MATH314 – HOMEWORK SOLUTIONS

HOMEWORK #6

Section 4.3: Problems 1(a)(b),3,4,6,15
Section 5.1: Problems 1(a)(b),3(a),5,9
Section 5.2: Problems 1(b),2,3,4
Section 5.3: Problems 1(a),3(a),5,7

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Problem 4.3.1

a) We have

$$L(\mathbf{e}_1) = L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} = -\mathbf{e}_1,\quad L(\mathbf{e}_2) = L\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = \mathbf{e}_2,$$

$$L(\mathbf{u}_1) = L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = \mathbf{u}_2,\quad L(\mathbf{u}_2) = L\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \mathbf{u}_1.$$

Hence,

$$\mathbb{A} = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix},\qquad \mathbb{B} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}.$$

b) We have

$$L(\mathbf{e}_1) = L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} = -\mathbf{e}_1,\quad L(\mathbf{e}_2) = L\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -\mathbf{e}_2,$$

$$L(\mathbf{u}_1) = L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix} = -\mathbf{u}_1,\quad L(\mathbf{u}_2) = L\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = -\mathbf{u}_2.$$

Hence,

$$\mathbb{A} = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix},\qquad \mathbb{B} = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}.$$

Problem 4.3.3

a) We have the transition matrix from $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ to $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ given by

$$\mathbb{U} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

b) Since the $L(\mathbf{x}) = (2x_1 - x_2 - x_3, 2x_2 - x - 1 - x_3, 2x_3 - x_1 - x_2)^T$ we get

$$\mathbb{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Hence,

$$\mathbb{B} = \mathbb{U}^{-1}\mathbb{A}\mathbb{U} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Problem 4.3.4 We have the transition matrix from $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ to $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ given by

$$\mathbb{V} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix}.$$

Since

$$\mathbb{A} = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix},$$

we have

$$\mathbb{B} = \mathbb{V}^{-1}\mathbb{A}\mathbb{V} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 4.3.6

a) Let $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [1, e^x, e^{-x}]$ and $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = [1, \frac{1}{2}(e^x + e^{-x}), \frac{1}{2}(e^x - e^{-x})]$. Since

$$\mathbf{v}_1 = \mathbf{u}_1,$$

$$\mathbf{v}_2 = \mathbf{u}_2 + \mathbf{u}_3,$$

$$\mathbf{v}_3 = \mathbf{u}_2 - \mathbf{u}_3,$$

The transition matrix from the basis $[1, e^x, e^{-x}]$ to the basis $[1, \frac{1}{2}(e^x + e^{-x}), \frac{1}{2}(e^x - e^{-x})]$ is

$$\mathbb{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

b) If the linear operator D is the differentiation we have

$$D(\mathbf{u}_1) = 0,$$

$$D(\mathbf{u}_2) = \mathbf{u}_3,$$

$$D(\mathbf{u}_3) = \mathbf{u}_2,$$

so that

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

c) On the other hand, clearly,

$$D(\mathbf{v}_1) = 0,$$

$$D(\mathbf{v}_2) = \mathbf{v}_2,$$

$$D(\mathbf{v}_3) = -\mathbf{v}_3,$$

so that

$$\mathbb{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

d) Showing that $\mathbb{B} = \mathbb{S}^{-1}\mathbb{A}\mathbb{S}$ is the same as showing that $\mathbb{S}\mathbb{B} = \mathbb{A}\mathbb{S}$. We get

$$\mathbb{S}\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

$$\mathbb{A}\mathbb{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Problem 4.3.15 Let

$$\mathrm{tr}(\mathbb{A}) = a_{11} + a_{22} + \cdots + a_{nn}$$

be the sum of the diagonal elements of a square matrix $\mathbb{A} = (a_{ij})$.

a) We have

$$(\mathbb{A}\mathbb{B})_{ii} = \sum_k a_{ik} b_{ki}$$

and

$$(\mathbb{BA})_{mm} = \sum_j b_{mj} a_{jm}.$$

Now, by definition, we have

$$\operatorname{tr}(\mathbb{AB}) = \sum_{ik} a_{ik} b_{ki},$$

$$\operatorname{tr}(\mathbb{BA}) = \sum_{mj} b_{mj} a_{jm}.$$

But these double sums are the same by renaming $m = k$ and $j = i$.

b) As $\mathbb{A} = \mathbb{S}^{-1}\mathbb{BS}$, using part (a), we have

$$\operatorname{tr}(\mathbb{A}) = \operatorname{tr}(\mathbb{S}^{-1}\mathbb{BS}) = \operatorname{tr}(\mathbb{BS}^{-1}\mathbb{S}) = \operatorname{tr}(\mathbb{B}).$$

Problem 5.1.1

a) Let $\mathbf{v} = (2, 1, 3)^T$, $\mathbf{w} = (6, 3, 9)^T$. We have

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||||\mathbf{w}||} = \frac{42}{\sqrt{14}\sqrt{126}} = \frac{42}{9\cdot 14} = \frac{1}{3}.$$

b) Let $\mathbf{v} = (2, -3)^T$, $\mathbf{w} = (3, 2)^T$. We have

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||||\mathbf{w}||} = 0.$$

Problem 5.1.3(a) Let $\mathbf{x} = (3, 4)^T$, $\mathbf{y} = (1, 0)^T$. Then $\mathbf{p} = (3, 0)^T$ and $\mathbf{x} - \mathbf{p} = (0, 4)^T$. Clearly, $\mathbf{p}\cdot(\mathbf{x} - \mathbf{p}) = (3, 0)^T \cdot (0, 4)^T = 0$.

Problem 5.1.5 Assume that the point in question is a vector with coordinates $(a, 2a)^T$. We must have that

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} a \\ 2a \end{pmatrix} = \begin{pmatrix} 5-a \\ 2-2a \end{pmatrix}$$

is perpendicular to $(a, 2a)^T$. But that means that

$$5 - a + 2(2 - 2a) = 0$$

which means that $9 = 5a$. Hence, the point in question has coordinates $(9/5, 18/5)$.

Problem 5.1.9 The normal vector to that plane has coordinates $\mathbf{n} = (2, 2, 1)$. Hence, the projection of the vector $\mathbf{v} = (1, 1, 1)^T$ on the normal

$$\mathbf{p} = \frac{\mathbf{v}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} = \frac{5}{9}(2, 2, 1)^T.$$

The shortest distance is the norm of $\mathbf{p}$ which is $\frac{5}{3}$.

Problem 5.2.1(b) We reduce $\mathbb{A}$ first

$$\mathbb{A} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix} \simeq \begin{pmatrix} 1 & 3 & 1 \\ 0 & -2 & -2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

$R(\mathbb{A}^T)$ is the row space of $\mathbb{A}$ and, hence, we can choose the basis $\mathcal{B} = [(1,3,1)^T, (2,4,0)^T]$.

$N(\mathbb{A})$ is the null space of $\mathbb{A}$, which is $(2t, -t, t)^T$ and, hence, we can choose the basis $\mathcal{B} = [(2,-1,1)^T]$. Note that $N(\mathbb{A})$ is orthogonal to $R(\mathbb{A}^T)$.

$R(\mathbb{A})$ is the column space of $\mathbb{A}$ and, hence, we can choose the basis $\mathcal{B} = [(1,0)^T, (0,1)^T]$.

As $N(\mathbb{A}^T) = R(\mathbb{A})^\perp$ we conclude that $N(\mathbb{A}^T) = \{\mathbf{0}\}$ is trivial.

Problem 5.2.2 Let $S = \text{span}\{(1,-1,1)^T\}$.

a) $S^\perp$ is the null space of $\mathbb{A} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$. These are all vectors $(s-t, s, t)^T = s(1,1,0)^T + t(-1,0,1)$.

b) S is the line through $(0,0,0)$ parallel to $\mathbf{x}$. $S^\perp$ is the plane through $(0,0,0)$ perpendicular to this line and it has equation $x_1 - x_2 + x_3 = 0$.

Problem 5.2.3 Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$. Let

$$\mathbb{A} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

a) The null space of $\mathbb{A}$ is defined to be all vectors in $\mathbb{Z} \in \mathbb{R}^3$ such that $\mathbb{A} \cdot \mathbf{z} = 0$. But, in particular, we get that

$$\mathbf{x} \cdot \mathbf{z} = 0 = \mathbf{y} \cdot \mathbf{z},$$

and, hence $\mathbb{Z}$ must be perpendicular to both $\mathbf{x}, \mathbf{y}$. As our S is the span of these two vectors any such $\mathbf{z}$ will be perpendicular to any vector in S and hence $\mathbf{z} \in S^\perp$. This shows that $N(\mathbb{A}) = S^\perp$.

b) Using part (a) we have

$$\mathbb{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/3 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 5/3 \\ 0 & 1 & -1/3 \end{pmatrix}.$$

Hence, the null space of $\mathbb{A}$ is spanned by $(-5,1,3)^T$. We get $S^\perp = \text{span}\{(-5,1,3)^T\}$.

Problem 5.2.4 We use the result of the previous problem and start with the matrix

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -2 \end{pmatrix}.$$

As it is already reduced we get the nulls space

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2s - t \\ 2t - 3s \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

and $S^{\perp} = N(\mathbb{A}) = \text{span}\{(2, -3, 1, 0)^T, (-1, 2, 0, 1)^T\}$.

Problem 5.3.1(a) We have

$$\mathbb{A} = \begin{pmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

To get the for the normal equation we need

$$\mathbb{A}^T \mathbb{A} = \begin{pmatrix} 5 & -5 \\ -5 & 10 \end{pmatrix}, \quad \mathbb{A}^T \mathbf{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Hence, the normal equation reads

$$\begin{pmatrix} 5 & -5 \\ -5 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Its solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbb{A}^{-1} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Problem 5.3.3(a) We have

$$\mathbb{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

To get the for the normal equation we need

$$\mathbb{A}^T \mathbb{A} = \begin{pmatrix} 6 & 12 \\ 12 & 24 \end{pmatrix}, \quad \mathbb{A}^T \mathbf{b} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

Hence, the normal equation reads

$$\begin{pmatrix} 6 & 12 \\ 12 & 24 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

As the coefficient matrix of the normal equation is singular there is no unique solution to the problem. All solutions are described by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5/6 - 2t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -5/6 \\ 0 \end{pmatrix}.$$

Problem 5.3.5

a) We have

$$\mathbb{A} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix},$$

$$\mathbb{A}^t\,\mathbb{A} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}, \qquad \mathbb{A}^T\,\mathbf{b} = \begin{pmatrix} 13 \\ 21 \end{pmatrix}.$$

Hence, the normal equation reads

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 13 \\ 21 \end{pmatrix}.$$

Its solution is

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \mathbb{A}^{-1}\begin{pmatrix} 13 \\ 21 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 13 \\ 21 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.9 \end{pmatrix}.$$

The "best fitting line" has the equation $y(x) = 1.8 + 2.9x$. Please, plot these 4 points and the line in the Cartesian plane.

Problem 5.3.7 Let

$$\mathbb{A} = (\mathbf{a} \quad \mathbf{x}),$$

where $\mathbf{a} = (1, \ldots, 1)^T$ and $\mathbf{x} = (x_1, \ldots, x_n)^T$. Then

$$\mathbb{A}^T\,\mathbb{A} = \begin{pmatrix} n & \sum_i x_i \\ \sum_{i=1}^n x_i & \mathbf{x}^T\mathbf{x} \end{pmatrix},$$

and

$$\mathbb{A}^T\,\mathbf{y} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \mathbf{x}^T\mathbf{y} \end{pmatrix}.$$

Hence, the associated normal equation reads:

$$\begin{pmatrix} n & \sum_i x_i \\ \sum_i x_i & \mathbf{x}^T\mathbf{x} \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sum_i y_i \\ \mathbf{x}^T\mathbf{y} \end{pmatrix}.$$

Now, assume that $\frac{1}{n}\sum_{i=1}^n x_i = 0$. Then $\sum_{i=1}^n x_i = 0$ and the normal equation simplifies to

$$\begin{pmatrix} n & 0 \\ 0 & \mathbf{x}^T\mathbf{x} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \mathbf{x}^T\mathbf{y} \end{pmatrix}.$$

This is easily solved

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \frac{1}{n\mathbf{x}^T\mathbf{x}} \begin{pmatrix} \mathbf{x}^T\mathbf{x} & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} \sum_{i=1}^n y_i \\ \mathbf{x}^T\mathbf{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n y_i \\ \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \bar{y} \\ \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{x}} \end{pmatrix}.$$