MATH314 – HOMEWORK SOLUTIONS

HOMEWORK #5

Section 4.1: Problems 1,2,3,5(a)(b),6(a),8,9
Section 4.2: Problems 2,3(a)(b),4,6,8

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Problem 4.1..1

a)

$$L(\mathbf{x}) = L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}.$$

We have

$$L(\alpha\mathbf{x}) = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = \begin{pmatrix} -\alpha x_1 \\ \alpha x_2 \end{pmatrix} = \alpha\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \alpha L(\mathbf{x}),$$

$$L(\mathbf{x}+\mathbf{y}) = \begin{pmatrix} -(x_1+y_1) \\ (x_2+y_2) \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} = L(\mathbf{x}) + L(\mathbf{y}).$$

This transformation is the reflection through the x_2-axis.

b)

$$L(\mathbf{x}) = -\mathbf{x}.$$

We have

$$L(\alpha\mathbf{x}+\beta\mathbf{y}) = -(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha(-\mathbf{x}) + \beta(-\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

This is rotation by π.

c)

$$L(\mathbf{x}) = L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

We have

$$L(\alpha\mathbf{x}) = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_2 \\ \alpha x_1 \end{pmatrix} = \alpha\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \alpha L(\mathbf{x}),$$

$$L(\mathbf{x}+\mathbf{y}) = \begin{pmatrix} x_2+y_2 \\ x_1+y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} + \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} = L(\mathbf{x}) + L(\mathbf{y}).$$

This transformation is the reflection through the $x_1 = x_2$-axis.

d)

$$L(\mathbf{x}) = \frac{1}{2}\mathbf{x}.$$

We have

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \frac{1}{2}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha(\frac{1}{2}\mathbf{x}) + \beta(\frac{1}{2}\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

This is scaling of each vector by $\frac{1}{2}$.

e)

$$L(\mathbf{x}) = x_2\mathbf{e}_2.$$

We have

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = (\alpha\mathbf{x} + \beta\mathbf{y})_2\mathbf{e}_2 = [\alpha x_2 + \beta y_2]\mathbf{e}_2 = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

This is the projection on the x_2-axis.

Problem 4.1.2 Let

$$L_\alpha(\mathbf{x}) = L_\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\cos\alpha - x_2\sin\alpha \\ x_1\sin\alpha + x_2\cos\alpha \end{pmatrix}.$$

In polar coordinates we have $x_1 = r\cos\theta, x_2 = r\sin\theta$, where $r = \sqrt{x_1^2 + x_2^2}$ and $\tan\theta = x_2/x_1$. Substituting we get:

$$L_\alpha(r\cos\theta, r\sin\theta) = (r\cos\theta\cos\alpha - r\sin\theta\sin\alpha, r\cos\theta\sin\alpha + r\sin\theta\cos\alpha) =$$

$$= (r\cos(\alpha+\theta), r\sin(\alpha+\theta)).$$

Hence, in polar coordinates L can be written as

$$L_\alpha(r,\theta) = (r, \theta + \alpha)$$

and it is a clockwise rotation by the angle α.

Problem 4.1.3 Let

$$L(\mathbf{x}) = \mathbf{x} + \mathbf{a}, \qquad \mathbf{a} \neq \mathbf{0}.$$

This is not a linear transformation as $L(\mathbf{0}) = \mathbf{a} \neq \mathbf{0}$. In particular, neither $L(\alpha\mathbf{x}) = \alpha L(\mathbf{x})$ nor $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$ for this translations. Geometrically, we add the non-zero vector $\mathbf{a}$ to a vector $\mathbf{x}$ using the parallelogram rule.

Problem 4.1.5

a) This transformation is a projection on the x_2x_3-coordinate plane. It is linear and it is generalization of the Problem 4.1.1(e). (Same proof shows linearity.)

b) This transformation is trivially linear. It maps every vector in $\mathbb{R}^3$ to the origin in $\mathbb{R}^2$.

Problem 4.1.6 $L : \mathbb{R}^2 \to \mathbb{R}^3$.

a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$ is not linear as $L(\mathbf{0}) = (0, 0, 1)^T \neq \mathbf{0}$.

Problem 4.1.8

a) Let $L(p(x)) = xp(x)$. We have

$$L(\alpha p(x) + \beta q(x)) = x(\alpha p(x) + \beta q(x)) = \alpha x p(x) + \beta x q(x) = \alpha L(p(x)) + \beta L(q(x)).$$

Hence, this transformation from P_2 to P_3 is linear.

b) Let $L(p(x)) = x^2 + p(x)$. This is not linear as $L(p(x) \equiv 0) = x^2 \not\equiv 0$.

c) Let $L(p(x)) = p(x) + xp(x) + x^2 p'(x)$. We have

$$L(\alpha p(x) + \beta q(x)) = \alpha p(x) + \beta q(x) + x(\alpha p(x) + \beta q(x)) + x^2(\alpha p(x) + \beta q(x))' =$$

$$\alpha(p(x) + xp(x) + x^2 p'(x)) + \beta(q(x) + xq(x) + x^2 q'(x)) = \alpha L(p(x)) + \beta L(q(x)).$$

Hence, this transformation from P_2 to P_3 is linear.

Problem 4.1.9 For any $f \in C[0, 1]$ we define

$$L(f) = \int_0^x f(t)dt.$$

This is a linear transformations as

$$L(\alpha f + \beta g) = \int_0^x [\alpha f(t) + \beta g(t)]dt = \alpha \int_0^x f(t)dt + \beta \int_0^x g(t)dt = \alpha L(f) + \beta L(g).$$

Also,

$$L(e^x) = \int_0^x e^t dt = e^t]_0^x = e^x - 1.$$

$$L(x^2) = \int_0^x t^2 dt = \frac{t^3}{3}\Big]_0^x = \frac{x^3}{3}.$$

Problem 4.2.2

a)

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix}.$$

b)

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

c)

$$L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-1&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_2-x_1\\x_3-x_2\end{pmatrix}.$$

Problem 4.2.3

a)

$$L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_3\\x_2\\x_1\end{pmatrix}.$$

b)

$$L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1\\x_1+x_2\\x_1+x_2+x_3\end{pmatrix}.$$

Problem 4.2.4 We get

$$L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2x_1-x_2-x_3\\2x_2-x_1-x_3\\2x_3-x_1-x_2\end{pmatrix}.$$

Hence, we have

a)

$$L\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

b)

$$L\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}2\\-1\\-1\end{pmatrix}.$$

c)

$$L\begin{pmatrix}-5\\3\\2\end{pmatrix}=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\begin{pmatrix}-5\\3\\2\end{pmatrix}=\begin{pmatrix}-15\\9\\6\end{pmatrix}.$$

Problem 4.2.8 Let

$$\mathbf{y}_1=\begin{pmatrix}1\\1\\1\end{pmatrix},\quad\mathbf{y}_2=\begin{pmatrix}1\\1\\0\end{pmatrix},\quad\mathbf{y}_3=\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Consider the following linear transformation

$$L(c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3) = (c_1 + c_2 + c_3)\mathbf{y}_1 + (2c_1 + c_3)\mathbf{y}_2 - (2c_2 + c_3)c_3\mathbf{y}_3.$$

a)

$$L\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + c_3 \\ 2c_1 + c_3 \\ -2c_2 - c_3 \end{pmatrix}.$$

b) We will write

$$\mathbf{x} = \begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix} = 2\mathbf{y}_1 + 3\mathbf{y}_2 + 2\mathbf{y}_3.$$

Hence, in the $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ basis the vector $\mathbf{x}$ has coordinates $(c_1, c_2, c_3) = (2, 3, 2)$ and we get

$$L(\mathbf{x}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix}.$$

This means that $L(\mathbf{x}) = 7\mathbf{y}_1 + 6\mathbf{y}_2 - 8\mathbf{y}_3$.

Similarly,

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3.$$

Hence, in the $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ basis the vector $\mathbf{x}$ has coordinates $(c_1, c_2, c_3) = (1, 1, 1)$ and we get

$$L(\mathbf{x}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}.$$

This means that $L(\mathbf{x}) = 3\mathbf{y}_1 + 3\mathbf{y}_2 - 3\mathbf{y}_3$.

Finally,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 3\mathbf{y}_1 - \mathbf{y}_2 - \mathbf{y}_3.$$

Hence, in the $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ basis the vector $\mathbf{x}$ has coordinates $(c_1, c_2, c_3) = (3, -1, -1)$ and we get

$$L(\mathbf{x}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}.$$

This means that $L(\mathbf{x}) = \mathbf{y}_1 + 5\mathbf{y}_2 + 3\mathbf{y}_3$.