MATHEMATICS 180

NAME SOLUTIONS

DATE

1.

2.

3.

4.

TOTAL

Problem 1.(*30 pts.*) Let $f(x) = x^3 - 3x + 2$.

a) Compute the first derivative $f'(x)$ and find all critical points. Determine the intervals when the function is increasing and when it is decreasing. Use the first derivative test to identify local extrema.

$$f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$$

$f'(x) = 0 \iff x = -1$ OR $x = +1$ (CRITICAL POINTS)

x	$(-\infty, -1)$	-1	$(-1, +1)$	$+1$	$(+1, +\infty)$
$f(x)$	$\nearrow$	4	$\searrow$	0	$\nearrow$
$f'(x)$	$+++$	0	$---$	0	$+++$

$f(-1) = 4$ is a relative maximum

$f(+1) = 0$ is a — " — minimum

b) Find the second derivative $f''(x)$, all inflection points and intervals where $f(x)$ is concave up and intervals where it is concave down.

$f''(x) = 6x$

$f''(x) = 0 \iff 6x = 0 \iff x = 0$

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f(x)$	DOWN	2	UP
$f''(x)$	$---$	0	$+++$

$f(0) = 2$ is an inflection point

c) Graph $f(x)$.

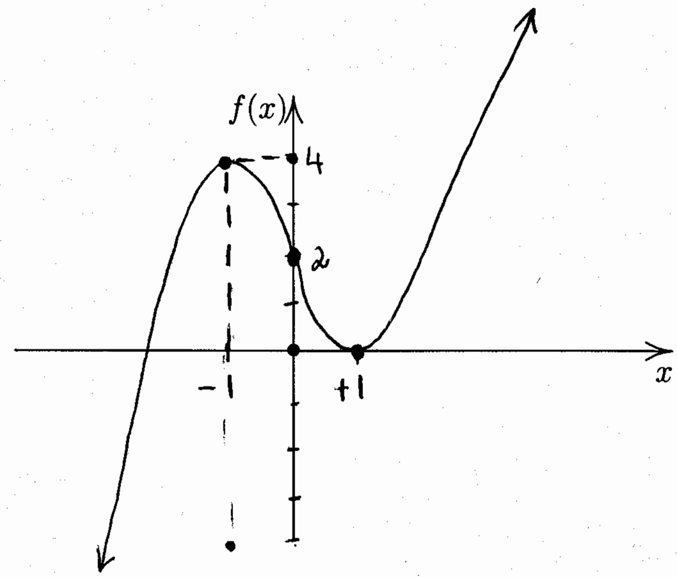


Fig. 1. The graph of $f(x) = x^3 - 3x + 2$

Problem 2. (*30 pts.*) Let $f(x) = 5x + \frac{35}{x}$.

a) Compute the first derivative $f'(x)$ and find all critical points. Determine the intervals when the function is increasing and when it is decreasing. Identify local minima and maxima using the first derivative test.

$$f'(x) = 5 - \frac{35}{x^2}$$

$$f'(x) = 0 \iff 5 - \frac{35}{x^2} = 0 \iff x^2 = 7 \iff x = \pm\sqrt{7}$$

$x = 0$ $f(x)$ is not defined.

x	$(-\infty, -\sqrt{7})$	$-\sqrt{7}$	$(-\sqrt{7}, 0)$		$(0, +\sqrt{7})$	$\sqrt{7}$	$(\sqrt{7}, +\infty)$
$f(x)$	↗	$-10\sqrt{7}$	↘		↘	$+10\sqrt{7}$	↗
$f'(x)$	$+++$	0	$---$		$---$	0	$+++$

$f(\sqrt{7}) = 10\sqrt{7}$ is a local minimum

$f(-\sqrt{7}) = -10\sqrt{7}$ is a local maximum

$x = 0$ is a vertical asymptote

b) Find the second derivative $f''(x)$, all inflection points and intervals where $f(x)$ is concave up and intervals where it is concave down.

$$f''(x) = \frac{70}{x^3} \qquad f''(x) = 0 \text{ has no solutions}$$

x	$(-\infty, 0)$	$(0, \infty)$
$f(x)$	DOWN	UP
$f''(x)$	$- - -$	$+ + +$

$f(x)$ has no inflection points but it changes its concavity through $x = 0$.

c) Graph $f(x)$.

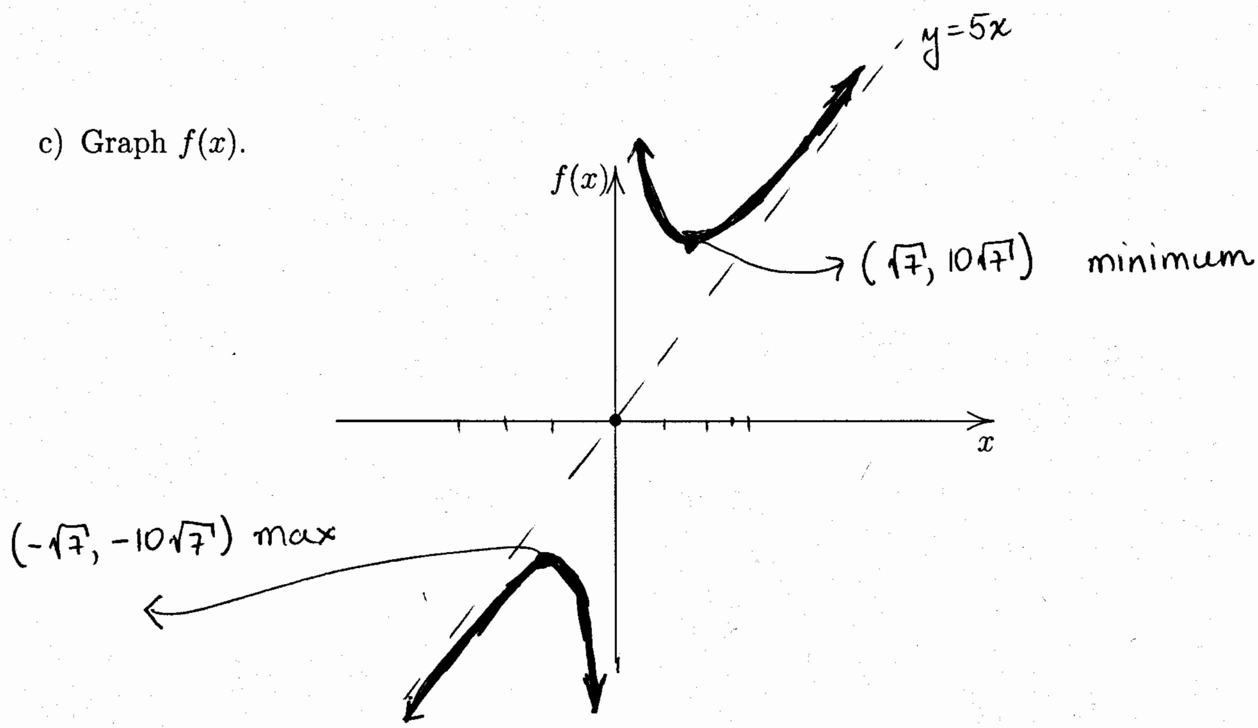


Fig. 2. The graph of $f(x) = 5x + \frac{35}{x}$

Problem 3.(*20 pts.*) Let $f(x) = \frac{x^2}{x^2+1}$. Determine the absolute minimum and the absolute minimum of $f(x)$ on the interval $[-3, 3]$.

$$f'(x) = \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

$$f'(x) = 0 \iff 2x = 0 \iff x = 0$$

$x = 0$ is the only critical point of $f(x)$

Since

$f(-3) = f(3) = \frac{9}{10}$

$f(0) = 0$

We get:

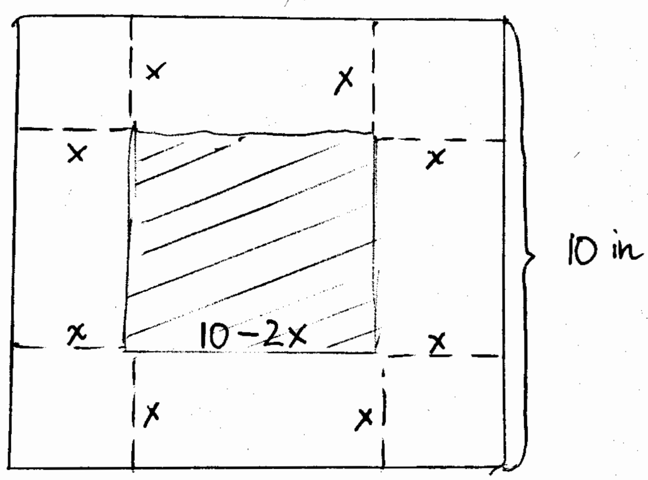
$f(-3) = 9/10$

$f(3) = 9/10$

are both absolute maxima

$f(0) = 0$ is the absolute minimum (and relative minimum).

Problem 4.(*20 pts.*) From a thin piece of cardboard 10 in. by 10 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?



$$\begin{cases} \text{base} : & 10-2x \\ \text{height} : & x \end{cases}$$

$$\text{Volume} = V(x) = (10-2x)^2 \cdot x = 4x^3 - 40x^2 + 100x$$

$$V'(x) = 12x^2 - 80x + 100 = 4(3x^2 - 20x + 25)$$

$$x_{1,2} = \frac{80 \pm \sqrt{80^2 - 4{,}800}}{24} = \frac{80 \pm 40}{24}$$

$$= \begin{cases} 40/24 = 10/6 = 5/3 \text{ in} \\ 120/24 = 30/6 = 5 \text{ in} \end{cases}$$

$$V''(x) = 24x - 80$$

$V''(5) > 0$ $\quad x = 5$ is a minimum, $V(5) = 0$

$V''(5/3) = \frac{120}{3} - 80 < 0$, $\quad x = 5/3$ is a maximum

Dimensions : $6\frac{2}{3} \times 6\frac{2}{3} \times 1\frac{2}{3}$

Volume $\quad V(\frac{2}{3}) = \left(\frac{20}{3}\right)^2 \cdot \frac{5}{3} = \frac{2000}{27}$ [in^3]

MATHEMATICS 180

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TOTAL

Problem 1.(*25 pts.*) Let $f(x) = -x^3 + 3x^2 + 1$.

a) Compute the first derivative $f'(x)$ and find all critical points. Determine the intervals when the function is increasing and when it is decreasing. Use the first derivative test to identify local extrema.

$$f'(x) = -3x^2 + 6x = 3x(2-x)$$

$$f'(x) = 0 \iff x = 0 \quad \text{OR} \quad x = 2$$

x	$(-\infty, 0)$	0	$(0,2)$	2	$(2,\infty)$
$f(x)$	$\searrow$	1	$\nearrow$	5	$\searrow$
$f'(x)$	$---$	0	$+++$	0	$---$

$f(0) = 1$ is a relative minimum

$f(2) = 5$ is a —"— maximum

b) Find the second derivative $f''(x)$, all inflection points and intervals where $f(x)$ is concave up and intervals where it is concave down.

$$f''(x) = -6x + 6 = 6(1-x)$$

$$f''(x) = 0 \iff x = 1$$

x	$(-\infty, 1)$	1	$(1, \infty)$
$f(x)$	UP	3	DOWN
$f''(x)$	$+++$	0	$---$

$f(1) = 3$ is an inflection point

c) Graph $f(x)$.

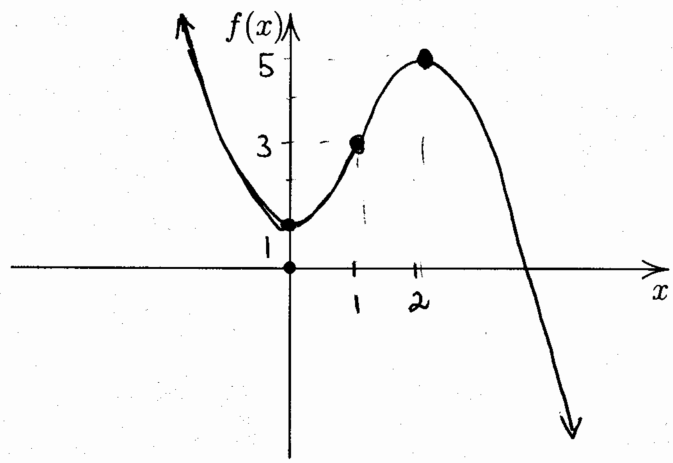


Fig. 1. The graph of $f(x) = -x^3 + 3x^2 + 1$

Problem 2.(*25 pts.*) Find the following derivatives

a)

$$\frac{d}{dx}[(x^2+3)(x^2-3)^{10}] = \quad 2x\,(x^2-3)^{10} + 20x\,(x^2-3)^9(x^2+3)$$

b)

$$\frac{d}{dx}\left[\frac{x^2+2x}{x+1}\right] = \quad \frac{(2x+2)(x+1) - x^2 - 2x}{(x+1)^2} = \frac{x^2+2x+2}{(x+1)^2}$$

c)

$$\frac{d}{dx}\left[\frac{x^4-4x^2+3}{x}\right] = \quad \frac{(4x^3-8x)x - x^4 + 4x^2 - 3}{x^2} = \frac{3x^4-4x^2-3}{x^2}$$

d)

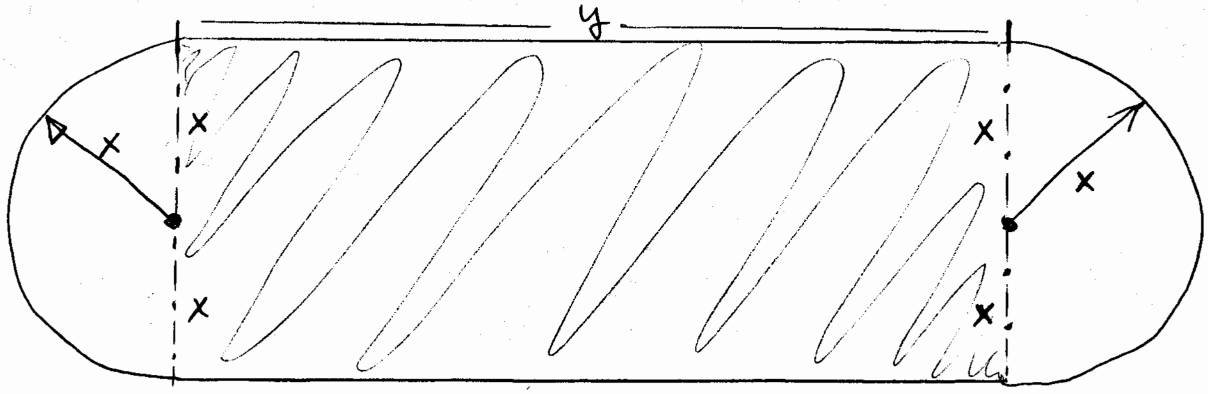
$$\frac{d}{dx}[(x+3)\sqrt{2x-3}]\,|_6 = \quad \sqrt{2x-3} + (x+3)\frac{1}{\sqrt{2x-3}}\Bigg|_{x=6} = 3 + \frac{9}{3} = 6$$

e)

$$\frac{d}{dx}\left[(x+1)\sqrt{x^2+3}\right] = \quad \sqrt{x^2+3} + (x+1)\frac{2x}{2\sqrt{x^2+3}}$$

$$= \sqrt{x^2+3} + \frac{x(x+1)}{\sqrt{x^2+3}}$$

Problem 3.(*25 pts.*) An athletic field consists of a rectangular region with semi-circular regions at each end. The perimeter will be used for a 440 yard track. Find the value of x for which the area of the **rectangular region** is as large as possible.



The perimeter $P = 2\pi x + 2y = 440$

the area of the rectangular region:

$$A = 2x \cdot y = x \cdot 2y = x(440 - 2\pi x)$$

$$A(x) = -2\pi x^2 + 440x$$

$$A'(x) = -4\pi x + 440$$

$$A'(x) = 0 \iff x = \frac{440}{4\pi} = \frac{110}{\pi} \text{ yards}$$

$A''(x) = -4\pi < 0$ so $x = 110/\pi$ is a maximum.

Problem 4. (20 pts.) A one product firm estimates its total const function is $C(x) = x^3 - 6x^2 + 13x + 15$ (in suitable units) and its total revenue function is $R(x) = 28x$. Find the value of x that maximizes the daily profit.

$$P(x) = R(x) - C(x)$$

$$= 28x - (x^3 - 6x^2 + 13x + 15)$$

$$= -x^3 + 6x^2 + 15x - 15$$

$$P'(x) = -3x^2 + 12x + 15$$

$$= -3(x^2 - 4x + 3) = -3(x-1)(x-3)$$

$$P'(x) = 0 \iff x = 1 \quad \text{or} \quad x = 3$$

But $\quad P''(x) = -6x + 12$

$P''(1) = -6 + 12 > 0 \quad$ minimum

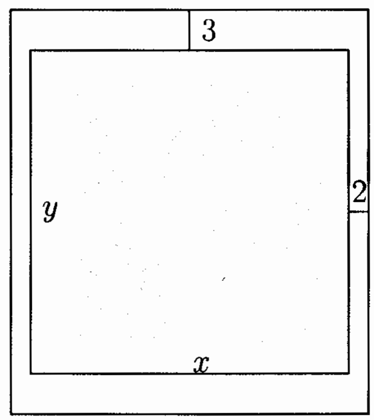
$P''(3) = -18 + 12 = -6 < 0 \quad$ maximum

Hence $\quad x = 3$ is $\quad$ is $\quad$ local maximum

$$P(3) = -3^3 + 6 \cdot 9 + 45 - 15 =$$

$$= 3 \cdot 9 + 30 = 57.$$

Bonus Problem (10 pts.) A mathematics book is to contain 90 square inches of printed matter per page, with margins of 2 inch along the sides and 3 inches along the top and bottom. Find the dimension of the page that will require the minimum amount of paper.



$x \cdot y = 90$ (PRINTED AREA)

$A = (y+6)(x+4)$ (WHOLE PAGE)

$A = 6x + 4y + xy + 24$

$= 6x + 4\dfrac{90}{x} + 90 + 24$

$A(x) = 6x + \dfrac{360}{x} + 114$

$A'(x) = 6 - \dfrac{360}{x^2}$, $A'(x) = 0 \Leftrightarrow x^2 = \dfrac{360}{6} = 60$

$x = +\sqrt{60}$ ($x = -\sqrt{60}$ DOES NOT MAKE SENSE).

$A''(x) = \dfrac{720}{x^3}$ $A''(+\sqrt{60}) > 0$ so $x = \sqrt{60}$ is a minimum

Dimensions: $x = \sqrt{60}$ $y = \frac{3}{2}\sqrt{60}$ (PRINTED AREA)

$x + 4 = 4 + \sqrt{60}$ $y + 6 = 6 + \frac{3}{2}\sqrt{60}$ (WHOLE PAGE).