MATH327 – HOMEWORK SOLUTIONS

HOMEWORK #1

Section 0: Problems 1,4,5,6,7,8,10,13

Krzysztof Galicki

Problem 0.1.

a) True, because $4 = 2 + 2$ is true and $7 < \sqrt{50}$ is true.
b) False, because $4 \neq 2 + 2$ is false.
c) True, because $4 = 2 + 2$ is true and $7 < \sqrt{50}$ is true.
d) True, because $4 \neq 2 + 2$ is false (the hypothesis).
e) False, because the hypothesis $4 = 2 + 2$ is true but the conclusion $7 > \sqrt{50}$ is false.
f) False, because the area of the circle is not $2\pi r$ and its circumference is not πr^2.
g) False, because the hypothesis $2 + 3 = 5$ is true but the conclusion $5 + 6 = 10$ is false.

Problem 0.4.

a) **Converse**: If a/c is an integer then a/b and b/c are integers. **Contrapositive:** If a/c is not an integer then either a/b or b/c is not an integer.
b) **Converse**: If $x \pm 1$ then $x^2 = 1$. **Contrapositive:** If $x \neq \pm 1$ then $x^2 \neq 1$.
c) **Converse**: If graph G is connected than G is Euclidean. **Contrapositive:** If graph G is not connected then it is not Euclidean.
d) **Converse**: If $a = 0$ or $b = 0$ then $ab = 0$. **Contrapositive:** If $a \neq 0$ and $b \neq 0$ then $ab \neq 0$.
e) **Converse**: If A is a four-sided figure then A is a square. **Contrapositive:** If A is not a four-sided figure than A is not a square.
f) **Converse**: If $a^2 = b^2 + c^2$ then ΔBAC is a right triangle. **Contrapositive:** If $a^2 \neq b^2 + c^2$ then ΔBAC is not a right triangle.

Problem 0.5.

a) There exists a continuous function which is not differentiable.
b) For all $x \in \mathbb{R}$, $2^x \geq 0$.
c) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y > x$.
d) For each prime number p, there exists a prime number q such that $q > p$.
e) For each positive $x \in \mathbb{R}$, x is a product of primes.
f) For each positive $x \in \mathbb{R}$, $\sqrt{x}$ is real.

Problem 0.6.

THEOREM A: For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y > x$.

Proof: Let $x \in \mathbb{R}$ be arbitrary. Consider $y = x + 1$. Then $y > x$. ∎

THEOREM B: There exists $y \in \mathbb{R}$ such that $y > x$ for every $x \in \mathbb{R}$.

Proof: (This statement is false. We will disprove it assuming that it is true via the "proof by contradiction".) Suppose THEOREM B is true. Let $y_{\max}$ be such a number. Set $x = y_{\max} + 100$. Then $x > y_{\max}$. The contradiction shows that the statement is false. ∎

Problem 0.7.

THEOREM A: If $a \in \mathbb{Z}$ then either a or $a+1$ is even.

Proof:

Case 1: Suppose a is even. Then the statement is true.

Case 2: Suppose a is odd. Then $a = 2k+1$ for some integer k. But then $a+1 = 2k+2 = 2(k+1)$ and, hence, $a+1$ is even. Again, the statement is true in this case. ∎

THEOREM B: If $n \in \mathbb{Z}$ then $n^2 + n$ is even.

Proof:

Case 1: Let $n = 2k$ be even. Then $n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$ is even.

Case 2: Let $n = 2k+1$ be odd. Then $n^2+n = (2k+1)^2+(2k+1) = 4k^2+4k+1+2k+1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$ is even. ∎

Problem 0.8.

THEOREM: If $n \in \mathbb{Z}$ then $n^2 - n + 5$ is odd.

Proof:

Case 1: Let $n = 2k$ be even. Then $n^2 - n + 5 = 4k^2 - 2k + 5 = 2(2k^2 - k + 2) + 1$ is odd.

Case 2: Let $n = 2k+1$ be odd. Then $n^2-n+5 = (2k+1)^2-(2k+1)+5 = 4k^2+2k+5 = 2(2k^2 + k + 2) + 1$ is odd. ∎

Problem 0.10.

THEOREM: $a^2 - b^2$ is odd if and only if a and b have opposite parity.

Proof:

Case 1: If $a = 2k$ and $b = 2l$ then $a^2 - b^2 = (2k)^2 - (2l)^2 = 2(2k^2 - 2l^2)$ is even.

Case 2: If $a = 2k+1$ and $b = 2l+1$ then $a^2-b^2 = (2k+1)^2-(2l+1)^2 = 2(2k^2+2k-2l^2-2l)$ is even.

Case 3: If $a = 2k+1$ and $b = 2l$ then $a^2 - b^2 = (2k+1)^2 - (2l)^2 = 2(2k^2 + 2k - 2l^2) + 1$ is odd.

Case 4: If $a = 2k$ and $b = 2l+1$ then $a^2 - b^2 = (2k)^2 - (2l+1)^2 = 2(2k^2 - 2l^2 - 2l - 1) + 1$ is odd. ∎

Problem 0.13.

THEOREM: There exists no smallest positive real number.

Proof: Suppose such a number does exist and let us denote it by $x_{\min}$. Consider $y = \frac{1}{2}x_{\min}$. Then $y < x$ and y is positive. But this contradicts the fact that $x_{\min}$ was the smallest real number. ∎