MATH314 – HOMEWORK SOLUTIONS

HOMEWORK #4

Section 3.3: Problems 1(a)(b)(c),2(c)(d),4(c),6(a)(c),10
Section 3.4: Problems 3,4,5,8
Section 3.5: Problems 1(a),4,5,7,9
Section 3.6: Problems 1(b),2(a),3,4(b),6

Krzysztof Galicki

Problem 3.3.1

a) $(2,1)$ is clearly not a multiple of $(3,2)$ and, hence, the two vectors are linearly independent in $\mathbb{R}^2$.

b) $2(2,3) = (4,6)$ and, hence, the two vectors are linearly dependent in $\mathbb{R}^2$.

c) Any three vectors must be linearly dependent in $\mathbb{R}^2$.

Problem 3.3.2

c) We have

$$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

and, hence, the three vectors are linearly dependent in $\mathbb{R}^3$.

d) We have

$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix}$$

and, hence, the three vectors are linearly dependent in $\mathbb{R}^3$.

Problem 3.3.4

c) We have

$$2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

and, hence, the three vectors are linearly dependent in $\mathcal{M}_{2\times 2}(\mathbb{R})$.

Problem 3.3.6

a) We have

$$x^2 - 2 = 1 \cdot x^2 - 2 \cdot 1$$

and, hence, the three vectors are linearly dependent in P_3.

b) Assume that

$$c_1(x+2) + c_2(x+1) + c_3(x^2-1) = 0$$

for all $x \in \mathbb{R}$. Then

$$c_3x^2 + (c_1+c_2)x + (2c_1+c_2-c_3) = 0.$$

This means that

$$c_3 = 0, \quad c_1 + c_2 = 0, \quad 2c_1 + c_2 - c_3 = 0.$$

Now, the first equation implies that $c_3 = 0$. Then subtracting the two remaining equations gives $c_1 = 0$. With $c_3 = c_1 = 0$ the second (or third) equation says that also $c_2 = 0$. Hence, the three vectors are linearly independent in P_3.

Problem 3.3.6

a) Consider $f_1(x) = 2x$ and $f_2(x) = |x|$ as continuous functions on the interval $[-1,1]$. Let

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all $x \in [-1,+1]$. Substitute $x = -1$ first and $x = +1$ next into this equation. One gets

$$-2c_1 + c_2 = 0$$

and

$$2c_1 + c_2 = 0.$$

But this means that $c_1 = c_2 = 0$ and, hence, f_1, f_2 are linearly independent in $C[-1,+1]$.

b) On the interval $[0,1]$ we have $f_1(x) = 2f_2(x)$ so the two functions are linearly dependent.

Problem 3.4.3 Let

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

a) As $\mathbf{x}_1$ is not a scalar multiple of $\mathbf{x}_2$ they must be linearly independent. To show that they also span $\mathbb{R}^2$ consider any vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ and let

$$\begin{pmatrix} a \\ b \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2c_1 + 4c_2 \\ c_1 + 3c_2 \end{pmatrix}.$$

It is easily seen that one has a unique solution here and

$$c_2 = \frac{1}{2}(2b - a), \quad c_1 = b - 3c_2 = b - \frac{3}{2}(2b - a) = \frac{3}{2}a - 2b.$$

Hence, $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis in $\mathbb{R}^2$.

b) Any three vectors in $\mathbb{R}^2$ will be linearly dependent. Here, using part (a) we can see that ($c_2 = -\frac{13}{2}, c_1 = \frac{33}{2}$)

$$\begin{pmatrix} 7 \\ -3 \end{pmatrix} = \frac{33}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{13}{2}\begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

c) As $\mathbf{x}_3$ is linear combination of the first two $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathbb{R}^2$ and its dimension is 2.

Problem 3.4.5 Let

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}.$$

a) It is easily seen that

$$4\mathbf{x}_1 - 2\mathbf{x}_2 = \mathbf{x}_3$$

and, therefore, the three vectors are linearly dependent.

b) $\mathbf{x}_1$ is not a scalar multiple of $\mathbf{x}_2$ so the two vectors are linearly independent.

c) Since $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$ and the first two are linearly independent the dimension of this space is 2.

d) $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is the plane through the origin $(0, 0, 0) \in \mathbb{R}^3$ spanned by the two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Problem 3.4.8 Let

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}.$$

a) These two vectors are linearly independent but they cannot span $\mathbb{R}^3$. It is easy to see that $(0, 0, 1)^T = \mathbf{k}$ is not a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

b) $\det(X)$ would have to be non-zero. If we set $\mathbf{x}_3 = (a, b, c)^T$ then one gets

$$\det(X) = \begin{vmatrix} 1 & 3 & a \\ 1 & -1 & b \\ 1 & 4 & c \end{vmatrix} = 5a - b - 4c \neq 0.$$

c) My remark in (a) shows that we can take $\mathbf{x}_3 = \mathbf{k} = (0, 0, 1)^T$. The calculation in (b) shows that any $x_3 = (a, b, c)^T$ as long as $5a - b - 4c \neq 0$.

Problem 3.5.1

a) The transition matrix is

$$\mathbb{U} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Problem 3.5.4 Let $[\mathbf{u}_1, \mathbf{u}_2] = [(5,3)^T, (3,2)^T]$. Then the transition matrix from $B = [\mathbf{u}_1, \mathbf{u}_2]$ to $E = [\mathbf{e}_1, \mathbf{e}_2]$ is

$$\mathbb{U} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

The transition matrix from $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$ is its inverse

$$\mathbb{U}^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.$$

Hence,

$$[\mathbf{x}]_B = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

$$[\mathbf{y}]_B = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix},$$

$$[\mathbf{z}]_B = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}.$$

Problem 3.5.5 Let $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = [(1,1,1)^T, (1,2,2)^T, (2,3,4)^T]$.

a) Then the transition matrix from $B = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ to $E = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ is

$$\mathbb{U} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

The transition matrix from $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ is $\mathbb{U}^{-1}$ and

$$\mathbb{U}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Hence,

b)

$$\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}_B = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_B = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}_B = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

Problem 3.5.7 Let $B = [\mathbf{v}_1, \mathbf{v}_2] = [(1,2)^T, (2,3)^T]$. Then the transition matrix from $B = [\mathbf{v}_1, \mathbf{v}_2]$ to $E = [\mathbf{e}_1, \mathbf{e}_2]$ is

$$\mathbb{V} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Suppose now $[\mathbf{w}_1, \mathbf{w}_2]$ has a transition matrix to $E = [\mathbf{e}_1, \mathbf{e}_2]$ which is $\mathbb{W}$. Then the transition matrix from $[\mathbf{w}_1, \mathbf{w}_2]$ to $[\mathbf{v}_1, \mathbf{v}_2]$ is $\mathbb{S}$ and it is related to $\mathbb{V}$ and $\mathbb{W}$ as follows

$$\mathbb{S} = (\mathbb{W}^{-1} \cdot \mathbb{V})^{-1} = \mathbb{V}^{-1} \cdot \mathbb{W}.$$

Hence,

$$\mathbb{W} = \mathbb{V} \cdot \mathbb{S} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 9 & 4 \end{pmatrix}.$$

We get $[\mathbf{w}_1, \mathbf{w}_2] = [(5,9)^T, (1,4)^T]$.

Problem 3.5.9 Let $V = [\mathbf{v}_1, \mathbf{v}_2] = [x, 1]$ and $W = [\mathbf{w}_1, \mathbf{w}_2] = [2x-1, 2x+1]$ be the two bases for P_2. It is easy to see that

$$\mathbf{w}_1 = 2\mathbf{v}_1 - \mathbf{v}_2$$

$$\mathbf{w}_2 = 2\mathbf{v}_1 + \mathbf{v}_2$$

and

$$\mathbf{v}_1 = \frac{1}{4}\mathbf{w}_1 + \frac{1}{4}\mathbf{w}_2$$

$$\mathbf{v}_2 = -\frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2.$$

Hence, the corresponding transition matrices are

$$\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1/4 & -1/2 \\ 1/4 & 1/2 \end{pmatrix}.$$

Problem 3.6.1(b) First we reduce

$$\begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ 0 & 7 & 1 & -2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 7 & 1 & -2 \end{pmatrix} \simeq$$

$$\simeq \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Basis for the row space: we can take $\{(1,0,0,-10/7), (0,1,0,-2/7), (0,0,1,0)\}$. Since the rank is 3 we could also take the basis to be the rows of the original matrix.

Basis for the null space: the null space is described by the one parameter family of solutions $(x_1, x_2, x_3, x_4) = (10/7, 2/7, 0, 1)t$. It is, therefore, one-dimensional and we can take the basis to be any non-zero multiple of the vector $(10/7, 2/7, 0, 1)$.

Basis for the column space: Since the rank is 3 the columns space is the whole $\mathbb{R}^3$. Hence we could take $\{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\}$ for the basis (or the first three rows of the original matrix).

Problem 3.6.2(a) We reduce

$$\begin{pmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 12 \end{pmatrix}.$$

This is enough to see that the column space of this matrix is 3-dimensional and, hence, the span is $\mathbb{R}^3$.

Problem 3.6.3 We reduce first

$$\mathbb{A} = \begin{pmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 & 2 \end{pmatrix} \simeq$$

$$\simeq \begin{pmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & 0 & 5 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{U}.$$

a) Columns 2,4,5 correspond to free variables and columns 1,3,6 to the lead variables. If we denote column vectors of $\mathbb{U}$ by $\mathbf{u}_i$ we get

$$\mathbf{u}_2 = 2\mathbf{u}_1$$

$$\mathbf{u}_4 = 5\mathbf{u}_1 - \mathbf{u}_3$$

$$\mathbf{u}_5 = -3\mathbf{u}_1 + 2\mathbf{u}_3.$$

b) Columns 1,3,6 correspond to the lead variables of $\mathbb{U}$. If we denote column vectors of $\mathbb{A}$ by $\mathbf{a}_i$ we get

$$\mathbf{a}_2 = 2\mathbf{a}_1$$

$$\mathbf{a}_4 = 5\mathbf{a}_1 - \mathbf{a}_3$$

$$\mathbf{a}_5 = -3\mathbf{a}_1 + 2\mathbf{a}_3.$$

Problem 3.6.3(b) We reduce first

$$\mathbb{A} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

The column space is therefore spanned by vector $(3,1)^T$ and $(1,1)^T$ is not in this span. The linear system is inconsistent.

Problem 3.6.6 There will be exactly one solution. As $\mathbf{b}$ is in the column space of $\mathbb{A}$ there must be a solution. But the columns of $\mathbb{A}$ are linearly independent and therefore any vector in the column space is a unique linear combination of the basis for the column space. This determines the coefficients $(x_1,\ldots,x_n)$ uniquely.