MATH180 – HOMEWORK SOLUTIONS

HOMEWORK #5

Section 4.4: 1-8 (all), 9, 11, 19, 29, 35, 39, 41

Section 4.5: 1, 3, 9, 15, 28

Section 5.1: 1-25 (odd only)

EXERCISES 4.4, page 311

1. f has no absolute extrema.

2. f has an absolute minimum at $(-2,-\frac{1}{2})$ and an absolute maximum at $(2, \frac{1}{2})$.

3. f has an absolute minimum at (0,0).

4. f has an absolute minimum at (0,0). f has no absolute maximum.

5. f has an absolute minimum at (0,-2) and an absolute maximum at (1,3).

6. f has no absolute extrema.

7. f has an absolute minimum at $(\frac{3}{2}, -\frac{27}{16})$ and an absolute maximum at (-1,3).

8. f has an absolute minimum at (0,-3) and an absolute maximum at (3,1).

9. The graph of $f(x) = 2x^2 + 3x - 4$ is a parabola that opens upward. Therefore, the vertex of the parabola is the absolute minimum of f. To find the vertex, we solve the equation $f'(x) = 4x + 3 = 0$ giving $x = -3/4$. We conclude that the absolute minimum value is $f(-\frac{3}{4}) = -\frac{41}{8}$.

10. The graph of $g(x) = -x^2 + 4x + 3$ is a parabola that opens downward. Therefore, the vertex of the parabola is the absolute maximum of f. To find the vertex, we solve the equation $g'(x) = -2x + 4 = 0$ giving $x = 2$. We conclude that the absolute maximum value is $f(2) = 7$.

11. Since $\lim_{x \to -\infty} x^{1/3} = -\infty$ and $\lim_{x \to \infty} x^{1/3} = \infty$, we see that h is unbounded. Therefore it has no absolute extrema.

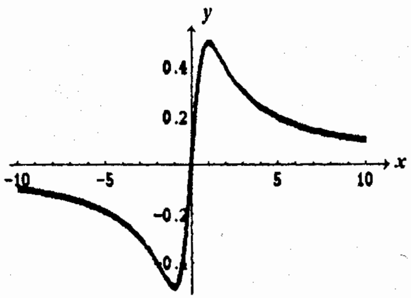
12. From the graph of f (see Fig. 4.19, page 276, in the text), we see that (0,0) affords absolute minimum of f. There is no absolute maximum since $\lim_{x \to \infty} x^{2/3} = \infty$.

13. $f(x) = \dfrac{1}{1+x^2}$.

Using the techniques of graphing, we sketch the graph of f (see Fig. 4.40, page 297, in the text). The absolute maximum of f is $f(0) = 1$. Alternatively, observe that $1 + x^2 \geq 1$ for all real values of x. Therefore, $f(x) \leq 1$ for all x, and we see that the absolute maximum is attained when $x = 0$.

14. $f(x) = \dfrac{x}{1+x^2}$. Since f is defined for all x in $(-\infty, \infty)$, we use the graphical method.

Using the techniques of graphing, we sketch the graph of f as follows.



From the graph we see that f has an absolute maximum at $(1,\frac{1}{2})$ and an absolute minimum at $(-1,-\frac{1}{2})$.

15. $f(x) = x^2 - 2x - 3$ and $f'(x) = 2x - 2 = 0$, so $x = 1$ is a critical point. From the table,

x	-2	1	3
$f(x)$	5	-4	0

we conclude that the absolute maximum value is $f(-2) = 5$ and the absolute minimum value is $f(1) = -4$.

16. $g(x) = x^2 - 2x - 3$; $g'(x) = 2x - 2 = 0$ so $x = 1$ is a critical point.

x	0	1	4
$f(x)$	-3	-4	5

So g has an absolute minimum at (1,-4) and an absolute maximum at (4, 5).

17. $f(x) = -x^2 + 4x + 6$; The function f is continuous and defined on the closed interval [0,5]. $f'(x) = -2x + 4$ and $x = 2$ is a critical point. From the table

x	0	2	5
$f(x)$	6	10	1

we conclude that $f(2) = 10$ is the absolute maximum value and $f(5) = 1$ is the absolute minimum value.

18. $f(x) = -x^2 + 4x + 6$; The function f is continuous and defined on the closed interval [3,6]. $f'(x) = -2x + 4$ and $x = 2$ is a critical point. But this point lies outside the given interval. From the table

x	3	6
$f(x)$	9	-6

we conclude that $f(3) = 9$ is the absolute maximum value and $f(6) = -6$ is the absolute minimum value.

19. The function $f(x) = x^3 + 3x^2 - 1$ is continuous and defined on the closed interval $[-3,2]$ and differentiable in $(-3,2)$. The critical points of f are found by solving

$$f'(x) = 3x^2 + 6x = 3x(x+2)$$

giving $x = -2$ and $x = 0$. Next, we compute the values of f given in the following table.

x	-3	-2	0	2
$f(x)$	-1	3	-1	19

From the table, we see that the absolute maximum value of f is $f(2) = 19$ and the absolute minimum value is $f(-3) = -1$ and $f(0) = -1$.

20. The function $g(x) = x^3 + 3x^2 - 1$ is continuous and defined on the closed interval $[-3,1]$ and differentiable in $(-3,1)$. The critical points of g are found by solving

$$g'(x) = 3x^2 + 6x = 3x(x+2) = 0$$

giving $x = -2$ and $x = 0$. We next compute the values given in the following table.

x	-3	-2	0	1
$g(x)$	-1	3	-1	3

From the table we see that the absolute maximum value of g is given by $g(1) = 3$ and $g(-2) = 3$ and the absolute minimum value of g is given by $g(-3) = -1$ and $g(0) = -1$.

21. The function $g(x) = 3x^4 + 4x^3$ is continuous and differentiable on the closed interval $[-2,1]$ and differentiable in $(-2,1)$. The critical points of g are found by solving

$$g'(x) = 12x^3 + 12x^2 = 12x^2(x + 1)$$

giving $x = 0$ and $x = -1$. We next compute the values of g shown in the following table.

x	-2	-1	0	1
$g(x)$	16	-1	0	7

From the table we see that $g(-2) = 16$ is the absolute maximum value of g and $g(-1) = -1$ is the absolute minimum value of g.

22. $f(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$ is continuous on the closed interval [-2,3] and differentiable in the open interval (-2,3). The critical points of f are found by solving

$$f'(x) = 2x^3 - 2x^2 - 4x = 2x(x^2 - x - 2) = 2x(x - 2)(x + 1) = 0$$

giving $x = -1$, 0, and 2 as critical points. We compute

x	-2	-1	0	2	3
$f(x)$	25/3	13/6	3	-7/3	15/2

From the table we see that the absolute maximum value of f is $f(-2) = 25/3$, and the absolute minimum value of f is $f(2) = -7/3$.

23. $f(x) = \dfrac{x+1}{x-1}$ on [2,4]. Next, we compute,

$$f'(x) = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} = -\frac{2}{(x-1)^2}.$$

Since there are no critical points, ($x = 1$ is not in the domain of f), we need only test the endpoints. From the table

x	2	4
$g(x)$	3	5/3

we conclude that $f(4) = 5/3$ is the absolute minimum value and $f(2) = 3$ is the absolute maximum value.

24. $g(t) = \dfrac{t}{t-1}$; $g'(t) = \dfrac{(t-1)-t}{(t-1)^2} = -\dfrac{1}{(t-1)^2}$. Since there are no critical points, ($t = 1$ is not in the domain of g), we need only test the endpoints. From the table

t	2	4
$g(t)$	2	4/3

we conclude that $g(2) = 2$ is the absolute maximum value and $g(4) = 4/3$ is the absolute minimum value.

25. $f(x) = 4x + \dfrac{1}{x}$ is continuous on [1,3] and differentiable in (1,3). To find the critical points of f, we solve $f'(x) = 4 - \frac{1}{x^2} = 0$, obtaining $x = \pm\frac{1}{2}$. Since these critical points lie outside the interval [1,3], they are not candidates for the absolute extrema of f. Evaluating f at the endpoints of the interval [1,3], we find that the absolute maximum value of f is $f(3) = \frac{37}{3}$, and the absolute minimum value of f is $f(1) = 5$.

26. $f(x) = 9x - \dfrac{1}{x}$ is continuous on [1,3] and differentiable in (1,3). To find the critical points of f, we solve $f'(x) = 9 + \frac{1}{x^2} = 0$, obtaining $x^2 = -1/9$ which has no solution. Evaluating f at the endpoints of the interval [1,3], we find that the absolute minimum value is $f(1) = 8$ and the absolute maximum value is $f(3) = \frac{80}{3}$.

27. $f(x) = \frac{1}{2}x^2 - 2\sqrt{x} = \frac{1}{2}x^2 - 2x^{1/2}$. To find the critical points of f, we solve

$$f'(x) = x - x^{-1/2} = 0, \quad \text{or} \qquad x^{3/2} - 1 = 0,$$

obtaining $x = 1$. From the table

x	0	1	3
$f(x)$	0	$-\frac{3}{2}$	$\frac{9}{2}-2\sqrt{3}\approx 1.04$

we conclude that $f(3)\approx 1.04$ is the absolute maximum value and $f(1) = -3/2$ is the absolute minimum value.

28. The function $g(x)=\frac{1}{8}x^2-4\sqrt{x}=\frac{1}{8}x^2-4x^{1/2}$ is continuous on the closed interval [0,9] and differentiable in (0,9). To find the critical points of g, we first compute

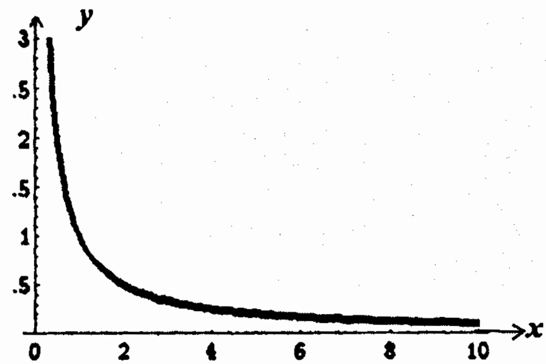
$$g'(x)=\tfrac{1}{4}x-2x^{-1/2}=\tfrac{1}{4}x^{-1/2}(x^{3/2}-8).$$

Setting $g'(x) = 0$, we have $x^{3/2} = 8$, or $x = 4$. Next, we compute the values of g shown in the following table.

x	0	4	9
$f(x)$	0	-6	-15/8

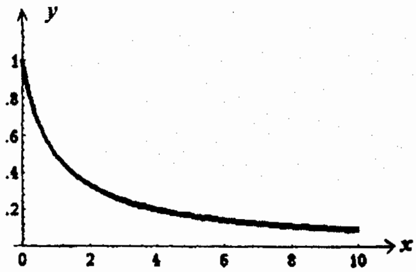
We conclude that $g(4) = -6$ is the absolute minimum value and $g(0) = 0$ is the absolute maximum value of g.

29. The graph of $f(x) = 1/x$ over the interval $(0,\infty)$ follows.



From the graph of f, we conclude that f has no absolute extrema.

30. The graph of $g(x) = \dfrac{1}{x+1}$ on $(0,\infty)$ follows.



From the graph of g, we conclude that g has no absolute extrema.

31. $f(x) = 3x^{2/3} - 2x$. The function f is continuous on $[0,3]$ and differentiable on $(0,3)$. To find the critical points of f, we solve

$$f'(x) = 2x^{-1/3} - 2 = 0$$

obtaining $x = 1$ as the critical point. From the table,

x	0	1	3
$f(x)$	0	1	$3^{5/3} - 6 \approx 0.24$

we conclude that the absolute maximum value is $f(1) = 1$ and the absolute minimum value is $f(0) = 0$.

32. $g(x) = x^2 + 2x^{2/3}$. $g'(x) = 2x + \frac{4}{3}x^{-1/3} = \frac{2}{3}x^{-1/3}(3x^{4/3} + 2)$ is never zero, but $g'(x)$ is not defined at $x = 0$, which is a critical point of g. From the following table,

x	-2	0	2
$f(x)$	$4 + 2^{5/3}$	0	$4 + 2^{5/3}$

we conclude that $g(-2) = 4 + 2^{5/3}$ and $g(2) = 4 + 2^{5/3}$ give the absolute maximum value and $g(0) = 0$ gives the absolute minimum value.

33. $f(x) = x^{2/3}(x^2 - 4)$. $f'(x) = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2 - 4) = \frac{2}{3}x^{-1/3}[3x^2 + (x^2 - 4)]$

$$= \frac{8(x^2-1)}{3x^{1/3}} = 0.$$

Observe that f' is not defined at $x = 0$. Furthermore, $f'(x) = 0$ at $x \pm 1$. So the critical points of f are -1, 0, 1. From the following table,

x	-1	0	1	2
$f(x)$	-3	0	-3	0

we see that f has an absolute minimum at (-1,-3) and (1,-3) and absolute maxima at (0,0) and (2,0).

34. The function is the same as that of Exercise 33. Using the results from Exercise 33, we have the following table.

x	-1	0	1	3
$f(x)$	-3	0	-3	$5\cdot 3^{2/3}$

We see that f has an absolute minimum at (-1,-3) and (1,-3) and an absolute maximum at $(3,\ 5\cdot 3^{2/3})$.

35. $f(x) = \dfrac{x}{x^2+2}$. To find the critical points of f, we solve

$$f'(x) = \frac{(x^2+2)-x(2x)}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2} = 0$$

obtaining $x = \pm\sqrt{2}$. Since $x = -\sqrt{2}$ lies outside [-1,2], $x = \sqrt{2}$ is the only critical point in the given interval. From the table

x	-1	$\sqrt{2}$	2
$f(x)$	$-\frac{1}{3}$	$\sqrt{2}/4 \approx 0.35$	$\frac{1}{3}$

we conclude that $f(\sqrt{2})) = \sqrt{2}/4 \approx 0.35$ is the absolute maximum value and

$f(-1) = -1/3$ is the absolute minimum value.

36. $f'(x) = \frac{d}{dx}(x^2+2x+5)^{-1} = -(x^2+2x+5)^{-2}(2x+2) = \dfrac{-2(x+1)}{(x^2+2x+5)^2}$.

Setting $f'(x) = 0$ gives $x = -1$ as a critical point.

x	-2	-1	1
$f(x)$	1/5	1/4	1/8

From the table, we see that f has an absolute minimum at $(1, \frac{1}{8})$ and an absolute maximum at $(-1, \frac{1}{4})$.

37. The function $f(x) = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x}{(x^2+1)^{1/2}}$ is continuous and defined on the closed interval [-1,1] and differentiable on (-1,1). To find the critical points of f, we first compute

$$f'(x) = \frac{(x^2+1)^{1/2}(1) - x(\frac{1}{2})(x^2+1)^{-1/2}(2x)}{[(x^2+1)^{1/2}]^2}$$

$$= \frac{(x^2+1)^{-1/2}[x^2+1-x^2]}{x^2+1} = \frac{1}{(x^2+1)^{3/2}}$$

which is never equal to zero. Next, we compute the values of f shown in the following table.

x	-1	1
$f(x)$	$-\sqrt{2}/2$	$\sqrt{2}/2$

We conclude that $f(-1) = -\sqrt{2}/2$ is the absolute minimum value and $f(1) = \sqrt{2}/2$ is the absolute maximum value.

38. $g(x) = x(4-x^2)^{1/2}$ on [0,2].

$g'(x) = (4-x^2)^{1/2} + x(\frac{1}{2})(4-x^2)^{-1/2}(-2x)$

$$= (4-x^2)^{-1/2}(4-x^2-x^2) = -\frac{2(x^2-2)}{\sqrt{4-x^2}}.$$

The critical points of g in $(0,2)$ is $\sqrt{2}$. Next, we compute the values of g shown in the following table.

x	0	$\sqrt{2}$	2
$f(x)$	0	2	0

We conclude that ($g(\sqrt{2})$) = 2 is the absolute maximum value and $g(0) = 0$ and $g(2) = 0$ gives the absolute minimum value.

39. $h(t) = -16t^2 + 64t + 80$. To find the maximum value of h, we solve

$$h'(t) = -32t + 64 = -32(t - 2) = 0$$

giving $t = 2$ as the critical point of h. Furthermore, this value of t gives rise to the absolute maximum value of h since the graph of h is parabola that opens downward. The maximum height is given by

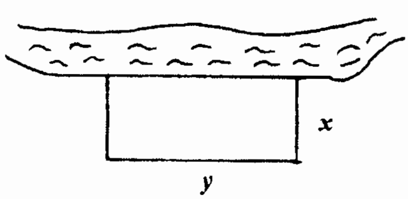
$$h(2) = -16(4) + 64(2) + 80 = 144, \text{ or } 144 \text{ feet.}$$

40. $P(x) = -10x^2 + 1760x - 50{,}000$; $P'(x) = -20x + 1760 = 0$ if $x = 88$ and this is a critical point of P. Now $P(88) = -10(88)^2 + 1760(88) - 50{,}000 = 27{,}440$. The graph of P is a parabola that opens downward. So, the point (88, 27,440) is an absolute maximum of P. So if 88 units are rented out, the maximum monthly profit realizable is \$27,440.

41. $P(x) = -0.04x^2 + 240x - 10{,}000$. We compute $P'(x) = -0.08x + 240$. Setting $P'(x) = 0$ gives $x = 3000$. The graph of P is a parabola that opens downward and so $x = 3000$ gives rise to the absolute maximum of P. Thus, to maximize profits, the company should produce 3000 cameras per month.

EXERCISES 4.5, page 325

1. Refer to the following figure.



We have $2x + y = 3000$ and we want to maximize the function

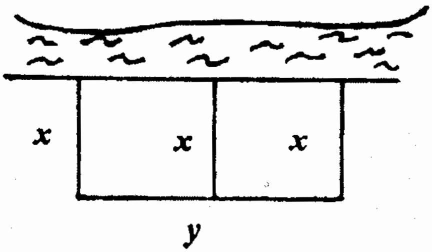
$$A = f(x) = xy = x(3000 - 2x) = 3000x - 2x^2$$

on the interval [0,1500]. The critical point of A is obtained by solving $f'(x) = 3000 - 4x = 0$, giving $x = 750$. From the table of values

x	0	750	1500
$f(x)$	0	1,125,000	0

we conclude that $x = 750$ yields the absolute maximum value of A. Thus, the required dimensions are 750×1500 yards. The maximum area is 1,125,000 sq yd.

2. Refer to the following figure.



Let x denote the length of one of the sides. Then $y = 3000 - 3x = 3(1000 - x)$. The area is $A(x) = xy = 3x(1000 - x) = -3x^2 + 3000x$ for $0 \le x \le 1000$. Next, $A'(x) = -6x + 3000 = -6(x - 500)$. Setting $A'(x) = 0$ gives $x = 500$ as the critical point. From the table of values

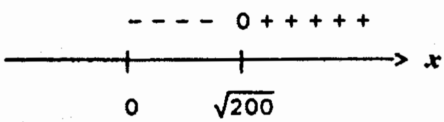
x	0	500	1000
$A(x)$	0	750,000	0

we see that $f(500) = 750{,}000$ is the absolute maximum value. Next, $y = 3(1000 - 500) = 1500$. Therefore, the required dimensions are 500 yd $\times$ 1500 yd. The area is 750,000 sq yd.

3. Let x denote the length of the side made of wood and y the length of the side made of steel. The cost of construction will be $C = 6(2x) + 3y$. But $xy = 800$. So $y = 800/x$ and therefore $C = f(x) = 12x + 3\left(\frac{800}{x}\right) = 12x + \frac{2400}{x}$. To minimize C, we compute

$$f'(x) = 12 - \frac{2400}{x^2} = \frac{12x^2 - 2400}{x^2} = \frac{12(x^2 - 200)}{x^2}.$$

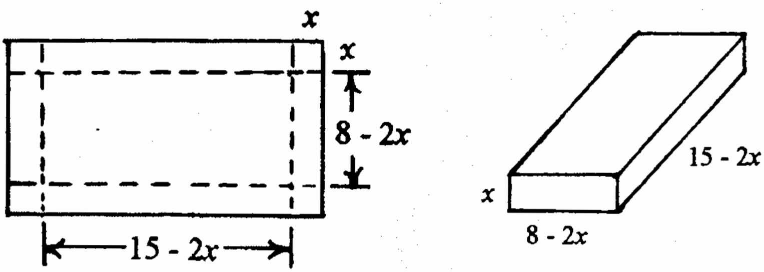
Setting $f'(x) = 0$ gives $x = \pm\sqrt{200}$ as critical points of f. The sign diagram of f'



shows that $x = \pm\sqrt{200}$ gives a relative minimum of f. $f''(x) = \dfrac{4800}{x^3} > 0$ if $x > 0$ and so f is concave upward for $x > 0$. Therefore $x = \sqrt{200} = 10\sqrt{2}$ actually yields the absolute minimum. So the dimensions of the enclosure should be

$$10\sqrt{2}\text{ ft} \times \frac{800}{10\sqrt{2}}\text{ ft, or } 14.1 \text{ ft} \times 56.6 \text{ ft.}$$

4. Refer to the following figures.



The volume of the box is given by

$$V = f(x) = (8 - 2x)(15 - 2x)x = 4x^3 - 46x^2 + 120x.$$

Since the sides of the box must be nonnegative, we must have

$$8 - 2x \geq 0 \text{ or } x \leq 4$$

and $$15 - 2x \geq 0 \text{ or } x \leq \tfrac{15}{2}.$$

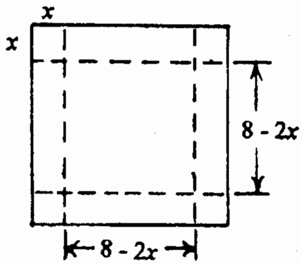
The problem is equivalent to the following: Find the absolute maximum of f on $[0,4]$. Now, $f'(x) = 12x^2 - 92x + 120 = 4(3x^2 - 23x + 30) = 4(3x - 5)(x - 6)$ so that $f'(x) = 0$ implies $x = 5/3$ or $x = 6$. Since $x = 6$ is outside the interval $[0,4]$, only $x = 5/3$ qualifies as the critical point of f. From the table of values

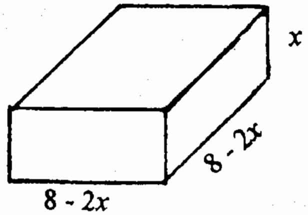
x	0	5/3	4
$f(x)$	0	2450/27	0

we see that $x = 5/3$ gives rise to an absolute maximum of f. Thus, the dimensions which yield the maximum volume are $\frac{14}{3}" \times \frac{35}{3}" \times \frac{5}{3}"$. The maximum volume is $\frac{2450}{27}$, or approximately 90.7 cubic inches.

5. Let the dimensions of each square that is cut out be $x" \times x"$. Refer to the following

diagram.





Then the dimensions of the box will be $(8 - 2x)''$ by $(8 - 2x)''$ by x''. Its volume will be $V = f(x) = x(8 - 2x)^2$. We want to maximize f on [0,4].

$$f'(x) = (8 - 2x)^2 + x(2)(8 - 2x)(-2) \qquad \text{[Using the Product Rule.]}$$

$$= (8 - 2x)[(8 - 2x) - 4x] = (8 - 2x)(8 - 6x) = 0$$

if $x = 4$ or 4/3. The latter is a critical point in (0,4).

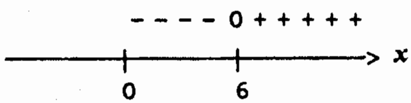
x	0	4/3	4
$f(x)$	0	1024/27	0

We see that $x = 4/3$ yields an absolute maximum for f. So the dimensions of the box should be $\frac{16}{3}'' \times \frac{16}{3}'' \times \frac{4}{3}''$.

6. Let the dimensions of the box be $x'' \times x'' \times y''$. Since its volume is 108 cubic inches, we have $x^2y = 108$. We want to minimize $S = x^2 + 4xy$. But $y = 108/x^2$ and so we want to minimize $S = x^2 + 4x\left(\frac{108}{x^2}\right) = x^2 + \frac{432}{x}$ $(x > 0)$. Now

$$S' = 2x - \frac{432}{x^2} = \frac{2(x^3 - 216)}{x^2}.$$

Setting $S' = 0$ gives $x = 6$ as a critical point of S. The sign diagram

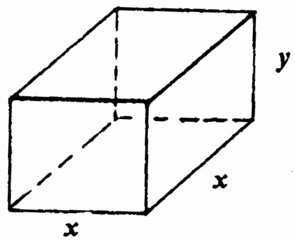


shows that $x = 6$ gives a relative minimum of S. Next,

$$S'' = 2 + \frac{864}{x^3} > 0 \text{ if } x > 0$$

and this says that S is concave upward on $(0,\infty)$. Therefore, $x = 6$ gives an absolute minimum. So the dimensions of the box should be 6" × 6" × 3".

7. Let x denote the length of the sides of the box and y denote its height. Referring to the following figure, we see that the volume of the box is given by $x^2y = 128$. The



amount of material used is given by

$$S = f(x) = 2x^2 + 4xy$$
$$= 2x^2 + 4x\left(\frac{128}{x^2}\right)$$
$$= 2x^2 + \frac{512}{x} \text{ square inches.}$$

We want to minimize f subject to the condition that $x > 0$. Now

$$f'(x) = 4x - \frac{512}{x^2} = \frac{4x^3 - 512}{x^2} = \frac{4(x^3 - 128)}{x^2}.$$

Setting $f'(x) = 0$ yields $x = 5.04$, a critical point of f. Next,

$$f''(x) = 4 + \frac{1024}{x^3} > 0$$

for all $x > 0$. Thus, the graph of f is concave upward and so $x = 5.04$ yields an absolute minimum of f. Thus, the required dimensions are 5.04" × 5.04" × 5.04".

8. From the given figure, we see that $x^2y = 20$ and $y = 20/x^2$, and

$$C = 30x^2 + 10(4xy) + 20x^2 = 50x^2 + 40x\left(\frac{20}{x^2}\right) = 50x^2 + \frac{800}{x}.$$

To find the critical points of C, we solve $C' = 100x - \frac{800}{x^2} = 0$, obtaining ,

$100x^3 = 800$, $x^3 = 8$, or $x = 2$. Next, $C'' = \frac{1600}{x^3} > 0$ for all $x > 0$, we see that $x = 2$ gives the absolute minimum value of C. Since $y = 20/4 = 5$, we see that the dimensions are 2 ft × 2 ft × 5 ft.

9. The length plus the girth of the box is $4x + h = 108$ and $h = 108 - 4x$. Then

$$V = x^2h = x^2(108 - 4x) = 108x^2 - 4x^3$$

and $V' = 216x - 12x^2$. We want to maximize V on the interval $[0,27]$. Setting $V'(x) = 0$ and solving for x, we obtain $x = 18$ and $x = 0$. Evaluating $V(x)$ at $x = 0$, $x = 18$, and $x = 27$, we obtain

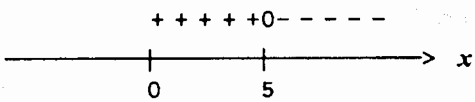
$$V(0) = 0,\ V(18) = 11{,}664, \text{ and } V(27) = 0$$

Thus, the dimensions of the box are 18" × 18" × 36" and its maximum volume is approximately 11,664 cu in.

10. $xy = 50$ and so $y = 50/x$. The area of the printed area is

$$A = (x-1)(y-2) = (x-1)(\tfrac{50}{x} - 2) = (x-1)(\tfrac{50-2x}{x}) = -2x + 52 - \tfrac{50}{x}.$$

$$A' = -2 + \frac{50}{x^2} = \frac{-2(x^2-25)}{x^2} = 0$$ if $x = \pm 5$. From the sign diagram



for A', we see that $x = 5$ yields a maximum. Since

$$A'' = -\frac{100}{x^3} < 0 \quad \text{if } x > 0$$

we see that the graph of A is concave downward on $(0,\infty)$ and so $x = 5$ yields an absolute maximum. The dimensions of the paper should, therefore, be 5" × 10".

11. We take $2\pi r + \ell = 108$. We want to maximize

$$V = \pi r^2 \ell = \pi r^2(-2\pi r + 108) = -2\pi^2 r^3 + 108\pi r^2$$

subject to the condition that $0 \le r \le \frac{54}{\pi}$. Now

$$V'(r) = -6\pi^2 r^2 + 216\pi r = -6\pi r(\pi r - 36).$$

Since $V' = 0$, we find $r = 0$ or $r = 36/\pi$, the critical points of V. From the table

r	0	$36/\pi$	$54/\pi$
V	0	$46{,}656/\pi$	0

we conclude that the maximum volume occurs when $r = 36/\pi \approx 11.5$ inches and $\ell = 108 - 2\pi\left(\frac{36}{\pi}\right) = 36$ inches and its volume is $46{,}656/\pi$ cu in .

12. Let r and h denote the radius and height of the container. Since its capacity is to be

36 cu in, we have $\pi r^2 h = 36$ or $h = 36/\pi r^2$. We want to minimize $S = 2\pi r^2 + 2\pi rh$ or

$$S = f(r) = 2\pi r^2 + 2\pi r\left(\frac{36}{\pi r^2}\right) = 2\pi r^2 + \frac{72}{r},$$

over the interval $(0, \infty)$. Now

$$f'(r) = 4\pi r - \frac{72}{\pi r^2} = 0 \text{ gives } 4\pi r^3 = 72, \text{ or } r = \left(\frac{18}{\pi}\right)^{1/3},$$

as the only critical point of f. Next, observe that $f''(r) = 4\pi + \frac{144}{\pi r} > 0$ for r in

$(0,\infty)$. So f is concave upward on $(0,\infty)$ and $r = \left(\frac{18}{\pi}\right)^{1/3}$ gives rise to the absolute

minimum of f. We find $h = \dfrac{36}{\pi\left(\frac{18}{\pi}\right)^{2/3}} = \dfrac{2 \cdot 18}{\pi^{1/3} 18^{2/3}} = 2\left(\dfrac{18}{\pi}\right)^{1/3}$ or twice the radius.

13. Let y denote the height and x the width of the cabinet. Then $y = (3/2)x$. Since the volume is to be 2.4 cu ft, we have $xyd = 2.4$, where d *is* the depth of the cabinet.

We have $x\left(\dfrac{3}{2}x\right)d = 2.4$ or $d = \dfrac{2.4(2)}{3x^2} = \dfrac{1.6}{x^2}$.

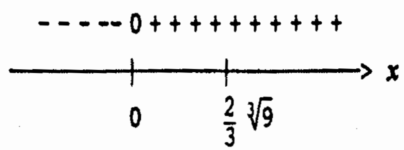
The cost for constructing the cabinet is

$$C = 40(2xd + 2yd) + 20(2xy) = 80\left[\frac{1.6}{x} + \left(\frac{3}{2}x\right)\left(\frac{1.6}{x^2}\right)\right] + 40x\left(\frac{3}{2}x\right)$$

$$= \frac{320}{x} + 60x^2.$$

$$C'(x) = -\frac{320}{x^2} + 120x = \frac{120x^3 - 320}{x^2} = 0 \text{ if } x = \sqrt[3]{\frac{8}{3}} = \frac{2}{\sqrt[3]{3}} = \frac{2}{3}\sqrt[3]{9}$$

Therefore, $x = \frac{2}{3}\sqrt[3]{9}$ is a critical point of C. The sign diagram



shows that $x = \frac{2}{3}\sqrt[3]{9}$ gives a relative minimum. Next, $C''(x) = \dfrac{640}{x^3} + 120 > 0$

for all $x > 0$ tells us that the graph of C is concave upward. So $x = \frac{2}{3}\sqrt[3]{9}$ yields an

absolute minimum. The required dimensions are $\frac{2}{3}\sqrt[3]{9}' \times \sqrt[3]{9}' \times \frac{2}{5}\sqrt[3]{9}'$.

14. Since the perimeter of the window is 28 ft, we have

$$2x + 2y + \pi x = 28 \quad \text{or} \quad y = \tfrac{1}{2}(28 - \pi x - 2x)$$

We want to maximize

$$A = 2xy + \tfrac{1}{2}\pi x^2 = \tfrac{1}{2}\pi x^2 + x(28 - \pi x - 2x) = \tfrac{1}{2}\pi x^2 + 28x - \pi x^2 - 2x^2$$
$$= 28x - \tfrac{\pi}{2}x^2 - 2x^2.$$

Now $A' = 28 - \pi x - 4x = 0$ gives $x = \frac{28}{4+\pi}$ as a critical point of A. Since $A'' = -\pi - 4 < 0$, the point yields a maximum of A. Finally,

$$y = \frac{1}{2}\left[28 - \frac{28\pi}{4+\pi} - \frac{56}{4+\pi}\right] = \frac{1}{2}\left[\frac{112 + 28\pi - 28\pi - 56}{4+\pi}\right] = \frac{28}{4+\pi}.$$

15. We want to maximize the function

$$R(x) = (200 + x)(300 - x) = -x^2 + 100x + 60000.$$
$$R'(x) = -2x + 100 = 0$$

gives $x = 50$ and this is a critical point of R. Since $R''(x) = -2 < 0$, we see that $x = 50$ gives an absolute maximum of R. Therefore, the number of passengers should be 250. The fare will then be \$250/passenger and the revenue will be \$62,500.

16. Let x denote the number of trees beyond 22 per acre. Then the yield is

$$Y = (36 - 2x)(22 + x) = -2x^2 - 8x + 792.$$

Next, $Y' = -4x - 8 = 0$ gives $x = -2$ as the critical point of Y. Now $Y'' = -4 < 0$ and so $x = -2$ gives the absolute maximum of Y. So we should plant 20 trees/acre.

17. Let x denote the number of people beyond 20 who sign up for the cruise. Then the revenue is $R(x) = (20 + x)(600 - 4x) = -4x^2 + 520x + 12{,}000$. We want to maximize R on the closed bounded interval [0, 70].

$$R'(x) = -8x + 520 = 0 \text{ implies } x = 65,$$

a critical point of R. Evaluating R at this critical point and the endpoints, we have

x	0	65	70
$R(x)$	12,000	28,900	28,800

From this table, we see that R is maximized if $x = 65$. Therefore, 85 passengers will result in a maximum revenue of \$28,900. The fare would be \$340/passenger.

18. Let x denote the number of bottles beyond 10,000. Then the profit is

$$P(x) = (10{,}000 + x)(5 - 0.0002x) = -0.0002x^2 + 3x + 50{,}000$$

We want to maximize P on $[0, \infty)$.

$$P'(x) = -0.0004x + 3 = 0$$

implies $x = 7500$. Since $P''(x) = -0.0004 < 0$, the graph of P is concave downward, and we see that $x = 7500$ gives the absolute maximum of P. So Phillip should produce 17,000 bottles of wine giving a profit of

$$P(7500) = -0.0002(7500)^2 + 3(7500) + 50{,}000 \quad \text{or } \$61{,}250.$$

The price will be $5 - 0.0002(7500)$ or \$3.50/bottle.

19. We want to maximize $S = kh^2w$. But $h^2 + w^2 = 24^2$ or $h^2 = 576 - w^2$. So $S = f(w) = kw(576 - w^2) = k(576w - w^3)$. Now, setting

$$f'(w) = k(576 - 3w^2) = 0$$

gives $w = \pm\sqrt{192} \approx \pm 13.86$. Only the positive root is a critical point of interest. Next, we find $f''(w) = -6kw$, and in particular,

$$f''(\sqrt{192}) = -6\sqrt{192}\,k < 0,$$

so that $w = \pm\sqrt{192} \approx \pm 13.86$ gives a relative maximum of f. Since $f''(w) < 0$ for $w > 0$, we see that the graph of f is concave downward on $(0, \infty)$ and so, $w = \sqrt{192}$ gives an absolute maximum of f. We find $h^2 = 576 - 192 = 384$ or $h \approx 19.60$. So the width and height of the log should be approximately 13.86 inches and 19.60 inches, respectively.

20. We want to minimize $S = 3\pi r^2 + 2\pi rh$. But $\pi r^2 h + \frac{2}{3}\pi r^3 = 504\pi$, or $h = \frac{1}{r^2}(504 - \frac{2}{3}r^3)$.

Therefore, $S = f(r) = 3\pi r^2 + 2\pi r \cdot \frac{1}{r^2}(504 - \frac{2}{3}r^3)$

$$= 3\pi r^2 + \frac{1008\pi}{r} - \frac{4\pi r^2}{3} = \frac{5\pi r^2}{3} + \frac{1008\pi}{r}.$$

Now, $f'(r) = \dfrac{10\pi r}{3} - \dfrac{1008\pi}{r^2} = \dfrac{10\pi r^3 - 3024\pi}{3r^2}$.

So $f'(r) = 0$ if $r^3 = \frac{3024\pi}{10\pi}$ or $r = \left(\frac{1512}{5}\right)^{1/3} \approx 6.7$ is a critical point of f. Since $f''(r) = \frac{10\pi}{3} + \frac{2016\pi}{r^3} > 0$ for all r in $(0, \infty)$, we see that $r \approx 6.7$ does yield an absolute minimum of h. Therefore, the radius should be approximately 6.7 ft and the height should be approximately 6.7 ft.

21. We want to minimize $C(x)=1.50(10{,}000-x)+2.50\sqrt{3000^2+x^2}$ subject to $0 \le x \le 10{,}000$. Now

$$C'(x)=-1.50+2.5(\tfrac{1}{2})(9{,}000{,}000+x^2)^{-1/2}(2x)=-1.50+\frac{2.50x}{\sqrt{9{,}000{,}000+x^2}}$$

$$C'(x)=0 \Rightarrow 2.5x=1.50\sqrt{9{,}000{,}000+x^2}$$

$$6.25x^2=2.25(9{,}000{,}000+x^2) \text{ or } 4x^2=20250000,\ x=2250.$$

x	0	2250	10000
$f(x)$	22500	21000	26101

From the table, we see that $x=2250$ gives the absolute minimum.

22. We need to minimize $\hat{V}=\dfrac{16r^2}{(r+\frac{1}{2})^2}-r^2$. Now,

$$\hat{V}'=\frac{(r+\frac{1}{2})^2(32r)-16r^2\cdot 2(r+\frac{1}{2})}{(r+\frac{1}{2})^4}-2r=\frac{32r(r+\frac{1}{2})(r+\frac{1}{2}-r)-2r(r+\frac{1}{2})^4}{(r+\frac{1}{2})^3}$$

$$=\frac{16r(r+\frac{1}{2})-2r(r+\frac{1}{2})^4}{(r+\frac{1}{2})^3}=\frac{2r\left[8-(r+\frac{1}{2})^3\right]}{(r+\frac{1}{2})^3}=0$$

implies $8-(r+\frac{1}{2})^3=0$, $(r+\frac{1}{2})^3=8$, $r+\frac{1}{2}=2$, or $r=\frac{3}{2}$.

Next, $\hat{V}(\frac{3}{2})=\dfrac{16(\frac{3}{2})^2}{2^2}-(\frac{3}{2})^2=(\frac{3}{2})^2(4-1)=3(\frac{9}{4})=\frac{27}{4}$

$$h=\frac{16}{(r+\frac{1}{2})^2}-1=\frac{16}{4}-1=3$$

So the dimensions are $r=\frac{3}{2}$, and $h=3$. From the table

r	0	$\frac{3}{2}$	$\frac{7}{2}$
$\hat{V}$	0	$\frac{27}{4}$	0

We see that V is maximized if $r=\frac{3}{2}$. So the radius is 1.5 ft, and the height is 3'.

23. The time taken for the flight is

$$T=f(x)=\frac{12-x}{6}+\frac{\sqrt{x^2+9}}{4}.$$

$$f'(x) = -\frac{1}{6} + \frac{1}{4}\left(\frac{1}{2}\right)(x^2+9)^{-1/2}(2x) = -\frac{1}{6} + \frac{x}{4\sqrt{x^2+9}}$$

$$= \frac{3x - 2\sqrt{x^2+9}}{12\sqrt{x^2+9}}.$$

Setting $f'(x) = 0$ gives $3x = 2\sqrt{x^2+9}$, $9x^2 = 4(x^2+9)$ or $5x^2 = 36$. Therefore, $x = \pm 6/\sqrt{5} = \pm 6\sqrt{5}/5$. Only the critical point $x = 6\sqrt{5}/5$ is of interest. The nature of the problem suggests $x \approx 2.68$ gives an absolute minimum for T.

24. The fuel cost is $x/400$ dollars per mile, and the labor cost is $8/x$ dollars per mile. Therefore, the total cost is $C(x) = \frac{8}{x} + \frac{x}{400}$; $C'(x) = -\frac{8}{x^2} + \frac{1}{400}$.

Setting $C'(x) = 0$ gives $-\frac{8}{x^2} = -\frac{1}{400}$; $x^2 = 3200$, and $x = 56.57$.

Next, $C''(x) = \frac{16}{x^3} > 0$ for all $x > 0$ so C is concave upward. Therefore, $x = 56.57$ gives the absolute minimum. So the most economical speed is 56.57 mph.

25. Let x denote the number of motorcycle tires in each order. We want to minimize

$$C(x) = 400\left(\frac{40{,}000}{x}\right) + x = \frac{16{,}000{,}000}{x} + x.$$

We compute $C'(x) = -\frac{16{,}000{,}000}{x^2} + 1 = \frac{x^2 - 16{,}000{,}000}{x^2}$.

Setting $C'(x) = 0$ gives $x = 4000$, a critical point of C. Since

$$C''(x) = \frac{32{,}000{,}000}{x^3} > 0 \text{ for all } x > 0,$$

we see that the graph of C is concave upward and so $x = 4000$ gives an absolute minimum of C. So there should be 10 orders per year, each order of 4000 tires.

26. Let x denote the number of bottles in each order. We want to minimize

$$C(x) = 200\left(\frac{2{,}000{,}000}{x}\right) + \frac{x}{2}(0.40) = \frac{400{,}000{,}000}{x} + 0.2x.$$

We compute $C'(x) = -\frac{400{,}000{,}000}{x^2} + 0.2$. Setting $C'(x) = 0$ gives

$$x^2 = \frac{400{,}000{,}000}{0.2} = 2{,}000{,}000{,}000$$, or $x = 44{,}721$, a critical point of C.

$$C'(x) = \frac{800{,}000{,}000}{x^3} > 0$$ for all $x > 0$, and we see that the graph of C is concave upward and so $x = 44{,}721$ gives an absolute minimum of C. Therefore, there should be $2{,}000{,}000/x \approx 45$ orders per year (since we can not have fractions of an order.) Then each order should be for $2{,}000{,}000/4.5 \approx 44{,}445$ bottles.

27. We want to minimize the function $C(x) = \frac{500{,}000{,}000}{x} + 0.2x + 500{,}000$ on the interval (0, 1,000,000). Differentiating C(x), we have $C'(x) = -\frac{500{,}000{,}000}{x^2} + 0.2$. Setting $C'(x) = 0$ and solving the resulting equation, we find $0.2x^2 = 500{,}000{,}000$ and $x = \sqrt{2{,}500{,}000{,}000}$ or $x = 50{,}000$. Next, we find

$$C''(x) = \frac{1{,}000{,}000{,}000}{x^3} > 0$$ for all x and so the graph of C is concave upward on $(0,\infty)$. Thus, $x = 50{,}000$ gives rise to the absolute minimum of C. So, the company should produce 50,000 containers of cookies per production run.

28. The area enclosed by the rectangular region of the racetrack is $A = (\ell)(2r) = 2r\ell$. The length of the racetrack is $2\pi r + 2\ell$, and is equal to 1760. That is,

$$2(\pi r + \ell) = 1760;\ \pi r + \ell = 880,\ \text{ or } \ell = 880 - \pi r.$$

Therefore, we want to maximize $A = f(r) = 2r(880 - \pi r) = 1760r - 2\pi r^2$.
The restricition on r is $0 \le r \le \frac{880}{\pi}$. To maximize A, we compute

$f'(r) = 1760 - 4\pi r$. Setting $f'(r) = 0$ gives $r = \frac{1760}{4\pi} = \frac{440}{\pi} \approx 140$. Since

$f(0) = f\left(\frac{880}{\pi}\right) = 0$, we see that the maximum rectangular area is enclosed if we take $r = \frac{440}{\pi}$ and $\ell = 880 - \pi\left(\frac{440}{\pi}\right) = 440$. So $r = 140$ and $\ell = 440$. The total area enclosed is $2r\ell + \pi r^2 = 2\left(\frac{440}{\pi}\right)(440) + \pi\left(\frac{440}{\pi}\right)^2 = \frac{2(440)^2}{\pi} + \frac{440^2}{\pi} = \frac{580{,}000}{\pi} \approx 184{,}874$ sq ft.

EXERCISES 5.1, page 337

1. a. $4^{-3} \times 4^{5} = 4^{-3+5} = 4^{2} = 16$ b. $3^{-3} \times 3^{6} = 3^{6-3} = 3^{3} = 27.$

2. a. $(2^{-1})^{3} = 2^{-3} = \frac{1}{2^{3}} = \frac{1}{8}.$ b. $(3^{-2})^{3} = 3^{-6} = \frac{1}{3^{6}} = \frac{1}{729}.$

3. a. $9(9)^{-1/2} = \frac{9}{9^{1/2}} = \frac{9}{3} = 3.$ b. $5(5)^{-1/2} = 5^{1/2} = \sqrt{5}.$

4. a. $[(-\frac{1}{2})^{3}]^{-2} = (-\frac{1}{2})^{-6} = \frac{(-1)^{-6}}{2^{-6}} = 2^{6} = 64.$

   b. $[(-\frac{1}{3})^{2}]^{-3} = (-\frac{1}{3})^{-6} = \frac{(-1)^{-6}}{3^{-6}} = 3^{6} = 729.$

5. a. $\frac{(-3)^{4}(-3)^{5}}{(-3)^{8}} = (-3)^{4+5-8} = (-3)^{1} = -3.$ b. $\frac{(2^{-4})(2^{6})}{2^{-1}} = 2^{-4+6+1} = 2^{3} = 8.$

6. a. $3^{1/4} \times 9^{-5/8} = 3^{1/4}(3^{2})^{-5/8} = 3^{1/4} \times 3^{-5/4} = 3^{(1/4)-(5/4)} = 3^{-1} = \frac{1}{3}.$

   b. $2^{3/4} \times 4^{-3/2} = 2^{3/4}(2^{2})^{-3/2} = 2^{3/4} \times 2^{-3} = 2^{(3/4)-3} = 2^{-9/4} = \frac{1}{2^{9/4}}.$

7. a. $\frac{5^{3.3} \cdot 5^{-1.6}}{5^{-0.3}} = \frac{5^{3.3-1.6}}{5^{-0.3}} = 5^{1.7+(0.3)} = 5^{2} = 25.$

   b. $\frac{4^{2.7} \cdot 4^{-1.3}}{4^{-0.4}} = 4^{2.7-1.3+0.4} = 4^{1.8} \approx 12.1257.$

8. a. $\left(\frac{1}{16}\right)^{-1/4}\left(\frac{27}{64}\right)^{-1/3} = (16)^{1/4}\left(\frac{64}{27}\right)^{1/3} = 2\left(\frac{4}{3}\right) = \frac{8}{3}.$

   b. $\left(\frac{8}{27}\right)^{-1/3}\left(\frac{81}{256}\right)^{-1/4} = \left(\frac{27}{8}\right)^{1/3}\left(\frac{256}{81}\right)^{1/4} = \frac{3}{2} \cdot \frac{4}{3} = 2.$

9. a. $(64x^{9})^{1/3} = 64^{1/3}(x^{9/3}) = 4x^{3}.$

   b. $(25x^{3}y^{4})^{1/2} = 25^{1/2}(x^{3/2})(y^{4/2}) = 5x^{3/2}y^{2} = 5xy^{2}\sqrt{x}.$

10. a. $(2x^3)(-4x^{-2}) = -8x^{3-2} = -8x$ b. $(4x^{-2})(-3x^5) = -12x^{-2+5} = -12x^3$

11. a. $\dfrac{6a^{-5}}{3a^{-3}} = 2a^{-5+3} = 2a^{-2} = \dfrac{2}{a^2}.$ b. $\dfrac{4b^{-4}}{12b^{-6}} = \dfrac{1}{3}b^{-4+6} = \dfrac{1}{3}b^2.$

12. a. $y^{-3/2}y^{5/3} = y^{(-3/2)+(5/3)} = y^{1/6}$ b. $x^{-3/5}x^{8/3} = x^{(-3/5)+(8/3)} = x^{31/15}.$

13. a. $(2x^3y^2)^3 = 2^3 \times x^{3(3)} \times y^{2(3)} = 8x^9y^6.$
b. $(4x^2y^2z^3)^2 = 4^2 \times x^{2(2)} \times y^{2(2)} \times z^{3(2)} = 16x^4y^4z^6.$

14. a. $(x^{r/s})^{s/r} = x^{(r/s)(s/r)} = x$ b. $(x^{-b/a})^{-a/b} = x^{(-b/a)(-a/b)} = x.$

15. a. $\dfrac{5^0}{(2^{-3}x^{-3}y^2)^2} = \dfrac{1}{2^{-3(2)}x^{-3(2)}y^{2(2)}} = \dfrac{2^6x^6}{y^4} = \dfrac{64x^6}{y^4}.$
b. $\dfrac{(x+y)(x-y)}{(x-y)^0} = (x+y)(x-y).$

16. a. $\dfrac{(a^m \cdot a^{-n})^{-2}}{(a^{m+n})^2} = \dfrac{a^{-2m} \cdot a^{2n}}{a^{2(m+n)}} = a^{-2m+2n-2(m+n)} = \dfrac{1}{a^{4m}}.$
b. $\left(\dfrac{x^{2n-2}y^{2n}}{x^{5n+1}y^{-n}}\right)^{1/3} = \left(\dfrac{y^{3n}}{x^{3n+3}}\right)^{1/3} = \dfrac{y^n}{x^{n+1}}.$

17. $6^{2x} = 6^4$ if and only if $2x = 4$ or $x = 2$. 18. $5^{-x} = 5^3$ if and only if $-x = 3$ or $x = -3$.

19. $3^{3x-4} = 3^5$ if and only if $3x - 4 = 5$, $3x = 9$, or $x = 3$.

20. $10^{2x-1} = 10^{x+3}$ if and only if $2x - 1 = x + 3$, or $x = 4$.

21. $(2.1)^{x+2} = (2.1)^5$ if and only if $x + 2 = 5$, or $x = 3$.

22. $(-1.3)^{x-2} = (-1.3)^{2x+1}$ if and only if $x - 2 = 2x + 1$, or $x = -3$.

23. $8^x = (\frac{1}{32})^{x-2}$, $(2^3)^x = (32)^{2-x} = (2^5)^{2-x}$, so $2^{3x} = 2^{5(2-x)}$, $3x = 10 - 5x$, $8x = 10$, or $x = 5/4$.

24. $3^{x-x^2} = \frac{1}{9^x} = (3^2)^{-x} = 3^{-2x}$. This is true if and only if $x - x^2 = -2x$, $x^2 - 3x = x(x-3) = 0$, so $x = 0$ or 3.

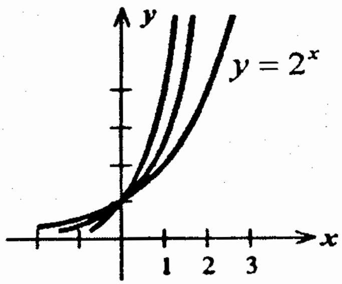
25. Let $y = 3^x$, then the given equation is equivalent to

$$y^2 - 12y + 27 = 0$$
$$(y-9)(y-3) = 0$$

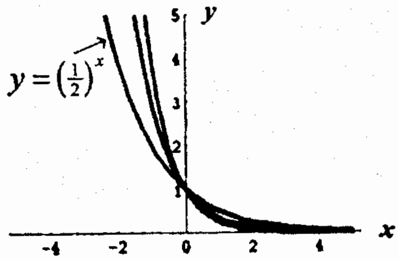
giving $y = 3$ or 9. So $3^x = 3$ or $3^x = 9$, and therefore, $x = 1$ or $x = 2$.

26. $2^{2x} - 4 \cdot 2^x + 4 = 0$, $(2^x)^2 - 4(2^x) + 4 = 0$. Let $y = 2^x$, then we have $y^2 - 4y + 4 = (y-2)^2 = 0$, or $y = 2$. So we have $2^x = 2$ or $x = 1$.

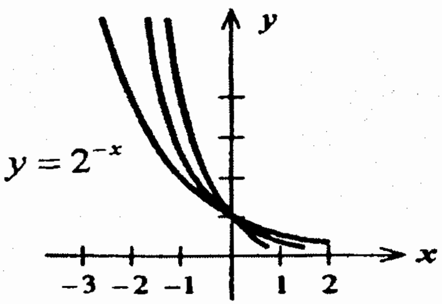
27. $y = 2^x, y = 3^x$, and $y = 4^x$



28. $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, and $y = \left(\frac{1}{4}\right)^x$.



29. $y = 2^{-x}, y = 3^{-x}$, and $y = 4^{-x}$



30. $y = 4^{0.5x}$ and $y = 4^{-0.5x}$