III geometries. Examples of simply-connected type I geometries that do not arise via Theorem 6.2.5 are the homogeneous Aloff-Wallach spaces $N_{k,l} = SU(3)/T^1_{k,l}$, with $\gcd(k,l) = 1$ and $(k,l) \neq (1,1)$ [**57, ?**] together with the homogeneous real Stiefel manifold $SO(5)/SO(3)$ [**56**]. All the known type II geometries are the 3 homogeneous examples from the list of Corollary 4.1.3(iii) (not 3-Sasakian) and the inhomogeneous simply-connected circle bundles over $P_k \times \mathbb{CP}^1$, where P_k is the del Pezzo surface with $2 < k < 9$ [**83**]. Actually, $N_{1,1}$ has three Einstein metrics. One is 3-Sasakian and is denoted by $\mathcal{S}(1,1,1)$ in section 8.4 below. The second is the proper G_2 metric of Theorem 6.2.5, while the third Einstein metric also has weak holonomy G_2 most likely of type I but we could not positively exclude type II as a possibility [**57**].

14.3.3. Nearly Kähler and G_2-Structure.

CHAPTER 15

Saskian Manifolds and Supersymmetry

In this chapter we will recall some basic properties of Kähler manifolds and Kähler metrics. We will also discuss how such properties generalize to the category of Kähler orbifolds.

15.1. Supersymmetry and Killing Spinors

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APPENDIX A

A.1. Preliminaries on Groupoids

We briefly review some properties of groupoids [**58, 59, 148, 157, 162**]. Here we follow [**157**] closely.

Definition A.1.1: *A* **groupoid** $\mathcal{G}$ *is a small category in which every morphism has an inverse.*

So a groupoid $\mathcal{G}$ consists of a set G_0 of *objects* and a set G_1 of *arrows* or *morphisms* together with certain structure maps. For any groupoid there are canonical maps $s : G_1 \longrightarrow G_0$ and $t : G_1 \longrightarrow G_0$, called the *source map* and *target map* (sometimes the latter is called the *range map*), respectively, and defined by sending each arrow $g : x \longrightarrow y$ to its source x and target y, respectively, i.e. $s(g) = x, t(g) = y$.

Next there is an associative multiplication, called *multiplication* or *composition*, defined for any two arrows $g, h \in G_1$ such that $t(g) = s(h)$ with $s(h \circ g) = s(g)$ and $t(h \circ g) = t(h)$. Alternatively, if $g : x \longrightarrow y$ and $h : y \longrightarrow z$ are arrows from x to y, and from y to z, respectively, then $h \circ g : x \longrightarrow z$ is an arrow from x to z. Since, composition is defined only when the target of the first arrow equals the source of the second, we consider the fibre product $G_1 \times_{G_0} G_1$ defined by

$$G_1 \times_{G_0} G_1 = \{(h, g) \in G_1 \times G_1 \mid s(h) = t(g)\}. \tag{A.1.1}$$

Then composition defines a map $m : G_1 \times_{G_0} G_1 \longrightarrow G_1$ by $m(h, g) = h \circ g$. For each object $x \in G_0$ there is an *unit* or *identity* arrow $1_x \in G_1$ defined by sending x to itself, i.e. $1_x x = x$. The identity is a two-sided unit for composition, i.e. $g \circ 1_x = g$ and $1_x \circ h = h$ for any $g, h \in G_1$ with $s(g) = x$ and $t(h) = x$. The unit arrows define a map $u : G_0 \longrightarrow G_1$ by $u(x) = 1_x$.

Finally for each arrow $g : x \longrightarrow y$ in G_1 there is an inverse arrow $g^{-1} : y \longrightarrow x$ which is a two-sided inverse for composition, i.e. $g \circ g^{-1} = 1_y$ and $g^{-1} \circ g = 1_x$. The inverse defines an involution $\iota : G_1 \longrightarrow G_1$ by $\iota(g) = g^{-1}$.

Thus, a groupoid $\mathcal{G} = (G_0, G_1, s, t, m, u, \iota)$ consists of two sets G_0 and G_1 together with five structure maps s, t, m, u, ι. For any groupoid $\mathcal{G}$ we have the following subsets of interest:

(A.1.2)

$$\mathcal{G}_x = \{g \in G_1 \mid s(g) = x\}, \qquad \mathcal{G}^y = \{g \in G_1 \mid t(g) = y\}$$

(A.1.3)

$$\mathcal{G}_x^y = \{g \in G_1 \mid s(g) = x,\ r(g) = y\}, \quad \mathcal{G}_Z^W = \{g \in G_1 \mid s(g) \in Z,\ t(g) \in W\}.$$

Here Z and W are subsets of G_0. Notice from this point of view that the composition map m becomes a map on the subsets $\mathcal{G}_x^y \times \mathcal{G}_y^z \longrightarrow \mathcal{G}_x^z$ and that the subset $\mathcal{G}_x^x \subset G_1$ is a group which acts on $\mathcal{G}_x^y$ from the left. Morever, the group $\mathcal{G}_y^y$ acts on $\mathcal{G}_x^y$ from the right. The group $\mathcal{G}_x^x$ of arrows from x to itself is called the *isotropy group* at x and often denoted by G_x.

Definition A.1.2: *A* **subgroupoid** $\mathcal{H}$ *of* $\mathcal{G}$ *is defined by a pair of subsets* $H_0 \subset G_0$ *and* $H_1 \subset G_1$ *such that* $s(H_1) \subset H_0$, $t(H_1) \subset H_0$, *and* H_1 *is closed under multiplication when it is defined and under inversion. If* $H_0 = G_0$ *then* $\mathcal{H}$ *is called a* **wide subgroupoid**. *If for all* $x, y \in H_0$, $\mathcal{H}_x^y = \mathcal{G}_x^y$ *then* $\mathcal{H}$ *is said to be a* **full subgroupoid**.

Notice that the fact that $\mathcal{H}$ is closed under inversion implies that $u(x) = 1_x$ is in H_1 for all $x \in H_0$, i.e. the restriction of u to H_0 defines a map from H_0 to H_1. Next we give some simple examples of groupoids.

Example A.1.3: Some simple examples of groupoids are:

(i) The *identity* or *trivial* or *unit* groupoid: All arrows are units, so $G_1 = G_0$ as well. Any subset of this groupoid is a subgroupoid.

(ii) A group G is a groupoid with $G_0 = \{\mathrm{pt}\}$. The subgroupoids of G are just the subgroups of G and the empty set.

(iii) The *pair groupoid* $\mathcal{P}$: $G_0 = X$ is any set, and $G_1 = X \times X$ with exactly one arrow between any two points $x, y \in G_0$. Then a subgroupoid of $\mathcal{P}$ is a relation on X which is symmetric and transitive. A wide subgroupoid is an *equivalence relation* on X.

The groupoids that we are interested in have a richer structure. In fact they are smooth manifolds. This leads to:

Definition A.1.4: *A* **topological groupoid** *is a groupoid* $\mathcal{G}$ *in which both* G_1 *and* G_0 *are topological spaces, and the structure maps are continuous. A* **Lie groupoid** *is a topological groupoid for which* G_1 *is a smooth, but not necessarily Hausdorff, manifold,* G_0 *is a smooth Hausdorff manifold, the above structure maps are differentiable, and both the source and target maps are submersions.*

For a complete treatment of Lie groupoids, see [**148**]. In almost all cases of interest to us, the groupoid $\mathcal{G}$ will be a Lie groupoid, and G_1 will be Hausdorff. By abuse of notation we sometimes refer to the manifold G_1 as the groupoid $\mathcal{G}$ and by the dimension of $\mathcal{G}$ we mean the dimension of G_1. For any Lie groupoid $\mathcal{G}$ the subsets defined by A.1.2 above are smooth manifolds, and $G_x = \mathcal{G}_x^x$ is a Lie group. Lie groupoids exist in abundance. The simple examples in Example A.1.3 have their Lie groupoid analogues.

(i) $G_0 = M$ is any manifold and all arrows are units, so $G_1 = M$ as well. $\mathcal{G}$ is called the *unit Lie groupoid.*

(ii) A Lie group is a Lie groupoid with $G_0 = \{\mathrm{pt}\}$.

(iii) $G_0 = M$ is any manifold, and $G_1 = M \times M$ with exactly one arrow between any two points $x, y \in G_0$. This is called the *pair Lie groupoid* and sometimes it is denoted by $M \times M$.

There are two other Lie groupoids that are of much interest to us. The first is the *holonomy groupoid* $\mathcal{G}(M, \mathcal{F})$ of a foliation $\mathcal{F}$ that is discussed in detail in Chapter 1. The second is the so-called action or translation groupoid.

Definition A.1.5: *Let* K *be a Lie group acting smoothly on the manifold* M *from the right*[1]. *We define a Lie groupoid* $M \rtimes K$ *called the* **action** *or* **translation**

[1] It is more common to use the left action, but we choose the action to be from the right in order to follow the standard convention for principal bundles.

groupoid *as follows: the space of objects* $(M \rtimes K)_0$ *is* M *and the space of arrows* $(M \rtimes K)_1$ *is* $M \times K$. *An arrow* $(x,k) \in (M \rtimes K)_1 = M \times K$ *from* $x \in M$ *to* $y \in M$ *is given by the right action* $y = xk$. *The source map is* $s : M \times K \longrightarrow M$ *is projection onto the first factor, while the target map* $t : M \times K \longrightarrow M$ *is defined by* $t(x,k) = xk$. *Composition is defined by* $(x,k) \cdot (y,k') = (x,kk')$ *when* $y = xk$. *The unit is* $u(x) = (x,e)$ *where* e *is the identity in* K.

Next we give two definitions that are essential to understanding orbifolds from the groupoid point of view .

Definition A.1.6: *A Lie (topological) groupoid* $\mathcal{G}$ *is called an* **étale groupoid** *if the source map* s *is a local diffeomorphism (homeomorphism).*

Note that the fact that the source map s is a local diffeomorphism (homeomorphism) implies that the other structure maps are diffeomorphisms (homeomorphisms).

Definition A.1.7: *A topological groupoid* $\mathcal{G}$ *is said to be* **proper** *if the map* $(s,t) : G_1 \longrightarrow G_0 \times G_0$ *is proper.*

These two definitions are enough to capture the essence of an orbifold. We can think of a proper étale groupoid as representing an orbifold.

We now describe the important notion of 'equivalence' of groupoids. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A homomorphism $\phi : \mathcal{G} \longrightarrow \mathcal{H}$ of groupoids is a pair $\phi_0 : G_0 \longrightarrow H_0$ and $\phi_1 : G_1 \longrightarrow H_1$ of maps that commute with all the structure maps, e.g. $\phi_0 \circ s = s \circ \phi_1$, *etc.* If $\mathcal{G}$ and $\mathcal{H}$ are Lie (topological) groupoids, we demand that the maps are smooth (continuous). In particular, ϕ is an *isomorphism* of Lie (topological) groupoids if the maps ϕ_0, ϕ_1 are diffeomorphisms (homeomorphisms). We call such groupoids *equivalent*. For groupoids it is important to have a weaker notion of equivalence.

Definition A.1.8: *A homomorphism of groupoids* $\phi : \mathcal{G} \longrightarrow \mathcal{H}$ *is called an* **essential** *or* **weak equivalence** *if:*

(i) *the map* $s \circ \pi_2 : G_0 \times_{H_0} H_1 \longrightarrow H_0$ *is a surjective submersion.*
(ii) *the square*

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi} & H_1 \\ \Big\downarrow{\scriptstyle (s,t)} & & \Big\downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow{\phi \times \phi} & H_0 \times H_0 \end{array}$$

is a fibered product.

Two such groupoids are said to be **Morita equivalent** *if there exists a third groupoid* $\mathcal{K}$ *such that* $\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}$ *are weak equivalences.*

One can check that these definitions define equivalence relations. Although Morita equivalent Lie groupoids can be very different, for example they can have different dimensions and different connectivity properties, Morita equivalence does preserve some essential geometric features as we shall see. For us the relationship between certain Lie groupoids and orbifolds is crucial. The following theorem essentially due of Moerdijk and Pronk [**158, 157**] plays an important role relating the holonomy groupoid to orbifolds.

Theorem A.1.9: *Let* $\mathcal{F}$ *be a Lie groupoid. Then the following are equivalent:*

(i) $\mathcal{G}$ *is Morita equivalent to an effective proper étale groupoid.*
(ii) $\mathcal{G}$ *is Morita equivalent to the holonomy groupoid of a foliation with compact leaves and finite holonomy groups.*
(iii) $\mathcal{G}$ *is Morita equivalent to an action groupoid* $M \rtimes K$ *where* K *is a compact Lie group acting on* M *such that each isotropy group* K_x *is finite and acts effectively on the normal bundle.*

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