Mathematics 327 **Exam #1 – SOLUTIONS** 9 October 2000

Problem 1. Let A, B, C be sets. Show that, in general, $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$.

Solution 1: Observe here that if $B = \emptyset$, $(A \setminus B) \setminus C = A \setminus C$ and $A \setminus (B \setminus C) = A$. So, any example with $B = \emptyset$ and C not disjoint with A will do: say $A = \{1, 2\}, B = \emptyset, C = \{1\}$, for instance.

Solution 2: We could also look for an example by setting $B = C$. Then $(A \setminus B) \setminus C = A \setminus B$ while $A \setminus (B \setminus C) = A$. If B is not disjoint with A we will always get an example. Take $A = \{1, 2\}, B = C = \{1\}$.

Problem 2. Determine whether or not each of the binary relations $\mathcal{R}$ is reflexive, symmetric, antisymmetric, or transitive:

a) $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (3, 4), (4, 3)\}$.

reflexive: NO, $(2, 2) \notin \mathcal{R}$.

symmetric: YES, for each $(a, b) \in \mathcal{R}$ also $(b, a) \in \mathcal{R}$.

antisymmetric: NO, $(2, 1) \in \mathcal{R}$ and $(1, 2) \in \mathcal{R}$ but $1 \neq 2$.

transitive: NO, $(2, 1) \in \mathcal{R}$ and $(1, 2) \in \mathcal{R}$ but $(2, 2) \notin \mathcal{R}$.

b) $A = \mathbb{R}$, $(a, b) \in \mathcal{R}$ if and only if $a - b \leq 3$.

reflexive: YES, $a - a = 0 \leq 3$ so $(a, a) \in \mathcal{R}$ for all a.

symmetric: NO, $(-10, -1) \in \mathcal{R}$ but $(-1, -10) \notin \mathcal{R}$.

antisymmetric: NO: if $a - b \leq 3$ and $b - a \leq 3$ that implies that $-3 \leq a - b \leq 3$. But this does not mean that $a = b$. For example, $(2, 1)$ and $(1, 2)$ are both in $\mathcal{R}$ but $1 \neq 2$.

transitive: NO, clearly, $(4, 1) \in \mathcal{R}$ and $(1, 0) \in \mathcal{R}$ but $(4, 0) \notin \mathcal{R}$.

c) $A = \mathbb{Z}$, $(a, b) \in \mathcal{R}$ if and only if $a + b = 10$.

reflexive: NO, $(6, 6) \notin \mathcal{R}$, for example.

symmetric: YES, if $(a, b) \in \mathcal{R}$ so is (b, a).

antisymmetric: NO, for example both $(4, 6)$ and $(6, 4)$ are in $\mathcal{R}$.

transitive: NO, for example $(11, -1) \in \mathcal{R}$ and $(-1, 11) \in \mathcal{R}$ but $(11, 11) \notin \mathcal{R}$.

d) $A = \mathbb{N}$, $(a, b) \in \mathcal{R}$ if and only if $\frac{a}{b} \in \mathbb{N}$.

reflexive: YES, $a/a = 1$ for all natural numbers a.

symmetric: NO, $(2, 1) \in \mathcal{R}$ but $(1, 2) \notin \mathcal{R}$.

antisymmetric: YES, if a/b is natural and b/a is natural that means that $a = b$.

transitive: YES, if a/b is natural and b/c is natural so is their product, which is simply a/c.

Problem 3. Let $S = \{1, 2, 3, 4, 5\}$, and let $f, g, h : S \to S$ be the function defined by

$$f = \{(1, 2), (2, 1), (3, 3), (4, 5), (5, 4)\},$$

$$g = \{(1, 5), (2, 3), (3, 1), (4, 2), (5, 4)\},$$

$$h = \{(1, 2), (2, 2), (3, 2), (4, 3), (5, 1)\}.$$

a) $f \circ g = \{(1, 4), (2, 3), (3, 2), (4, 1), (5, 5)\}$, $g \circ f = \{(1, 3), (2, 5), (3, 1), (4, 4), (5, 2)\}$.

b) $f^{-1} = \{(1, 2), (2, 1), (3, 3), (4, 5), (5, 4)\}$, $g^{-1} = \{(1, 3), (2, 4), (3, 2), (4, 5), (5, 1)\}$, h^{-1} does not exist as h is not bijective.

c) $f^2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\} = \text{id}$, $f^3 = f$, $f^4 = \text{id}$. What is $f^{2k} = \text{id}$, and $f^{2k+1} = f$.

Problem 4. In each case determine if the function is injective (1-1) and/or surjective (onto):

a) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = 3n$: injective but not surjective because $f(n) = 3n = 1$ has no solutions in natural numbers.

b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x$: injective and surjective.

c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x$: surjective but not injective as $f(1, 0) = f(1, 1) = 1$.

d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4 - x^2$: not injective as $f(0) = f(1) = 0$. Not surjective as $x^4 - x^2 = x^2(x^2 - 1) > -1$ and hence $f(x) = -1$ has no solutions. To see that take $x^4 - x^2 = -1$. This means that $(x^2 - 1)^2 + x^2 = 0$. But this is impossible.

Problem 5.

a) Write the number 10001 in base $b = 2$: $10001 = (10011100010001)_2$.

b) Let $x = (10001)_3$ and $y = (111)_3$. Compute the sum $x + y$ and the product $x \cdot y$ in base $b = 3$: As $x = 3^4 + 1 = 82$ and $y = 3^2 + 3 + 1 = 13$. We have

$$x + y = 95 = 3^4 + 3^2 + 3 + 2 = (10112)_3,$$

$$xy = 1066 = 3^6 + 3^5 + 3^4 + 3^2 + 3 + 1 = (1110111)_3.$$

BONUS PROBLEM. Suppose $m, n \in \mathbb{Z}$ and $n^2 + 1 = 2m$. Prove that m is a sum of two squares (i.e., $m = p^2 + q^2$ where p, q are some integers).

Proof: Note that n has to be odd for $n^2 + 1$ to be even. Take $n = 2k + 1$. We get $(2k + 1)^2 + 1 = 2m$ which means that $4k^2 + 4k + 2 = 2m$. let us divide by 2:

$$m = 2k^2 + 2k + 1 = k^2 + (k^2 + 2k + 1) = k^2 + (k + 1)^2 \blacksquare$$