MATH314 – HOMEWORK SOLUTIONS

HOMEWORK #2

Section 1.3: Problems 1,2(a)(b),4(c),10
Section 1.4: Problems 1,3(b),4(c),6(a),7(a),9(e)
Section 1.5: Problems 1,5(a)(b),11,12
Section 2.1: Problems 1,2,3(c)(g),6

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Problem 1.3.1

(a)

$$2\mathbb{A} = 2\begin{pmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 8 \\ -4 & 0 & 2 \\ 2 & 4 & 4 \end{pmatrix}.$$

(b)

$$\mathbb{A} + \mathbb{B} = \begin{pmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 6 \\ -5 & 1 & 2 \\ 3 & -2 & 3 \end{pmatrix}.$$

(c)

$$2\mathbb{A} - 3\mathbb{B} = \begin{pmatrix} 6 & 2 & 8 \\ -4 & 0 & 2 \\ 2 & 4 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 6 \\ -9 & 3 & 3 \\ 6 & -12 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ 5 & -3 & -1 \\ -4 & 16 & 1 \end{pmatrix}.$$

(d)

$$(2\mathbb{A})^T - (3\mathbb{B})^T = (2\mathbb{A} - 3\mathbb{B})^T = \begin{pmatrix} 3 & 2 & 2 \\ 5 & -3 & -1 \\ -4 & 16 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 5 & -4 \\ 2 & -3 & 16 \\ 2 & -1 & 1 \end{pmatrix}.$$

(e)

$$\mathbb{A} \cdot \mathbb{B} = \begin{pmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{pmatrix}.$$

(f)

$$\mathbb{B}\cdot\mathbb{A} = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 8 \\ -10 & -1 & -9 \\ 15 & 4 & 6 \end{pmatrix}.$$

(g)

$$\mathbb{A}^T \cdot \mathbb{B}^T = (\mathbb{B}\cdot\mathbb{A})^T = \begin{pmatrix} 5 & 5 & 8 \\ -10 & -1 & -9 \\ 15 & 4 & 6 \end{pmatrix}^T = \begin{pmatrix} 5 & -10 & 15 \\ 5 & -1 & 4 \\ 8 & -9 & 6 \end{pmatrix}.$$

(h) Same as in (g).

Problem 1.3.2

a)

$$\begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 19 \\ 4 & 0 \end{pmatrix}.$$

b) The first matrix is 3×2 and the second 1×3 so the product is not defined.

Problem 1.3.4(c)

c)

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & -2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}.$$

Problem 1.3.10 Note that $\mathbb{A}^2 = \mathbb{A}$. That means that $\mathbb{A}^3 = \mathbb{A}$ and in fact $\mathbb{A}^n = \mathbb{A}$. Such a matrix is called a projector and it describes a projection of a 2-vector on some axis in $\mathbb{R}^2$. Clearly, repeating the projection is the same as applying it only once.

Problem 1.4.1

a) Yes, it corresponds to switching the first and the second row; type I.

b) No, it is a product of two elementary row matrices of the second type (scale first row by 2, and then scale the second row by 3).

c) Yes, it corresponds to adding 5 times the first row to the third row, type III.

d) Yes, it corresponds to the scaling of the second row by 5, type II.

Problem 1.4.3(b) Switch the second and the third row, i.e.,

$$\mathbb{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Problem 1.4.4(c) Multiply the first column by 1/2, i.e.,

$$\mathbb{E} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 1.4.6(a) We will reduce $\mathbb{A}$ to an upper-diagonal form by using a sequence of three elementary row operation of type III.

$$\mathbb{A} = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix} \simeq \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix} \simeq \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} = \mathbb{U}.$$

The corresponding elementary matrices are

$$\mathbb{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad \mathbb{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

with $\mathbb{A}_3 \mathbb{E}_2 \mathbb{E}_1 \mathbb{A} = \mathbb{U}$.

Problem 1.4.7(a) We apply just one elementary row operation to this matrix (subtract 3 times the first row from the second) to obtain an upper-triangular $\mathbb{U}$. Hence

$$\mathbb{E}\mathbb{A} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 9 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \mathbb{U}.$$

Hence, $\mathbb{A} = \mathbb{E}^{-1}\mathbb{U}$, or

$$\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}.$$

Problem 1.4.9(e) One can reduce this matrix by applying a sequence of only 2 elementary row operations of type III.

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding elementary matrices are

$$\mathbb{E}_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

As $\mathbb{E}_2\mathbb{E}_1\mathbb{A} = \mathbb{I}_3$ then the inverse is simply the product

$$\mathbb{E}_2\mathbb{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 1.5.1

a) $\mathbb{A}^{-1}\,(\,\mathbb{A} \quad \mathbb{I}\,) = (\,\mathbb{I} \quad \mathbb{A}^{-1}\,)$.

$[n \times n$ times $n \times (n, n)$ equals $n \times (n, n)]$

b) $\begin{pmatrix} \mathbb{A} \\ \mathbb{I} \end{pmatrix} \mathbb{A}^{-1} = \begin{pmatrix} \mathbb{I} \\ \mathbb{A} \end{pmatrix}.$

$[(n, n) \times n$ times $n \times n$ equals $(n, n) \times n]$

c) $(\,\mathbb{A} \quad \mathbb{I}\,)^T\,(\,\mathbb{A} \quad \mathbb{I}\,) = \begin{pmatrix} \mathbb{A}^T \\ \mathbb{I} \end{pmatrix}(\,\mathbb{A} \quad \mathbb{I}\,) = \begin{pmatrix} \mathbb{A}^T\mathbb{A} & \mathbb{A}^T \\ \mathbb{A} & \mathbb{I} \end{pmatrix}.$

$[(n, n) \times n$ times $n \times (n, n)$ equals $(n, n) \times (n, n)]$

d) $(\,\mathbb{A} \quad \mathbb{I}\,)(\,\mathbb{A} \quad \mathbb{I}\,)^T = (\,\mathbb{A} \quad \mathbb{I}\,)\begin{pmatrix} \mathbb{A}^T \\ \mathbb{I} \end{pmatrix} = \mathbb{A}\mathbb{A}^T + \mathbb{I}.$

$[n \times (n, n)$ times $(n, n) \times n$ equals $n \times n]$

d) $\begin{pmatrix} \mathbb{A}^{-1} \\ \mathbb{I} \end{pmatrix}(\,\mathbb{A} \quad \mathbb{I}\,) = \begin{pmatrix} \mathbb{I} & \mathbb{A}^{-1} \\ \mathbb{A} & \mathbb{I} \end{pmatrix}.$

$[(n, n) \times n$ times $n \times (n, n)$ equals $(n, n) \times (n, n)]$

Problem 1.5.5

a) $[2 \times (3, 1)$ times $(3, 1) \times 3$ gives $2 \times 3]$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & -1 \end{array}\right) \cdot \begin{pmatrix} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \\ -- & -- & -- \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 1 \\ 11 & -1 & 4 \end{pmatrix}.$$

b) $[(3, 1) \times 2$ times $2 \times (3, 1)$ gives $(3, 1) \times (3, 1)]$

$$\begin{pmatrix} 4 & -2 \\ 2 & 3 \\ 1 & 1 \\ -- & -- \\ 1 & 2 \end{pmatrix} \cdot \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & -1 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 2 & 0 & -2 \\ 8 & 5 & 8 & -5 \\ 3 & 2 & 3 & -2 \\ -- & -- & -- & -- \\ 5 & 3 & 5 & -3 \end{array}\right).$$

Problem 1.5.11 We have

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{O} & \mathbb{A}_{22} \end{pmatrix}$$

with the diagonal $n \times n$ matrices $\mathbb{A}_{11}, \mathbb{A}_{22}$ non-singular. Assume that the matrix

$$\begin{pmatrix} \mathbb{M} & \mathbb{C} \\ \mathbb{N} & \mathbb{P} \end{pmatrix}$$

is the inverse. Then

$$\begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{O} & \mathbb{A}_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{M} & \mathbb{C} \\ \mathbb{N} & \mathbb{P} \end{pmatrix} = \begin{pmatrix} \mathbb{A}_{11}\mathbb{M} + \mathbb{A}_{12}\mathbb{N} & \mathbb{A}_{11}\mathbb{C} + \mathbb{A}_{12}\mathbb{P} \\ \mathbb{A}_{22}\mathbb{N} & \mathbb{A}_{22}\mathbb{P} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix}.$$

This gives the following four equations

$$\mathbb{A}_{11}\mathbb{M} + \mathbb{A}_{12}\mathbb{N} = \mathbb{I},$$

$$\mathbb{A}_{11}\mathbb{C} + \mathbb{A}_{12}\mathbb{P} = \mathbb{O},$$

$$\mathbb{A}_{22}\mathbb{N} = \mathbb{O},$$

$$\mathbb{A}_{22}\mathbb{P} = \mathbb{I}.$$

The fourth equation implies that $\mathbb{P} = \mathbb{A}_{22}^{-1}$. Since $\mathbb{A}_{22}$ is non-singular, the third equation can be true only if all the columns of $\mathbb{N}$ are zero vectors and, hence, the whole $\mathbb{N} = \mathbb{O}$. Then the first equation, by putting $\mathbb{N} = \mathbb{O}$, says that $\mathbb{M} = \mathbb{A}_{11}^{-1}$. This establishes part (a). To prove (b) it is enough to solve the second equation and we get

$$\mathbb{C} = -\mathbb{A}_{11}^{-1}\mathbb{A}_{12}\mathbb{A}_{22}^{-1}.$$

Problem 1.5.12 Note that

$$\mathbb{A}^2 = \mathbb{A} \cdot \mathbb{A} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{B} & \mathbb{O} \end{pmatrix} \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{B} & \mathbb{O} \end{pmatrix} = \begin{pmatrix} \mathbb{B} & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix}.$$

Hence,

$$\mathbb{A}^4 = \begin{pmatrix} \mathbb{B} & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{B} & \mathbb{O} \\ \mathbb{O} & \mathbb{B} \end{pmatrix} = \begin{pmatrix} \mathbb{B}^2 & \mathbb{O} \\ \mathbb{O} & \mathbb{B}^2 \end{pmatrix}.$$

Problem 2.1.1

a)

$$\det(\mathbb{M}_{21}) = \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = -8 \quad \det(\mathbb{M}_{22}) = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = -2 \quad \det(\mathbb{M}_{23}) = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5.$$

b) $A_{21} = 8, A_{22} = -2, A_{23} = -5$.

c) $\det(\mathbb{A}) = 1 \cdot 8 + (-2) \cdot (-2) + 3 \cdot (-5) = -3$.

Problem 2.1.2

a) non-singular, determinant equals 2.

b) singular, determinant equals 0.

c) non-singular, determinant equals 24.

Problem 2.1.3

c) Determinant vanishes as the second and third rows are equal.

g) We will expand with respect to the second row:

$$\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} = 2 \begin{vmatrix} 2 & 0 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 2 \cdot 6 - 4 = 8.$$

Problem 2.1.2 We have

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 12 = (\lambda - 6)(\lambda + 1).$$

Hence, the determinant vanishes for $\lambda = 6$ and $\lambda = -1$.