1. Find an orthonormal basis for the row space of $\mathbb{A}$ containing the first row.

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 1 \\ -2 & -2 & 1 \\ 2 & 0 & -1 \end{pmatrix}.$$

SOLUTION: It is easy to see that $\det(\mathbb{A}) \neq 0$. Hence, the row space is $\mathbb{R}^3$. As the first row is simply $\mathbf{k}$ so that we can take $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as an orthonormal basis containing the first row.

2. Consider the following linear homogeneous equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - x_2 + 2x_3 + 2x_4 &= 0 \\ x_2 + x_4 &= 0. \end{aligned}$$

a) Find all the solutions of the system and write it as a k-flat W in $\mathbb{B}^4$:

SOLUTION:

We reduce

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

Hence, the space of solutions is a line (1-flat) and it is given by $\text{span}\{(3, -1, -3, 1)\}$.

b) The space $W \subset \mathbb{R}^4$ found in (a) is a linear subspace in $\mathbb{R}^4$. Find an orthonormal basis for W.

SOLUTION:

Clearly, as W is 1-dimensional its orthonormal basis is simply $\frac{1}{\sqrt{20}}(3, -1, -3, 1)$.

c) Find $W^{\perp}$ and write an orthonormal basis for it.

SOLUTION: $W^{\perp}$ is simply the the row space of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

which is also the row space of the reduced matrix

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

We apply the G-S process to get

$$\mathbf{u}_1 = \frac{1}{\sqrt{10}}(1,0,0,-3),$$

$$\mathbf{x}_2 - < \mathbf{x}_2, \mathbf{u}_1 > \mathbf{u}_1 = (0,1,0,1) + \frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}}(1,0,0,-3) = (0,1,0,1) + \frac{3}{10}(1,0,0,-3) =$$

$$= (3/10, 1, 0, 1/10) = \frac{1}{10}(3,10,0,1).$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{110}}(3,10,0,1),$$

$$\mathbf{x}_3 - < \mathbf{x}_3, \mathbf{u}_1 > \mathbf{u}_1 - < \mathbf{x}_3, \mathbf{u}_2 > \mathbf{u}_2 =$$

$$= (0,0,1,3) + \frac{9}{\sqrt{10}}\frac{1}{\sqrt{10}}(1,0,0,-3) - \frac{3}{\sqrt{110}}\frac{1}{\sqrt{110}}(3,10,0,1) =$$

$$= (0,0,1,3) + \frac{9}{10}(1,0,0,-3) - \frac{3}{110}(3,10,0,1) =$$

$$= \frac{1}{110}[(0,0,110,330) + (99,0,0,-297) - (9,30,0,3)] = \frac{1}{11}(9,-3,11,3).$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{220}}(9,-3,11,3).$$

3. Given the following symmetric matrix

$$\mathbb{A} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

find

a) the characteristic polynomial of $\mathbb{A}$:

SOLUTION: Note that

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix} = \det \begin{pmatrix} 2-\lambda & \lambda-2 & 0 \\ -1 & 1-\lambda & -1 \\ 0 & \lambda-2 & 2-\lambda \end{pmatrix} =$$

$$= (2-\lambda)\left(\det \begin{pmatrix} 1-\lambda & -1 \\ \lambda-2 & 2-\lambda \end{pmatrix} + \det \begin{pmatrix} -1 & -1 \\ 0 & 2-\lambda \end{pmatrix} \right) =$$

$$= (2-\lambda)\Big((2-\lambda)[(1-\lambda)-1] - (2-\lambda) \Big) =$$

$$= (2-\lambda)^2(-1-\lambda) = -(2-\lambda)^2(1+\lambda).$$

b) all eigenvalues of $\mathbb{A}$:

SOLUTION: Hence, $\mathbb{A}$ has two real eigenvalues $\{-1, 2\}$.

c) all eigenvectors of $\mathbb{A}$:

SOLUTION: Consider $\lambda = -1$ first. We have to find the null space of the following matrix:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 0 & 3 & -3 \\ -1 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, the eigenspace

$$E_{\lambda=-1} = \text{span}\{(1,1,1)\}.$$

Next, consider $\lambda = 2$. Here we have to find the null space of the following matrix:

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The null space is give by all vectors $(-s-t,s,t) = s(-1,1,0)+t(-1,0,1)$. Hence, the eigenspace

$$E_{\lambda=2} = \text{span}\{(-1,1,0),(-1,0,1)\}.$$

4. Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}, \quad T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Find the associated matrix $\mathbb{A}$ of T. Compute $T\begin{pmatrix}1\\2\end{pmatrix}$.

SOLUTION: We have

$$T\begin{pmatrix}0\\1\end{pmatrix} = T\left(\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}\right) = T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}0\\-2\\1\end{pmatrix}.$$

Hence, the associated matrix of T is

$$\mathbb{A}_T = \begin{pmatrix}1 & 0\\2 & -2\\-1 & 1\end{pmatrix}.$$

In particular,

$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1 & 0\\2 & -2\\-1 & 1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}.$$

We can also get this as

$$T\begin{pmatrix}1\\2\end{pmatrix} = T\begin{pmatrix}1\\0\end{pmatrix} + 2T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + 2\begin{pmatrix}0\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}.$$