MATH180 – HOMEWORK SOLUTIONS

HOMEWORK #3

Section 3.3: Problems 1-63 (odd), 71, 75, 77

Section 3.4: Problems 1, 5, 7, 13, 15, 19, 21

Section 3.5: Problems 1, 5, 9, 15, 19, 29, 31, 41, 43, 45

Section 3.6: Problems 1, 5, 9, 15, 19, 27, 40, 41, 49, 51, 55

Krzysztof Galicki

Note: In problems 3.3.1-45 we always use the formula

$$y(x) = [f(x)]^n, \qquad y'(x) = n[f(x)]^{n-1}f'(x),$$

which is a special case of the chain rule.

Problem 3.3.1 Let $f(x) = (2x-1)^4 4$. Then $f'(x) = 4(2x-1)^3 \cdot 2 = 8(2x-1)3$.

Problem 3.3.3 Let $f(x) = (x^2+2)^5$. Then $f'(x) = 5(x^2+2)^4 \cdot (2x) = 10x(x^2+2)^4$.

Problem 3.3.5 Let $f(x) = (2x-x^2)^3$. Then $f'(x) = 3(2x-x^2)^2 \cdot (2-2x) = = 6x^2(2-x)^2(1-x)$.

Problem 3.3.7 Let $f(x) = (2x+1)^{-2}$. Then $f'(x) = -2(2x+1)^{-3} \cdot (2) = -4(2x+1)^{-3}$.

Problem 3.3.9 Let $f(x) = (x^2-4)^{3/2}$. Then $f'(x) = \frac{3}{2}(x^2-4)^{1/2} \cdot (2x) = 3x\sqrt{x^2-4}$.

Problem 3.3.11 Let $f(x) = \sqrt{3x-2} = (3x-2)^{1/2}$. Then $f'(x) = \frac{1}{2}(3x-2)^{-1/2} \cdot (3) = \frac{3}{2}(3x-2)^{-1/2}$.

Problem 3.3.13 Let $f(x) = (1-x^2)^{1/3}$. Then $f'(x) = \frac{1}{3}(1-x^2)^{-2/3} \cdot (-2x)$.

Problem 3.3.15 Let $f(x) = (2x+3)^{-3}$. Then $f'(x) = -3(2x+3)^{-4} \cdot (2) = -6(2x+3)^{-4}$.

Problem 3.3.17 Let $f(t) = (2t-3)^{-1/2}$. Then $f'(t) = -\frac{1}{2}(2t-3)^{-3/2} \cdot (2) = (2t-3)^{-3/2}$.

Problem 3.3.19 Let $f(x) = (4x^4 + x)^{-3/2}$. Then $f'(x) = -\frac{3}{2}(4x^4 + x)^{-5/2} \cdot (16x^3 + 1)$.

Problem 3.3.21 Let $f(x) = (3x^2+2x+1)^{-2}$. Then $f'(x) = -2(3x^2+2x+1)^{-3} \cdot (6x+2)$.

Problem 3.3.23 Let $f(x) = (x^2 + 1)^3 - (x^3 + 1)^2$. Then $f'(x) = 3(x^2 + 1)^2 \cdot (2x) - 2(x^3 + 1) \cdot (3x^2)$.

Problem 3.3.25 Let $f(t) = (t^{-1} - t^{-2})^3$. Then $f'(t) = 3(t^{-1} - t^{-2})^2(-t^{-2} + 2t^{-3})$.

Problem 3.3.27 Let $f(x) = \sqrt{x+1} + \sqrt{x-1} = (x+1)^{1/2} + (x-1)^{1/2}$. Then $f'(x) = \frac{1}{2}(x+1)^{-1/2} + \frac{1}{2}(x-1)^{-1/2}$.

Problem 3.3.29 Let $f(x) = 2x^2(3-4x)^4$. (Here we also have to use the product rule):

$$f'(x) = 4x(3-4x)^4 + 2x^2\frac{d}{dx}[(3-4x)^4] = 4x(3-4x)^4 + 2x^2 \cdot (4(3-4x)^3) \cdot (-4) =$$

$$= 4x(3-4x)^4 - 32x^2(3-4x)^3 = 4(3-4x)^3(x(3-4x) - 8x^2) = 4(3-4x)^3(3x - 12x^2) =$$

$$= 12(3-4x)^3(x-4x^2) = 12x(3-4x)^3(1-4x).$$

Problem 3.3.31 Let $f(x) = (x-1)^2(2x+1)^4$. (Here we also have to use the product rule):

$$f'(x) = 2(x-1)(2x+1)^4 + (x-1)^2 \cdot (4(2x+1)^3 \cdot (2)) = 2(x-1)(2x+1)^4 + 8(x-1)^2(2x+1)^3.$$

Problem 3.3.33 Let $f(x) = \left(\frac{x+3}{x-2}\right)^3$. (Here we also have to use the quotient rule):

$$f'(x) = 3\left(\frac{x+3}{x-2}\right)^2 \cdot \frac{d}{dx}\left(\frac{x+3}{x-2}\right) = 3\left(\frac{x+3}{x-2}\right)^2 \frac{(x-2)-(x+3)}{(x-2)^2} = 3\left(\frac{x+3}{x-2}\right)^2 \frac{-5}{(x-2)^2}.$$

Problem 3.3.35 Let $s(t) = \left(\frac{t}{2t+1}\right)^{3/2}$. (Here we also have to use the quotient rule):

$$s'(t) = \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2} \cdot \frac{d}{dt}\left(\frac{t}{2t+1}\right) = \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2} \frac{(2t+1)-2t}{(2t+1)^2} =$$

$$= \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2} \frac{1}{(2t-1)^2}.$$

Problem 3.3.37 Let $g(u) = \left(\frac{u+1}{3u+2}\right)^{1/2}$. (Here we also have to use the quotient rule):

$$g'(u) = \frac{1}{2}\left(\frac{u+1}{3u+2}\right)^{-1/2} \cdot \frac{d}{du}\left(\frac{u+1}{3u+2}\right) = \frac{1}{2}\left(\frac{u+1}{3u+2}\right)^{-1/2} \frac{(3u+2)-3(u+1)}{(2t+1)^2} =$$

$$= \frac{1}{2}\left(\frac{u+1}{3u+2}\right)^{-1/2} \frac{-1}{(3u+2)^2}.$$

Problem 3.3.39 Let $f(x) = \frac{x^2}{(x^2-1)^4}$. (Here we also have to use the quotient rule):

$$f'(x) = \frac{2x(x^2-1)^4 - x^2 \cdot (8x(x^2-1)^3)}{(x^2-1)^8} = \frac{2x(x^2-1) - 8x^3}{(x^2-1)^5} = \frac{-2x-6x^3}{(x^2-1)^5}.$$

Problem 3.3.41 Let $f(x) = \frac{(3x^2+1)^3}{(x^2-1)^4}$. (Here we also have to use the quotient rule):

$$f'(x) = \frac{3(3x^2+1)^2(6x)(x^2-1)^4 - (3x^2+1)^3 \cdot (8x(x^2-1)^3)}{(x^2-1)^8}$$

$$= \frac{2x(3x^2+1)^2(x^2-1)^3[9(x^2-1) - 4(3x^2+1)}{(x^2-1)^8} =$$

$$= \frac{2x(3x^2+1)^2(-13-3x^2)}{(x^2-1)^5}.$$

Problem 3.3.43 Let $f(x) = \frac{(2x+1)^{1/2}}{x^2-1}$. (Here we also have to use the quotient rule):

$$f'(x) = \frac{(2x+1)^{-1/2}(x^2-1) - 2x(2x+1)^{1/2}}{(x^2-1)^2}.$$

Problem 3.3.45 Let $g(t) = \frac{(t+1)^{1/2}}{(t^2+1)^{1/2}}$. (Here we also have to use the quotient rule):

$$g'(t) = \frac{\frac{1}{2}(t+1)^{-1/2}(t^2+1)^{1/2} - t(t+1)^{1/2}(t^2+1)^{-1/2}}{(t^2+1)}.$$

Problem 3.3.47 Let $f(x) = (3x+1)^4(x^2-x+1)^3$. (Here we also have to use the product rule):

$$f'(x) = 12(3x+1)^3(x^2-x+1)^3 + 3(3x+1)^4(x^2-x+1)^2(2x-1).$$

Problem 3.3.49 Let $y = u^{4/3}$ and $u = 3x^2 - 1$. Then

$$\frac{dy}{du} = \frac{4}{3}u^{1/3}, \qquad \frac{du}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{4}{3}u^{1/3}(6x) = 8x(3x^2-1)^{1/3}.$$

Problem 3.3.51 Let $y = u^{-2/3}$ and $u = 2x^3 - x + 1$. Then

$$\frac{dy}{du} = -\frac{2}{3}u^{-5/3}, \qquad \frac{du}{dx} = 6x - 1$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{2}{3}u^{-5/3}(6x-1) = -\frac{2}{3}(2x^3-x+1)^{-5/3}(6x-1).$$

Problem 3.3.53 Let $y = u^{1/2} + u^{-1/2}$ and $u = x^3 - x$. Then

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} - \frac{1}{2}u^{-3/2}, \qquad \frac{du}{dx} = 3x^2 - 1$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}(u^{-1/2} - u^{-3/2})(3x^2-1) = \frac{1}{2}((x^3-x)^{-1/2} - (x^3-x)^{-3/2})(3x^2-1).$$

Problem 3.3.55 Since $F(x) = g(f(x))$ and $f(2) = 3, f'(2) = -3, g(3) = 5, g'(3) = 4$ we get

$$F'(x) = g'(f(x))f'(x),$$

$$F'(2) = g'(f(2)) \cdot f'(2) = g'(3) \cdot (-3) = 4 \cdot (-3) = -12.$$

Problem 3.3.57 Let $F(x) = f(x^2+1)$ and $f'(2) = 3$. We get

$$F'(x) = 2xf'(x^2+1),$$
$$F'(1) = 2f'(1+1) = 6.$$

Problem 3.3.59 When $h(x) = g(f(x))$ then $h'(x) = g'(f(x))\cdot f'(x)$ which **is not** $g'(f'(x))$.

Problem 3.3.61 Let $f(x) = (1-x)(x^2-1)^2$, $P = (2,-9)$. We get

$$f'(x) = -(x^2-1)^2 + (1-x)\cdot(4x)\cdot(x^2-1),$$
$$m = f'(2) = -(4-1)^2 + (1-2)\cdot(8)\cdot(4-1) = -9-24 = -33,$$

and the equation of the line is

$$y + 9 = -33(x-2).$$

Problem 3.3.63 Let $f(x) = x\sqrt{2x^2+7}$, $P = (3,15)$. We get

$$f'(x) = \sqrt{2x^2+7} + x\cdot(4x)\cdot\frac{1}{2\sqrt{2x^2+7}},$$
$$f'(3) = \sqrt{18+7} + 3\cdot(12)\cdot\frac{1}{2\sqrt{15+7}} =$$
$$f'(3) = 5 + \frac{18}{5} = 43/5,$$

and the equation of the line is

$$y - 15 = \frac{43}{5}(x-3).$$

Problem 3.3.71 Since $A(t) = 0.03t^3(t-7)^4 + 60.2$ we get

$$A'(t) = 0.09t^2(t-7)^4 + 0.12t^3(t-7)^3 = t^2(t-7)^3[0.09(t-7)+0.12t] = t^2(t-7)^3(0.21t-0.63).$$

Hence

$$A'(1) = (-6)^3\cdot(-0.42) = 90.72$$
$$A'(3) = 9\cdot(-4)^3\cdot(0) = 0,$$
$$A'(4) = 16\cdot(-27)\cdot(0.21) = -90.72$$

At 8 a.m. the level is increasing. At 10 a.m. it stops increasing and starts decreasing afterwards.

Problem 3.3.75 Since the are of the circular disc is $A(r) = \pi r^2$ we get

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}.$$

As $dr/dt = 2$ ft/sec is constant, when $r = 40$ feet we obtain

$$\frac{dA}{dt}\;|_{r=40} = 80\pi \cdot 2 = 180\pi \text{ ft}^2/\text{sec}.$$

Problem 3.3.77 We have

$$x = f(t) = 6.25t^2 + 19.75t + 74.75, \qquad 0 \le t \le 5,$$

and

$$S = g(x) = -.00075x^2 + 67.5, \qquad\qquad 75 \le x \le 350.$$

Since beginning of 1959 is $t = 0$, the beginning of 1999 corresponds to $t = 4$. We have $x = f(4) = 100 + 79 + 74.75 = 253.75$. Using the chain rule, we have

$$\frac{dS}{dt} = \frac{dS}{dx} \cdot \frac{dx}{dt} = -.0015x \cdot (12.5t + 19.75) = -0.38 \cdot 72.25 = -27.45$$

and

$$S(253.75) = 19.21.$$

EXERCISES 3.4, page 212

1. a. $C(x)$ is always increasing because as x, the number of units produced, increases, the greater the amount of money that must be spent on production.
   b. This occurs at $x = 4$, or a production level of 4000. You can see this by looking at the slopes of the tangent lines for x less than, equal to, and a little larger then $x = 4$.

2. a. If very few units of the commodity are produced then the cost/unit of production will be very large. Next, if x is very large the typical total cost is very large due to overtime, excessive cost of raw material, breakdowns of machines, etc., so that $A(x)$ is very large as well; in fact, for a typical total cost function, $C(x)$ ultimately $A(x)$ grows faster than x, that is, $\lim_{x\to\infty} \dfrac{C(x)}{x} = \infty$.
   b. The average cost per unit is smallest, ($\$y_0$) when the level of production is x_0 units.

3. a. The actual cost incurred in the production of the 1001st record is given by
$$\begin{aligned} C(1001) - C(1000) &= [2000 + 2(1001) - 0.0001(1001)^2] \\ &\quad -[2000 + 2(1000) - 0.0001(1000)^2] \\ &= 3901.7999 - 3900 = 1.7999, \end{aligned}$$
or \$1.80. The actual cost incurred in the production of the 2001st record is given by
$$\begin{aligned} C(2001) - C(2000) &= [2000 + 2(2001) - 0.0001(2001)^2] \\ &\quad -[2000 + 2(2000) - 0.0001(2000)^2] \\ &= 5601.5999 - 5600 = 1.5999, \text{ or } \$1.60. \end{aligned}$$
   b. The marginal cost is $C'(x) = 2 - 0.0002x$. In particular

$$C'(1000) = 2 - 0.0002(1000) = 1.80$$
and $$C'(2000) = 2 - 0.0002(2000) = 1.60.$$

4. a. $C(101) - C(100) = [0.0002(101)^3 - 0.06(101)^2 + 120(101) + 5000]$
$\qquad - [0.0002(100)^3 - 0.06(100)^2 + 120(100) + 5000]$
$\qquad \approx 114$, or approximately \$114.

Similarly, we find $C(201) - C(200) \approx$ \$120.16; $C(301) - C(300) \approx$ \$138.12.

b. We compute $C'(x) = 0.0006x^2 - 0.12x + 120$. So the required quantities are
$C'(100) = 0.0006(100)^2 - 0.12(100) + 120 = 114$, or \$114,
$C'(200) = 0.0006(200)^2 - 0.12(200) + 120 = 120$, or \$120,
and $C'(300) = 0.0006(300)^2 - 0.12(300) + 120 = 138$, or \$138.

5. a. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{100x + 200{,}000}{x} = 100 + \dfrac{200{,}000}{x}$.

b. $\overline{C}'(x) = \dfrac{d}{dx}(100) + \dfrac{d}{dx}(200{,}000x^{-1}) = -200{,}000x^{-2} = -\dfrac{200{,}000}{x^2}$.

c. $\lim\limits_{x\to\infty} \overline{C}(x) = \lim\limits_{x\to\infty}\left[100 + \dfrac{200{,}000}{x}\right] = 100$

and this says that the average cost approaches \$100 per unit if the production level is very high.

6. a. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{5000}{x} + 2$. b. $\overline{C}'(x) = -\dfrac{5000}{x^2}$.

c. Since the marginal average cost function is negative for $x > 0$, the rate of change of the average cost function is negative for all $x > 0$.

7. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{2000 + 2x - 0.0001x^2}{x} = \dfrac{2000}{x} + 2 - 0.0001x$.

$\overline{C}'(x) = -\dfrac{2000}{x^2} + 0 - 0.0001 = -\dfrac{2000}{x^2} - 0.0001$.

8. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{0.0002x^3 - 0.06x^2 + 120x + 5000}{x} = 0.0002x^2 - 0.06x + 120 + \dfrac{5000}{x}$.

$\overline{C}'(x) = 0.0004x - 0.06 - \dfrac{5000}{x^2}$.

9. a. $R'(x) = \dfrac{d}{dx}(8000x - 100x^2) = 8000 - 200x.$

b. $R'(39) = 8000 - 200(39) = 200.$ $R'(40) = 8000 - 200(40) = 0$
$R'(41) = 8000 - 200(41) = -200$

This suggests the total revenue is maximized if the price charged/ passenger is \$40.

10. a. $R(x) = px = x(-0.04x + 800) = -0.04x^2 + 800x$

b. $R'(x) = -0.08x + 800$ c. $R'(5000) = -0.08(5000) + 800 = 400.$

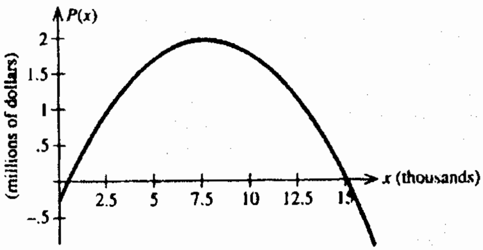
This says that when the level of production is 5000 units the production of the next speaker system will bring an additional revenue of \$400.

11. a. $P(x) = R(x) - C(x) = (-0.04x^2 + 800x) - (200x + 300{,}000)$
$= -0.04x^2 + 600x - 300{,}000.$

b. $P'(x) = -0.08x + 600$

c. $P'(5000) = -0.08(5000) + 600 = 200$ $P'(8000) = -0.08(8000) + 600 = -40.$

d.



The profit realized by the company increases as production increases, peaking at a level of production of 7500 units. Beyond this level of production, the profit begins to fall.

12. a. $P(x) = -10x^2 + 1760x - 50{,}000$. To find the actual profit realized from renting the 51st unit, assuming that 50 units have already been rented is given by
$P(51) - P(50) = [-10(51)^2 + 1760(51) - 50{,}000] - [-10(50)^2 + 1760(50) - 50{,}000]$
$= -26{,}010 + 89{,}760 - 50{,}000 + 25{,}000 - 88{,}000 + 50{,}000 = 750$, or \$750.

b. The marginal profit is given by $P'(x) = -20x + 1760$. When $x = 50$,
$P'(50) = -20(50) + 1760 = 760$, or \$760.

13. a. The revenue function is $R(x) = px = (600 - 0.05x)x = 600x - 0.05x^2$ and the profit function is
$P(x) = R(x) - C(x)$
$= (600x - 0.05x^2) - (0.000002x^3 - 0.03x^2 + 400x + 80{,}000)$

$$= -0.000002x^3 - 0.02x^2 + 200x - 80{,}000.$$

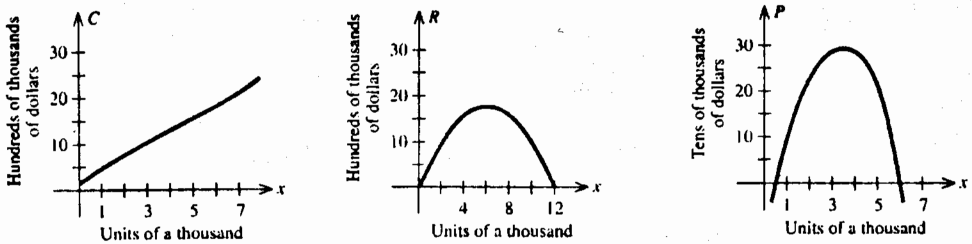
b. $C'(x) = \dfrac{d}{dx}(0.000002x^3 - 0.03x^2 + 400x + 80{,}000) = 0.000006x^2 - 0.06x + 400.$

$R'(x) = \dfrac{d}{dx}(600x - 0.05x^2) = 600 - 0.1x.$

$P'(x) = \dfrac{d}{dx}(-0.000002x^3 - 0.02x^2 + 200x - 80{,}000) = -0.000006x^2 - 0.04x + 200.$

c. $C'(2000) = 0.000006(2000)^2 - 0.06(2000) + 400 = 304$, and this says that at a level of production of 2000 units, the cost for producing the 2001st unit is \$304. $R'(2000) = 600 - 0.1(2000) = 400$ and this says that the revenue realized in selling the 2001st unit is \$400. $P'(2000) = R'(2000) - C'(2000) = 400 - 304 = 96$, and this says that the revenue realized in selling the 2001st unit is \$96.

d.



14. a. $R(x) = xp(x) = -0.006x^2 + 180x$

$P(x) = R(x) - C(x)$

$= -0.006x^2 + 180x - (0.000002x^3 - 0.02x^2 + 120x + 60{,}000)$

$= -0.000002x^3 + 0.014x^2 + 60x - 60{,}000.$

b. $C'(x) = 0.000006x^2 - 0.04x + 120$; $R'(x) = -0.012x + 180$

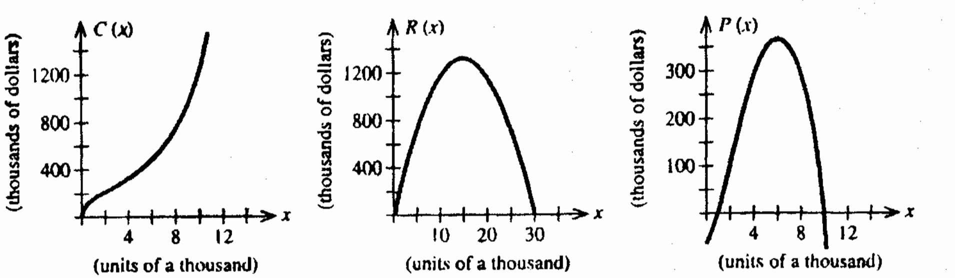
$P'(x) = -0.000006x^2 + 0.028x + 60.$

c. $C'(2000) = 0.000006(2000)^2 - 0.04(2000) + 120 = 64$;

$R'(2000) = -0.012(2000) + 180 = 156$

$P'(2000) = -0.000006(2000)^2 + 0.028(2000) + 60 = 92.$

d.



15. $\overline{C}(x)=\dfrac{C(x)}{x}=\dfrac{0.000002x^3-0.03x^2+400x+80{,}000}{x}$

$$=0.000002x^2-0.03x+400+\frac{80{,}000}{x}.$$

a. $\overline{C}'(x)=0.000004x-0.03-\dfrac{80{,}000}{x^2}.$

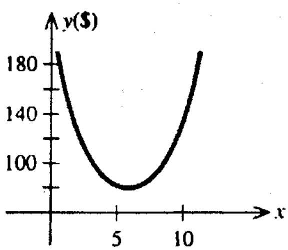
b. $\overline{C}'(5000)=0.000004(5000)-0.03-\dfrac{80{,}000}{5000^2}\approx-0.0132,$

and this says that, at a level of production of 5000 units, the average cost of production is dropping at the rate of approximately a penny per unit.

$$\overline{C}'(10{,}000)=0.000004(10000)-0.03-\frac{80{,}000}{10{,}000^2}\approx 0.0092,$$

and this says that, at a level of production of 10,000 units, the average cost of production is increasing at the rate of approximately a penny per unit.

c.



16. $C(x)=0.000002x^3-0.02x^2+120x+60{,}000$

$\overline{C}(x)=0.000002x^2-0.02x+120+\dfrac{60{,}000}{x}.$

a. The marginal average cost function is given by

$$\overline{C}'(x)=0.000004x-0.02-\frac{60{,}000}{x^2}$$

b. $\overline{C}'(5000)=0.000004(5000)-0.02-\dfrac{60{,}000}{(5000)^2}=0.02-0.02-0.0024=-0.0024.$

$\overline{C}'(10000)=0.000004(10000)-0.02-\dfrac{60{,}000}{(10000)^2}=0.04-0.02-0.0006=0.0194.$

We conclude that the average cost is decreasing when 5000 TV sets are produced and increasing when 10,000 units are produced.

17. a. $R(x) = px = \dfrac{50x}{0.01x^2 + 1}$. b. $R'(x) = \dfrac{(0.01x^2+1)50 - 50x(0.02x)}{(0.01x^2+1)^2} = \dfrac{50 - 0.5x^2}{(0.01x^2+1)^2}$

c. $R'(2) = \dfrac{50 - 0.5(4)}{[0.01(4)+1]^2} \approx 44.379.$

This result says that at a level of sale of 2000 units, the revenue increases at the rate of approximately \$44,379 per sales of 1000 units.

18. $\dfrac{dC}{dx} = \dfrac{d}{dx}(0.712x + 95.05) = 0.712.$

19. $C(x) = 0.873x^{1.1} + 20.34$; $C'(x) = 0.873(1.1)x^{0.1}$
$C'(10) = 0.873(1.1)(10)^{0.1} = 1.21$, or \$1.21 billion per billion dollars.

20. $\dfrac{dS}{dx} = \dfrac{d}{dx}[x - C(x)] = 1 - \dfrac{dC}{dx}.$

21. The consumption function is given by $C(x) = 0.712x + 95.05$. The marginal propensity to consume is given by $\dfrac{dC}{dx} = 0.712$. The marginal propensity to save is given by $\dfrac{dS}{dx} = 1 - \dfrac{dC}{dx} = 1 - 0.712 = 0.288$, or \$0.288 billion per billion dollars.

22. Here $C(x) = 0.873x^{1.1} + 20.34$. So $C'(x) = 0.9603x^{0.1}$ and

$$\frac{dS}{dx} = 1 - \frac{dC}{dx} = 1 - 0.9603x^{0.1}.$$

When $x = 10$, we have $\dfrac{dS}{dx} = 1 - 0.9603(10)^{0.1} = -0.209$,

or –\$0.209 billion per billion dollars.

23. Here $x = f(p) = -\frac{5}{4}p + 20$ and so $f'(p) = -\frac{5}{4}$. Therefore,

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p(-\frac{5}{4})}{-\frac{5}{4}p + 20} = \frac{5p}{80 - 5p}.$$

$$E(10) = \frac{5(10)}{80 - 5(10)} = \frac{50}{30} = \frac{5}{3} > 1,$$ and so the demand is elastic.

EXERCISES 3.5, page 219

1. $f(x)=4x^2-2x+1;\ f'(x)=8x-2;\ f''(x)=8.$

2. $f(x)=-0.2x^2+0.3x+4;\ f'(x)=-0.4x+0.3$ and $f''(x)=-0.4.$

3. $f(x)=2x^3-3x^2+1;\ f'(x)=6x^2-6x;\ f''(x)=12x-6=6(2x-1).$

4. $g(x)=-3x^3+24x^2+6x-64;\ g'(x)=-9x^2+48x+6;\ g''(x)=-18x+48.$

5. $h(t)=t^4-2t^3+6t^2-3t+10;\ h'(t)=4t^3-6t^2+12t-3$
   $h''(t)=12t^2-12t+12=12(t^2-t+1).$

6. $f(x)=x^5-x^4+x^3-x^2+x-1;\ f'(x)=5x^4-4x^3+3x^2-2x+1;$
   $f''(x)=20x^3-12x^2+6x-2.$

7. $f(x)=(x^2+2)^5;\ f'(x)=5(x^2+2)^4(2x)=10x(x^2+2)^4$ and
   $$f''(x)=10(x^2+2)^4+10x(x^2+2)^3(2x)$$
   $$=10(x^2+2)^3[(x^2+2)+8x^2]=10(9x^2+2)(x^2+2)^3.$$

8. $g(t)=t^2(3t+1)^4;$
   $$g'(t)=2t(3t+1)^4+t^2(4)(3t+1)^3(3)=2t(3t+1)^3[(3t+1)+6t]$$
   $$=(3t+1)^3(18t^2+2t);$$
   $$g''(t)=2t(9t+1)^3(3t+1)^2(3)+(3t+1)^3(36t+2)$$
   $$=2(3t+1)^2[9t(9t+1)+(3t+1)(18t+1)]$$

$$= 2(3t+1)^2(81t^2+9t+54t^2+3t+18t+1) = 2(135t^2+30t+1)(3t+1)^2.$$

9. $g(t) = (2t^2-1)^2(3t^2);$

$$g'(t) = 2(2t^2-1)(4t)(3t^2) - (2t^2-1)^2(6t)$$
$$= 6t(2t^2-1)[4t^2+(2t^2-1)] = 6t(2t^2-1)(6t^2-1)$$
$$= 6t(12t^4-8t^2+1) = 72t^5-48t^3+6t.$$
$$g''(t) = 360t^4-144t^2+6 = 6(60t^4-24t^2+1)$$

10. $h(x) = (x^2+1)^2(x-1).$

$$h'(x) = 2(x^2+1)(2x)(x-1)+(x^2+1)^2(1)$$
$$= (x^2+1)[4x(x-1)+(x^2+1)] = (x^2+1)(5x^2-4x+1);$$
$$h''(x) = 2x(5x^2-4x+1)+(x^2+1)(10x-4)$$
$$= 10x^3-8x^2+2x+10x^3-4x^2+10x-4$$
$$= 20x^3-12x^2+12x-4 = 4(5x^3-3x^2+3x-1).$$

11. $f(x) = (2x^2+2)^{7/2};\ f'(x) = \frac{7}{2}(2x^2+2)^{5/2}(4x) = 14x(2x^2+2)^{5/2};$

$$f''(x) = 14(2x^2+2)^{5/2}+14x(\tfrac{5}{2})(2x^2+2)^{3/2}(4x)$$
$$= 14(2x^2+2)^{3/2}[(2x^2+2)+10x^2] = 28(6x^2+1)(2x^2+2)^{3/2}.$$

12. $h(w) = (w^2+2w+4)^{5/2};$

$$h'(w) = \tfrac{5}{2}(w^2+2w+4)^{3/2}(2w+2) = 5(w+1)(w^2+2w+4)^{3/2};$$
$$h''(w) = 5(w^2+2w+4)^{3/2}+5(w+1)(\tfrac{3}{2})(w^2+2w+4)^{1/2}(2w+2)$$
$$= 5(w^2+2w+4)^{1/2}[(w^2+2w+4)+3(w+1)^2]$$
$$= 5(4w^2+8w+7)(w^2+2w+4)^{1/2}.$$

13. $f(x) = x(x^2+1)^2;$

$$f'(x) = (x^2+1)^2+x(2)(x^2+1)(2x)$$
$$= (x^2+1)[(x^2+1)+4x^2] = (x^2+1)(5x^2+1);$$
$$f''(x) = 2x(5x^2+1)+(x^2+1)(10x) = 2x(5x^2+1+5x^2+5)$$
$$= 4x(5x^2+3).$$

14. $g(u)=u(2u-1)^3;$

$$g'(u)=(2u-1)^3+u(3)(2u-1)^2(2)=(2u-1)^2[(2u-1)+6u]=(8u-1)(2u-1)^2;$$

$$g''(u)=8(2u-1)^2+(8u-1)(2)(2u-1)(2)$$

$$=4(2u-1)[2(2u-1)+(8u-1)]=12(2u-1)(4u-1).$$

15. $f(x)=\dfrac{x}{2x+1};\ f'(x)=\dfrac{(2x+1)(1)-x(2)}{(2x+1)^2}=\dfrac{1}{(2x+1)^2};$

$$f''(x)=\frac{d}{dx}(2x+1)^{-2}=-2(2x+1)^{-3}(2)=-\frac{4}{(2x+1)^3}.$$

16. $g(t)=\dfrac{t^2}{t-1};\quad g'(t)=\dfrac{(t-1)(2t)-t^2(1)}{(t-1)^2}=\dfrac{t^2-2t}{(t-1)^2}=\dfrac{t(t-2)}{(t-1)^2};$

$$g''(t)=\frac{(t-1)^2(2t-2)-t(t-2)2(t-1)}{(t-1)^4}=\frac{2(t-1)[(t-1)^2-t(t-2)]}{(t-1)^4}=\frac{2}{(t-1)^3}.$$

17. $f(s)=\dfrac{s-1}{s+1};\ f'(s)=\dfrac{(s+1)(1)-(s-1)(1)}{(s+1)^2}=\dfrac{2}{(s+1)^2}.$

$$f''(s)=2\frac{d}{ds}(s+1)^{-2}=-4(s+1)^{-3}=-\frac{4}{(s+1)^3}.$$

18. $f(u)=\dfrac{u}{u^2+1};\ f'(u)=\dfrac{(u^2+1)(1)-(u)(2u)}{(u^2+1)^2}=\dfrac{-u^2+1}{(u^2+1)^2};$

$$f'(u)=\frac{(u^2+1)(1)-(u)(2u)}{(u^2+1)^2}=\frac{-u^2+1}{(u^2+1)^2};$$

$$f''(u)=\frac{(u^2+1)^2(-2u)-(-u^2+1)(2)(u^2+1)(2u)}{(u^2+1)^4}$$

$$=\frac{2u(u^2+1)(-u^2-1+2u^2-2)}{(u^2+1)^4}=\frac{2u(u^2-3)}{(u^2+1)^3};$$

19. $f(u)=\sqrt{4-3u}=(4-3u)^{1/2}.\ f'(u)=\frac{1}{2}(4-3u)^{-1/2}(-3)=-\dfrac{3}{2\sqrt{4-3u}}.$

$$f''(u) = -\frac{3}{2}\cdot\frac{d}{du}(4-3u)^{-1/2} = -\frac{3}{2}\left(-\frac{1}{2}\right)(4-3u)^{-3/2}(-3) = -\frac{9}{4(4-3u)^{3/2}}.$$

20. $f(x) = \sqrt{2x-1} = (2x-1)^{1/2}$.

$$f'(x) = \frac{1}{2}(2x-1)^{-1/2}(2) = (2x-1)^{-1/2} = \frac{1}{\sqrt{2x-1}}.$$

$$f''(x) = -\frac{1}{2}(2x-1)^{-3/2}(2) = -(2x-1)^{-3/2} = -\frac{1}{\sqrt{(2x-1)^3}}.$$

21. $f(x) = 3x^4 - 4x^3$; $f'(x) = 12x^3 - 12x^2$; $f''(x) = 36x^2 - 24x$; $f'''(x) = 72x - 24$.

22. $f(x) = 3x^5 - 6x^4 + 2x^2 - 8x + 12$; $f'(x) = 15x^4 - 24x^3 + 4x - 8$;
$f''(x) = 60x^3 - 72x^2 + 4$; $f'''(x) = 180x^2 - 144x$.

23. $f(x) = \frac{1}{x}$; $f'(x) = \frac{d}{dx}(x^{-1}) = -x^{-2}$; $f''(x) = 2x^{-3}$; $f'''(x) = -6x^{-4} = -\frac{6}{x^4}$.

24. $f(x) = \frac{2}{x^2}$; $f'(x) = 2\frac{d}{dx}(x^{-2}) = -4x^{-3}$; $f''(x) = 12x^{-4}$; $f'''(x) = -48x^{-5} = -\frac{48}{x^5}$.

25. $g(s) = (3s-2)^{1/2}$; $g'(s) = \frac{1}{2}(3s-2)^{-1/2}(3) = \frac{3}{2(3s-2)^{1/2}}$;

$$g''(s) = \frac{3}{2}\left(-\frac{1}{2}\right)(3s-2)^{-3/2}(3) = -\frac{9}{4}(3s-2)^{-3/2} = -\frac{9}{4(3s-2)^{3/2}};$$

$$g'''(s) = \frac{27}{8}(3s-2)^{-5/2}(3) = \frac{81}{8}(3s-2)^{-5/2} = \frac{81}{8(3s-2)^{5/2}}.$$

26. $g(t) = \sqrt{2t+3}$; $g'(t) = \frac{1}{2}(2t+3)^{-1/2}(2) = (2t+3)^{-1/2}$;
$g''(t) = -\frac{1}{2}(2t+3)^{-3/2}(2) = -(2t+3)^{-3/2}$; $g'''(t) = \frac{3}{2}(2t+3)^{-5/2}(2) = \frac{3}{(2t+3)^{5/2}}$.

27. $f(x) = (2x-3)^4$; $f'(x) = 4(2x-3)^3(2) = 8(2x-3)^3$
$f''(x) = 24(2x-3)^2(2) = 48(2x-3)^2$; $f'''(x) = 96(2x-3)(2) = 192(2x-3)$.

28. $g(t)=(\frac{1}{2}t^2-1)^5;\ g'(t)=5(\frac{1}{2}t^2-1)^4(t)=5t(\frac{1}{2}t^2-1)^4$

$g''(t)=5(\frac{1}{2}t^2-1)^4+5t(4)(\frac{1}{2}t^2-1)^3(t)=5(\frac{1}{2}t^2-1)^3[(\frac{1}{2}t^2-1)+4t^2]$

$=\frac{5}{2}(9t^2-2)(\frac{1}{2}t^2-1)^3.$

$g'''(t)=\frac{5}{2}[18t(\frac{1}{2}t^2-1)^3+(9t^2-2)3(\frac{1}{2}t^2-1)^2(t)]$

$=\frac{15}{2}t(\frac{1}{2}t^2-1)^2[6(\frac{1}{2}t^2-1)+(9t^2-2)]=30t(3t^2-2)(\frac{1}{2}t^2-1)^2.$

29. Its velocity at any time t is $v(t)=\dfrac{d}{dt}(16t^2)=32t$. The hammer strikes the ground when $16t^2=256$ or $t=4$ (we reject the negative root). Therefore, its velocity at the instant it strikes the ground is $v(4)=32(4)=128$ ft/sec. Its acceleration at time t is $a(t)=\dfrac{d}{dt}(32t)=32$. In particular, its acceleration at $t=4$ is 32 ft/sec^2.

30. $s(t)=20t+8t^2-t^3;\ s'(t)=20+16t-3t^2;\ s''(t)=16-6t.$

$s''(\frac{8}{3})=16-6(\frac{8}{3})=16-\frac{48}{3}=0.$

We conclude that the acceleration of the car at $t=8/3$ seconds is zero and that the car will start to decelerate at that point in time.

31. $N(t)=-0.1t^3+1.5t^2+100.$

a. $N'(t)=-0.3t^2+3t=0.3t(10-t)$. Since $N'(t)>0$ for $t=0, 1, 2, \ldots, 7$, it is evident that $N(t)$ (and therefore the crime rate) was increasing from 1988 through 1995.

b. $N''(t)=-0.6t+3=0.6(5-t)$. Now $N''(4)=0.6>0$, $N''(5)=0$, $N''(6)=-0.6<0$ and $N''(7)=-1.2<0$. This shows that the rate of the rate of change was decreasing beyond $t=5$ (1990). This shows that the program was working.

32. $G(t)=-0.2t^3+2.4t^2+60.$

a. $G'(t)=-0.6t^2+4.8t=0.6t(8-t)$

$G'(0)=0$, $G'(1)=4.2$, $G'(2)=7.2$, $G'(3)=9$, $G'(4)=9.6$, $G'(5)=9$, $G'(6)=7.2$, $G'(7)=4.2$, $G'(8)=0$.

b. $G''(t)=-1.2t+4.8=1.2(4-t)$

$G''(0)=4.8$, $G''(1)=3.6$, $G''(2)=2.4$, $G''(3)=1.2$, $G''(4)=0$, $G''(5)=-1.2$, $G''(6)=-2.4$, $G''(7)=-3.6$, $G''(8)=-4.8$.

c. Our computations show that the GDP is increasing at an increasing rate in the first five years. Even though the GDP continues to rise from that point on, the negativity of $G'(t)$ shows that the rate of increase is slowing down.

33. $N(t) = 0.00037t^3 - 0.0242t^2 + 0.52t + 5.3 \quad (0 \le t \le 10)$

$N'(t) = 0.00111t^2 - 0.0484t + 0.52$

$N''(t) = 0.00222t - 0.0484$

So $N(8) = 0.00037(8)^3 - 0.0242(8)^2 + 5.3 = 8.1$

$N'(8) = 0.00111(8)^2 - 0.0484(8) + 0.52 \approx 0.204.$

$N''(8) = 0.00222(8) - 0.0484 = -0.031.$

We conclude that at the beginning of 1998, there were 8.1 million persons receiving disability benefits, the number is increasing at the rate of 0.2 million/yr, and the rate of the rate of change of the number of persons is decreasing at the rate of 0.03 million persons/yr^2.

34. a. $h(t) = \frac{1}{16}t^4 - t^3 + 4t^2$. $h'(t) = \frac{1}{4}t^3 - 3t^2 + 8t$

b. $h'(0) = 0$ or zero feet per second.

$h'(4) = \frac{1}{4}(64) - 3(16) + 8(4) = 0,$ or zero feet per second.

$h'(8) = \frac{1}{4}(8)^3 - 3(64) + 8(8) = 0,$ or zero feet per second.

c. $h''(t) = \frac{3}{4}t^2 - 6t + 8$

d. $h''(0) = 8$ ft/sec^2; $h''(4) = \frac{3}{4}(16) - 6(4) + 8 = -4$ ft/sec^2.

$h''(8) = \frac{3}{4}(64) - 6(8) + 8 = 8$ ft/sec^2.

e. $h(0) = 0$ feet; $h(4) = \frac{1}{16}(4)^4 - (4)^3 + 4(4)^2 = 16$ feet.

$h(8) = \frac{1}{16}(8)^4 - (8)^3 + 4(8)^2 = 0$ feet.

35. $N(t) = -0.00233t^4 + 0.00633t^3 - 0.05417t^2 + 1.3467t + 25$

$N'(t) = -0.00932t^3 + 0.01899t^2 - 0.10834t + 1.3467$

$N''(t) = -0.02796t^2 + 0.03798t - 0.10834$

So $N'(10) = -7.158$ and $N''(10) = -2.5245$.

Our computations show that at the beginning of the year 2000, the number of Americans aged 45 to 54 was decreasing at the rate of 7 million people per year, and the number decreases at a rate of approximately 2.5 million people per year per year.

36. $A(t) = 100 - 17.63t + 1.915t^2 - 0.1316t^3 + 0.00468t^4 - 0.00006t^5$

$A'(t) = -17.63 + 3.83t - 0.3948t^2 + 0.01872t^3 - 0.0003t^4$
$A''(t) = 3.83 - 0.7896t + 0.05616t^2 - 0.0012t^3$.
So, $A'(10) = -3.09$ and $A''(10) = 0.35$.
Our computations show that 10 minutes after the start of the test,the smoke remaining is decreasing at a rate of 3 percent per minute but the rate at which the rate of smoke is decreasing is increasing at the rate of 0.35 percent per minute per minute.

37. $f(t) = 10.72(0.9t+10)^{0.3}$.
$f'(t) = 10.72(0.3)(0.9t+10)^{-0.7}(0.9) = 2.8944(0.9t+10)^{-0.7}$
$f''(t) = 2.8944(-0.7)(0.9t+10)^{-1.7}(0.9) = -1.823472(0.9t+10)^{-1.7}$
So $f''(10) = -1.823472(19)^{-1.7} \approx -0.01222$. And this says that the rate of the rate of change of the population is decreasing at the rate of $0.01\%/\,\text{yr}^2$.

38. $P(t) = 33.55(t+5)^{0.205}$; $P'(t) = 33.55(0.205)(t+5)^{-0.795} = 6.87775(t+5)^{-0.795}$
$P''(t) = 6.87775(-0.795)(t+5)^{-1.795} = -5.46781125(t+5)^{-1.795}$
So $P''(20) = 6.87775(-0.795)(t+5)^{-1.795} = -5.46781125(t+5)^{-1.795}$
And this says that the rate of the rate of change of such mothers is decreasing at the rate of $0.02\%/\,\text{yr}^2$.

39. False. If f has derivatives of order two at $x = a$, then $f''(a) = [f'(a)]^2$.

40. True. If $h = fg$ where f and g have derivatives of order 2. Then
$$h''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x).$$

41. True. If $f(x)$ is a polynomial function of degree n, then $f^{(n+1)}(x) = 0$.

42. True. Suppose $P(t)$ represents the population of bacteria at time t and suppose $P'(t) > 0$ and $P''(t) < 0$, then the population is increasing at time t but at a decreasing rate.

43. True. Using the chain rule, $h'(x) = f'(2x)\cdot\frac{d}{dx}(2x) = f'(x)\cdot 2 = 2f'(2x)$
Using the chain rule again, $h''(x) = 2f''(2x)\cdot 2 = 4f''(2x)$.

44. $f'(x) = \frac{7}{3}x^{5/3}$, $f''(x) = \frac{35}{9}x^{2/3}$, and so f' and f'' exist everywhere.

But $f'''(x) = \frac{70}{27}x^{-1/3} = \dfrac{70}{27x^{1/3}}$ is not defined at $x = 0$.

45. Consider the function $f(x) = x^{(2n+1)/2} = x^{n+(1/2)}$.

Then
$$f'(x) = (n + \tfrac{1}{2})x^{n-(1/2)}$$
$$f''(x) = (n + \tfrac{1}{2})(n - \tfrac{1}{2})x^{n-(3/2)}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f^{(n)}(x) = (n + \tfrac{1}{2})(n - \tfrac{1}{2}) \cdots \tfrac{3}{2}x^{1/2}$$
$$f^{(n+1)}(x) = (n + \tfrac{1}{2})(n - \tfrac{1}{2}) \cdots \tfrac{1}{2}x^{-1/2}.$$

The first n derivatives exist at $x = 0$, but the $(n + 1)$st derivative fails to be defined there.

46. Let $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$. Then
$$P'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots a_{n-1}.$$
Eventually, $P^{(n)}(x) = a_0$, $P^{(n+1)}(x) = P^{(n+2)}(x) = P^{(n+3)}(x) = \cdots = 0$.
So P has derivatives of all orders.

EXERCISES 3.6, page 2231

1. a. Solving for y in terms of x, we have $y = -\frac{1}{2}x + \frac{5}{2}$. Therefore, $y' = -\frac{1}{2}$.
   b. Next, differentiating $x + 2y = 5$ implicitly, we have $1 + 2y' = 0$, or $y' = -\frac{1}{2}$.

2. a. Solving for y in terms of x, we have $y = -\frac{3}{4}x + \frac{3}{2}$. Therefore, $y' = -\frac{3}{4}$.
   b. Next, differentiating $3x + 4y = 6$ implicitly, we obtain $3 + 4y' = 0$, or $y' = -\frac{3}{4}$.

3. a. $xy = 1, y = \dfrac{1}{x}$, and $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$.
   b.
$$x\frac{dy}{dx} + y = 0$$
$$x\frac{dy}{dx} = -y$$
$$\frac{dy}{dx} = -\frac{y}{x} = \frac{-\frac{1}{x}}{x} = -\frac{1}{x^2}.$$

4. a. Solving for y, we have $y(x-1) = 1$ or $y = (x-1)^{-1}$. Therefore,
$$y' = -(x-1)^{-2} = -\frac{1}{(x-1)^2}.$$
   b. Next, differentiating $xy - y - 1 = 0$ implicitly, we obtain
$$y + xy' - y' = 0, \text{ or } y'(x-1) = -y \quad \text{or} \quad y' = -\frac{y}{x-1} = -\frac{1}{(x-1)^2}.$$

5. $x^3 - x^2 - xy = 4$.
   a. $-xy = 4 - x^3 + x^2$
$$y = -\frac{4}{x} + x^2 - x \text{ and } y' = \frac{4}{x^2} + 2x - 1.$$
   b. $x^3 - x^2 - xy = 4$
$$-x\frac{dy}{dx} = -3x^2 + 2x + y$$
$$\frac{dy}{dx} = 3x - 2 - \frac{y}{x}$$
$$= 3x - 2 - \frac{1}{x}(-\frac{4}{x} + x^2 - x) = 3x - 2 + \frac{4}{x^2} - x + 1$$
$$= \frac{4}{x^2} + 2x - 1.$$

6. $x^2y - x^2 + y - 1 = 0$.

a. $(x^2 + 1)y = 1 + x^2$, or $y = \dfrac{1+x^2}{1+x^2} = 1$. Therefore, $\dfrac{dy}{dx} = 0$.

b. Differentiating implicitly,

$$x^2y' + 2xy - 2x + y' = 0;\quad (x^2+1)y' = 2x(1-y);\ y' = \frac{2x(1-y)}{x^2+1}.$$

But from part (1), we know that $y = 1$, so $y' = \dfrac{2x(1-1)}{x^2+1} = 0$.

7. a. $\dfrac{x}{y} - x^2 = 1$ is equivalent to $\dfrac{x}{y} = x^2 + 1$, or $y = \dfrac{x}{x^2+1}$. Therefore,

$$y' = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

b. Next, differentiating the equation $x - x^2y = y$ implicitly, we obtain

$$1 - 2xy - x^2y' = y',\ y'(1 + x^2) = 1 - 2xy, \text{ or } y' = \frac{1-2xy}{(1+x^2)}.$$

(This may also be written in the form $-2y^2 + \dfrac{y}{x}$.) To show that this is equivalent to the results obtained earlier, use the value of y obtained before, to get

$$y' = \frac{1-2x\left(\dfrac{x}{x^2+1}\right)}{1+x^2} = \frac{x^2+1-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

8. a. $\dfrac{y}{x} - 2x^3 = 4$ is equivalent to $y = 2x^4 + 4x$. Therefore, $y' = 8x^3 + 4$.

b. Next, differentiating the equation $y - 2x^4 = 4x$ implicitly, we obtain

$$y' - 8x^3 = 4 \quad \text{and so } y' = 8x^3 + 4$$

as obtained earlier.

9. $x^2 + y^2 = 16$. Differentiating both sides of the equation implicitly, we obtain

$$2x + 2yy' = 0 \text{ and so } y' = -\frac{x}{y}.$$

10. $2x^2 + y^2 = 16$, $4x + 2y\dfrac{dy}{dx} = 0$ and $\dfrac{dy}{dx} = -\dfrac{2x}{y}$.

11. $x^2 - 2y^2 = 16$. Differentiating implicitly with respect to x, we have

$$2x - 4y\frac{dy}{dx} = 0 \text{ and } \frac{dy}{dx} = \frac{x}{2y}.$$

12. $x^3 + y^3 + y - 4 = 0$. Differentiating both sides of the equation implicitly, we obtain $3x^2 + 3y^2y' + y' = 0$ or $y'(3y^2 + 1) = -3x^2$. Therefore, $y' = -\dfrac{3x^2}{3y^2+1}$.

13. $x^2 - 2xy = 6$. Differentiating both sides of the equation implicitly, we obtain $2x - 2y - 2xy' = 0$ and so $y' = \dfrac{x-y}{x} = 1 - \dfrac{y}{x}$.

14. $x^2 + 5xy + y^2 = 10$. Differentiating both sides of the equation implicitly, we obtain $2x + 5y + 5xy' + 2yy' = 0$, $2x + 5y + y'(5x + 2y) = 0$ and so $y' = -\dfrac{2x+5y}{5x+2y}$.

15. $x^2y^2 - xy = 8$. Differentiating both sides of the equation implicitly, we obtain

$$2xy^2 + 2x^2yy' - y - xy' = 0,\ 2xy^2 - y + y'(2x^2y - x) = 0$$

and so

$$y' = \frac{y(1-2xy)}{x(2xy-1)} = -\frac{y}{x}.$$

16. $x^2y^3 - 2xy^2 = 5$. Differentiating both sides of the equation implicitly, we obtain $2xy^3 + 3x^2y^2y' - 2y^2 - 4xyy' = 0$, $2y^2(xy - 1) + xy(3xy - 4)y' = 0$,

So

$$y' = \frac{2y(1-xy)}{x(3xy-4)}.$$

17. $x^{1/2} + y^{1/2} = 1$. Differentiating implicitly with respect to x, we have

$$\tfrac{1}{2}x^{-1/2} + \tfrac{1}{2}y^{-1/2}\frac{dy}{dx} = 0. \text{ Therefore, } \frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

18. $x^{1/3} + y^{1/3} = 1$. Differentiating both sides of the equation implicitly, we obtain $\tfrac{1}{3}x^{-2/3} + \tfrac{1}{3}y^{-2/3}y' = 0$ and so $y' = -\dfrac{x^{-2/3}}{y^{-2/3}} = -\dfrac{y^{2/3}}{x^{2/3}} = -\left(\dfrac{y}{x}\right)^{2/3}$.

19. $\sqrt{x+y}=x$. Differentiating both sides of the equation implicitly, we obtain

$$\tfrac{1}{2}(x+y)^{-1/2}(1+y')=1,\ 1+y'=2(x+y)^{1/2},$$

or
$$y'=2\sqrt{x+y}-1.$$

20. $(2x+3y)^{1/3}=x^2$. Differentiating both sides of the equation implicitly, we obtain

$$\tfrac{1}{3}(2x+3y)^{-2/3}(2+3y')=2x,\ 2+3y'=6x(2x+3y)^{2/3}$$

or
$$y'=\tfrac{2}{3}[3x(2x+3y)^{2/3}-1].$$

21. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=1$. Differentiating both sides of the equation implicitly, we obtain

$$-\frac{2}{x^3}-\frac{2}{y^3}y'=0,\ \text{or}\ y'=-\frac{y^3}{x^3}.$$

22. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=5$. Differentiating both sides of the equation implicitly, we obtain

$$-\frac{3}{x^4}-\frac{3}{y^4}y'=0,\ \text{or}\ y'=-\frac{y^4}{x^4}.$$

23. $\sqrt{xy}=x+y$. Differentiating both sides of the equation implicitly, we obtain

$$\tfrac{1}{2}(xy)^{-1/2}(xy'+y)=1+y'$$

$$xy'+y=2\sqrt{xy}(1+y')$$

$$y'(x-2\sqrt{xy})=2\sqrt{xy}-y$$

or
$$y'=-\frac{(2\sqrt{xy}-y)}{(2\sqrt{xy}-x)}=\frac{2\sqrt{xy}-y}{x-2\sqrt{xy}}.$$

24. $\sqrt{xy}=2x+y^2$. Differentiating both sides of the equation implicitly, we obtain

$$\tfrac{1}{2}(xy)^{-1/2}(xy'+y)=2+2yy'$$

$$xy'+y=4\sqrt{xy}+4\sqrt{xy}\,yy' \Rightarrow y'(x-4y\sqrt{xy})=4\sqrt{xy}-y$$

and
$$y'=\frac{4\sqrt{xy}-y}{x-4y\sqrt{xy}}.$$

25. $\dfrac{x+y}{x-y}=3x$, or $x+y=3x^2-3xy$. Differentiating both sides of the equation implicitly, we obtain $1+y'=6x-3xy'-3y$ or $y'=\dfrac{6x-3y-1}{3x+1}$.

26. $\dfrac{x-y}{2x+3y}=2x$, or $x-y=4x^2+6xy$. Differentiating both sides of the equation implicitly, we have $1-y'=8x+6y+6xy'$ or $y'=-\dfrac{8x+6y-1}{6x+1}$.

27. $xy^{3/2}=x^2+y^2$. Differentiating implicitly with respect to x, we obtain

$$y^{3/2}+x\left(\tfrac{3}{2}\right)y^{1/2}\frac{dy}{dx}=2x+2y\frac{dy}{dx}$$

$$2y^{3/2}+3xy^{1/2}\frac{dy}{dx}=4x+4y\frac{dy}{dx}$$ (Multiplying by 2.)

$$(3xy^{1/2}-4y)\frac{dy}{dx}=4x-2y^{3/2}$$

$$\frac{dy}{dx}=\frac{2(2x-y^{3/2})}{3xy^{1/2}-4y}.$$

28. $x^2y^{1/2}=x+2y^3$. Differentiating implicitly with respect to x, we have

$$2xy^{1/2}+\tfrac{1}{2}x^2y^{-1/2}y'=1+6y^2y'\Rightarrow 4xy+x^2y'=2y^{1/2}+12y^{5/2}y'$$

or

$$y'=\frac{2\sqrt{y}-4xy}{x^2-12y^{5/2}}.$$

29. $(x+y)^3+x^3+y^3=0$. Differentiating implicitly with respect to x, we obtain

$$3(x+y)^2\left(1+\frac{dy}{dx}\right)+3x^2+3y^2\frac{dy}{dx}=0$$

$$(x+y)^2+(x+y)^2\frac{dy}{dx}+x^2+y^2\frac{dy}{dx}=0$$

$$[(x+y)^2+y^2]\frac{dy}{dx}=-[(x+y)^2+x^2]$$

$$\frac{dy}{dx}=-\frac{2x^2+2xy+y^2}{x^2+2xy+2y^2}.$$

30. $(x+y^2)^{10}=x^2+25$. Differentiating both sides of the equation with respect to x, we obtain $10(x+y^2)^9(1+2yy')=2x$, or $y'=\frac{x-5(x+y^2)^9}{10y(x+y^2)^9}$.

31. $4x^2+9y^2=36$. Differentiating the equation implicitly, we obtain

$$8x+18yy'=0.$$

At the point $(0,2)$, we have $0+36y'=0$ and the slope of the tangent line is 0. Therefore, an equation of the tangent line is $y=2$.

32. $y^2-x^2=16$. Differentiating both sides of the equation implicitly, we obtain $2yy'-2x=0$. At the point $(2,2\sqrt{5})$, we have $4\sqrt{5}y'-4=0$, or $m=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$. Using the point–slope form of an equation of a line, we have $y=\frac{\sqrt{5}}{5}x+\frac{8\sqrt{5}}{5}$.

33. $x^2y^3-y^2+xy-1=0$. Differentiating implicitly with respect to x, we have

$$2xy^3+3x^2y^2\frac{dy}{dx}-2y\frac{dy}{dx}+y+x\frac{dy}{dx}=0.$$

At $(1,1)$, $2+3\frac{dy}{dx}-2\frac{dy}{dx}+1+\frac{dy}{dx}=0$, and

$$2\frac{dy}{dx}=-3 \text{ and } \frac{dy}{dx}=-\frac{3}{2}.$$

Using the point-slope form of an equation of a line, we have

$$y-1=-\tfrac{3}{2}(x-1)$$

and the equation of the tangent line to the graph of the function f at $(1,1)$ is

$$y=-\tfrac{3}{2}x+\tfrac{5}{2}.$$

34. $(x-y-1)^3=x$. Differentiating both sides of the given equation implicitly, we obtain $3(x-y-1)^2(1-y')=1$. At the point $(1,-1)$, $3(1+1-1)^2(1-y')=1$ or $y'=\frac{2}{3}$ Using the point–slope form of an equation of a line, we have

$$y+1=\tfrac{2}{3}(x-1) \quad\text{or}\quad y=\tfrac{2}{3}x-\tfrac{5}{3}.$$

35. $xy = 1$. Differentiating implicitly, we have $xy' + y = 0$, or $y' = -\dfrac{y}{x}$.

Differentiating implicitly once again, we have $xy'' + y' + y' = 0$.

Therefore, $$y'' = -\frac{2y'}{x} = \frac{2\left(\frac{y}{x}\right)}{x} = \frac{2y}{x^2}.$$

36. $x^3 + y^3 = 28$. Differentiating implicitly, we have

$$3x^2 + 3y^2y' = 0, \text{ or } y' = -\frac{x^2}{y^2}.$$

Differentiating again, we have $6x + 3y^2y'' + 6y(y')^2 = 0$.

So $y'' = -\dfrac{2y(y')^2 + 2x}{y^2}$. But $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$, and, therefore,

$$y'' = -\frac{2y\left(\frac{x^4}{y^4}\right) + 2x}{y^2} = -\frac{2\left(\frac{x^4}{y^3} + x\right)}{y^2} = -\frac{2x(x^3 + y^3)}{y^5}.$$

37. $y^2 - xy = 8$. Differentiating implicitly we have $2yy' - y - xy' = 0$

and $y' = \dfrac{y}{2y - x}$. Differentiating implicitly again, we have

$$2(y')^2 + 2yy'' - y' - y' - xy'' = 0, \quad \text{or} \quad y'' = \frac{2y' - 2(y')^2}{2y - x}.$$

Then $$y'' = \frac{2\left(\frac{y}{2y - x}\right)\left(1 - \frac{y}{2y - x}\right)}{2y - x} = \frac{2y(2y - x - y)}{(2y - x)^3} = \frac{2y(y - x)}{(2y - x)^3}.$$

38. Differentiating implicitly, we have $\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$ and $y' = -\dfrac{y^{2/3}}{x^{2/3}}$.

Differentiating implicitly once again, we have

$$y'' = -\frac{x^{2/3}\left(\frac{2}{3}\right)y^{-1/3}y' - y^{2/3}\left(\frac{2}{3}\right)x^{-1/3}}{x^{4/3}} = \frac{-\frac{2}{3}x^{2/3}y^{-1/3}\left(-\frac{y^{2/3}}{x^{2/3}}\right) + \frac{2}{3}y^{2/3}x^{-1/3}}{x^{4/3}}$$

$$= \frac{2}{3}\left(\frac{y^{1/3} + y^{2/3}x^{-1/3}}{x^{4/3}}\right) = \frac{2y^{1/3}(x^{1/3} + y^{1/3})}{3x^{4/3}x^{1/3}} = \frac{2y^{1/3}}{3x^{5/3}}.$$

39. a. Differentiating the given equation with respect to t, we obtain

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh\frac{dr}{dt} = \pi r\left(r\frac{dh}{dt} + 2h\frac{dr}{dt}\right).$$

b. Substituting $r = 2$, $h = 6$, $\frac{dr}{dt} = 0.1$ *and* $\frac{dh}{dt} = 0.3$ into the expression for $\frac{dV}{dt}$

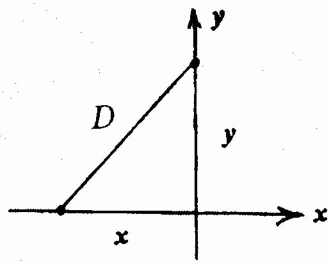
we obtain $\frac{dV}{dt} = \pi(2)[2(0.3) + 2(6)(0.1)] = 3.6\pi$

and so the volume is increasing at the rate of 3.6π cu in/sec.

40. Let $(x,0)$ and $(0,y)$ denote the position of the two cars. Then $D^2 = x^2 + y^2$.
Differentiating with respect to t, we obtain

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or $$D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$



When $t = 4$, $x = -20$, and $y = 28$,

$$\frac{dx}{dt} = -9, \text{ and } \frac{dy}{dt} = 11.$$

Therefore

$$\left(\sqrt{(-20)^2 + (28)^2}\right)\frac{dD}{dt} = (-20)(-9) + (28)(11) = 488$$

and $$\frac{dD}{dt} = \frac{488}{\sqrt{1184}} = 14.18 \text{ ft/sec.}$$

That is, the distance is changing at the rate of 14.18 ft/sec.

41. We are given $\frac{dp}{dt} = 2$ and are required to find $\frac{dx}{dt}$ when $x = 9$ and $p = 63$.
Differentiating the equation $p + x^2 = 144$ with respect to t, we obtain

$$\frac{dp}{dt} + 2x\frac{dx}{dt} = 0.$$

When $x = 9$, $p = 63$, and $\dfrac{dp}{dt} = 2$,

$$2+2(9)\frac{dx}{dt} = 0$$

and $$\frac{dx}{dt} = -\frac{1}{9} \approx -0.111,$$

or the quantity demanded is decreasing at the rate of 111 tires per week.

42. $p = \frac{1}{2}x^2 + 48$. Differentiating implicitly, we have

$$\frac{dp}{dt} - x\frac{dx}{dt} = 0, \text{ and } \quad -x\frac{dx}{dt} = -\frac{dp}{dt}, \quad \text{or } \frac{dx}{dt} = \frac{\frac{dp}{dt}}{x}.$$

When $x = 6$, $p = 66$, and $\dfrac{dp}{dt} = -3$, we have $\dfrac{dx}{dt} = -\dfrac{3}{6} = -\dfrac{1}{2}$,

or $(-\frac{1}{2})(1000) = -500$ tires/week.

43. $100x^2 + 9p^2 = 3600$. Differentiating the given equation implicitly with respect to t, we have $200x\dfrac{dx}{dt} + 18p\dfrac{dp}{dt} = 0$. Next, when $p = 14$, the given equation yields

$$100x^2 + 9(14)^2 = 3600$$
$$100x^2 = 1836,$$

or $x = 4.2849$. When $p = 14$, $\dfrac{dp}{dt} = -0.15$, and $x = 4.2849$, we have

$$200(4.2849)\frac{dx}{dt} + 18(14)(-0.15) = 0$$
$$\frac{dx}{dt} = 0.0441.$$

So the quantity demanded is increasing at the rate of 44 ten–packs per week.

44. $625p^2 - x^2 = 100$. Differentiating the given equation implicitly with respect to t, we have $1250p\dfrac{dp}{dt} - 2x\dfrac{dx}{dt} = 0$. To find p when $x = 25$, we solve the equation

$625p^2 - 625 = 100$ giving $\quad p = \sqrt{\dfrac{725}{625}} \approx 1.0770.$

Therefore, $1250(1.077)(-0.02) - 2(25)\dfrac{dx}{dt} = 0$ and $\dfrac{dx}{dt} = -0.5385.$

We conclude that the supply is falling at the rate of 539 dozen eggs per week.

45. From the results of Problem 44, we have

$$1250p\frac{dp}{dt} - 2x\frac{dx}{dt} = 0.$$

When $p = 1.0770$, $x = 25$, and $\dfrac{dx}{dt} = -1$, we find that

$$1250(1.077)\frac{dp}{dt} - 2(25)(-1) = 0,$$

and $$\frac{dp}{dt} = -\frac{50}{1250(1.077)} = -0.037.$$

We conclude that the price is decreasing at the rate of 3.7 cents per carton.

46. $p = -0.01x^2 - 0.1x + 6$. Differentiating the given equation with respect to t, we obtain

$$1 = -0.02x\frac{dx}{dp} - 0.1\frac{dx}{dp}$$
$$= -(0.02x + 0.1)\frac{dx}{dp}.$$

When $x = 10$, we have

$$1 = -[0.02(10) + 0.1]\frac{dx}{dp} \text{ ,or } \qquad \frac{dx}{dp} = -\frac{1}{0.3} = -\frac{10}{3}.$$

Also, for this value of x, $p = -0.01(100) - 0.1(10) + 6 = 4.$

Therefore, for these values of x and p,

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{4\left(-\frac{10}{3}\right)}{10} = \frac{4}{3} > 1 \qquad \left[f'(p) = \frac{dx}{dp}\right]$$

and the demand is elastic.

47. $p = -0.01x^2 - 0.2x + 8$. Differentiating the given equation implicitly with respect to p, we have

$$1 = -0.02x\frac{dx}{dp} - 0.2\frac{dx}{dp} = [0.02x + 0.2]\frac{dx}{dp}$$

or $$\frac{dx}{dp} = -\frac{1}{0.02x + 0.2}.$$

When $x = 15$, $p = -0.01(15)^2 - 0.2(15) + 8 = 2.75$

and $$\frac{dx}{dp} = -\frac{1}{0.02(15) + 0.2} = -2.$$

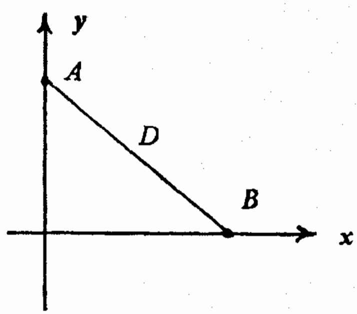
Therefore, $E(p) = -\frac{pf'(p)}{f(p)} = -\frac{(2.75)(-2)}{15} = 0.37 < 1,$

and the demand is inelastic.

48. $V = x^3$. $\frac{dV}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dV}{dt} = 3(25)(0.1) = 7.5$ cu in/sec.

49. $A = \pi r^2$. Differentiating with respect to t, we obtain
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}.$$
When the radius of the circle is 40 ft and increasing at the rate of 2 ft/sec,
$$\frac{dA}{dt} = 2\pi(40)(2) = 160\pi \text{ ft}^2/\text{sec}.$$

50. Let D denote the distance between the two ships, x the distance that Ship A traveled north, and y the distance that Ship B traveled east. Then $D^2 = x^2 + y^2$. Differentiating implicitly, we have



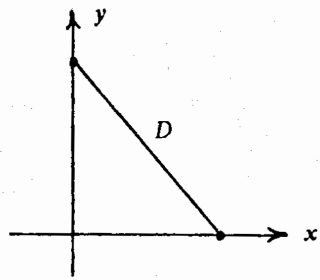
$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

or $$D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

At 1 P.M., $x = 12$, and $y = 15$, so
$$\sqrt{144+225}\,\frac{dD}{dt} = (12)(12) + (15)(15)$$
$$\frac{dD}{dt} = \frac{369}{\sqrt{369}} \approx 19.21 \text{ ft/sec}.$$

51. Let D denote the distance between the two cars, x the distance traveled by the car heading east, and y the distance traveled by the car heading north as shown in the

diagram at the right. Then
$D^2 = x^2 + y^2$.Differentiating with respect to t, we have



$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

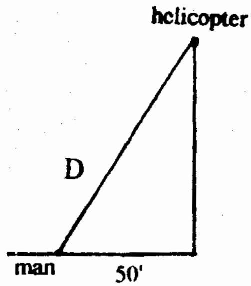
$$\text{or} \quad \frac{dD}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{D}$$

When $t = 5$, $x = 30, y = 40, \dfrac{dx}{dt} = 2(5) + 1 = 11$, and $\dfrac{dy}{dt} = 2(5) + 3 = 13$.

Therefore, $\dfrac{dD}{dt} = \dfrac{(30)(11) + (40)(13)}{\sqrt{900 + 1600}} = 17$ ft/sec.

52. $D^2 = x^2 + (50)^2 = x^2 + 2500$. Differentiating implicitly with respect to t, we have

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt}, \quad \frac{dD}{dt} = \frac{x\frac{dx}{dt}}{D}.$$



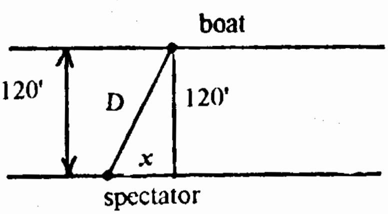
When $x = 120$ and $\dfrac{dx}{dt} = 44$,

$$\frac{dD}{dt} = \frac{(120)(44)}{\sqrt{(120)^2 + (50)^2}} \approx 40.6$$

and so the distance between the helicopter and the man is increasing at the rate of 40.6 ft/sec.

53. Referring to the diagram at the right, we see that

$$D^2 = 120^2 + x^2.$$

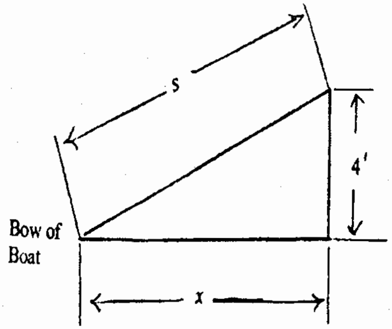


Differentiating this last equation with respect to t, we have

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} \quad \text{and} \quad \frac{dD}{dt} = \frac{x\frac{dx}{dt}}{D}.$$

When $x = 50$, $D = \sqrt{120^2 + 50^2} = 130$ and
$\dfrac{dD}{dt} = \dfrac{(20)(50)}{130} \approx 7.69$, or 7.69 ft/sec.

54. Refer to the diagram at the right. By the Pythagorean Theorem,
$s^2 = x^2 + 4^2 = x^2 + 16$.



We want to find $\dfrac{dx}{dt}$ when $x = 25$, given that $\dfrac{ds}{dt} = -3$. Differentiating both sides of the equation with respect to t yields

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} \qquad \text{or} \qquad \frac{dx}{dt} = \frac{s\dfrac{ds}{dt}}{x}.$$

Now, when $x = 25$, $s^2 = 25^2 + 16 = 241$ and $s = \sqrt{241}$. Therefore, when $x = 25$, we have $\dfrac{dx}{dt} = \dfrac{\sqrt{241}(-3)}{25} \approx -1.86$; that is, the boat is approaching the dock at the rate of approximately 1.86 ft/sec.

55. Let V and S denote its volume and surface area. Then we are given that $\dfrac{dV}{dt} = -kS$, where k is the constant of proportionality. But from $V = \left(\dfrac{4}{3}\right)\pi r^3$, we find, upon differentiating both sides with respect to t, that

$$\frac{dV}{dt} = \left(\frac{4}{3}\right)\pi(3\pi r^2)\frac{dr}{dt} = 4\pi^2 r^2\frac{dr}{dt}$$

and using the fact stated earlier,

$$\frac{dV}{dt} = 4\pi^2 r^2\frac{dr}{dt} = -kS = -k(4\pi r^2).$$

Therefore, $\dfrac{dr}{dt} = -\dfrac{k(4\pi r^2)}{4\pi^2 r^2} = -\dfrac{k}{\pi}$ and this proves that the radius is decreasing at the constant rate of (k/π) units/unit time.