MATH314 – HOMEWORK SOLUTIONS

HOMEWORK #7

Section 5.4: Problems 1,2,4(a)(b),7
Section 5.5: Problems 1,2,6,8,25
Section 5.6: Problems 1,3,4,5

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Problem 5.4.1 Let $\mathbf{x} = (-1,-1,1,1)^T$ and $\mathbf{y} = (1,1,5,-3)^T$. Clearly,

$$\mathbf{x}\cdot\mathbf{y} = -1-1+5-3 = 0,$$

$$||\mathbf{x}||_2 = \sqrt{4} = 2,$$

$$||\mathbf{y}||_2 = \sqrt{36} = 6,$$

$$||\mathbf{x}-\mathbf{y}||_2 = ||(-2,-2,-4,4)^T||_2 = \sqrt{40},$$

$$4+36 = 40.$$

Problem 5.4.2 Let $\mathbf{x} = (1,1,1,1)^T$, $\mathbf{y} = (8,2,2,0)^T$.

a)

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||_2||\mathbf{y}||_2} = \frac{12}{\sqrt{4}\sqrt{72}} = \sqrt{2}/2.$$

b)

$$\mathbf{p} = \frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y} = \frac{1}{6}(8,2,2,0)^T = (4/3,1/3,1/3,0)^T.$$

c) $\mathbf{x}-\mathbf{p} = (-1/3,2/3,2/3,1)^T$ is clearly perpendicular to $(4/3,1/3,1/3,0)^T$ as $-4/9+2/9+2/9 = 0$.

d)

$$||\mathbf{x}-\mathbf{p}||_2 = \frac{1}{3}\sqrt{18} = \sqrt{2},$$

$$||\mathbf{p}||_2 = \frac{1}{6}\sqrt{72} = \sqrt{2},$$

$$||\mathbf{x}||_2 = 2,$$

$$||\mathbf{x}-\mathbf{p}||_2^2 + ||\mathbf{p}||_2^2 = ||\mathbf{x}||_2^2.$$

Problem 5.4.4 Let

$$\mathbb{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{pmatrix}.$$

We have

$$\mathbb{A} \cdot \mathbb{B}^T = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{pmatrix}^T = \begin{pmatrix} 0 & 7 & -7 \\ -2 & 1 & -3 \\ -10 & -4 & -1 \end{pmatrix}.$$

Hence,

a)

$$< \mathbb{A}, \mathbb{B} >= \operatorname{tr}(\mathbb{A} \cdot \mathbb{B}^T) = 0.$$

b) As

$$< \mathbb{A}, \mathbb{A} >= \operatorname{tr}(\mathbb{A} \cdot \mathbb{A}^T) = \operatorname{tr} \begin{pmatrix} 9 & 5 & 7 \\ 5 & 5 & 5 \\ 7 & 5 & 11 \end{pmatrix} = 25.$$

we get $||\mathbb{A}||_F = \sqrt{25} = 5$.

Problem 5.4.7 We have

a)

$$< e^x, e^{-x} >= \int_0^1 dx = 1.$$

b)

$$< x, \sin \pi x >= \int_0^1 x \sin \pi x dx = -\frac{1}{\pi} \int_0^1 x(\cos \pi x)' dx =$$

$$= -\frac{1}{\pi} x(\cos \pi x)]_0^1 + \frac{1}{\pi} \int_0^1 (\cos \pi x) dx = \frac{1}{\pi}.$$

c)

$$< x^2, x^3 >= \int_0^1 x^5 dx = 1/6.$$

Problem 5.5.1

a) Yes, they form an orthonormal basis.

b) No, both vectors are of unit length but they are not perpendicular.

c) No, the two vectors are perpendicular but neither is of unit length.

d) Yes, they form an orthonormal basis.

Problem 5.5.2 Let

$$\mathbf{u}_1 = (1/3\sqrt{2}, 1/3\sqrt{2}, -4/3\sqrt{2})^T, \quad \mathbf{u}_2 = (2/3, 2/3, 1/3)^T, \quad \mathbf{u}_1 = (1/\sqrt{2}, -1/\sqrt{2}, 0)^T.$$

a) We have

$$\begin{aligned}
||\mathbf{u}_1||^2 &= \frac{1}{18}(1+1+16) = 1,\\
||\mathbf{u}_2||^2 &= \frac{1}{9}(4+4+1) = 1,\\
||\mathbf{u}_3||^2 &= \frac{1}{2}(1+1+0) = 1,\\
\mathbf{u}_3 \cdot \mathbf{u}_2 &= \frac{1}{\sqrt{2}}(1/3\sqrt{2} - 1/3\sqrt{2}) = 0,\\
\mathbf{u}_3 \cdot \mathbf{u}_1 &= \frac{1}{\sqrt{2}}(2/3 - 2/3) = 0,\\
\mathbf{u}_2 \cdot \mathbf{u}_1 &= \frac{1}{9\sqrt{2}}(2+2-4) = 0.
\end{aligned}$$

b) Let $\mathbf{x} = (1,1,1)^T$. We get

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{u}_1 &= 1/3\sqrt{2} + 1/3\sqrt{2} - 4/3\sqrt{2} = -2/3\sqrt{2},\\
\mathbf{x} \cdot \mathbf{u}_2 &= 2/3 + 2/3 + 1/3 = 5/3,\\
\mathbf{x} \cdot \mathbf{u}_3 &= 1/\sqrt{2} - 1/\sqrt{2} = 0.
\end{aligned}$$

Hence, $\mathbf{x} = -\frac{2}{3\sqrt{2}}\mathbf{u}_1 + \frac{5}{3}\mathbf{u}_2$, and

$$||\mathbf{x}||^2 = \frac{4}{18} + \frac{25}{9} = \frac{54}{18} = 3.$$

Thus $||\mathbf{x}|| = \sqrt{3}$.

Problem 5.5.6 Let

$$\mathbf{u} = \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3, \qquad \mathbf{v} = \mathbf{u}_1 + 7\mathbf{u}_3.$$

Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis we get

a) $\langle \mathbf{u}, \mathbf{v} \rangle = 1 + 14 = 15.$

b) $||\mathbf{u}|| = \sqrt{9} = 3, \quad ||\mathbf{v}|| = \sqrt{50}.$

c) $\cos\theta = \frac{15}{3\sqrt{50}} = \frac{1}{\sqrt{2}}, \qquad \theta = \pi/4.$

Problem 5.5.8 We have

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (3\cos x + 2\sin x)(\cos x - \sin x) dx =$$

$$= \frac{1}{\pi}\left(3\int_{-\pi}^{\pi} \cos^2 x dx - 2\int_{-\pi}^{\pi} \sin^2 x dx i - \int_{-\pi}^{\pi} \cos x \sin x dx \right) = 3 - 2 + 0 = 1.$$

Problem 5.5.25

a)

$$\langle 1, x \rangle = \int_{-1}^{1} x dx = 0.$$

b)

$$||1||^2 = \int_{-1}^{1} dx = 2, \qquad ||1|| = \sqrt{2},$$

$$||x||^2 = \int_{-1}^{1} x^2 dx = 2/3, \qquad ||x|| = \sqrt{\frac{2}{3}},$$

Hence, we see that

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}, \qquad \mathbf{u}_2 = \sqrt{\frac{3}{2}} x,$$

is an orthonormal set in $C[-1, 1]$.

c) We consider $\mathbf{u} = x^{\frac{1}{3}}$.

$$c_1 = \langle x^{1/3}, \mathbf{u}_1 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} x^{\frac{1}{3}} dx = 0.$$

$$c_2 \langle x^{1/3}, \mathbf{u}_2 \rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} x^{\frac{4}{3}} dx = \frac{6\sqrt{3}}{7\sqrt{2}}.$$

Hence, the least square linear approximation of the cube root $\mathbf{u} = x^{1/3}$ is

$$\mathbf{u} = x^{\frac{1}{3}} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 = \sqrt{\frac{3}{2}} \frac{6\sqrt{3}}{7\sqrt{2}} x = \frac{9}{7} x.$$

Please, plot both $x^{1/3}$ and $\frac{9x}{7}$ for $-1 \leq x \leq 1$ to compare.

Problem 5.6.1

a) We start with

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

and apply the G-S process to get

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$\mathbf{x}_2 - \mathbf{p}_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

b) We start with

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 5 \\ 10 \end{pmatrix},$$

and apply the G-S process to get

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$\mathbf{x}_2 - \mathbf{p}_1 = \begin{pmatrix} 5 \\ 10 \end{pmatrix} - 4\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Problem 5.6.3 Let

$$\mathbf{x}_1 = (1,2,-2)^T, \quad \mathbf{x}_2 = (4,3,2)^T, \quad \mathbf{x}_3 = (1,2,1)^T.$$

Applying the G-S process gives

$$\mathbf{u}_1 = \frac{1}{3}(1,2,-2)^T,$$

$$\mathbf{x}_2 - \mathbf{p}_1 = (4,3,2)^T - \frac{2}{3}(1,2,-2)^T = (10/3, 5/3, 10/3)^T = \frac{5}{3}(2,1,2)^T,$$

$$\mathbf{u}_2 = \frac{1}{3}(2,1,2)^T,$$

$$\mathbf{x}_3 - \mathbf{p}_2 = (1,2,1)^T - \frac{1}{3}(1,2,-2)^T - \frac{2}{3}(2,1,2)^T = \frac{1}{3}(-2,2,1)^T,$$

$$\mathbf{u}_3 = \frac{1}{3}(-2,2,1)^T.$$

Problem 5.6.4

$$\mathbf{x}_1 = 1, \quad \mathbf{x}_2 = x, \quad \mathbf{x}_3 = x^2.$$

Applying the G-S process gives

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}},$$

$$\mathbf{x}_2 - \mathbf{p}_1 = x - \frac{1}{2}\int_{-1}^{1} x dx = x - 0 = x,$$

$$\mathbf{u}_2 = \sqrt{\frac{3}{2}}x = \frac{\sqrt{6}}{2}x,$$

$$\mathbf{x}_3 - \mathbf{p}_2 = x^2 - \langle \mathbf{x}_3, \mathbf{u}_1\rangle \mathbf{u}_1 - \langle \mathbf{x}_3, \mathbf{u}_2\rangle \mathbf{u}_2 = x^2 - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\int_{-1}^{1} x^2 dx - 0 = x^2 - \frac{1}{2}\frac{2}{3} = x^2 - \frac{1}{3}.$$

Since

$$||\mathbf{x}_3 - \mathbf{p}_2||^2 = \int_{-1}^{1}(x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1}(x^4 - \frac{2}{3}x^2 + \frac{1}{9})dx = (\frac{2}{5} - \frac{2}{3}\cdot\frac{2}{3} - \frac{2}{9}) = \frac{8}{45},$$

we have

$$\mathbf{u}_3 = \frac{\sqrt{45}}{\sqrt{8}}(x^2 - 1/3) = \frac{3\sqrt{10}}{4}(x^2 - 1/3).$$

Problem 5.6.5

a) We have

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Applying the G-S process gives

$$\mathbf{u}_1 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

$$\mathbf{x}_2 - \mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/9 \\ 4/9 \\ -1/9 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix},$$

and, hence,

$$\mathbf{u}_1 = \frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}.$$

b) From part (a) we have

$$\mathbb{Q} = [\mathbf{u}_1 \ \mathbf{u}_2] = \begin{pmatrix} 2/3 & -\sqrt{2}/6 \\ 1/3 & 2\sqrt{2}/3 \\ 2/3 & -\sqrt{2}/6 \end{pmatrix}.$$

Now, to compute $\mathbf{R}$ observe that

$$r_{11} = ||\mathbf{x} - 1|| = 3,$$

$$r_{12} = \langle \mathbf{x}_2, \mathbf{u}_1 \rangle = \frac{5}{3},$$

$$r_{22} = ||\mathbf{x}_2 - \mathbf{p}_1|| = \frac{1}{9}\sqrt{18} = \frac{\sqrt{2}}{3},$$

and, hence,

$$\mathbf{R} = \begin{pmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{pmatrix},$$

with $\mathbb{A} = \mathbb{Q} \cdot \mathbf{R}$.

c) We have

$$\mathbb{A}^T \mathbb{A} = \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix},$$

$$\mathbb{A}^T \mathbf{b} = \begin{pmatrix} 66 \\ 36 \end{pmatrix},$$

so that the normal equation reads

$$\begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 66 \\ 36 \end{pmatrix}.$$

Its solutions is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix} \begin{pmatrix} 66 \\ 36 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}.$$