MATHEMATICS 314

NAME: ____________________

SSN: ____________________

1.____________________

2.____________________

3.____________________

4.____________________

TOTAL____________________

1. (30 pts.) Solve the following linear system

$$\begin{array}{rcrcrcrl} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = 4 \\ 2x_1 & - & 3x_2 & + & 2x_3 & - & 3x_4 & = -1 \\ 3x_1 & - & 5x_2 & + & 3x_3 & - & 4x_4 & = 3 \\ -x_1 & + & x_2 & - & x_3 & + & 2x_4 & = 5. \end{array}$$

2. (30 pts.) Let

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 3 & -5 & 3 \end{pmatrix}.$$

(a) Write down a basis for the null space of $\mathbf{A}$.

(b) What are the nullity and the rank of $\mathbf{A}$?

3. (30 pts.) Let $\mathbf{A} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ be a square matrix and

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 5 & -3 \\ -3 & 2 & -4 \end{pmatrix}.$$

Find $\mathbf{A}^{-1}$.

4. (30 pts.) Consider the following three vectors in $\mathbb{R}^4$

$$\vec{v}_1 = (1, 2, 1, 1), \quad \vec{v}_2 = (2, 1, 0, 2), \quad \vec{v}_3 = (-1, 4, 3, 3).$$

Determine if the set of these three vectors is linearly independent.

5. (**20 bonus points**) Let

$$\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix}.$$

Express **A** as a product of elementary matrices.

SOLUTIONS

PROBLEM 1.

The partition matrix for this system is

$$[\mathbf{A}|\vec{b}] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 2 & -3 & 2 & -3 & -1 \\ 3 & -5 & 3 & -4 & 3 \\ -1 & 1 & -1 & 2 & 5 \end{array}\right] \simeq \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & -1 & 0 & 1 & 9 \end{array}\right] \simeq \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 4 \\ 0 & 1 & 0 & -1 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \simeq$$

$$\simeq \left[\begin{array}{cccc|c} \mathbf{1} & 0 & 1 & -3 & -14 \\ 0 & \mathbf{1} & 0 & -1 & -9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is clearly consistent. There are pivots in first two columns; columns number 3 and 4 are pivot-free. Hence, there is a two parameter family of solutions. We can introduce our free parameters as follows

$$x_4 = t, \qquad x_3 = s.$$

Then the first two equations of the reduced system give

$$x_1 = -x_3 + 3x_4 - 14 = -14 + 3t - s, \qquad x_2 = x_4 - 9 = -9 + t.$$

We can write

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -14 + 3t - s \\ -9 + t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -14 \\ -9 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

PROBLEM 2.

(a) The null space of **A** is the space of solutions of the corresponding homogeneous linear system. We get

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 3 & -5 & 3 \end{pmatrix} \simeq \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \mathbf{1} & 0 & 1 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are pivots in first two columns; column number 3 is pivot-free. Hence, there is a one-parameter family of solutions. We can introduce our free parameter as follows

$$x_3 = t.$$

Then the second equation says that $X_2 = 0$ and the first one that $x_1 = -t$. Hence, we have the general solution in the form

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

This the basis for the null space is $\{(-1, 0, 1)\}$.

(b) As nullity of **A** equals 1, rank of **A** must also equal to 2 so that the sum of rank and nullity adds up to 3.

PROBLEM 3.

We find the determinant of $\mathbf{A}$ first:

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & -3 \\ -3 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -4 & 3 \\ 0 & 11 & -10 \end{vmatrix} = 7.$$

Then we compute the adjoint

$$\text{adj}(\mathbf{A}) = \begin{pmatrix} -14 & 21 & 21 \\ 8 & -10 & -11 \\ 1 & -3 & -4 \end{pmatrix}^T.$$

Hence,

$$\mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} -14 & 8 & 1 \\ 21 & -10 & -3 \\ 21 & -11 & -4 \end{pmatrix}.$$

PROBLEM 4.

Consider the following 3×4 matrix

$$\begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 4 & 3 & 3 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & -2 & 0 \\ 0 & 6 & 4 & 4 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, the all three rows have pivots so that the vectors must be linearly independent.

PROBLEM 5.

First we use the Gauss-Jordan reduction method to reduce $\mathbf{A}$ to the identity matrix:

$$\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1/5 \\ 4 & 2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1/5 \\ 0 & 6/5 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1/5 \\ 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We list, in order, all **four** elementary row operations performed in the reduction, the corresponding elementary matrices, and their inverses:

$$\frac{1}{5}(R1): \quad \mathbf{E}_1 = \begin{pmatrix} 1/5 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{E}_1^{-1} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(R2) - 4(R1): \quad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, \qquad \mathbf{E}_2^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},$$

$$\frac{5}{6}(R2): \quad \mathbf{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 5/6 \end{pmatrix}, \qquad \mathbf{E}_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 6/5 \end{pmatrix},$$

$$(R1) - \frac{1}{5}(R2): \quad \mathbf{E}_4 = \begin{pmatrix} 1 & -1/5 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{E}_4^{-1} = \begin{pmatrix} 1 & 1/5 \\ 0 & 1 \end{pmatrix}.$$

Now, we can write (check all matrix multiplications):

$$\mathbf{A} = \mathbf{E}_1^{-1} \cdot \mathbf{E}_2^{-1} \cdot \mathbf{E}_3^{-1} \cdot \mathbf{E}_4^{-1} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 6/5 \end{pmatrix}\begin{pmatrix} 1 & 1/5 \\ 0 & 1 \end{pmatrix}.$$