Mathematics 327 Sample Exam #1 - SOLUTIONS 3 October 2000

Problem 1. Suppose a is rational and b is irrational. Prove that $a+b$ and ab are both irrational.

Proof: $(a+b)$ Suppose $a \in \mathbb{Q}$ and $b \in \mathbb{R} \backslash \mathbb{Q}$. Further suppose $a = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$. We proceed by contradiction. Suppose that $a+b \in \mathbb{Q}$. By definition that means that $a+b = \frac{n}{m}$ for some integers $n, m \in \mathbb{Z}, m \neq 0$. But then we have

$$b = \frac{n}{m} - \frac{p}{q} = \frac{nq - mp}{mq}.$$

Since both $nq - mp, mq \in \mathbb{Z}, mq \neq 0$ this means that b is rational. This contradict the hypothesis. ∎

Proof: (ab) Note that this second statement **is not true as stated**. For example, $a = 0$, $b = \sqrt{2}$, and $ab = 0$ shows that it is false because $0 \in \mathbb{Q}, \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ but their product iz zero which is rational.. However, the statement is true if we assume that $a \neq 0$. Congratulations if you were able to notice that!

The proof is almost identical to the one for the sum. Suppose $a \in \mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$. Further suppose $a = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$. Since $a \neq 0$ this means that $p \neq 0$ as well. We proceed by contradiction. Suppose that $ab \in \mathbb{Q}$. By definition, that means that $ab = \frac{n}{m}$ for some integers $n, m \in \mathbb{Z}, m \neq 0$. But then we have

$$b = \frac{n}{m} \cdot \frac{q}{p} = \frac{nq}{mp}.$$

Since both $nq, mp \in \mathbb{Z}, mp \neq 0$ this means that b is rational. This contradict the hypothesis. ∎

Problem 2. Suppose $A \cap B = A$. Determine $A \cup B = ?$

Note that, by definition of the intersection $A \cap B$, for any two sets we have

$$A \cap B \subseteq A, \qquad A \cap B \subseteq B.$$

Since $A \cap B = A$ the second inclusion becomes $A \subseteq B$. But if A is a subset of B then $A \cup B = B$.

Problem 3. Determine whether or not each of the binary relations $\mathcal{R}$ is reflexive symmetric, antisymmetric, or transitive:

a) $A = \{1, 2\}$, $\mathcal{R} = \{(1, 2)\}$.

not reflexive (does not contain $(1,1)$), not symmetric (contains $(1,2)$ but not $(2,1)$), antisymmetric and transitive (any 1-element relation is always antisymmetric and transitive)

b) $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1,1), (1,2), (2,1), (3,4)\}$.

not reflexive (does not contain $(2,2)$), not symmetric (does not contain $(4,3)$ while it does contain $(3,4)$), not antisymmetric(contains the two pairs $(1,2), (2,1)$ whith $2 \neq 1$), transitive

c) $A = \mathbb{Z}$, $(a, b) \in \mathcal{R}$ if and only if $ab \geq 0$.

reflexive (because $a^2 \geq 0$ for all $a \in \mathbb{R}$), symmetric (because $ab \geq 0$ if and only if $ba \geq 0$), not antisymmetric (because $(1, 2)$ and $(2, 1)$ are both in $\mathcal{R}$), not transitive (becasue if $(-1) \cdot 0 \geq 0$ and $0 \cdot 1 \geq 0$ but $(-1) \cdot 1$ is negative).

d) $A = \mathbb{R}$, $(a, b) \in \mathcal{R}$ if and only if $a^2 = b^2$.

reflexive because $((a, a) \in \mathcal{R}$ $(a^2 = a^2)$ for all $a \in \mathbb{R})$, symmetric (because $(a, b) \in \mathcal{R}$ is true if and only if $(b, a) \in \mathcal{R}$ ($a^2 = b^2$ is the same as $b^2 = a^2$)), transitive (because $a^2 = b^2$ and $b^2 = c^2$ implies that $a^2 = c^2$). This is an equivalence relation. The equivalence class of 0 is $[0] = \{0\}$ and equivalence class of every other $a \in \mathbb{R}$ is $[a] = \{-a, +a\}$. The quotient set can be identified with the interval $[0, \infty)$.

Problem 4. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For $a, b \in A$ define $a \sim b$ if ab is a perfect square.

a) List ordered pairs of this relation.

Each elemend of A is in relation with itself (relation is reflexive) so we have all the diagonal elements $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)$ in " $\sim$ " $\subseteq A \times A$. Since $1 \sim 4 \sim 9$, and $2 \sim 8$ we have $(1, 4), (4, 1), (1, 9), (9, 1), (4, 9), (9, 4)$ and $(2, 8)$, $(8, 2)$ in as well. By inspection, it is easy to see that this is all.

b) For each $a \in A$ find the $[a] \equiv \bar{a} = \{x \in A \mid x \sim a\}$

From part (a) we get $[1] = [4] = [9] = \{1, 4, 9\}$ and $[2] = [8] = \{2, 8\}$, with $[3] = \{3\}$, $[5] = \{5\}$, $[6] = \{6\}$, $[7] = \{7\}$, for all the others.

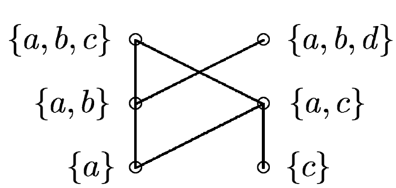
c) Explain why $\sim$ is an equivalence relation on A.

It is reflexive as a^2 is a perfect square. It is symmetric as ab being a perfect square is equivalent to ba being also one. It is also transitive which is obvious by inspection.

Note: This relation is an equivalence realtion not just on A but on $\mathbb{N}$ as well. If $a \cdot b = k^2$ and $b \cdot c = l^2$ then, in fact $a \cdot c = \frac{k^2 l^2}{b^2} = (kl/b)^2$. One only has to show that b divides the product kl. We will prove it in class when we discuss factorization of integers.

Problem 5. Draw the Hesse diagram for the following partial order:

$$(\{\{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}, \subseteq).$$



Problem 6. Let $f : A \to A$ be defined by $f(x) = x^2 + 2$.

a) Let $A = \mathbb{Z}$. Determine if f is injective and surjective.

$f(-1) = f(1) = 3$ so f is not injective. $f(x) = 4$ has no solutions for $x \in \mathbb{Z}$ so that f is not surjective.

b) Repeat part (a) for $A = \mathbb{R}$.

$f(-1) = f(1) = 3$ so f is not injective. $f(x) = 0$ has no solutions for $x \in \mathbb{Z}$ so that f is not surjective.

Problem 7. Let $X = \{a, b\}$ and $Y = \{1, 2, 3\}$.

a) List all the functions from X to Y.

$$f_1 = \{(a,1),(b,1)\}, \quad f_2 = \{(a,1),(b,2)\}, \quad f_3 = \{(a,1),(b,3)\},$$
$$f_4 = \{(a,2),(b,1)\}, \quad f_5 = \{(a,2),(b,2)\}, \quad f_6 = \{(a,2),(b,3)\},$$
$$f_7 = \{(a,3),(b,1)\}, \quad f_8 = \{(a,3),(b,2)\}, \quad f_9 = \{(a,3),(b,3)\}.$$

b) List all the functions from Y to X.

$$g_1 = \{(1,a),(2,a),(3,a)\}, \quad g_2 = \{(1,a),(2,a),(3,b)\},$$
$$g_3 = \{(1,a),(2,b),(3,a)\}, \quad g_4 = \{(1,a),(2,b),(3,b)\},$$
$$g_5 = \{(1,b),(2,a),(3,a)\}, \quad g_6 = \{(1,b),(2,a),(3,b)\},$$
$$g_7 = \{(1,b),(2,b),(3,a)\}, \quad g_8 = \{(1,b),(2,b),(3,b)\}.$$

c) List all the injective functions from X to Y.

$f_2, f_3, f_4, f_6, f_7, f_8$ are all injective while f_1, f_5, f_9 are not.

d) List all the surjective functions from X to Y.

There are none.

Problem 8. Let $S = \{1, 2, 3, 4, 5\}$, and let $f, g, h : S \to S$ be the function defined by

$$f = \{(1,2),(2,1),(3,4),(4,5),(5,3)\},$$
$$g = \{(1,3),(2,5),(3,1),(4,2),(5,4)\},$$
$$h = \{(1,2),(2,2),(3,4),(4,3),(5,1)\}.$$

a) Find $f \circ g$ and $g \circ f$.

$$f \circ g = \{(1,4),(2,3),(3,2),(4,1),(5,5)\},$$
$$g \circ f = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}.$$

b) Find f^{-1}, g^{-1}, and h^{-1} (if they exist).

$$f^{-1} = \{(2,1),(1,2),(4,3),(5,4),(3,5)\},$$
$$g^{-1} = \{(3,1),(5,2),(1,3),(2,4),(4,5)\},$$

h^{-1} does not exist as h is not injective ($h(1) = h(2) = 2$).

c) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$.

$$(f \circ g)^{-1} = \{(4,1),(3,2),(2,3),(1,4),(5,5)\},$$

$$g^{-1} \circ f^{-1} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\},$$

$$f^{-1} \circ g^{-1} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}.$$

Problem 9.

a) Find the one-to-one correspondence between the intervals $(1,\infty)$ and $(3,\infty)$

Take $f(x) = x + 2$. This is injective and surjective map $f : (1,\infty) \to (3,\infty)$ as desired.

b) Find the one-to-one correspondence between the intervals $(0,1)$ and (a,b).

We will construct a liner function $f(x) = \alpha x + \beta$ that does the job. How should we choose the constants? Well, $f(0) = a = \beta$ and $f(1) = b = \alpha + \beta$. Hence, $\alpha = b - a$. So take $f(x) = (b-a)x + a$. This is a liner function with non-zero slope. Any such function is injective. But $f\big((0,1)\big) = (a,b)$ by our choice of the constants so it must also be surjective.

Problem 10.

a) Write the number 1001 in base $b = 2, 8$.

Since $1001 = 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3 + 1 = 512 + 256 + 128 + 64 + 32 + 8 + 1$ we get $1001 = (1111101001)_2$. Since $1001 = 8^3 + 7 \cdot 8^2 + 5 \cdot 8 + 1$ we get $1001 = (1751)_8$.

b) Suppose in base 12 we use A to denote 10 and B to denote 11. What is $1BBA$?

$(1BBA)_{12} = 10 + 11 \cdot (12) + 11 \cdot (12)^2 + (12)^3 = 3,454.$