MATH314 – HOMEWORK SOLUTIONS

HOMEWORK #3

Section 2.2: Problems 1(a),2(a),5,6,10
Section 2.3: Problems 1(b)(c),2(b)(c),8
Section 3.1: Problems 4,5,6,11
Section 3.2: Problems 1(a)(e),2(a)(b),3(b)(c),4(b),9(a), 13,14

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Problem 2.2.1(a) Expanding with respect to the first column gives:

$$\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 0 & 3 \\ 4 & 1 \end{vmatrix} = 2 \cdot (-12) = -24.$$

Problem 2.2.2(a) We start by adding twice the second row to the third [(3)+2(2)] and subtracting it from the fourth [(4)-(2)]. Then we switch the first two rows [(1) $\leftrightarrow$ (2)].

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \end{pmatrix} \simeq$$

Next, we subtract the second row from the fourth [(4)-(2)] after which we add rows three and four [(3)+(4)]

$$\simeq \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -5 & -7 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

The last matrix is upper-triangular and it has determinant -10. As we got it from $\mathbb{A}$ only by using elementary row operations of the third kind with **one row flip**, it follows that $\det(\mathbb{A}) = 10$.

Problem 2.2.5 We only use the fact that $\det(\mathbb{B} \cdot \mathbb{A}) = \det(\mathbb{B}) \det(\mathbb{A})$. Let $\mathbb{B} = \alpha \mathbb{I}_n$. Clearly, $\det(\alpha \mathbb{I}_n) = \alpha^n$. Then

$$\det(\alpha\mathbb{A}) = \det(\alpha \mathbb{I}_n \cdot \mathbb{A}) = \det(\alpha \mathbb{I}_n) \det(\mathbb{A}) = \alpha^n \det(\mathbb{A}).$$

Problem 2.2.6 We only use the fact that $\det(\mathbb{B} \cdot \mathbb{A}) = \det(\mathbb{B}) \det(\mathbb{A})$. Let $\mathbb{B} = \mathbb{A}^{-1}$. Then we have

$$\det(\mathbb{B} \cdot \mathbb{A}) = \det(\mathbb{A}^{-1} \cdot \mathbb{A}) = \det(\mathbb{I}_n) = 1$$

on one hand and

$$\det(\mathbb{B} \cdot \mathbb{A}) = \det(\mathbb{A}^{-1} \cdot \mathbb{A}) = \det(\mathbb{A}^{-1}) \det(\mathbb{A}),$$

on the other. Hence,

$$1 = \det(\mathbb{A}^{-1}) \det(\mathbb{A}),$$

or

$$\det(\mathbb{A}^{-1}) = \frac{1}{\det(\mathbb{A})}.$$

Problem 2.2.10

a) We get

$$\det(\mathbb{V}) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1^2 \\ x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix} =$$

$$= (x_2 - x_1)(x_3^2 - x_1^2) - (x_3 - x_1)(x_2^2 - x_1^2) = (x_2 - x_1)(x_3^2 - x_1^2) - (x_2 - x_1)(x_3 - x_1)(x_2 + x_1)$$

$$= (x_2 - x_1)[(x_3 - x_1)(x_3 + x_1) - (x_3 - x_1)(x_2 + x_1)]$$

$$= (x_2 - x_1)(x_3 - x_1)[(x_3 + x_1) - (x_2 + x_1)] = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

b) $\det(\mathbb{V}) \neq 0$ if and only if all three numbers x_1, x_2, x_3 are different from one another.

Problem 2.3.1

b)

$$\mathbb{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}, \quad \operatorname{adj}(\mathbb{A}) = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}, \quad \mathbb{A}^{-1} = \frac{1}{10} \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}.$$

c) Let

$$\mathbb{A} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{pmatrix}.$$

Then we have

$$\operatorname{adj}(\mathbb{A}) = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \end{pmatrix}^T =$$

$$\begin{pmatrix} -3 & 0 & 6 \\ 5 & 1 & -8 \\ 2 & 1 & -5 \end{pmatrix}^T = \begin{pmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{pmatrix}.$$

Since

$$\det(\mathbb{A}) = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ 0 & 8 & 1 \end{vmatrix} = \begin{vmatrix} -5 & -1 \\ 8 & 1 \end{vmatrix} = 3,$$

we have

$$\mathbb{A}^{-1} = \frac{1}{3}\begin{pmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{pmatrix}.$$

Problem 2.3.3

b) Here we have

$$\mathbb{A} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Hence, $\det(\mathbb{A}) = -5$ and we get

$$x_1 = -\frac{1}{5}\begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} = \frac{11}{5},$$

$$x_2 = -\frac{1}{5}\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} = -\frac{4}{5}.$$

c) Here we have

$$\mathbb{A} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 8 \\ 2 \end{pmatrix}.$$

Since

$$\det(\mathbb{A}) = \begin{vmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -3 \\ 4 & 5 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} = 6,$$

$$\begin{vmatrix} 0 & 1 & -3 \\ 8 & 5 & 1 \\ 2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -3 \\ 0 & 9 & -15 \\ 2 & -1 & 4 \end{vmatrix} = 2(-15+27) = 24,$$

$$\begin{vmatrix} 2 & 0 & -3 \\ 4 & 8 & 1 \\ -2 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -3 \\ 0 & 8 & 7 \\ 0 & 2 & 1 \end{vmatrix} = 2(8-14) = -12,$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 4 & 5 & 8 \\ -2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 8 \\ 0 & 0 & 2 \end{vmatrix} = 12.$$

Hence,

$$x_1 = \frac{24}{6} = 4, x_2 = \frac{-12}{6} = -2, x_3 = \frac{12}{6} = 2.$$

Problem 2.3.8 We use the identity

$$\text{adj}(\mathbb{A}) \cdot \mathbb{A} = \det(\mathbb{A})\mathbb{I}_n.$$

Taking determinant on both sides (and using Problem 2.2.5 of this homework) we get

$$\det(\text{adj}(\mathbb{A}) \cdot \mathbb{A}) = \det(\text{adj}(\mathbb{A}))\det(\mathbb{A}) = \det(\det(\mathbb{A})\mathbb{I}_n) = [\det(\mathbb{A})]^n.$$

Hence

$$\det(\text{adj}(\mathbb{A})) = [\det(\mathbb{A})]^{n-1}.$$

Problem 3.1.4 Let $V = \mathcal{M}_{m\times n}(\mathbb{R})$ with the usual matrix addition and multiplication of matrices by real numbers. We check all the axioms of the vector space:

A1) $\mathbb{A} + \mathbb{B} = \mathbb{B} + \mathbb{A}$ is obvious,

A2) $(\mathbb{A} + \mathbb{B}) + \mathbb{C} = \mathbb{A} + (\mathbb{B} + \mathbb{C})$ is obvious,

A3) the zero vector in $V = \mathcal{M}_{m\times n}(\mathbb{R})$ is the matrix with all zero entries,

A4) the inverse matrix under addition is the matrix $-\mathbb{A}$ with all the entries having the opposite sign to the original matrix $\mathbb{A}$,

A5) $\alpha(\mathbb{A} + \mathbb{B}) = \alpha\mathbb{B} + \alpha\mathbb{A}$ is obvious,

A6) $(\alpha + \beta)\mathbb{A} = \alpha\mathbb{A} + \beta\mathbb{A}$ is obvious,

A7) $(\alpha\beta)\mathbb{A} = \alpha(\beta\mathbb{A})$ is obvious,

A8) $1 \cdot \mathbb{A} = \mathbb{A}$ is obvious.

Problem 3.1.5 Let $V = C[a, b]$ with the usual pointwise addition of functions and multiplication of functions by real numbers. We check all the axioms of the vector space:

A1) $f(x) + g(x) = g(x) + f(x)$ is obvious,

A2) $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$ is obvious, A3) the zero vector in $V = [a, b]$ is $f(x) \equiv 0$ – the function vanishing at every $x \in [a, b]$,

A4) the inverse function under addition is the function $-f(x)$,

A5) $\alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x)$ is obvious,

A6) $(\alpha + \beta)f(x) = \alpha f(x) + \beta f(x)$ is obvious,

A7) $(\alpha\beta)f(x) = \alpha(\beta f(x))$ is obvious,

A8) $1 \cdot f(x) = f(x)$ is obvious.

Problem 3.1.6 Let $V = P$ be the space of polynomial functions with the usual pointwise addition of polynomials and multiplication of polynomials by real numbers. Clearly, the sum of two polynomials is a polynomial and multiplying a polynomial by a number gives a polynomial (of the same degree). As in the previous problem we check all the axioms of the vector space:

A1) $f(x) + g(x) = g(x) + f(x)$ is obvious,

A2) $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$ is obvious,

A3) the zero vector in $V = [a, b]$ is $f(x) \equiv 0$ – the polynomial vanishing at every x,

A4) the inverse function under addition is the function $-f(x)$,

A5) $\alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x)$ is obvious,

A6) $(\alpha + \beta)f(x) = \alpha f(x) + \beta f(x)$ is obvious,

A7) $(\alpha\beta)f(x) = \alpha(\beta f(x))$ is obvious,

A8) $1 \cdot f(x) = f(x)$ is obvious.

Problem 3.1.11 Axioms (A1-4) are clearly satisfied. Now, we check the remaining axioms:

A5)

$$\alpha \circ ((x_1, x_2) + (y_1, y_2)) = \alpha \circ (x_1 + y_1, x_2 + y_2) = (\alpha(x_1 + y_1), x_2 + y_2),$$

$$\alpha \circ (x_1, x_2) + \alpha \circ (y_1, y_2)) = (\alpha x_1, x_2) + (\alpha y_1, y_2) = (\alpha x_1 + \alpha y_1), x_2 + y_2).$$

This axiom is fine.

A6)

$$(\alpha + \beta) \circ (x_1, x_2) = ((\alpha + \beta)x_1, x_2),$$

$$\alpha \circ (x_1, x_2) + \beta \circ (x_1, x_2) = (\alpha x_1, x_2) + (\beta x_1, x_2) = (\alpha x_1 + \beta x_1, 2x_2).$$

This is false for $x_2 \neq 0$. It is easy to see that this is the only axiom that fails. Hence, V **is not** a vector space.

Problem 3.2.1

a) This is a line in $\mathbb{R}^2$ through the origin, and, hence, a vector subspace.

e) This is **not** a line in $\mathbb{R}^2$ through the origin, and, hence, **not** a vector subspace. These are two lines through the origin: $x_1 = x_2$ and $x_1 = -x_2$. For example, both $(1, 1)$ and $(1, -1)$ are in this set, but the sum $(2, 0)$ **is not**.

Problem 3.2.2

a) Not a vector subspace as $(0,0,0)$ is not in it.

b) This is a line through the origin in $\mathbb{R}^3$ and, hence, a vector subspace.

Problem 3.2.3

b) This is a vector subspace as it is closed under the two vector space operations and the zero matrix is lower triangular.

c) This is not a vector subspace. The zero matrix is not in this subset. It is not even closed under the two vector space operations.

Problem 3.2.4(b) We reduce first

$$\begin{pmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We introduce free parameters for pivot-free columns $x_2 = t, x_3 = s$. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3s - 2t \\ t \\ s \\ 0 \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 3.2.9(a) This set is spanning. Consider any vector in $\mathbb{R}^2$ say $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We want to find $a, b \in \mathbb{R}$ such that

$$a \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

But we can write this as

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This always has a solution as

$$\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 \neq 0.$$

Problem 3.2.13 For arbitrary real numbers $a, b, c, d \in \mathbb{R}$ we have

$$a\mathbb{E}_{11} + b\mathbb{E}_{12} + c\mathbb{E}_{21} + d\mathbb{E}_{22} =$$

$$= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Problem 3.2.14 $P_3 = \{p(x) \mid p(x) = a + bx + cx^2\}$. We have

a) $p(x) = x$ is clearly not in the span of $\{1, x^2, x^2 - 2\}$.

b) $\{2, x^2, x, 2x + 3\}$ is spanning for P_3 as is the subset $\{2, x^2, x\}$.

c) $\{x + 2, x + 1, x^2 - 1\}$ is spanning for P_3 as we can see from the following calculation

$$\alpha(x + 2) + \beta(x + 1) + \gamma(x^2 - 1) = \gamma x^2 + (\alpha + \beta)x + 2\alpha + \beta - \gamma.$$

For an arbitrary polynomial $p(x) = a+bx+cx^2$ choose $\gamma = c$. Then choose $\alpha = a-b+c$ and $\beta = 2b - a + c$.

d) $p(x) = x^2 + x$ is clearly not in the span of $\{x + 2, x^2 - 1\}$.