# MATH327 – HOMEWORK SOLUTIONS

## HOMEWORK #3

Section 1.3: Problems 1, 3, 4, 5
Section 1.4: Problems 2, 3, 5, 6, 15, 16

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**Problem 1.3.1.**

The Cartesian product $S \times B$ is a set of all ordered pairs $(s, b)$, where $s$ is any student of the college and $b$ is any book in the library. As an example we can consider the following binary relation $\mathcal{R} \subset S \times B$

$$\mathcal{R} = \{(s, b) \in S \times B \mid s \text{ read } b\}.$$

**Problem 1.3.3.**

a) not reflexive, not symmetric, not transitive.

b) reflexive, symmetric, but not transitive (here there is a slight problems as the word *friend* is not well defined; other answers can be argued for).

c) not reflexive, not symmetric, but transitive.

d) reflexive, symmetric, and transitive.

e) not reflexive, not symmetric, not transitive.

**Problem 1.3.4.** $A = \{a, b, c, d\}$. We give 4 examples of binary relations $\mathcal{R} \subset A \times A$ such that each example has at least elements and is:

a) not reflexive and symmetric

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | | $x$ | $x$ | $x$ |
| $b$ | $x$ | | $x$ | $x$ |
| $c$ | $x$ | $x$ | | $x$ |
| $d$ | $x$ | $x$ | $x$ | |

b) not symmetric and not antisymmetric

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | | $x$ | $x$ | $x$ |
| $b$ | | | $x$ | $x$ |
| $c$ | $x$ | $x$ | | $x$ |
| $d$ | $x$ | $x$ | $x$ | |

c) not symmetric but antisymmetric

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | | $x$ | $x$ | $x$ |
| $b$ | | | $x$ | $x$ |
| $c$ | | | | $x$ |
| $d$ | | | | |

d) transitive

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | | $x$ | $x$ | $x$ |
| $b$ | | | $x$ | $x$ |
| $c$ | | | | $x$ |
| $d$ | | | | $x$ |

**Problem 1.3.5.** Let $A = \{1, 2, 3\}$. Below are some examples of binary relations with the required properties. They are not unique.

a) $\{(1, 2), (2, 3), (1, 1)\}$.
b) $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$.
c) $\{1, 2), (2, 1)\}$.
d) $\{1, 2), (2, 3), (1, 3)\}$.
e) $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$.
f) $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$.
g) $\{(1, 1), (1, 2), (2, 1)\}$.
h) $\{(1, 1), (2, 2), (3, 3)\}$.

**Problem 1.4.2.**

a) $\mathcal{R}$ is clearly not reflexive as $(2, 2) \notin \mathcal{R}$. It is not transitive either.
b) $\mathcal{R}$ is not symmetric as $(2, 3) \in \mathcal{R}$ but $(3, 2) \notin \mathcal{R}$.
c) $\mathcal{R}$ is not symmetric as $(1, 3) \in \mathcal{R}$ but $(3, 1) \notin \mathcal{R}$.

**Problem 1.4.3.** This relations is the equality of elements

$$\text{“} = \text{”} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}.$$

**Problem 1.4.5.** Let $A = \{\sqrt{3}, -2, 1/2, \pi, 6, \sqrt{12}\}$.

a) define a relation “ $\sim$ ” by $a \sim b$ if $a/b \in \mathbb{Q}$. We show that this is an equivalence relations:
   1. It is reflexive: For any $a \in A$ we have $a/a = 1$ and $1 \in Q$.
   2. It is symmetric: If $a/b$ is rational then so is $b/a$. (Note that $A$ does not contain 0).
   3. It is transitive: If $a/b \in \mathbb{Q}$ and $b/c \in \mathbb{Q}$ than the product $\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c} \in \mathbb{Q}$.
b) The equivalence classes here are the subsets

$$\{-2, 1/2, 6\}, \quad \{\sqrt{3}, \sqrt{12}\}, \quad \{\pi\}.$$

Actually, the fact that $\sqrt{3}$ and $\pi$ are not equivalent, or that $\frac{\sqrt{3}}{\pi}$ is not rational is not trivial. It follows from the fact that $\pi$ is not an algebraic number (it is not a root of any polynomial with integer coefficients).

**Problem 1.4.6.** Let $A = \mathbb{N}$. Define

$$a \sim b \quad if \quad a^2 + b \text{ is even.}$$

1. It is reflexive: For each $a \in \mathbb{N}$ we have $a^2 + a$ even.

2. It is symmetric: Suppose $a \sim b$. Then $a^2 + b$ is even. But that means that either both $a$ and $b$ are odd, or both $a$ and $b$ are even. In each case $b^2 + a$ must be even and, hence, $b \sim a$.

3. It is transitive: If $a \sim b$ and $b \sim c$ then $a^2 + b$ is even and $b^2 + c$ is even. But then $a^2 + c = (a^2 + b) + (b^2 + c) - (b^2 + b)$ is a sum of three even numbers and hence even. The parity of the first two follows from the hypothesis and the third is even by reflexivity of the relation "$\sim$".

Thus we have an equivalence relation. The equivalence classes are

$$[1] = \{1, 3, 5, 7, \ldots\}, \quad [2] = \{2, 4, 6, 8, \ldots\}$$

the odd numbers (all equivalent to 1) and the even numbers (all equivalent to 2).

**Problem 1.4.15.**

a) It is an equivalence relation.

1. It is reflexive: For any $(a, b) \in \mathbb{R}^2$ we have $a+2b = a+2b$ and, hence $(a, b) \sim (a, b)$.

2. It is symmetric: if $(a, b) \sim (c, d)$ then $a + 2b = c + 2d$. But then $c + 2d = a + 2b$ which says that $(c, d) \sim (a, b)$.

3. It is transitive: if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then $a + 2b = c + 2d$ and $c + 2d = e + 2f$ which implies that $a + 2b = e + 2f$, hence, $(a, b) \sim (e, f)$.

Here, the equivalence classes can be labeled by a real number $\lambda$ and an equivalence class corresponding to $\lambda$ consists of all point on the line $x + 2y = \lambda$ in the $XY$-plane. We can also denote it by $[(-\lambda, -2\lambda)$.

b) It is an equivalence relation and the proof all all three properties is identical to the one in (a). Here, the equivalence classes can be labeled by a real number $\lambda$ and an equivalence class corresponding to $\lambda$ consists of all point on the hyperbola $xy = \lambda$ in the $XY$-plane. We can also denote it by $[(\lambda, 1)]$. The equivalence class $[0, 1)$ consists of the two axes.

c) This is not not equivalence relation as it is not reflexive. For example, it is not It is not true that for any $(a, b)$ we have $(a, b) \sim (a, b)$, that is $a^2 + b = a + b^2$. Take $(1, 2)$, for instance.

d) It is an equivalence relation.

1. It is reflexive: for any $(a,b) \in \mathbb{R}^2$ we have $a = a$ and, hence $(a,b) \sim (a,b)$.

2. It is symmetric: if $(a,b) \sim (c,d)$ then $a = c$. But then $c = a$ which says that $(c,d) \sim (a,b)$.

3. It is transitive: if $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ then $a = c$ and $c = e$ which implies that $a = e$, hence, $(a,b) \sim (e,f)$.

Here, the equivalence classes can be labeled by a real number $\lambda$ and an equivalence class corresponding to $\lambda$ consists of all point on the line $x = \lambda$ in the $XY$-plane (lines parallel to the $y$-axis). We can also denote it by $[(\lambda, 0)]$

e) This is not and equivalence relation as it is not reflexive. For example, it is not true that for any $(a,b) \in \mathbb{R}^2$ we have $ab = a^2$. Take $(1,2)$, for instance.

**Problem 1.4.16.** For any $a, b \in \mathbb{R}$, define $a \simeq b$ if $a - b \in \mathbb{Z}$.

a) This is an equivalence relation because:

1. It is reflexive. For any real number $a$ the difference $a - a = 0 \in \mathbb{Z}$.

2. It is symmetric. If for some $a, b$ the difference $a - b$ is an integer then so is $b - a = -(a - b)$.

3. It is transitive. Suppose $a - b$ is an integer and $b - c$ is an integer. Then $a - c = a - b + b - c$ must also be an integer.

b) The equivalence class of 5 are all integers $[5] = \mathbb{Z}$. The equivalence class of $5\frac{1}{2}$ are all "half-integers", that is all real numbers obtained from integers by adding 1/2.

c) The quotient set is the half-closed interval $[0,1)$. For any $\lambda \in [0,1)$ define

$$[\lambda] = [0 + \lambda] = \{x \in \mathbb{R} \mid x = n + \lambda, \quad n \in \mathbb{Z}\}.$$