MATHEMATICS – 314 Section 002

NAME: ______________________________

SSN: ______________________________

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1. ______________________

2. ______________________

3. ______________________

4. ______________________

TOTAL ______________________

1. (30 pts.) Solve the following linear system

$$
\begin{aligned}
x_1 - 2x_2 + x_3 - x_4 + x_5 &= 4 \\
2x_1 - 3x_2 + 2x_3 - 3x_4 + 2x_5 &= 7 \\
3x_1 - 5x_2 + 3x_3 - 4x_4 + 3x_5 &= 11.
\end{aligned}
$$

2. (30 pts.) Let

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -6 & 4 \\ 3 & -5 & 3 \end{pmatrix}.$$

(a) Write down a basis for the null space of $\mathbf{A}$.

(b) What are the nullity and the rank of $\mathbf{A}$?

3. (30 pts.)

a) Compute $\mathbb{A}^{-1}$ by Gauss-Jordan reduction on $(\mathbb{A}|\mathbb{I}_3)$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Write both $\mathbb{A}$ and $\mathbb{A}^{-1}$ as a product of two elementary row matrices.

4. (30 pts.) Consider the following three vectors in $\mathbb{R}^3$

$$\vec{v}_1 = (1, 2, 1), \quad \vec{v}_2 = (2, 1, 0), \quad \vec{v}_3 = (-1, 4, 3).$$

Determine if the set of these three vectors is linearly independent.

5. (**20 bonus points; no partial credit on the bonus problem)**

a) Write down a non-zero 2×2 matrix $\mathbb{A}$ such that $\mathbb{A}^2 = \mathbb{O}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

b) Write down a non-zero 3×3 matrix $\mathbb{A}$ such that $\mathbb{A}^2 \neq \mathbb{O}_3$ but $\mathbb{A}^3 = \mathbb{O}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

SOLUTIONS

PROBLEM 1.

The partition matrix for this system is

$$[\mathbf{A}|\vec{b}] = \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 4 \\ 2 & -3 & 2 & -3 & 2 & 7 \\ 3 & -5 & 3 & -4 & 3 & 11 \end{array}\right] \simeq \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 4 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \end{array}\right] \simeq$$

$$\simeq \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 4 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \simeq \left[\begin{array}{ccccc|c} \mathbf{1} & 0 & 1 & -3 & 1 & 2 \\ 0 & \mathbf{1} & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is clearly consistent. There are pivots in first two columns; columns number 3 through 5 are pivot-free. Hence, there is a three-parameter family of solutions. We can introduce our free parameters as follows

$$x_5 = r, \qquad x_4 = t, \qquad x_3 = s.$$

Then the first two equations of the reduced system give

$$x_1 = -x_3 + 3x_4 - x_5 + 2 = 2 - s + 3t - r, \qquad x_2 = x_4 - 1 = -1 + t.$$

We can write

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 + 3t - s - r \\ -1 + t \\ s \\ t \\ r \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

PROBLEM 2.

(a) The null space of $\mathbf{A}$ is the space of solutions of the corresponding homogeneous linear system. We get

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 3 & -5 & 3 \end{pmatrix} \simeq \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \mathbf{1} & 0 & 1 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are pivots in first two columns; column number 3 is pivot-free. Hence, there is a one-parameter family of solutions. We can introduce our free parameter as follows

$$x_3 = t.$$

Then the second equation says that $x_2 = 0$ and the first one that $x_1 = -t$. Hence, we have the general solution in the form

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

This the basis for the null space is $\{(-1, 0, 1)\}$, the null space $N(\mathbf{A}) = \text{span}(\{-1, 0, 1)\})$.

(b) As nullity of $\mathbf{A}$ equals 1, rank of $\mathbf{A}$ must also equal to 2 so that the sum of rank and nullity adds up to 3.

PROBLEM 3.

We reduce the matrix $\mathbb{A}$ in two steps:

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with the same two operations applied to the identity

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{A}^{-1}$$

The two elementary matrices responsible for these two operations (in order) are

$$\mathbb{E}_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{E}_1^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbb{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{E}_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Now

$$\mathbb{A}^{-1} = \mathbb{E}_2 \cdot \mathbb{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbb{A} = \mathbb{E}_1^{-1} \cdot \mathbb{E}_2^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

PROBLEM 4.

Consider the following 3×3 matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, where all three vectors are considered as the column vectors. We get

$$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ ] = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & -2 & 4 \end{pmatrix} \simeq$$

$$\simeq \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \simeq \begin{pmatrix} \mathbf{1} & 2 & -1 \\ 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, the first two columns have pivots, the last one is pivot-free. Hence, the third vector is a linear combinations of the first two, and the set is **not** linearly independent. Indeed, we can write

$$\vec{v}_3 = (-1, 4, 3) = 3\vec{v}_1 - 2\vec{v}_2 = 3(1, 2, 1) - 2(2, 1, 0).$$

PROBLEM 5.

a) Take

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

b) Take

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\mathbb{A}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathbb{A}^3 = \mathbb{A}^2 \cdot \mathbb{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{O}_3.$$