HOMEWORK #8

Chapter 6

SECTION 1

2. If A is triangular then $A - a_{ii}I$ will be a triangular matrix with a zero entry in the (i,i) position. Since the determinant of a triangular matrix is the product of its diagonal elements it follows that

$$\det(A - a_{ii}I) = 0$$

Thus the eigenvalues of A are $a_{11}, a_{22}, \ldots, a_{nn}$.

3. A is singular if and only if $\det(A) = 0$. The scalar 0 is an eigenvalue if and only if

$$\det(A - 0I) = \det(A) = 0$$

Thus A is singular if and only if one of its eigenvalues is 0.

4. If A is a nonsingular matrix and λ is an eigenvalue of A, then there exists a nonzero vector $\mathbf{x}$ such that

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A^{-1}A\mathbf{x} &= \lambda A^{-1}\mathbf{x} \end{aligned}$$

It follows from Exercise 3 that $\lambda \neq 0$. Therefore

$$A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$$

and hence $1/\lambda$ is an eigenvalue of A^{-1}.

5. In the case where $m = 1$, $\lambda^1 = \lambda$ is an eigenvalue of A with eigenvector $\mathbf{x}$. Suppose λ^k is an eigenvalue of A^k and $\mathbf{x}$ is an eigenvector belonging to λ^k.

$$A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\lambda^k\mathbf{x}) = \lambda^k A\mathbf{x} = \lambda^{k+1}\mathbf{x}$$

Thus λ^{k+1} is an eigenvalue of A^{k+1} and $\mathbf{x}$ is an eigenvector belonging to λ^{k+1}. This completes the induction proof.

6. If A is idempotent and λ is an eigenvalue of A with eigenvector $\mathbf{x}$, then

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A^2\mathbf{x} &= \lambda A\mathbf{x} = \lambda^2\mathbf{x} \end{aligned}$$

and

$$A^2\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x}$$

Therefore

$$(\lambda^2 - \lambda)\mathbf{x} = \mathbf{0}$$

Since $\mathbf{x} \neq \mathbf{0}$ it follows that

$$\lambda^2 - \lambda = 0$$
$$\lambda = 0 \quad \text{or} \quad \lambda = 1$$

7. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k (Exercise 5). If $A^k = O$, then all of its eigenvalues are 0. Thus $\lambda^k = 0$ and hence $\lambda = 0$.

9. $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$. Thus A and A^T have the same characteristic polynomials and consequently must have the same eigenvalues. The eigenspaces however will not be the same. For example if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then the eigenvalues of A and A^T are both given by

$$\lambda_1 = \lambda_2 = 1$$

The eigenspace of A corresponding to $\lambda = 1$ is spanned by $(1,\ 0)^T$ while the eigenspace of A^T is spanned by $(0,\ 1)^T$. Exercise 24 shows how the eigenvectors of A and A^T are related.

10. $\det(A - \lambda I) = \lambda^2 - (2\cos\theta)\lambda + 1$. The discriminant will be negative unless θ is a multiple of π. The matrix A has the effect of rotating a real vector $\mathbf{x}$ about the origin by an angle of θ. Thus $A\mathbf{x}$ will be a scalar multiple of $\mathbf{x}$ if and only if θ is a multiple of 180°.

12. Since tr A equals the sum of the eigenvalues the result follows by solving

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$$

for λ_j.

13.
$$\begin{aligned} \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) \\ &= \lambda^2 - (\operatorname{tr} A)\lambda + \det(A) \end{aligned}$$

14. $A(A^m\mathbf{x}) = A^{m+1}\mathbf{x} = A^m(A\mathbf{x}) = A^m(\lambda\mathbf{x}) = \lambda(A^m\mathbf{x})$

15. If $A - \lambda_0 I$ has rank k then $N(A - \lambda_0 I)$ will have dimension $n - k$.

16. The subspace spanned by $\mathbf{x}$ and $A\mathbf{x}$ will have dimension 1 if and only if $\mathbf{x}$ and $A\mathbf{x}$ are linearly dependent and $\mathbf{x} \neq \mathbf{0}$. The vectors $\mathbf{x}$ and $A\mathbf{x}$ will be linearly dependent if and only if $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

17. (a) If $\alpha = a + bi$ and $\beta = c + di$, then

$$\overline{\alpha + \beta} = \overline{(a+c) + (b+d)i} = (a+c) - (b+d)i$$

and

$$\overline{\alpha} + \overline{\beta} = (a - bi) + (c - di) = (a+c) - (b+d)i$$

Therefore $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.

Next we show that the conjugate of the product of two numbers is the product of the conjugates.

$$\overline{\alpha\beta} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i$$

$$\overline{\alpha}\overline{\beta} = (a - bi)(c - di) = (ac - bd) - (ad + bc)i$$

Therefore $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$.

(b) If $A \in R^{m\times n}$ and $B \in R^{n\times r}$, then the (i,j) entry of $\overline{AB}$ is given by

$$\overline{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}} = \overline{a_{i1}}\overline{b_{1j}} + \overline{a_{i2}}\overline{b_{2j}} + \cdots + \overline{a_{in}}\overline{b_{nj}}$$

The expression on the right is the (i,j) entry of $\overline{A}\,\overline{B}$. Therefore

$$\overline{AB} = \overline{A}\,\overline{B}$$

18. If $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_r\mathbf{x}_r$ is an element of S, then

$$A\mathbf{x} = (c_1\lambda_1)\mathbf{x}_1 + (c_2\lambda_2)\mathbf{x}_2 + \cdots + (c_r\lambda_r)\mathbf{x}_r$$

Thus $A\mathbf{x}$ is also an element of S.

19. Since $\mathbf{x} \neq \mathbf{0}$ and S is nonsingular it follows that $S\mathbf{x} \neq \mathbf{0}$. If $B = S^{-1}AS$, then $AS = SB$ and it follows that

$$A(S\mathbf{x}) = (AS)\mathbf{x} = SB\mathbf{x} = S(\lambda\mathbf{x}) = \lambda(S\mathbf{x})$$

Therefore $S\mathbf{x}$ is an eigenvector of A belonging to λ.

20. If $\mathbf{x}$ is an eigenvector of A belonging to the eigenvalue λ and $\mathbf{x}$ is also an eigenvector of B corresponding to the eigenvalue μ, then

$$(\alpha A + \beta B)\mathbf{x} = \alpha A\mathbf{x} + \beta B\mathbf{x} = \alpha\lambda\mathbf{x} + \beta\mu\mathbf{x} = (\alpha\lambda + \beta\mu)\mathbf{x}$$

Therefore $\mathbf{x}$ is an eigenvector of $\alpha A + \beta B$ belonging to $\alpha\lambda + \beta\mu$.

21. If $\lambda \neq 0$ and $\mathbf{x}$ is an eigenvector belonging to λ, then

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ \mathbf{x} &= \frac{1}{\lambda}A\mathbf{x} \end{aligned}$$

Since $A\mathbf{x}$ is in $R(A)$ it follows that $\frac{1}{\lambda}A\mathbf{x}$ is in $R(A)$.

22. If

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

then for $i = 1, \ldots, n$

$$A\mathbf{u}_i = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T \mathbf{u}_i + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T \mathbf{u}_i + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \mathbf{u}_i$$

Since $\mathbf{u}_j^T \mathbf{u}_i = 0$ unless $j = i$, it follows that

$$A\mathbf{u}_i = \lambda_i \mathbf{u}_i \mathbf{u}_i^T \mathbf{u}_i = \lambda_i \mathbf{u}_i$$

and hence λ_i is an eigenvalue of A with eigenvector $\mathbf{u}_i$. The matrix A is symmetric since each $\lambda_i \mathbf{u}_i \mathbf{u}_i^T$ is symmetric and any sum of symmetric matrices is symmetric.

23. If the columns of A each add up to a fixed constant δ then the row vectors of $A - \delta I$ all add up to $(0, 0, \ldots, 0)$. Thus the row vectors of $A - \delta I$ are linearly dependent and hence $A - \delta I$ is singular. Therefore δ is an eigenvalue of A.

24. Since $\mathbf{y}$ is an eigenvector of A^T belonging to λ_2 it follows that

$$\mathbf{x}^T A^T \mathbf{y} = \lambda_2 \mathbf{x}^T \mathbf{y}$$

The expression $\mathbf{x}^T A^T \mathbf{y}$ can also be written in the form $(A\mathbf{x})^T \mathbf{y}$. Since $\mathbf{x}$ is an eigenvector of A belonging to λ_1, it follows that

$$\mathbf{x}^T A^T \mathbf{y} = (A\mathbf{x})^T \mathbf{y} = \lambda_1 \mathbf{x}^T \mathbf{y}$$

Therefore

$$(\lambda_1 - \lambda_2)\mathbf{x}^T \mathbf{y} = 0$$

and since $\lambda_1 \neq \lambda_2$, the vectors $\mathbf{x}$ and $\mathbf{y}$ must be orthogonal.

25. (a) If λ is a nonzero eigenvalue of AB with eigenvector $\mathbf{x}$, then let $\mathbf{y} = B\mathbf{x}$. Since

$$A\mathbf{y} = AB\mathbf{x} = \lambda \mathbf{x} \neq \mathbf{0}$$

it follows that $\mathbf{y} \neq \mathbf{0}$ and

$$BA\mathbf{y} = BA(B\mathbf{x}) = B(AB\mathbf{x}) = B\lambda\mathbf{x} = \lambda\mathbf{y}$$

Thus λ is also an eigenvalue of BA with eigenvector $\mathbf{y}$.

(b) If $\lambda = 0$ is an eigenvalue of AB, then AB must be singular. Since

$$\det(BA) = \det(B)\det(A) = \det(A)\det(B) = \det(AB) = 0$$

it follows that BA is also singular. Therefore $\lambda = 0$ is an eigenvalue of BA.

26. If $AB - BA = I$, then $BA = AB - I$. If the eigenvalues of AB are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then it follows from Exercise 8 that the eigenvalues of BA are $\lambda_1 - 1$, $\lambda_2 - 1, \ldots, \lambda_n - 1$. This contradicts the result proved in Exercise 25 that AB and BA have the same eigenvalues.

27. (a) If λ_i is a root of $p(\lambda)$, then

$$\lambda_i^n = a_{n-1}\lambda_i^{n-1} + \cdots + a_1\lambda_i + a_0$$

Thus if $\mathbf{x} = (\lambda_i^{n-1}, \lambda_i^{n-2}, \ldots, \lambda_i, 1)^T$, then

$$C\mathbf{x} = (\lambda_i^n, \lambda_i^{n-1}, \ldots, \lambda_i^2, \lambda_i)^T = \lambda_i \mathbf{x}$$

and hence λ_i is an eigenvalue of C with eigenvector $\mathbf{x}$.

(b) If $\lambda_1, \ldots, \lambda_n$ are the roots of $p(\lambda)$, then

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$$

If $\lambda_1, \ldots, \lambda_n$ are all distinct then by part (a) they are the eigenvalues of C. Since the characteristic polynomial of C has lead coefficient $(-1)^n$ and roots $\lambda_1, \ldots, \lambda_n$, it must equal $p(\lambda)$.

28. Let

$$D_m(\lambda) = \begin{pmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 \\ 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & -\lambda \end{pmatrix}$$

It can be proved by induction on m that

$$\det(D_m(\lambda)) = (-1)^m(a_m\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0)$$

If $\det(C - \lambda I)$ is expanded by cofactors along the first column one obtains

$$\begin{aligned} \det(C - \lambda I) &= (a_{n-1} - \lambda)(-\lambda)^{n-1} - \det(D_{n-2}) \\ &= (-1)^n(\lambda^n - a_{n-1}\lambda^{n-1}) - (-1)^{n-2}(a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0) \\ &= (-1)^n[(\lambda^n - a_{n-1}\lambda^{n-1}) - (a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0)] \\ &= (-1)^n[\lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \cdots - a_1\lambda - a_0] \\ &= p(\lambda) \end{aligned}$$

SECTION 2

3. (a) If

$$\mathbf{Y}(t) = c_1e^{\lambda_1 t}\mathbf{x}_1 + c_2e^{\lambda_2 t}\mathbf{x}_2 + \cdots + c_ne^{\lambda_n t}\mathbf{x}_n$$

then

$$\mathbf{Y}_0 = \mathbf{Y}(0) = c_1\mathbf{x}_2 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

(b) It follows from part (a) that

$$\mathbf{Y}_0 = X\mathbf{c}$$

If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent then X is nonsingular and we can solve for $\mathbf{c}$

$$\mathbf{c} = X^{-1}\mathbf{Y}_0$$

7. It follows from the initial condition that

$$\begin{aligned} x_1'(0) &= a_1\sigma = 2 \\ x_2'(0) &= a_2\sigma = 2 \end{aligned}$$

and hence

$$a_1 = a_2 = 2/\sigma$$

Substituting for x_1 and x_2 in the system

$$\begin{aligned} x_1'' &= -2x_1 + x_2 \\ x_2'' &= x_1 - 2x_2 \end{aligned}$$

yields

$$\begin{aligned} -a_1\sigma^2 \sin\sigma t &= -2a_1 \sin\sigma t + a_2 \sin\sigma t \\ -a_2\sigma^2 \sin\sigma t &= a_1 \sin\sigma t - 2a_2 \sin\sigma t \end{aligned}$$

Replacing a_1 and a_2 by $2/\sigma$ we get

$$\sigma^2 = 1$$

Using either $\sigma = -1$, $a_1 = a_2 = -2$ or $\sigma = 1$, $a_1 = a_2 = 2$ we obtain the solution

$$\begin{aligned} x_1(t) &= 2\sin t \\ x_2(t) &= 2\sin t \end{aligned}$$

9. $m_1 y_1'' = k_1 y_1 - k_2(y_2 - y_1) - m_1 g$
$m_2 y_2'' = k_2(y_2 - y_1) - m_2 g$

11. If

$$y^{(n)} = a_0 y + a_1 y' + \cdots + a_{n-1} y^{(n-1)}$$

and we set

$$y_1 = y,\ y_2 = y_1' = y'',\ y_3 = y_2' = y''', \ldots, y_n = y_{n-1}' = y^n$$

then the nth order equation can be written as a system of first order equations of the form $\mathbf{Y}' = A\mathbf{Y}$ where

$$A = \begin{pmatrix} 0 & y_2 & 0 & \cdots & 0 \\ 0 & 0 & y_3 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & y_n \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}$$

SECTION 3

1. The factorization XDX^{-1} is not unique. However the diagonal elements of D must be eigenvalues of A and if λ_i is the ith diagonal element of D, then $\mathbf{x}_i$ must be an eigenvector belonging to λ_i

(a) $\det(A - \lambda I) = \lambda^2 - 1$ and hence the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. $\mathbf{x}_1 = (1,\ 1)^T$ and $\mathbf{x}_2 = (-1,\ 1)^T$ are eigenvectors belonging to λ_1 and λ_2, respectively. Setting

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have

$$A = XDX^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

(b) The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$. If we take $\mathbf{x}_1 = (-2,\ 1)^T$ and $\mathbf{x}_2 = (-3,\ 2)^T$, then

$$A = XDX^{-1} = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}$$

(c) $\lambda_1 = 0$, $\lambda_2 = -2$. If we take $\mathbf{x}_1 = (4,\ 1)^T$ and $\mathbf{x}_2 = (2,\ 1)^T$, then

$$A = XDX^{-1} = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1/2 & -1 \\ -1/2 & 2 \end{pmatrix}$$

(d) The eigenvalues are the diagonal entries of A. The eigenvectors corresponding to $\lambda_1 = 2$ are all multiples of $(1,\ 0,\ 0)^T$. The eigenvectors belonging to $\lambda_2 = 1$ are all multiples of $(2,\ -1,\ 0)$ and the eigenvectors corresponding to $\lambda_3 = -1$ are multiples $(1,\ -3,\ 3)^T$.

$$A = XDX^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

(e) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -2$

$\mathbf{x}_1 = (3,\ 1,\ 2)^T$, $\mathbf{x}_2 = (0,\ 3,\ 1)^T$, $\mathbf{x}_3 = (0,\ -1,\ 1)^T$

$$A = XDX^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{5}{12} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

(f) $\lambda_1 = 2, \lambda_2 = \lambda_3 = 0, \mathbf{x}_1 = (1,\ 2,\ 3)^T, \mathbf{x}_2 = (1,\ 0,\ 1)^T, \mathbf{x}_3 = (-2,\ 1,\ 0)^T$

$$A = XDX^{-1} = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{3}{2} & -3 & \frac{5}{2} \\ -1 & -1 & 1 \end{pmatrix}$$

2. If $A = XDX^{-1}$, then $A^6 = XD^6X^{-1}$.

(a) $D^6 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^6 = I$

$A^6 = XD^6X^{-1} = XX^{-1} = I$

(b) $A^6 = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^6 \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 253 & 378 \\ -126 & -190 \end{pmatrix}$

(c) $A^6 = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}^6 \begin{pmatrix} 1/2 & -1 \\ -1/2 & 2 \end{pmatrix} = \begin{pmatrix} -64 & 256 \\ -32 & 128 \end{pmatrix}$

(d) $A^6 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^6 \begin{pmatrix} 1 & 2 & 5/3 \\ 0 & -1 & -1 \\ 0 & 0 & 1/3 \end{pmatrix}$

$$= \begin{pmatrix} 64 & 126 & 105 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) $$A^6 = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}^6 \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{5}{12} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -21 & 64 & 0 \\ -42 & 0 & 64 \end{pmatrix}$$

(f) $$A^6 = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^6 \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{3}{2} & -3 & \frac{5}{2} \\ -1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 32 & 64 & -32 \\ 64 & 128 & -64 \\ 96 & 192 & -96 \end{pmatrix}$$

3. If $A = XDX^{-1}$ is nonsingular, then $A^{-1} = XD^{-1}X^{-1}$

(a) $A^{-1} = XD^{-1}X^{-1} = XDX^{-1} = A$

(b) $$A^{-1} = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 1 & \frac{5}{2} \end{pmatrix}$$

(d) $$A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -1 & -\frac{3}{2} \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

(e) $$A^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{5}{12} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \end{pmatrix}$$

4. (a) The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 0$

$$A = XDX^{-1}$$

Since $D^2 = D$ it follows that

$$A^2 = XD^2X^{-1} = XDX^{-1} = A$$

(b) $$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = XD^{1/2}X^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

5. If X diagonalizes A, then

$$X^{-1}AX = D$$

where D is a diagonal matrix. It follows that

$$D = D^T = X^TA^T(X^{-1})^T = Y^{-1}A^TY$$

Therefore Y diagonalizes A^T.

6. If $A = ADA^{-1}$ where D is a diagonal matrix whose diagonal elements are all either 1 or -1, then $D^{-1} = D$ and

$$A^{-1} = XD^{-1}X^{-1} = XDX^{-1} = A$$

7. If $\mathbf{x}$ is an eigenvector belonging to the eigenvalue a, then

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & b-a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and it follows that

$$x_2 = x_3 = 0$$

Thus the eigenspace corresponding to $\lambda_1 = \lambda_2 = a$ has dimension 1 and is spanned by $(1,\ 0,\ 0)^T$. The matrix is defective since a is a double eigenvalue and its eigenspace only has dimension 1.

8. (a) The characteristic polynomial of the matrix factors as follows.

$$p(\lambda) = \lambda(2-\lambda)(\alpha-\lambda)$$

Thus the only way that the matrix can have a multiple eigenvalue is if $\alpha = 0$ or $\alpha = 2$. In the case $\alpha = 0$, we have that $\lambda = 0$ is an eigenvalue of multiplicity 2 and the corresponding eigenspace is spanned by $\mathbf{x}_1 = (-1, 1, 0)^T$ and $\mathbf{x}_2 = \mathbf{e}_3$. Since $\lambda = 0$ has two linearly independent eigenvectors, the matrix is not defective. Similarly in the case $\alpha = 2$ the matrix will not be defective since the eigenvalue $\lambda = 2$ possesses two linearly independent eigenvectors $\mathbf{x}_1 = (1, 1, 0)^T$ and $\mathbf{x}_2 = \mathbf{e}_3$.

9. If $A - \lambda I$ has rank 1, then

$$\dim N(A - \lambda I) = 4 - 1 = 3$$

Since λ has multiplicity 3 the matrix is not defective.

10. (a) The proof is by induction. In the case $m = 1$,

$$A\mathbf{x} = \sum_{i=1}^{n} \alpha_i A\mathbf{x}_i = \sum_{i=1}^{n} \alpha_i \lambda_i \mathbf{x}_i$$

If

$$A^k\mathbf{x} = \sum_{i=1}^{n} \alpha_i \lambda_i^k \mathbf{x}_i$$

then

$$A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\sum_{i=1}^{n} \alpha_i \lambda_i^k \mathbf{x}_i) = \sum_{i=1}^{n} \alpha_i \lambda_i^k A\mathbf{x}_i = \sum_{i=1}^{n} \alpha_i \lambda_i^{k+1} \mathbf{x}_i$$

(b) If $\lambda_1 = 1$, then

$$A^m\mathbf{x} = \alpha_1\mathbf{x}_1 + \sum_{i=2}^{n} \alpha_i \lambda_i^m \mathbf{x}_i$$

Since $0 < \lambda_i < 1$ for $i = 2, \ldots, n$, it follows that $\lambda_i^m \to 0$ as $m \to \infty$. Hence

$$\lim_{m\to\infty} A^m\mathbf{x} = \alpha_1\mathbf{x}_1$$

11. If A is an $n \times n$ matrix and λ is an eigenvalue of multiplicity n then A is diagonalizable if and only if

$$\dim N(A - \lambda I) = n$$

or equivalently

$$\operatorname{rank}(A - \lambda I) = 0$$

The only way the rank can be 0 is if

$$\begin{aligned} A - \lambda I &= O \\ A &= \lambda I \end{aligned}$$

12. If A is nilpotent, then 0 is an eigenvalue of multiplicity n. It follows from Exercise 11 that A is diagonalizable if and only if $A = O$.

13. Let A be a diagonalizable $n \times n$ matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the nonzero eigenvalues of A. The remaining eigenvalues are all 0.

$$\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0$$

If $\mathbf{x}_i$ is an eigenvector belonging to λ_i, then

$$\begin{aligned} A\mathbf{x}_i &= \lambda_i\mathbf{x}_i & i &= 1, \ldots, k \\ A\mathbf{x}_i &= \mathbf{0} & i &= k+1, \ldots, n \end{aligned}$$